Design of all-optical high-order temporal integrators based on multiple-phase-shifted Bragg gratings

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Abstract: In exact analogy with their electronic counterparts, photonic temporal integrators are fundamental building blocks for constructing alloptical circuits for ultrafast information processing and computing. In this work, we introduce a simple and general approach for realizing all-optical arbitrary-order temporal integrators. We demonstrate that the Nth cumulative time integral of the complex field envelope of an input optical waveform can be obtained by simply propagating this waveform through a single uniform fiber/waveguide Bragg grating (BG) incorporating N π -phase shifts along its axial profile. We derive here the design specifications of photonic integrators based on multiple-phase-shifted BGs. We show that the phase shifts in the BG structure can be arbitrarily located along the grating length provided that each uniform grating section (sections separated by the phase shifts) is sufficiently long so that its associated peak reflectivity reaches nearly 100%. The resulting designs are demonstrated by numerical simulations assuming all-fiber implementations. Our simulations show that the proposed approach can provide optical operation bandwidths in the tensof-GHz regime using readily feasible photo-induced fiber BG structures.

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1. Introduction

The cumulative time integral of a given waveform is a fundamental signal processing functionality with a wide range of potential applications, e.g. in communications, information processing and computing. Photonics temporal integrators, capable of processing optical signals, can provide operation bandwidths well beyond the reach of present electronic technologies typically limited to operation bandwidth less than 1 GHz. Very recently, a photonic temporal integrator has been experimentally demonstrated for the first time [1]. Some alternative designs for performing temporal integration of arbitrary signals in the all-optical domain have been also proposed [2]-[5]. This includes solutions based on Bragg gratings (BGs), which are especially attractive due to their simplicity, potential for low cost and full compatibility with fiber optics and/or integrated waveguide systems. As a particularly interesting solution, it has been shown that first-order time integration of optical waveforms can be achieved using a strong-coupling π -phase-shifted BG (uniform BG with a single π -phase shift in the middle of the grating) operating in transmission [3].

All the previously proposed solutions for all-optical integration can provide only the first cumulative time integral of the input signal (i.e. *first-order integration*) [1]-[4]. However, higher-order integrators, capable of providing the successive cumulative time integrals of the input signal, are also key building blocks in a large number of signal processing circuits. A very relevant example is that of computing systems devoted to solving ordinary differential equations (ODEs) [7]. It is well known that linear ODEs can be solved in real time using a suitable combination of integrators -first and higher-order integrators, in general-, adders and multipliers. The possibility of realizing these computations all-optically translates into potential processing speeds well beyond the reach of present electronic digital computers. An N^{th} -order integrator (N = 1, 2, 3 ... identifies the integration order) could be realized by concatenating in series N single first-order integrators [5]. A particularly attractive solution to implement this extension is that based on a π -phase-shifted BG [3] since in this approach, the BGs are operated in transmission. While in principle it would be relatively straightforward to concatenate N individual π -phase-shifted BGs, this solution would lead in general to an erratic device performance due to Fabry-Perot (FP) effects in the resonance cavities formed between the concatenated BG devices. These detrimental FP effects are particularly pronounced due to the required high reflectivity of the uniform gratings in the concatenated π -phase-shifted BGs [3], which effectively behaves like bandwidth-limited strong-reflection mirrors. These difficulties could be avoided by use of optical isolators between the concatenated BG integrators; however, this would translate into an increased complexity, longer devices, and a higher cost.

In this manuscript, we demonstrate that an Nth-order temporal integrator can be realized using a design based on a single-unit uniform BG incorporating N π -phase-shifts along its grating profile, i.e. multiple-phase-shifted BG (MPS-BG). We investigate for the first time the general design rules of this proposed design for arbitrary-order temporal integration. Our research on this issue has resulted in some interesting, unexpected findings. First, in contrast to the original design proposed in [3], we show that the required π phase-shift in a first-order integrator does not have to be necessarily located at the middle of the uniform grating; specifically, we show that the required π phase-shift may be located at any *arbitrary position* along the grating profile provided that each uniform grating section in the BG structure (sections separated by the phase shift) is sufficiently long so that the peak reflectivity associated with each of these uniform sections reaches nearly 100%. As a generalization of this idea, we demonstrate that an Nth-order photonic integrator can be realized using a single-

