WEIGHTED-NORM FIRST-ORDER SYSTEM LEAST-SQUARES (FOSLS) FOR DIV-CURL SYSTEM WITH THREE DIMENSIONAL EDGE SINGULARITIES

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Abstract. A weighted-norm first-order system least-squares (FOSLS) method for div/curl problems with edge singularities is presented. Traditional finite element methods, including least-squares methods, often suffer from a global loss of accuracy due to the influence of a nonsmooth solution near polyhedral edges. By minimizing a modified least-squares functional, optimal accuracy in weighted and non-weighted norms is recovered. Error estimates with and without mesh refinements are presented and numerical results are given to confirm the theory.

Key words. Singularities, weighted Sobolev spaces, least-squares, finite element methods

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1. Introduction.

In this paper, a weighted-norm first-order system least-squares (FOSLS) method for problems with edge singularities is studied. The least-squares finite element method is well-suited for many problems with first-order differential operators. Such systems often arise naturally from physical laws or can be formulated from a higher-order system (cf. [18]). The numerical solution is found by minimizing the norm of the residual of the system over an appropriate finite element space. Thus, to approximate a solution to the first-order system, the goal of a least-squares method is largely to choose the correct norm and finite element space. Under sufficient smoothness assumptions, many least-squares functionals induce a norm that can be shown to be equivalent to the product H^1-norm, indicating that H^1-conforming finite elements and multigrid can be used to solve the equations efficiently (cf. [9],[10]). However, the presence of boundary singularities may cause a loss of H^1-equivalence, and the use of H^1-finite element spaces may cause a loss of global accuracy in the approximate solution.

We focus on problems posed in polyhedral domains where the boundary contains edges with inner angle larger than π or edges upon which different types of boundary conditions meet with an inner angle larger than $\pi/2$. Such problems generally require special consideration and have been the focus of study in both Galerkin and least-squares methods. One of the most common approaches is to use $H(div)$ or $H(curl)$-conforming finite elements, for example, Raviart-Thomas or Nédélec's edge elements (see [29]). In papers [15] and [7], finite element spaces having a grid decomposition property and $H(div)$-conforming finite element spaces to get optimal convergence were investigated. In the classical paper [14], singular functions are added to standard finite element spaces, admitting optimal approximations at minimal additional cost. This approach in the context of a FOSLS formulation was explained in [5] and [6]. A weighted regularization in which weight functions were used in the divergence integral was presented in [13]. In [8], the H^{-1}-norm least-squares approach in discrete space was introduced in a weak variational formulation. In [24] and [22],

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two different types of modified first-order system LL* methods were developed that allow an accurate approximation using H^1-conforming finite elements for equations having singular boundaries in two and three dimensions, respectively. The latter of these employed partially weighted functionals.

In [23], two dimensional div/curl problems with boundary singularities are solved by minimizing a functional weighted according to the distance from the singular point. Here, we extend our earlier work to three dimensional problems with edge singularities and further consider local mesh refinement. Since many properties do not follow the two dimensional ones, this extension requires the investigation of the three dimensional singular solutions in the proper weighted Sobolev spaces and the establishment of certain new Poincaré-type bounds in scalar and vector forms. The goal of this paper is retaining the global L^2 accuracy and achieving the optimal order discretization accuracy in the induced weighted Sobolev spaces. In section 2, we introduce notation, definitions and our model problem. Some regularity results in weighted Sobolev spaces are described in section 3. In section 4, we investigate Poincaré inequalities in weighted Sobolev spaces. We prove optimal L^2- and H^1-error convergence away from the singularities and and the optimal L^2-error convergence globally by error estimates in weighted norms and L^2-norms in section 5. Establishing error estimates in weighted L^2- and H^1-norms with graded mesh refinement are presented in section 6. In section 7, we report several numerical examples. The theory applies for a range of values for the power of the weight function. Our numerical results show remarkable agreement with theory within this range and somewhat outside this range as well, indicating the possibility that the theory can be extended.

2. Weighted-norm least squares. Let Q be a polygon in $\mathbb{R}^2$ and $I \subset \mathbb{R}$ be a bounded interval. Then, consider the prototype domain

$$\Omega := Q \times I \subset \mathbb{R}^3, \tag{2.1}$$

which is a polyhedral cylinder. In this paper, we restrict ourselves to the case where the domain has one singular edge, which means

- the polygon Q has a corner with inner angle $> \pi$, or
- different types of boundary condition meet at the edge with inner angle $> \pi/2$;

however, the general case follows by consider our approach as a local result. We refer to the edge that causes the boundary singularity as E and the inner angle of the edge E as ω. From now on, the value λ indicates π/ω. We assume that E does not lie on top and bottom of the boundary, that is, $E = (x_0, y_0) \times I \subset \partial\Omega$. By translation and rotation, we may suppose that E coincides with the z-axis. Without loss of generality, we further assume $diam(\Omega) \leq\ 1$.

We denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ the L^2-inner product and norm, respectively. Let $H^k(\Omega)$ be the standard Sobolev space with the norm $\| \cdot \|_k$ and the semi-norm $| \cdot |_k$. Denote by $H^k_\beta(\Omega)$ the weighted Sobolev space of functions u such that

$$\|u\|^2_{k,\beta} = \sum_{|m|=0}^{k} \int_\Omega r^{2(\beta+|m|-k)} \, |D^m u|^2 \, d\Omega < \infty,$$

where $r := r(\mathbf{x})$ is the distance of $\mathbf{x} \in \Omega$ from the singular edge, E, and $|D^m u|^2 = \sum_{a_1+a_2+a_3=m} |\partial_x^{a_1} \partial_y^{a_2} \partial_z^{a_3} u|^2$. The corresponding semi-norm is $| \cdot |_{k,\beta} := \|D^k \cdot \|_{0,\beta}$. Characters in bold represent vector functions and components are subscripted by integers. For example, $\mathbf{u} = (u_1, u_2, u_3)^t$, and the vector L^2-norm is given by $\|\mathbf{u}\| =$

$(\|u_1\|^2 + \|u_2\|^2 + \|u_3\|^2)^{\frac{1}{2}}$. Define Hilbert spaces $H(\nabla\cdot)$ and $H(\nabla\times)$ as the spaces of L^2 vector functions, $\mathbf{u}$, satisfying $\nabla \cdot \mathbf{u} \in L^2(\Omega)$ and $\nabla \times \mathbf{u} \in L^2(\Omega)^3$, respectively.

For convenience, we omit the superscript t for vector transpose. Throughout this paper, c is a generic constant that is used to denote various constants and its dependence on other quantities will be indicated if necessary.