unit uniform BG incorporating N π phase shifts *arbitrarily located* along its length; again, the only important requirement in this design is that the peak reflectivity of each uniform grating section (in between successive phase shifts) must reach nearly 100%. Thus, the resulting designs offer an outstanding degree of freedom (concerning the location of the phase shifts along the grating profile); this should be contrasted with designs for first and higher-order photonic temporal differentiation based on MPS-BGs [8], [9], in which the required phase shifts must be located at very precise positions along the BG profiles. The method proposed here is demonstrated by numerically testing in-fiber MPS-BG designs for up to third-order photonic temporal integration. The resulting grating profiles are remarkably simple and could be readily realized with present technologies (e.g. using photoinduced phase-shifted fiber BGs [8], [10]).

2. MPS-BG arbitrary-order integrators: theory and design specifications

In exact analogy with its electronic counterpart [7], an Nth-order photonic temporal integrator is a device that performs the Nth cumulative time integral of the complex temporal envelope of an input arbitrary optical waveform. The spectrum associated with the Nth time cumulative integral of the temporal envelope of a given optical signal centered at frequency ω_0 (optical carrier frequency) $E(\omega-\omega_0)$ (represented in the Fourier domain) is given by $E(\omega-\omega_0)/[j(\omega-\omega_0)]^N$, where ω is the optical frequency variable, and $(\omega-\omega_0)$ is the base-band frequency variable. Thus, a Nth-order photonic temporal integrator is essentially a linear optical filtering device providing a spectral transfer function of the form $H(\omega-\omega_0) =$ $1/[j(\omega-\omega_0)]^N$ [2], [3]. An important feature of this spectral filtering function is that an exact π phase shift across the central frequency ω_0 is necessary for odd-order integrators (i.e. for N = 1, 3, 5...) (notice that the spectral transfer function of an even order integrator should exhibit no phase shift across the central frequency ω_0).



Fig. 1. Schematic of the proposed BG - based designs for first-order, second-order and Nth-order photonic temporal integrators. The vertical red lines indicate π phase shifts between the uniform grating sections.

We anticipate that the required spectral features of an Nth-order time integrator can be provided by the transmission response of a uniform BG with multiple (N) π -phase shifts along its grating profile, as illustrated in Fig. 1. The desired spectral response is achieved over a limited bandwidth centered at the grating Bragg frequency (ω_0). In what follows, we will first

derive the design specifications of a first-order integrator and we will then generalize the obtained results for the design of higher-order integrators. The obtained general design specifications will be subsequently proved through the theoretical analysis of the transmission spectral response of a uniform BG incorporating two π -phase shifts, according to the design shown in Fig. 1, for implementing a second-order integrator.

An arbitrary MPS-BG can be modeled using the transfer matrix method combined with coupled-mode theory [8]-[11]. Specifically, the transfer (2×2) matrix $T(z_0,L_i)$ relates the optical fields corresponding to the forward (transmission) $E_A(z_0)$ and backward (reflection) $E_B(z_0)$ propagating modes at the BG input end (z=z_0) with the fields corresponding to these same modes at the BG output end (z = z_0 + L_i), i.e. $E_A(z_0+L_i)$ and $E_B(z_0+L_i)$, where L_i is the grating length:

$$\begin{bmatrix} E_A (z_0 + L_i) \\ E_B (z_0 + L_i) \end{bmatrix} = T (z_0 + L_i) \begin{bmatrix} E_A (z_0) \\ E_B (z_0) \end{bmatrix}$$

$$= \begin{bmatrix} T_{11} (L_i) & T_{12} (z_0, L_i) \\ T_{21} (z_0, L_i) & T_{22} (L_i) \end{bmatrix} \begin{bmatrix} E_A (z_0) \\ E_B (z_0) \end{bmatrix}$$
(1)