Consider the div/curl system

$$\begin{aligned} \nabla \times \mathbf{u} &= \mathbf{f}, && \text{in} \quad \Omega, \\ \nabla \cdot \mathbf{u} &= g, && \text{in} \quad \Omega, \\ \mathbf{n} \times \mathbf{u} &= \mathbf{0}, && \text{on} \quad \Gamma_D, \\ \mathbf{n} \cdot \mathbf{u} &= 0, && \text{on} \quad \Gamma_N, \end{aligned} \tag{2.2}$$

where $\mathbf{n}$ is the outward unit normal and Γ_D, Γ_N are a finite number of connected pieces satisfying $\Gamma_D \cap \Gamma_N = \emptyset$, $\Gamma_D \cup \Gamma_N = \partial\Omega$. The above system can be rewritten as

$$L\mathbf{u} = \begin{bmatrix} \nabla\times \\ \nabla\cdot \end{bmatrix} \mathbf{u} = \begin{bmatrix} \mathbf{f} \\ g \end{bmatrix} = \mathbf{F}, \tag{2.3}$$

where L is a linear operator from $\mathcal{U}$ to $L^2(\Omega)^4$ with

$$\mathcal{U} = \{\mathbf{v} \in H(\nabla\cdot) \cap H(\nabla\times) : \mathbf{n} \times \mathbf{v} = 0 \text{ on } \Gamma_D,\ \mathbf{n} \cdot \mathbf{v} = 0 \text{ on } \Gamma_N\}$$

This type of div/curl system appears in many applications, including electromagnetics, porous media flow, and solid and fluid mechanics. Thus, solving $L\mathbf{u} = \mathbf{F}$ in nonsmooth domains is of wide interest. The traditional least squares method solves system (2.3) by minimizing the residual functional

$$G(\mathbf{v}; \mathbf{F}) = \|L\mathbf{v} - \mathbf{F}\|^2 = \|\nabla \times \mathbf{v} - \mathbf{f}\|^2 + \|\nabla \cdot \mathbf{v} - g\|^2$$

over $\mathcal{U}$, in the weak sense : find $\mathbf{u} \in \mathcal{U}$ satisfying

$$\langle L\mathbf{u}, L\mathbf{v}\rangle = \langle \mathbf{F}, L\mathbf{v}\rangle, \qquad \text{for all } \mathbf{v} \in \mathcal{U}.$$

The FOSLS method includes the design of functionals whose bilinear part is equivalent to an appropriate norm, often the H^1-norm when possible. As briefly mentioned in the introduction, if the domain is not convex and the boundary is not $\mathcal{C}^{1,1}$ or different types of boundary conditions meet at an edge with inner angle larger than $\pi/2$, then the space $\mathcal{U}$ is not continuously imbedded into $H^1(\Omega)^3$, therefore, the solution may not be in $H^1(\Omega)^3$. This would seem to preclude the use H^1-conforming finite element spaces even though the solution is smooth away from the edge. To overcome this difficulty, we introduce the following weighted-norm functional:

$$G_w(\mathbf{u}; \mathbf{F}) = \|w(L\mathbf{u} - \mathbf{F})\|^2 = \|w\nabla \times (\mathbf{u} - \mathbf{f})\|^2 + \|w(\nabla \cdot \mathbf{u} - g)\|^2, \tag{2.4}$$

where the weight function has the form $w = r^\beta$ for some $\beta > 0$, and define

$$\|L\mathbf{u}\|^2_{0,\beta} = \|r^\beta \nabla \times \mathbf{u}\|^2 + \|r^\beta \nabla \cdot \mathbf{u}\|^2.$$

Of course, one can restrict the weight function to some small region by use of scaling or a smooth cut-off function. In this presentation, we assumed $diam(\Omega) \leq 1$ and use the weight function as above for convenience of presentation.

In the following, we investigate minimizing (2.4) for appropriate values of β.

3. Coercivity of L in weighted Sobolev spaces. In this section, we show the operator L defined in (2.3) is coercive in certain weighted Sobolev spaces, leading to the norm equivalence between $\|L\mathbf{u}\|_{0,\beta}$ and $\|\nabla\mathbf{u}\|_{0,\beta}$.

LEMMA 3.1. *Let the operator L be defined as in (2.3) and $(\mathbf{f},g) \in H^0_\beta(\Omega)^4$.*

i) If $\mathbf{f} = \mathbf{0}$, then $L : H^1_\beta(\Omega)^3 \cap \mathcal{U} \to H^0_\beta(\Omega)^4$ is injective for

$$\beta \in \begin{cases} (1-\lambda, 1+\lambda), & \textit{when } \partial\Omega = \Gamma_D, \\ (1-\lambda, 1), & \textit{when } \partial\Omega = \Gamma_N, \\ (1-\lambda/2, 1+\lambda/2), & \textit{when } \Gamma_D \neq \emptyset \textit{ and } \Gamma_N \neq \emptyset. \end{cases} \tag{3.1}$$

ii) If $g = 0$, then $L : H^1_\beta(\Omega)^3 \cap \mathcal{U} \to H^0_\beta(\Omega)^4$ is injective for

$$\beta \in \begin{cases} (1-\lambda, 1+\lambda), & \textit{when } \partial\Omega = \Gamma_D, \\ (1-\lambda, 1), & \textit{when } \partial\Omega = \Gamma_N \textit{ and the sides of the domain} \\ & \quad \textit{are parallel to the coordinate axes}, \\ (1-\lambda/2, 1+\lambda/2), & \textit{when } \Gamma_D \neq \emptyset \textit{ and } \Gamma_N \neq \emptyset \textit{ and the sides of the} \\ & \quad \textit{domain are parallel to the coordinate axes.} \end{cases} \tag{3.2}$$

iii) If $\mathbf{f} \neq \mathbf{0}$ and $g \neq 0$, then $L : H^1_\beta(\Omega)^3 \cap \mathcal{U} \to H^0_\beta(\Omega)^4$ is injective for

$$\beta \in \begin{cases} (1-\lambda, 1+\lambda), & \textit{when } \partial\Omega = \Gamma_D, \\ (1-\lambda, 1), & \textit{when } \partial\Omega = \Gamma_N \textit{ and the sides of the domain} \\ & \quad \textit{are parallel to the coordinate axes.} \end{cases} \tag{3.3}$$

Proof. The result is proved by establishing a decomposition of $\mathbf{u} \in H^1_\beta(\Omega)^3 \cap \mathcal{U}$ and then demonstrating that components of the decomposition each satisfy a Poisson equation, for which known results apply (cf. [25],[26],[28]). The 2-dimensional orthogonal decomposition of $\mathbf{u}$ with boundary conditions in (2.2) naturally provides Poisson equations with Dirichlet or Neumann boundary conditions ([23]). But, it is not true that each component of the orthogonal decomposition of $\mathbf{u}$ can be easily induced to Dirichlet or Neumann Poisson equations in 3-dimensional space. So, we consider things in several cases as presented in lemma 3.1.

i) If $\mathbf{f} = \mathbf{0}$, then there exists a unique $p \in H^1(\Omega)$ satisfying $\mathbf{u} = \nabla p$ and

$$\begin{cases} \Delta p \;=\; \nabla\cdot\mathbf{u} \quad \textit{in } \Omega, \\ p \;=\; c \quad \textit{on } \Gamma_D, \quad \mathbf{n}\cdot\nabla p \;=\; 0 \quad \textit{on } \Gamma_N, \end{cases} \tag{3.4}$$

where c is any constant. The above Poisson problem (3.4) is an isomorphism from $H^2_\beta(\Omega)$ to $H^0_\beta(\Omega)$ for $\beta \in (1-\lambda, 1+\lambda)$ if $\partial\Omega = \Gamma_D$ ([26]), for $\beta \in (1-\lambda, 1)$ if $\partial\Omega = \Gamma_N$ ([26]), and for $\beta \in (1-\lambda/2, 1+\lambda/2)$ if $\Gamma_D \neq \emptyset$ and $\Gamma_N \neq \emptyset$ ([25],[28]). Also, there exists a constant C satisfying

$$\|p\|_{2,\beta} \;\leq\; C\|\nabla\cdot\mathbf{u}\|_{0,\beta} \tag{3.5}$$

under the above assumptions on β and boundary conditions. The inequality (3.5) implies $\|\mathbf{u}\|_{1,\beta} \leq c\|L\mathbf{u}\|_{0,\beta}$, therefore, L is injective.