Assuming a uniform BG, the elements of the corresponding transfer matrix can be obtained by solving the coupled mode equations [11], considering the following boundary conditions at the grating input and output ends: $E_A(z_0)=1$ and $E_B(z_0+L_i)=0$. The analytical expressions of the corresponding transfer matrix elements are as follows [8]:

$$T_{11} = T_{22}^{*} = \left[\cosh\left(\gamma L_{i}\right) + j\frac{\sigma}{\gamma}\sinh\left(\gamma L_{i}\right)\right]\exp\left[j\frac{2\pi n_{eff}L_{i}}{\lambda_{B}}\right],$$

$$T_{12} = T_{21}^{*} = -\frac{\kappa}{\gamma}\sinh\left(\gamma L_{i}\right)\exp\left[-j\frac{2\pi n_{eff}L_{i}}{\lambda_{B}}\right],$$
(2)

where $j=\sqrt{-1}$, $\kappa=\pi\Delta n_{eff}/\lambda_B$ is the coupling coefficient (defined as the grating coupling strength per unit length), Δn_{eff} is the amplitude of the refractive index modulation, $\lambda_B=2n_{eff}$ Λ is the Bragg resonance wavelength characteristic of the considered uniform grating, n_{eff} is the effective refractive index, Λ is the grating period, $\sigma = \beta - \pi/\Lambda$ is the mismatch factor, $\beta \approx 2\pi n_{eff}/\lambda$ is the mode propagation constant, $\gamma = (\kappa^2 - \sigma^2)^{1/2}$, and λ is the optical wavelength. The symbol * denotes complex conjugation. It is also well known that the elements of the transfer matrix Φ corresponding to a phase shift ϕ in the grating perturbation are given by the following expressions [8]:

$$\Phi_{11} = \Phi_{22}^{*} = \exp(-j\phi/2),$$

$$\Phi_{12} = \Phi_{21} = 0$$
(3)

The total transfer matrix T_{Σ} of an arbitrary BG profile (e.g. MPS-BG) can be obtained by multiplying, in the appropriate order, the transfer matrices T_j corresponding to its compound uniform grating sections and the transfer matrices Φ corresponding to the discrete phase shifts along the grating profile. The complex field transmission coefficient, *H*, of a Bragg grating

structure (spectral transfer function of the BG operated in transmission) can be found from the elements of its total transfer matrix T_{Σ} using the following expression [8]:

$$H = \frac{1}{T_{22}^{\Sigma}} = \left| H \right| \exp\left(j\phi_{\tau} \right) \tag{4}$$

2.1. General design specifications of a first-order photonic integrator

A BG structure incorporating a single π -phase shift, operated in transmission, can provide the spectral response corresponding to a first-order integrator, $H(\omega-\omega_0)\approx 1/[j(\omega-\omega_0)]$, over a certain optical bandwidth around the uniform Bragg grating frequency (ω_0). Here we show that the specification imposed in [3] (phase shift located at the exact center of the BG structure) is not strictly necessary and we provide the generalized conditions for this operation. A single phase-shifted BG structure can be described by the following matrix product:

$$T_{\Sigma} = T\left(L_1, L_2\right) \Phi T\left(0, L_1\right) \tag{5}$$

where L_i (i = 1, 2) is the length of the ith uniform grating section in the BG (see Fig. 1). The elements of T_{Σ} in Eq. (5) can be derived from the results in Eq. (2) and Eq. (3). The frequency dependence of the transfer function has its origin in the frequency dependence of the mismatch factor, i.e. $\sigma = n_{eff}(\omega - \omega_0)/c$ where $\omega = 2\pi c/\lambda$, $\omega_0 = 2\pi c/\lambda_B$, and c is the speed of light in vacuum. Here we assume that (i) the mismatch factor σ is much smaller than the coupling coefficient, $|\omega - \omega_0| << c\kappa/n_{eff}$, and (ii) the length of each uniform grating section is an integer number of grating periods. Under these conditions, one can approximate $\gamma \approx \kappa$ and the transfer function of this BG device can be calculated from Eq. (4):

$$H(\omega) \approx \frac{A_0}{j\frac{n_{eff}}{ck} (|r_1| + |r_2|)(\omega - \omega_0) + \frac{n_{eff}^2}{c^2k^2} |r_1| |r_2|(\omega - \omega_0)^2 - |r_1| |r_2| \exp(-j\phi) - 1}$$
(6)