ii) If $g = 0$, then there exists a vector potential $\phi = (\phi_1, \phi_2, \phi_3) \in H(\nabla\times)$ such that $\mathbf{u} = \nabla\times\phi$ and $\nabla\cdot\phi = 0$. Suppose $\partial\Omega = \Gamma_D$, then $\nabla\times\mathbf{u} = \nabla\times\nabla\times\phi = -\Delta\phi + \nabla\nabla\cdot\phi = -\Delta\phi$ and the boundary conditions $\mathbf{n}\times\phi = \mathbf{0}$, $\mathbf{n}\times\nabla\times\phi = \mathbf{0}$ yield, for $i = 1, 2$,

$$\begin{cases} -\Delta\phi_i \;=\; (\nabla\times\mathbf{u})_i & \textit{in } \Omega \\ \phi_i \;=\; 0 & \textit{on } \partial\Omega \end{cases} \quad \textit{and} \quad \begin{cases} -\Delta\phi_3 \;=\; (\nabla\times\mathbf{u})_3 & \textit{in } \Omega \\ \phi_3 \;=\; 0 & \textit{on the side walls} \\ \mathbf{n}\cdot\nabla\phi_3 \;=\; 0 & \textit{on Top and Bottom,} \end{cases}$$

where $(\nabla \times \mathbf{u})_i$ *is the i-th component of* $\nabla \times \mathbf{u}$*. Since the inner angle between top and side walls is* $\pi/2$*, there are no singularities around those edges. Therefore, L is injective for* $\beta \in (1-\lambda, 1+\lambda)$*. Analogously, L is injective for* $\beta \in (1-\lambda, 1)$ *when* $\partial\Omega = \Gamma_N$ *and the sides of the domain are parallel to the coordinate axes, in which* $\mathbf{n}$ *is of the form* $(n_1, 0, 0), (0, n_2, 0)$*, and* $(0, 0, n_3)$*. For the case when* $\Gamma_D \neq \emptyset$ *and* $\Gamma_N \neq \emptyset$*, we use the following decomposition in [1]: a function* $\mathbf{u}$ *satisfies* $\nabla \cdot \mathbf{u} = 0$ *and* $\mathbf{n} \cdot \mathbf{u} = 0$ *on* Γ_N *if and only if there exists a function* $\phi \in H(\nabla\times)$ *such that*

$$\begin{cases} \mathbf{u} = \nabla \times \phi, \quad \nabla \cdot \phi = 0 \quad in\ \Omega \\ \mathbf{n} \times \phi = \mathbf{0} \ on\ \Gamma_N, \quad and \quad \mathbf{n} \cdot \phi = 0 \ on\ \Gamma_D. \end{cases} \tag{3.6}$$

If the sides of the domain are parallel to the coordinate axes, then we can derive a Poisson equation with mixed boundary conditions from (3.6). Hence, L is injective for $\beta \in (1-\lambda/2, 1+\lambda/2)$*.*

iii) Let $\mathbf{u} \in H^1_\beta(\Omega)^3 \cap \mathcal{U}$*, then* $\mathbf{u}$ *has the orthogonal decomposition ([16])*

$$\mathbf{u} = \nabla \times \phi + \nabla p,$$

where $\phi \in H(\nabla\times)$ *with* $\nabla \cdot \phi = 0$ *and* $\mathbf{n} \times \phi = \mathbf{0}$*, and* $p \in H^1_0(\Omega)$ *is the solution of*

$$\langle \nabla p, \nabla \xi \rangle = \langle \mathbf{u}, \nabla \xi \rangle, \quad \forall \xi \in H^1_0(\Omega).$$

Similarly to i) and ii), one can establish the results. □

REMARK 3.2. *Note that the above restrictions on* β *are sufficient, but may not be necessary. In the remainder of the paper, we establish results that depend on L being injective. While the above theorem establishes sufficient conditions, the operator L may be injective for much wider range of values for the weight* β*. Extension of the range is beyond the scope of this paper, but we remark that numerical results seem to indicate that it may be possible.*

If the div/curl system is derived from the Poisson equation, then the problem easily matches to the first case in lemma 3.1.

LEMMA 3.3. *Let* $\mathbf{u} \in H^1_\beta(\Omega)^3 \cap \mathcal{U}$*, then there exists a constant c satisfying*

$$\|\mathbf{u}\|_{1,\beta} \le c \ (\ \|L\mathbf{u}\|_{0,\beta} + \|\mathbf{u}\|_{0,\beta-1}), \tag{3.7}$$

for any $\beta > 0$*.*

Proof. Since the norms $\|u\|_{1,\beta}$ and $\|r^\beta u\|_1$ are equivalent (see [25]) and $\|r^\beta u\| \le \|u\|_{0,\beta-1}$, it is enough to show that $\|\nabla(r^\beta \mathbf{u})\|$ is less than the right-hand side of (3.7). Since $r^\beta \mathbf{u} \in H^1(\Omega)^3$, the result in [12] and the triangle inequality yield $\|\nabla(r^\beta \mathbf{u})\| \le c\ \|L(r^\beta \mathbf{u})\| \le c\ (\|L\mathbf{u}\|_{0,\beta} + \|\mathbf{u}\|_{0,\beta-1})$. □

Based on lemmas 3.1 and 3.3, we achieve the goal of this section in the following theorem by using a modified compactness argument.

THEOREM 3.4. *Let* $L : H^1_\beta(\Omega)^3 \cap \mathcal{U} \to H^0_\beta(\Omega)^4$ *be injective and* $\mathbf{u} \in H^1_\beta(\Omega)^3 \cap \mathcal{U}$*. Then, there exists a constant* $c = c(\Omega, \beta)$ *such that*

$$\|\mathbf{u}\|_{1,\beta} \le c\ \|L\mathbf{u}\|_{0,\beta}. \tag{3.8}$$

Proof. First, we let

$$\Omega_R = \{\mathbf{x} = (r, \theta, z) \in \Omega \ : \ r > R\}$$

and define a norm $\|\mathbf{u}\|^2_{1,\beta,\Omega_R} = |\mathbf{u}|^2_{1,\beta,\Omega_R} + \|\mathbf{u}\|^2_{0,\beta-1,\Omega_R}$, for any $R \ge 0$. Define Ω_{2R} and a norm $\|\mathbf{u}\|_{1,\beta,\Omega_{2R}}$ in the same way by replacing R with $2R$. Let $\delta(r)$ be a smooth cut-off function,

$$\delta(r) = \begin{cases} 0, & \text{if} \quad r \le R, \\ 1, & \text{if} \quad r \ge 2R, \end{cases}$$

where $|\delta| \le 1$, $|\delta'| \le cR^{-1}$ and δ' has support only for $r \in (R, 2R)$. By the fact that, for any $\mathbf{u} \in H^1_\beta(\Omega)^3$, we have $\delta r^\beta \mathbf{u} \in H^1(\Omega)^3$ and the triangle inequality, we have the first (cf. [12]) and the second inequalities, respectively, of the following:

$$\begin{aligned} \|\nabla(\delta r^\beta \mathbf{u})\| &\le c\|L(\delta r^\beta \mathbf{u})\| \le c(\|\delta r^\beta L\mathbf{u}\| + \|\nabla(\delta r^\beta)\cdot \mathbf{u}\|) \\ &\le c(\|L\mathbf{u}\|_{0,\beta,\Omega_R} + c\|\mathbf{u}\|_{0,\beta-1,\Omega_R}), \end{aligned} \tag{3.9}$$

where c is independent on R. We also have

$$\begin{aligned} \|\nabla \mathbf{u}\|_{0,\beta,\Omega_{2R}} &\le \|\delta r^\beta \nabla \mathbf{u}\| \le \|\delta r^\beta \nabla \mathbf{u} + \nabla(\delta r^\beta)\cdot \mathbf{u}\| + \|\nabla(\delta r^\beta)\cdot \mathbf{u})\| \\ &\le \|\nabla(\delta r^\beta \mathbf{u})\| + c\|\mathbf{u}\|_{0,\beta-1,\Omega_R}, \end{aligned} \tag{3.10}$$

where c is independent on R. Putting (3.9) and (3.10) together we have

$$\|\nabla \mathbf{u}\|_{0,\beta,\Omega_{2R}} \le c(\|L\mathbf{u}\|_{0,\beta,\Omega_R} + \|\mathbf{u}\|_{0,\beta-1,\Omega_R}). \tag{3.11}$$

Here, we use a modified compactness argument: Assume that (3.8) is not true, that is, there is a sequence $\{\mathbf{u}_j\}_{j=1,\infty} \subset H^1_\beta(\Omega)^3 \cap \mathcal{U}$ such that

$$\|\mathbf{u}_j\|_{1,\beta} = 1 \quad \text{and} \quad \|L\mathbf{u}_j\|_{0,\beta} \to 0, \quad \text{as } n \to \infty. \tag{3.12}$$

Since $H^1_\beta(\Omega)$ is compactly embedded in $H^0_\beta(\Omega)$ ([25]), there is a subsequence $\{\mathbf{u}_{j_\ell}\}_{\ell=1,\infty}$, which we shall denote by $\{\mathbf{u}_\ell\}_{\ell=1,\infty}$, that is Cauchy in $H^0_\beta(\Omega)^3$ and, thus, has limit $\tilde{\mathbf{u}} \in H^0_\beta(\Omega)^3$. That is, $\|\mathbf{u}_\ell - \tilde{\mathbf{u}}\|_{0,\beta} \to 0$. Now, for any $R > 0$, the norms $\|\cdot\|_{0,\beta,\Omega_R}$ and $\|\cdot\|_{0,\beta-1,\Omega_R}$ are equivalent. This implies that $\{\mathbf{u}_\ell\}_{\ell=1,\infty}$ is Cauchy in $H^0_{\beta-1}(\Omega_R)$ and

$$\|\mathbf{u}_\ell - \tilde{\mathbf{u}}\|_{0,\beta-1,\Omega_R} \to 0.$$

Using (3.11) we have

$$|\mathbf{u}_\ell - \mathbf{u}_m|_{1,\beta,\Omega_{2R}} \le c\left(\|L(\mathbf{u}_\ell - \mathbf{u}_m)\|_{0,\beta,\Omega_R} + \|\mathbf{u}_\ell - \mathbf{u}_m\|_{0,\beta-1,\Omega_R}\right), \tag{3.13}$$

which implies that $\{\mathbf{u}_\ell\}_{\ell=1,\infty}$ is Cauchy in $H^1_\beta(\Omega_{2R})^3$ and, thus, $\tilde{\mathbf{u}} \in H^1_\beta(\Omega_{2R})^3$ and

$$\|\mathbf{u}_\ell - \tilde{\mathbf{u}}\|_{1,\beta,\Omega_{2R}} \to 0. \tag{3.14}$$

The triangle inequality yields

$$\begin{aligned} \|L\tilde{\mathbf{u}}\|_{0,\beta,\Omega_{2R}} &\le c\|L(\tilde{\mathbf{u}} - \mathbf{u}_\ell)\|_{0,\beta,\Omega_{2R}} + c\|L\mathbf{u}_\ell\|_{0,\beta,\Omega_{2R}} \\ &\le c\|(\tilde{\mathbf{u}} - \mathbf{u}_\ell)\|_{1,\beta,\Omega_{2R}} + c\|L\mathbf{u}_\ell\|_{0,\beta,\Omega_{2R}}, \end{aligned}$$

for every ℓ, which implies $L\tilde{\mathbf{u}} = 0$ on Ω_{2R}. Since this is true for every $R > 0$, we have

$$L\tilde{\mathbf{u}} = 0 \qquad \text{on} \quad \Omega. \tag{3.15}$$

To summarize what we now know, $\tilde{\mathbf{u}} \in H^0_\beta(\Omega)^3$, $\tilde{\mathbf{u}} \in H^1_\beta(\Omega_R)^3$ for every $R > 0$ and $L\tilde{\mathbf{u}} = 0$ on Ω. If $\tilde{\mathbf{u}} \in H^0_{\beta-1}(\Omega)^3$, using lemma 3.3 and the fact that $L\tilde{\mathbf{u}} = 0$, we have $\tilde{\mathbf{u}} \in H^1_\beta(\Omega)^3$, which would contradict the assumption that L is injective on $H^1_\beta(\Omega)^3 \cap \mathcal{U}$. So, we assume that $\tilde{\mathbf{u}} \notin H^0_{\beta-1}(\Omega)^3$, that is,

$$\lim_{R\to 0} \|\tilde{\mathbf{u}}\|_{0,\beta-1,\Omega_R} = \infty.$$

However,

$$\begin{aligned}\|\tilde{\mathbf{u}}\|_{0,\beta-1,\Omega_R} &\le \|\tilde{\mathbf{u}}-\mathbf{u}_\ell\|_{0,\beta-1,\Omega_R} + \|\mathbf{u}_\ell\|_{0,\beta-1,\Omega_R} \le \|\tilde{\mathbf{u}}-\mathbf{u}_\ell\|_{0,\beta-1,\Omega_R} + \|\mathbf{u}_\ell\|_{0,\beta-1,\Omega} \\ &\le \|\tilde{\mathbf{u}}-\mathbf{u}_\ell\|_{0,\beta-1,\Omega_R} + \|\mathbf{u}_\ell\|_{1,\beta,\Omega} \le \|\tilde{\mathbf{u}}-\mathbf{u}_\ell\|_{0,\beta-1,\Omega_R} + 1,\end{aligned}$$

for every ℓ independent of R. Thus, $\|\tilde{\mathbf{u}}\|_{0,\beta-1,\Omega_R} \le 1$, for all $R > 0$. This is a contradiction and the result is proved. □

The above theorem implies that the corresponding weak formulation of the least squares method is coercive, whenever L is injective, in the proper weighted Sobolev space. The continuity is obtained easily by the triangle inequality. Since the solution is in $H^1_\beta(\Omega)$, that is, the weighted L^2-norm of gradient of the solution is finite, it is reasonable to minimize the residual functional in the weighted norm to approximately solve the problem when using H^1-conforming finite elements.

4. Poincaré inequalities. In this section, we develop several types of Poincaré inequalities, which are useful in various settings. Recall that E is the singular edge and $diam(\Omega) < 1$. From now on, we consider the domain in the cylindrical coordinate system for convenience. The domain, Ω, can be rewritten as

$$\Omega = \{\mathbf{x} = (r,\theta,z) : 0 < r < R(\theta),\ 0 < \theta < \omega < 2\pi,\ a < z < b\},$$

where $R(\theta)$ is the distance from the singular edge, E, to boundary points depending on the angle θ. Then, $\partial\Omega = \{(r,\theta,z) : r = R(\theta), 0 \le \theta \le \omega\} \cup \{(r,\theta,z) : \theta = 0, 0 \le r \le R(0)\} \cup \{(r,\theta,z) : \theta = \omega, 0 \le r \le R(\omega)\} \cup \{(r,\theta,z) : z = a \text{ or } z = b\}$. The first lemma is known from [20].