where $r_i=j \tanh(\kappa L_i)$ (i = 1, 2) is the reflection peak of each uniform BG section of length L_i , and A_0 is a constant. From Eq. (6), it can be easily inferred that in order to approach the spectral response that is required for first-order integration, the two following specifications are necessary: (i) $|r_1| \times |r_2| \approx 1$ (or $|r_1| \approx |r_2| \approx 1$), which implies that the two uniform grating sections should exhibit a very high peak reflectivity of nearly 100%; and (ii) $\phi=\pi$, which means that an exact π -phase shift between these two uniform sections is needed. Under these conditions and around the grating Bragg frequency, Eq. (6) can be approximated by $H(\omega) \approx A/[j(\omega-\omega_0)]$ (A is a new constant), which is the spectral transfer function of a first-order integrator. Notice that this approximation is valid over the limited spectral bandwidth (centered at the Bragg frequency ω_0) defined above: $|\omega-\omega_0| < \kappa/n_{eff}$; the reader can easily prove that under this bandwidth condition, the second-order frequency term, $(\omega-\omega_0)^2$, in the denominator of Eq. (6) can be neglected as compared with the first-order frequency term.

Our analysis allows us to conclude that the required π -phase shift in this BG structure can be located at any *arbitrary position* along the grating length provided that each of the uniform grating sub-sections is sufficiently long so that to ensure that its associated reflectivity satisfies the condition given above ($|r_1|\approx|r_2|\approx1$). We reiterate that this solution should be contrasted with the design in Ref. [3], where a *symmetric* π -phase shifted BG was assumed.

2.2. General design specifications of an Nth-order photonic integrator

An Nth-order integrator could be realized by concatenating in series N single first-order integration devices. Based on the above results, one can anticipate that an Nth-order integrator can be implemented using a high-reflectivity uniform BG incorporating N π -phase shifts *arbitrarily located* along the grating profile, see Fig. 1. The only needed condition is that the peak reflectivity of the N+1 uniform grating sub-sections should satisfy the condition derived above: $|r_1|\approx|r_2|\approx...\approx|r_{N+1}|\approx|1$. As discussed in Ref. [3] (and as numerically demonstrated below), a grating peak reflectivity >99.999% ensures the proper operation of the proposed integrator designs. This specification implies that each grating sub-section should be longer than $L_i \ge \tanh^{-1}(0.99999)/\kappa \approx 6/\kappa$, where we reiterate that κ is the grating coupling coefficient. Finally, as discussed above for a first-order integrator, the transmission spectrum of the MPS-BG structure approximates the transfer function of an Nth-order integrator only over a limited bandwidth around the grating Bragg frequency, i.e. $\Delta\omega << 2ck/n_{eff}$, where $\Delta\omega$ refers to the full-width operation bandwidth of the device. Hence, a broader operation bandwidth can be achieved by use of a higher coupling coefficient (which in turns would imply the need for shorter grating sections).

For completeness, the validity of the design specifications anticipated above for a general Nth-order photonic integrator are first theoretically proved for the specific case of a BG structure incorporating two π -phase shifts (see Fig. 1) to be used as a second-order integrator (N =2). We remind the reader that the spectral transfer function of a second-order integrator is $H(\omega-\omega_0)=1/[j(\omega-\omega_0)]^2$. The considered MPS-BG structure can be described by the following matrix product (see Fig. 1):

$$T_{\Sigma} = T(L_1 + L_2, L_3) \Phi T(L_1, L_2) \Phi T(0, L_1)$$
(7)

The elements of T_{Σ} in Eq. (7) can be derived from the results in Eq. (2) and Eq. (3) (for $\phi=\pi$). We assume the same conditions as for the first-order integrator ($\sigma << \kappa$ and the length L_i of each sub-section is an integer number of grating periods). It can be easily proved that under these conditions and considering the anticipated specification $|r_1|\approx|r_2|\approx|r_3|\approx1$, the transfer function of this device (in transmission), can be easily calculated from Eq. (4):

$$H(\omega) \approx \frac{B_0}{-3(\omega - \omega_0)^2 + j\frac{n_{eff}}{ck}(\omega - \omega_0)^3}$$
(8)

where B_0 is a new constant. Eq. (8) approximates the spectral response of a second-order integrator, $H(\omega)\approx B/[j(\omega-\omega_0)]^2$, over a certain limited bandwidth around the grating's Bragg frequency, $|\omega-\omega_0| \ll \kappa/n_{eff}$, (the third-order frequency term in the denominator of Eq. (8) can be neglected as compared with the second-order frequency term over this limited bandwidth). Since the integrator operation bandwidth depends on the index strength (grating coupling coefficient) but not on the grating section length, the use of high index strength gratings is suggested in order to achieve a large device operation bandwidth.