LEMMA 4.1. *If $q \in H^1_\beta(\Omega)$ vanishes on $\partial\Omega$, then, for any β,*

$$\|q\|_{0,\beta-1} \ \le\ c\ \|\nabla q\|_{0,\beta}.$$

Now, we show several generalized Poincaré-type inequalities.

LEMMA 4.2. *For $p \in H^1_\beta(\Omega)$, there exists a constant c such that*

$$\|p\|_{0,\beta-1} \ \le\ c\ (\|p\|_{0,\beta} + \|\nabla p\|_{0,\beta}),$$

when $\beta > 0$.

Proof. Let $R_0 = (\min_{0<\theta<\omega} R(\theta))/4$ and let χ be a smooth function defined in Ω such that $\chi(r) = 1$ when $r < R_0$ and $\chi(r) = 0$ when $r > 2R_0$ and $|\chi'| \le cR_0^{-1}$ for some constant c. Since $1 = \chi + 1 - \chi$,

$$\int_0^{R(\theta)} r^{2\beta-2}|p|^2 r dr = \int_0^{R(\theta)} r^{2\beta-2}\,|\chi p + (1-\chi)p|^2\, r dr \le 2\int_0^{R(\theta)} r^{2\beta-2}\left(|\chi p|^2 + |(1-\chi)p|^2\right) r dr.$$

By the modified Hardy's inequality in [19], for $\beta > 0$,

$$\int_0^{R(\theta)} r^{2\beta-2}|\chi p|^2 r dr \le c\int_0^{R(\theta)} r^{2\beta}\left|\frac{\partial(\chi p)}{\partial r}\right|^2 r dr \le c\int_0^{2R_0} r^{2\beta}\left(\frac{1}{R_0^2}|p|^2 + \left|\frac{\partial p}{\partial r}\right|^2\right) r dr.$$

Since $(1-\chi)p$ has nonzero values only on $(R_0, R(\theta))$,

$$\begin{aligned}\int_0^{R(\theta)} r^{2\beta-2}|(1-\chi)p|^2 r dr &= \int_{R_0}^{R(\theta)} r^{2\beta-2}|(1-\chi)p|^2 r dr \\ &\le {R_0}^{-2}\int_{R_0}^{R(\theta)} r^{2\beta}|(1-\chi)p|^2 r dr \le {R_0}^{-2}\int_0^{R(\theta)} r^{2\beta}|p|^2 r dr.\end{aligned}$$

Hence, by Fubini's theorem,

$$\int_\Omega r^{2\beta-2}|p|^2 d\Omega \le c\ R_0^{-2}\int_\Omega r^{2\beta}|p|^2 d\Omega + c\int_\Omega r^{2\beta}|\nabla p|^2 d\Omega.$$

□

The above lemma is useful in establishing weighted-norm Poincaré inequalities since they are guaranteed if we can show

$$\|p\|_{0,\beta} \le c\|\nabla p\|_{0,\beta}. \tag{4.1}$$

Combining lemma 4.2 with the following lemma provides a Poincaré inequality when there are no given boundary conditions.

LEMMA 4.3. *Let $\epsilon > 0$, $\beta > -1$, and $\beta+1-\epsilon > 0$. If $p \in H^1_{\beta+1-\epsilon}(\Omega)$, then there exist constants b and $c = c(\Omega,\beta,\epsilon)$ such that*

$$\|p-b\|_{0,\beta}\ \le\ c\ \|\nabla p\|_{0,\beta+1-\epsilon}.$$

Proof. Here, we show an outline of the proof. The details can be found in [21]. If $p \in H^1_{\beta+1-\epsilon}(\Omega)$ when $\beta+1-\epsilon > 0$, then $p \in H^1(S)$, where $S \subset \Omega$ and $\bar{S}\cap E = \emptyset$. Then, we can consider the following expression for p: for $(r,\theta,z),(r_0,\theta_0,z_0) \in \Omega$,

$$\begin{aligned}
&p(r,\theta,z) - p(r_0,\theta_0,z_0)\\
&= p(r,\theta,z) - p(r,\theta_0,z) + p(r,\theta_0,z) - p(r_0,\theta_0,z) + p(r_0,\theta_0,z) - p(r_0,\theta_0,z_0)\\
&= \int_{\theta_0}^{\theta} \frac{\partial p}{\partial\tilde\theta}(r,\tilde\theta,z)\ d\tilde\theta\ +\ \int_{r_0}^{r}\frac{\partial p}{\partial\tilde r}(\tilde r,\theta_0,z)\ d\tilde r\ +\ \int_{z_0}^{z}\frac{\partial p}{\partial\tilde z}(r_0,\theta_0,\tilde z)\ d\tilde z.
\end{aligned}$$

Multiply by $r_0^{\beta+\frac12}$ and perform the integration $\int_\Omega r_0 dr_0 d\theta_0 dz_0$ on both sides :

$$\begin{aligned}
c_1 p(r,\theta,z) = &\int_\Omega r_0^{\beta+\frac12} p(r_0,\theta_0,z_0) r_0 dr_0 d\theta_0 dz_0 + \int_\Omega r_0^{\beta+\frac12}\left\{\int_{t_0}^{\theta}\frac{\partial p}{\partial\tilde\theta}(r,\tilde\theta,z)\ d\tilde\theta\right.\\
&\left.+\int_{r_0}^{r}\frac{\partial p}{\partial\tilde r}(\tilde r,\theta_0,z)\ d\tilde r\ +\ \int_{z_0}^{z}\frac{\partial p}{\partial\tilde z}(r_0,\theta_0,\tilde z)\ d\tilde z\right\} r_0 dr_0 d\theta_0 dz_0,
\end{aligned}\tag{4.2}$$

where $c_1 = \int_\Omega r_0^{\beta+\frac12} r_0 dr_0 d\theta_0 dz_0$. Let

$$b = \frac{1}{c_1}\int_\Omega r_0^{\beta+\frac12} p(r_0,\theta_0,z_0)\ r_0 dr_0 d\theta_0 dz_0,$$

then $|b| \le c\|p\| < \infty$. Subtracting b from both sides in (4.2), applying Fubini's theorem, inserting $\tilde r^{\frac{-1+2\epsilon}{2}} \cdot \tilde r^{\frac{1-2\epsilon}{2}} = 1$ to group $\tilde r^{\frac{1-2\epsilon}{2}}$ with the $\frac{\partial p}{\partial \tilde r}$ term, using the Cauchy-Schwarz inequality, and squaring both sides yields

$$\begin{aligned}
|p(r,\theta,z)-b|^2 \le c&\left\{\int_0^\omega \left|\frac{\partial p}{\partial\tilde\theta}(r,\tilde\theta,z)\right|^2 d\tilde\theta + \int_0^\omega\int_0^R \tilde r^{2\beta+3}\left|\frac{\partial p}{\partial\tilde r}(\tilde r,\theta_0,z)\right|^2 d\tilde r d\theta_0\right.\\
&\left.+\int_0^\omega\left\{\frac{R^{2\epsilon}}{\epsilon}\int_r^R \tilde r^{1-2\epsilon}\left|\frac{\partial p}{\partial\tilde r}(\tilde r,\theta_0,z)\right|^2 d\tilde r + \int_0^R r_0^{2\beta+3}\int_a^b\left|\frac{\partial p}{\partial\tilde z}(r_0,\theta_0,\tilde z)\right|^2 d\tilde z dr_0\right\} d\theta_0\right\}.
\end{aligned}$$

To establish the weighted L^2-norm of $|p-b|$, multiply by $r^{2\beta}$ and take an integration over Ω. Then, we have

$$\int_\Omega r^{2\beta}|p(r,\theta,z)-b|^2 d\Omega \le c\int_\Omega r^{2\beta+2}\left(\left|\frac{1}{r}\frac{\partial p}{\partial\theta}\right|^2+\left|\frac{\partial p}{\partial r}\right|^2+\left|\frac{\partial p}{\partial z}\right|^2\right)+r^{2\beta+2-2\epsilon}\left|\frac{\partial p}{\partial r}\right|^2 d\Omega$$
$$\le c\int_\Omega r^{2\beta+2-2\epsilon}|\nabla p\,|^2 d\Omega,$$

where $c=c(\Omega,\beta,\epsilon,(\beta+1)^{-1},\epsilon^{-1})\to\infty$ as $\epsilon\to 0$ and $\beta\to -1$. □

The next result follows from setting $\epsilon=1$ in lemma 4.3.

THEOREM 4.4. *Let $\beta>0$ and $p\in H^1_\beta(\Omega)$, then there exist constants b, c satisfying*

$$\|p-b\|_{0,\beta-1}\le c\|\nabla p\|_{0,\beta}.$$

Proof. From lemmas 4.2 and 4.3 (set $\epsilon=1$), we deduce the result. □

COROLLARY 4.5. *If $p\in H^1_\beta(\Omega)\cap H^1(\Omega)/\mathbb{R}$ is the solution of Poisson equation with Neumann boundary condition, then for $\beta>0$, by theorem 4.4, p satisfies*

$$\|p\|_{0,\beta-1}\le c\|\nabla p\|_{0,\beta}. \tag{4.3}$$

We now consider functions with zero boundary conditions given on a non empty portion of the boundary. The following theorem shows that the weighted-norm Poincaré inequality holds wherever the zero boundary conditions are located.

THEOREM 4.6. *Let $\beta>0$ and $p\in H^1_\beta(\Omega)$. If $p=0$ on $\Gamma_0\neq\emptyset$, then*

$$\|p\|_{0,\beta-1}\ \le\ c\ \|\nabla p\|_{0,\beta}. \tag{4.4}$$

Proof. In order to prove (4.4), by lemma 4.2, it is enough to show (4.1). Here, we show (4.1) for p satisfying $p=0$ on $\Gamma_0\subset\{\mathbf{x}\in\partial\Omega : z=a\}$; however, the other cases can be proved in the same manner. Consider the following expression for p:

$$p(r,\theta,z)=\int_{r_0}^{r}\frac{\partial p(\tilde r,\theta,z)}{\partial\tilde r}d\tilde r+\int_{\theta_0}^{\theta}\frac{\partial p(r_0,\tilde\theta,z)}{\partial\tilde\theta}d\tilde\theta+\int_a^z\frac{\partial p(r_0,\theta_0,\tilde z)}{\partial\tilde z}d\tilde z,$$

where $(r_0,\theta_0,a)\in\Gamma_0$. If $r\ge r_0$, then squaring both sides, applying the Cauchy inequality, and multiplying by $r_0^{2\beta+1}$ yield

$$r_0^{2\beta+1}|p(r,\theta,z)|^2\le c\int_{r_0}^{r}\tilde r^{2\beta+1}\left|\frac{\partial p(\tilde r,\theta,z)}{\partial\tilde r}\right|^2 d\tilde r+c\int_{\theta_0}^{\theta}r_0^{2\beta+1}\left|\frac{1}{r_0}\frac{\partial p(r_0,\tilde\theta,z)}{\partial\tilde\theta}\right|^2 d\tilde\theta$$
$$+c\int_a^z r_0^{2\beta+1}\left|\frac{\partial p(r_0,\theta_0,\tilde z)}{\partial\tilde z}\right|^2 d\tilde z$$
$$\le c\int_0^{R(\theta)}\tilde r^{2\beta+1}\left|\frac{\partial p(\tilde r,\theta,z)}{\partial\tilde r}\right|^2 d\tilde r+c\int_0^{\omega}r_0^{2\beta+1}\left|\frac{1}{r_0}\frac{\partial p(r_0,\tilde\theta,z)}{\partial\tilde\theta}\right|^2 d\tilde\theta$$
$$+c\int_a^b r_0^{2\beta+1}\left|\frac{\partial p(r_0,\theta_0,\tilde z)}{\partial\tilde z}\right|^2 d\tilde z.$$

First, integrate with respect to r_0, θ_0 and then, multiply by $r^{2\beta+1}$ on both sides and integrate over Ω with respect to r, θ, and z, respectively. Then, by Fubini's theorem,

$$\int_\Omega r^{2\beta}|p|^2 d\Omega \le c\left(\int_\Omega \tilde{r}^{2\beta}\left|\frac{\partial p}{\partial \tilde{r}}\right|^2 d\Omega + \int_\Omega r_0^{2\beta}\left|\frac{1}{r_0}\frac{\partial p}{\partial \tilde{\theta}}\right|^2 d\Omega + \int_\Omega r_0^{2\beta}\left|\frac{\partial p}{\partial \tilde{z}}\right|^2 d\Omega\right)$$

which yields (4.1).

If $r < r_0$, similarly, we square both sides, use the Cauchy inequality, and multiply by $r_0^{2\beta+1}r^{2\beta+1}$. Then, we have

$$\begin{aligned}
&r_0^{2\beta+1}r^{2\beta+1}|p(r,\theta,z)|^2 \le cr_0^{2\beta+1}\int_r^{r_0}\tilde{r}^{2\beta+1}\left|\frac{\partial p(\tilde{r},\theta,z)}{\partial \tilde{r}}\right|^2 d\tilde{r}\\
&+cr^{2\beta+1}\int_{\theta_0}^{\theta} r_0^{2\beta+1}\left|\frac{1}{r_0}\frac{\partial p(r_0,\tilde{\theta},z)}{\partial\tilde{\theta}}\right|^2 d\tilde{\theta} + cr^{2\beta+1}\int_a^z r_0^{2\beta+1}\left|\frac{\partial p(r_0,\theta_0,\tilde{z})}{\partial \tilde{z}}\right|^2 d\tilde{z}\\
&\le cr_0^{2\beta+1}\int_0^{R(\theta)}\tilde{r}^{2\beta+1}\left|\frac{\partial p}{\partial\tilde{r}}\right|^2 d\tilde{r} + cr^{2\beta+1}\int_0^{\omega} r_0^{2\beta+1}\left|\frac{1}{r_0}\frac{\partial p}{\partial\tilde{\theta}}\right|^2 d\tilde{\theta} + cr^{2\beta+1}\int_a^b r_0^{2\beta+1}\left|\frac{\partial p}{\partial\tilde{z}}\right|^2 d\tilde{z}.
\end{aligned}$$

Taking integrations with respect to r_0, θ_0, r, θ, and z implies (4.1). □

So far, Poincaré inequalities in weighted Sobolev spaces for scalar function have been established. We now extend these to vector functions.