2.3. Physical interpretation of the obtained design specifications

The fact that the required π phase shifts can be arbitrarily located along the grating profile is a very interesting feature of the proposed BG integrators, which should also facilitate the practical realization of these devices. As mentioned in the Introduction, this result should be contrasted with designs for first and higher-order photonic temporal differentiation based on MPS-BGs [8], [9], in which the required phase shifts must be located at very precise positions along the BG axial profiles. However, this interesting result, which has been directly derived from the evaluation of the BG device equation, also admits a very simple physical interpretation. As discussed in detail in Refs. [1-3], [5,6], a first-order photonic temporal integrator can be implemented using a resonant cavity providing a round-trip field loss/gain of nearly 1, e.g. a passive resonant cavity with negligible losses and with mirrors providing field reflectivities approaching 100%. The π phase shifted BG design is just a particular implementation of this general solution. On the basis of the general specifications required for a resonant cavity-based optical integrator, it can be inferred that the precise position of the phase shift along the BG length is of no relevance as long as each of the surrounding uniform grating sections ("cavity mirrors") provide a sufficiently high reflectivity (nearly 100%). As a generalization of this basic idea and considering that the resonant cavity-based integrator (e.g. phase-shifted BG) needs to be operated in transmission, an Nth-order photonic integrator can be simply implemented by concatenating in series N resonant cavities, each designed according to the above given conditions. In our specific solution, this would translate into a BG device incorporating N successive π phase shifts, where again the location of these phase shifts is of no relevance as long as each of the uniform grating sections ("cavity mirrors") in between the phase shifts provide a reflectivity of nearly 100%.

3. Simulations results and discussions

In this Section we provide some numerical results which confirm the proposed design for Nthorder photonic temporal integration. The simulated BGs are assumed to be photo-induced in a single-mode fiber with an effective refractive index of 1.452. These fiber BG structures have been simulated using coupled-mode theory and transfer-matrix techniques [8]-[11].

We consider first the design of a second-order temporal integrator which should operate over an optical spectrum centered at a frequency (ω_0) of 193 THz; this requires a grating period in the BG structure of Λ =534.89 nm. A constant (uniform) coupling coefficient of κ =5300 m⁻¹ (from [3]) is assumed, which implies that the length of each grating section should be $L_{26}/5300=1.1$ mm. A coupling coefficient of k = 5300 1/m corresponds with a peak-to-peak refractive index modulation of 2.6×10^{-3} [11], which is a value that can be readily achieved in practice using conventional FBG fabrication techniques [8], [10]. To illustrate the degree of flexibility offered by this method, we assume that the BG device exhibits two exact π phase shifts, respectively located at 1.1mm and 2.5mm from the grating input end (i.e. the length of the first and the second grating sub-sections are $L_1 = 1.1$ mm and $L_2 = 1.4$ mm, respectively) and we also set the length of the third section to $L_3 = 1.8$ mm. Figure 2(a) (solid, red curve) shows the normalized amplitude of the transmission spectral response of the simulated BG structure; for comparison, the frequency response of an ideal second-order integrator is also shown in the same plot (dotted, blue curve). There is a very good agreement between the two curves over a full bandwidth (integrator operation bandwidth) of \approx 40GHz, over which the deviation between the simulated and ideal frequency responses keeps < 4%. As discussed above, the integrator operation bandwidth in this case should be limited to $\Delta f \ll k/\pi n_{eff} \approx 350$ -GHz, which agrees very well with the operation bandwidth estimated from our numerical simulations.

The inset in Fig. 2(a) shows the phase of the transmission spectral response of the simulated BG structure; as expected for an even-order temporal integrator, the spectral phase

response shows no phase shift (2π) at the integrator's resonance frequency. To examine the behavior of this device as a 2nd-order temporal integrator, we have simulated the device's temporal response to an input complex envelope with a shape given by the 2nd-order time derivative of an ideal Gaussian pulse with a full-width-at-half-maximum (FWHM) duration of 40 ps (FWHM bandwidth \approx 22-GHz). This input optical signal is assumed to be centered at the grating Bragg frequency defined above, 193 THz. The input pulse is assumed to propagate through the FBG (the device is used in transmission mode) and the output pulse shape is calculated as the inverse Fourier transform of the result of multiplying the input pulse spectrum by the frequency transfer function of the BG. The input pulse shape in the time domain is plotted in Fig. 3(a) with a dotted, black curve. The simulated output pulse shape (amplitude of the field complex envelope) is shown in the same plot with a solid, red curve. As predicted, there is an excellent agreement between the obtained temporal profile at the BG output and the ideal 2nd-order time cumulative integral of the considered input waveform (Gaussian pulse), which is also shown in the same plot using a dashed, blue curve. For this specific example, the integration error (relative average deviation between the simulated and ideal output pulses) was $\approx 0.83\%$ and the energetic efficiency (ratio between the output pulse energy and the input pulse energy) was $\approx 5.82 \times 10^{-4}$ %.

Since the proposed integration device works in high reflection regime and the device is used in transmission, the achieved energetic efficiency is typically low; if needed, optical amplifiers could be used to increase the system energetic efficiency. In principle, the energetic efficiency could be improved by reducing the grating reflectivity but this would also translate into a corresponding increase of the integration error (the integration error is increased as the reflectivity is decreased because the deviation from the ideal design condition, i.e. reflectivity = 100%, becomes more significant). To give a reference, for a second-order integrator with a reflectivity amplitude in each grating section of |r|≈99.999% (corresponding to k=5300 1/m and Li=1.1mm, i =1, 2, 3), the energetic efficiency is $\approx 5.82 \times 10^{-4}$ % and the integration error is 0.83% (assuming the same input pulse we used above for second-order integrator); our simulations show that (for the same input pulse) if the reflectivity amplitude is decreased to $|\mathbf{r}| \approx 99.99\%$ (corresponding to k=4500 1/m and Li=1.1mm, i = 1, 2, 3), then the energetic efficiency will increase to 0.61% but the integration error will be also increased to 3.43%. Thus, there is an important trade-off between energetic efficiency and integration error. This has been investigated in greater detail for π -phase-shifted first-order integrators in the original paper by Ngo [3].

Notice also that an integrator basically emphasizes the frequencies around the central frequency and filters out the rest. Thus the energetic efficiency is strongly dependent on the input signal spectral energy distribution. For signals in which the spectral energy density is mainly distributed outside the central frequency lobe (e.g. second-order Gaussian derivative used as the input signal in the example shown in the paper), the energetic efficiency of the process is particularly low. In contrast, a higher energetic efficiency would be achieved when processing signals in which the spectral energy density is more concentrated around the central frequency. To give some examples, assuming the same FBG design parameters as in our previous example for second-order integration, if we use the first-order derivative (instead of the second-order derivative) of the same Gaussian (40-ps FWHM) as the input pulse, in which a higher energy fraction is concentrated around the central part of the spectrum, the energetic efficiency would be significantly increased to 6.8×10^{-3} %. In the same line, if we use the second-order derivative of the Gaussian pulse as the input signal but assuming a longer pulse duration (FWHM of 60ps), which translates into a narrower input spectral distribution, i.e. a higher energy concentration towards the central pulse frequency, the energetic efficiency of the same integration process would be increased to 3.1×10^{-3} %. Since the reflection from the

grating depends on the product of length and coupling coefficient and the device needs to be operated in the high reflection regime (with a minimum required peak reflectivity from each grating section of approximately 99.999%), the grating lengths in any alternative given design (e.g. equi-spaced phase-shifted grating design or any alternative grating-length distribution) need to be fixed according to this condition and no fundamental difference in terms of device performance among the different possible alternative designs is expected (e.g. as discussed, in any given device configuration, increasing the gratings' lengths will always translate into an increased reflectivity, i.e. degraded energetic efficiency but improved integration error).



Fig. 2. Numerically simulated spectral amplitude responses of the tested BG-based designs for second-order (a) and third-order (b) integration compared with the spectral profiles of the ideal second and third-order photonic time integrators. The insets show the numerically simulated spectral phase responses of the tested BG-based designs for second-order (a) and third-order (b) integration.