THEOREM 4.7. *Let $\beta > 0$ and $\mathbf{u} \in H^1_\beta(\Omega)^3$ with $\mathbf{n}\times\mathbf{u} = \mathbf{0}$ on Γ_D and $\mathbf{n}\cdot\mathbf{u} = 0$ on Γ_N, then there exists a constant c such that*

$$\|\mathbf{u}\|_{0,\beta-1} \le c\,\|\nabla\mathbf{u}\|_{0,\beta} \tag{4.5}$$

except for the case with $\Gamma_D = \{\mathbf{x}\in\partial\Omega : z = a \text{ and } b\}$ and $\Gamma_N = \partial\Omega\backslash\Gamma_D$.

Proof. Here, we prove (4.5) only for the case $\partial\Omega = \Gamma_N$. However, all the other cases can be proved easily by using same methodology.

First, we rotate the domain so that the side $\theta = 0$ lies on the x-axis. Since $\mathbf{n}\cdot\mathbf{u} = 0$ on $\partial\Omega$, $\mathbf{n}\cdot\mathbf{u} = (n_1,n_2,n_3)\cdot(u_1,u_2,u_3) = n_3u_3 = 0$ on the top and bottom of the domain, that is, where $z = a$ and b. Then, by theorem 4.6, $\|u_3\|_{0,\beta-1} \le c\|\nabla u_3\|_{0,\beta}$. From $\mathbf{n} = (0,n_2,0)$ on $\theta = 0$, we have $\mathbf{n}\cdot\mathbf{u} = n_2u_2 = 0$ which yields $u_2 = 0$, therefore, theorem 4.6 yields $\|u_2\|_{0,\beta-1} \le c\|\nabla u_2\|_{0,\beta}$. On the boundary that has nonzero n_1-component, we have $n_1u_1 + n_2u_2 = 0$. Then, the triangle inequality leads

$$\begin{aligned}
\|\mathbf{u}\|^2_{0,\beta-1} &= \|u_1\|^2_{0,\beta-1} + \|u_2\|^2_{0,\beta-1} + \|u_3\|^2_{0,\beta-1}\\
&\le c\frac{1}{n_1^2}\|n_1u_1+n_2u_2\|^2_{0,\beta-1} + c\frac{n_2^2}{n_1^2}\|u_2\|^2_{0,\beta-1} + \|u_2\|^2_{0,\beta-1} + \|u_3\|^2_{0,\beta-1}\\
&\le c\frac{1}{n_1^2}\|\nabla(n_1u_1+n_2u_2)\|^2_{0,\beta} + c\|u_2\|^2_{0,\beta-1} + \|u_3\|^2_{0,\beta-1}\\
&\le c(\|\nabla u_1\|^2_{0,\beta} + \|\nabla u_2\|^2_{0,\beta} + \|\nabla u_3\|^2_{0,\beta}) = c\|\nabla\mathbf{u}\|^2_{0,\beta}.
\end{aligned} \tag{4.6}$$

If $\Gamma_D = \{\mathbf{x}\in\partial\Omega : z = a,\ b\}$ and $\Gamma_N = \partial\Omega\backslash\{\mathbf{x}\in\partial\Omega : z = a,\ b\}$, then $\mathbf{n}\times\mathbf{u} = \mathbf{0}$ on Γ_D and $\mathbf{n}\cdot\mathbf{u} = 0$ on Γ_N do not provide any information about boundary condition for u_3. Therefore, we exclude this case. □

5. Error bounds. Let $\mathcal{T}_h$ be a quasi-uniform partition of the domain $\bar{\Omega} = \cup_{\tau\in\mathcal{T}_h}\tau$, and each finite element $\tau\in\mathcal{T}_h$ be a closed subset of $\bar{\Omega}$ with $h := \max\{h_\tau : h_\tau = diam(\tau)\}$. Assume that the partition $\mathcal{T}_h$ is regular so that we may choose a finite element basis that is conforming and satisfies the standard approximation properties (see [11]). We also assume that there exists a constant, ρ, satisfying $h \leq \rho \min_\tau h_\tau$. Let $\mathcal{P}^h$ denote the space of $\mathcal{C}^0$ piecewise polynomials on each finite element and $\mathcal{W}^h$ be a subspace of $\mathcal{P}^h$ such that

$$\mathcal{W}^h = \{\mathbf{v}^h \in \mathcal{P}^h : \mathbf{n} \times \mathbf{v}^h = \mathbf{0} \text{ on } \Gamma_D, \quad \mathbf{n} \cdot \mathbf{v}^h = 0 \text{ on } \Gamma_N\}.$$

The discrete weighted-norm least-squares approximation minimizes the functional

$$G_w(\mathbf{u}^h; f) = \min_{\mathbf{v}^h \in \mathcal{W}^h} G_w(\mathbf{v}^h; f). \tag{5.1}$$

For ease of discussion, we will consider cubes. However, arbitrary triangulations can be handled in a similar manner. Let $\mathcal{I}^h$ be a standard polynomial interpolation operator such that $\mathcal{I}^h p = p$, for any $p \in \mathbb{Q}_k(\tau)$, where $\mathbb{Q}_k(\tau)$ is a set of k-th order polynomials with respect to each variable on τ. Define $\mathcal{I}_0^h$ by,

$$\mathcal{I}_0^h u|_\tau \;=\; \begin{cases} \mathcal{I}^h u|_\tau = \sum_{i=1}^{(k+1)^3} u(a_i)\phi_i \;, & \text{if } \tau \text{ does not meet the singular edge, } E, \\ \sum_{a_i \notin N} u(a_i)\phi_i \;, & \text{if } \tau \text{ meets the singular edge, } E, \end{cases}$$

where ϕ_i are the basis functions, a_i are the nodal points corresponding to ϕ_i, and $N = \{a_j : a_j = (0,0,z_j)\}$ is the set of $k+1$ nodes seating along the singular edge, E. Here, it is easy to see that $\mathcal{I}_0^h \mathbf{u} \in \mathcal{W}^h$ for $\mathbf{u} \in H^1_\beta(\Omega)^3$ with $\mathbf{n} \times \mathbf{u} = \mathbf{0}$ on Γ_D and $\mathbf{n} \cdot \mathbf{u} = 0$ on Γ_N since we can consider $\mathcal{I}_0^h \mathbf{u}$ as $\mathcal{I}^h(\xi \mathbf{u})$, where the smooth function ξ satisfies $\xi = 0$ when $r < \epsilon$ and $\xi = 1$ when $r \geq 2\epsilon$ for sufficiently small positive ϵ.

Lemma 5.1. *Let $u \in H^m_\beta(\Omega)$, then*

$$\|u - \mathcal{I}_0^h u\|_{1,\beta} \;\leq\; c\; h^{m-1} \|u\|_{m,\beta}, \tag{5.2}$$

for $3/2 < m$ and any $0 < \beta$, where $\mathcal{I}_0^h$ is the modified interpolation operator into piecewise polynomials of degree $m-1 \leq k$.