To estimate the required tolerances for the grating phase shifts in the proposed designs, we have calculated how much change in the phase shifts can be tolerated around their nominal value of π in such a way that the integration error keeps below 5% (estimations made for the 2nd-order integrator example). The simulation shows that the integration error will be below 5% as long as the phase shift change around its nominal value (π) is less than \approx +/-0.2% (i.e. +/- 0.0063 rad). Finally, we also estimated how much deviation from the nominal Bragg frequency is acceptable for the input pulses' central (carrier) frequency in our proposed designs. As a reference, estimations were obtained for the reported second-order integration error keeps below 5%, then the deviation in the pulse's central frequency from the nominal Bragg frequency should be lower than \approx 0.5GHz (i.e. 1.25% relative frequency deviation frequency over operation bandwidth).

Figures 2(b) and 3(b) show the results corresponding to a 3rd-order integrator design. For this design, we have assumed the same grating period, Λ =534.89 nm, coupling coefficient, κ =5300 m⁻¹, and thus (from L_i≥6/ κ =1.1mm) basic grating lengths of L₁ = 1.4mm, L₂ = 1.8mm, L₃ = 1.3mm, and L₄ = 1.6mm, resulting in a total grating length of 6.1 mm. Figure 2(b) shows that the tested design approximates very closely the amplitude spectral response of a 3rd-order temporal integrator over a total optical bandwidth (integrator operation bandwidth)

of ≈ 40 GHz (the deviation between the simulated and ideal frequency responses over this bandwidth keeps < 8%), again in good agreement with the theoretical predictions. Moreover, this design also provides the π phase-shift (3 π) at the integrator resonance frequency that is required for odd-order temporal integration (see inset in Fig. 2(b)). As for the previous design, the behavior of this device as a 3rd-order temporal integrator was confirmed by simulating the response to an input complex envelope with a shape given by the 3rd-order time derivative of an ideal Gaussian pulse with a FWHM duration of 40 ps. This input optical signal is again assumed to be centered at the integrator's resonance frequency, 193 THz, and its amplitude envelope is plotted in Fig. 3(b) with a dotted, black curve. The simulated output pulse shape (amplitude of the field complex envelope) is shown in the same plot with a solid, red curve. As predicted, there is an excellent agreement between the obtained temporal profile (Gaussian pulse) and the ideal 3rd-order time cumulative integral of the considered input waveform (shown with a dashed, blue curve). The integration error in this case was $\approx 5.2\%$ and the energetic efficiency was $\approx 6.6 \times 10^{-7}$ %. The observed tendency in the two shown examples is a general one: the integration error typically increases with the integration order (the BG frequency response deviates further from the ideal frequency transfer function as the integration order increases) whereas the energetic efficiency is generally lower for higherorder integrators.

In principle, there is no fundamental, theoretical limitation to extend the presented design for Nth order integration (e.g. in terms of required coupling constant, detuning etc.). However, in practice, the number of possible integration stages will be limited by the energetic efficiency of the process. As evidenced by the examples presented in the paper, the energetic efficiency is degraded for higher-order integration.



Fig. 3. Time-domain response of the MPS-BG (a) second-order and (b) third-order integrators compared with the ideal (a) second and (b) third time cumulative integrals of the considered input pulse waveforms. The input amplitude envelopes are defined as the (a) second and (b) third time derivatives of an ideal Gaussian pulse with a FWHM of 40 ps.

4. Conclusions

We have proposed and numerically demonstrated a simple and general approach for implementing arbitrary-order time integration of optical waveforms. The basis of the approach

investigated here can be found in the previous proposal of a first-order optical integrator based on a single π phase-shifted BG [3]. We have first generalized the design proposed in [3] by showing that a symmetric phase-shifted BG is not strictly necessary to achieve the spectral features that are required for first-order time integration. Based on this generalization, we have shown that the Nth cumulative time integral of an input optical waveform can be obtained by simple transmission of this waveform through a single uniform BG incorporating N π phase shifts, which can be *arbitrarily located* along the grating profile as long as each of the uniform grating sub-sections (in between the π phase shifts) has a minimum length of $\approx 6/\kappa$ (where κ is the grating coupling coefficient); this latter condition ensures that the peak reflectivity of each uniform grating sub-section is nearly 100% (>99.999%). This BG design provides the required spectral characteristics over a limited bandwidth around the Bragg frequency of the uniform grating (in other words, the signal to be processed should be spectrally centered at this Bragg frequency). Our numerical simulations show that this approach can provide optical operation bandwidths in the tens-of-GHz regime, well beyond the reach of present electronic technologies, using readily feasible BG technologies (e.g. photo-induced in-fiber BGs).

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