Proof. We rewrite

$$\|u - \mathcal{I}_0^h u\|^2_{1,\beta} \;=\; \sum_{\tau \in \mathcal{T}_h} \|u - \mathcal{I}_0^h u\|^2_{1,\beta,\tau},$$

where $\|u\|_{1,\beta,\tau} = \left(\int_\tau r^{2\beta-2}|u|^2 + r^{2\beta}|\nabla u|^2 d\tau\right)^{\frac{1}{2}}$. Consider $\|u - \mathcal{I}_0^h u\|^2_{1,\beta,\tau}$ on each element τ. If τ does not touch the singular edge, then, by the definition of $\mathcal{I}_0^h$, $\mathcal{I}_0^h u|_\tau = \mathcal{I}^h u|_\tau$. Let $r_{min} = \inf\{r : (x,y,z) \in \tau\}$ and $r_{max} = \sup\{r : (x,y,z) \in \tau\}$ in τ. Then, $ch \leq r_{min} \leq r = (x^2+y^2)^{\frac{1}{2}} \leq r_{max} \leq r_{min} + \sqrt{3}h$ and the standard interpolation property yield

$$\begin{aligned}
&\|u - \mathcal{I}_0^h u\|^2_{1,\beta,\tau} = \|u - \mathcal{I}^h u\|^2_{1,\beta,\tau} = \int_\tau r^{2\beta}|\nabla(u - \mathcal{I}^h u)|^2 + r^{2(\beta-1)}|u - \mathcal{I}^h u|^2 d\tau \\
&\leq r^{2\beta}_{max} \int_\tau |\nabla(u - \mathcal{I}^h u)|^2 d\tau + r^{2\beta}_{max} r^{-2}_{min} \int_\tau |u - \mathcal{I}^h u|^2 d\tau \\
&\leq c r^{2\beta}_{max} h^{2(m-1)} |u|^2_{m,\tau} + c r^{2\beta}_{max} r^{-2}_{min} h^{2m} |u|^2_{m,\tau} = c r^{2\beta}_{max} h^{2(m-1)} (1 + r^{-2}_{min} h^2) |u|^2_{m,\tau} \\
&\leq c r^{2\beta}_{max} h^{2(m-1)} |u|^2_{m,\tau} \leq c h^{2(m-1)} r^{2\beta}_{max} r^{-2\beta}_{min} \int_\tau r^{2\beta} |D^m u|^2 d\tau \\
&\leq c h^{2(m-1)} \left(\frac{r_{min} + \sqrt{3}h}{r_{min}}\right)^{2\beta} \int_\tau r^{2\beta} |D^m u|^2 d\tau \leq c h^{2(m-1)} \|D^m u\|^2_{0,\beta,\tau}.
\end{aligned} \tag{5.3}$$

Next, consider the case in which $\tau \cap E \neq \emptyset$. Let $\delta \in \mathcal{C}^\infty$ be a cut-off function defined by

$$\delta(r) = \begin{cases} 1, & \text{if} \quad r \leq \frac{h}{3k}, \\ 0, & \text{if} \quad r > \frac{h}{2k}, \end{cases} \tag{5.4}$$

with $|\delta^{(m)}| \leq ch^{-m}$, where $\delta^{(m)}$ the m-th derivative of δ. By the triangle inequality,

$$\|u - \mathcal{I}_0^h u\|_{1,\beta,\tau}^2 \leq c(\|\delta u - \mathcal{I}_0^h(\delta u)\|_{1,\beta,\tau}^2 + \|(1-\delta)u - \mathcal{I}_0^h((1-\delta)u)\|_{1,\beta,\tau}^2). \tag{5.5}$$

On these τ, $\mathcal{I}_0^h(\delta u) = 0$. Thus, for the first term in (5.5), the properties in δ and Fubini's theorem imply

$$\|\delta u - \mathcal{I}_0^h(\delta u)\|_{1,\beta,\tau}^2 = \|\delta u\|_{1,\beta,\tau}^2 = \int_\tau r^{2\beta}|\nabla(\delta u)|^2 + r^{2(\beta-1)}|\delta u|^2 d\tau$$
$$\leq c\int_\tau r^{2\beta}(|\nabla\delta \cdot u|^2 + |\delta\nabla u|^2) + r^{2(\beta-1)}|\delta u|^2 d\tau$$
$$\leq c\iiint_{\frac{h}{3k}}^{\frac{h}{2k}} r^{2\beta}h^{-2}|u|^2 d\tau + c\iiint_0^{\frac{h}{3k}} r^{2\beta}|\nabla u|^2 + r^{2(\beta-1)}|u|^2 d\tau$$
$$\leq c\iiint_{\frac{h}{3k}}^{\frac{h}{2k}} r^{2(\beta-1)}|u|^2 d\tau + c\iiint_0^{\frac{h}{3k}} r^{2\beta}|\nabla u|^2 + r^{2(\beta-1)}|u|^2 d\tau$$
$$\leq c\int_\tau r^{2(m-1)}(r^{2(\beta-m+1)}|\nabla u|^2 + r^{2(\beta-m)}|u|^2)d\tau \leq ch^{2(m-1)}\|u\|_{m,\beta,\tau}^2.$$

For the second term in (5.5), use $\mathcal{I}_0^h((1-\delta)u) = \mathcal{I}^h((1-\delta)u)$ and theorem 4.6 to obtain

$$\|(1-\delta)u - \mathcal{I}_0^h((1-\delta)u)\|_{1,\beta,\tau}^2 \leq c|(1-\delta)u - \mathcal{I}^h((1-\delta)u)|_{1,\beta,\tau}^2.$$

Since $(1-\delta)u \in H^m(\tau)$, we use the standard interpolation property, Fubini's theorem, and the properties of δ to obtain

$$|(1-\delta)u - \mathcal{I}^h((1-\delta)u)|_{1,\beta,\tau}^2 \leq ch^{2\beta}\int_\tau |\nabla((1-\delta)u - \mathcal{I}^h((1-\delta)u))|^2 d\tau$$
$$\leq ch^{2\beta}h^{2(m-1)}\int_\tau |D^m((1-\delta)u)|^2 d\tau \leq ch^{2(\beta+m-1)}\int_\tau \sum_{j=0}^m |D^{m-j}(1-\delta)D^j u|^2 d\tau$$
$$\leq ch^{2(\beta+m-1)}\left(\iiint_{\frac{h}{3k}}^{\frac{h}{2k}} \sum_{j=0}^{m-1} |h^{j-m}D^j u|^2 d\tau + \iiint_{\frac{h}{3k}}^{r(\theta)} |(1-\delta)D^m u|^2 d\tau\right)$$
$$\leq ch^{2(\beta+m-1)}\left(\sum_{j=0}^{m-1}\iiint_{\frac{h}{3k}}^{\frac{h}{2k}} h^{-2\beta}r^{2(\beta+j-m)}|D^j u|^2 d\tau + \iiint_{\frac{h}{3k}}^{r(\theta)} h^{-2\beta}r^{2\beta}|D^m u|^2 d\tau\right)$$
$$= ch^{2(m-1)}\sum_{j=0}^m \int_\tau r^{2(\beta+j-m)}|D^j u|^2 d\tau = ch^{2(m-1)}\|u\|_{m,\beta,\tau}^2.$$

Hence, we have

$$\|u - \mathcal{I}_0^h u\|_{1,\beta}^2 = \sum_{\tau\in\mathcal{T}_h} \|u - \mathcal{I}_0^h u\|_{1,\beta,\tau}^2 \leq ch^{2(m-1)}\sum_{\tau\in\mathcal{T}_h} \|u\|_{m,\beta,\tau}^2 \leq ch^{2(m-1)}\|u\|_{m,\beta}^2.$$

□

LEMMA 5.2. *Let $\mathbf{u}^h \in \mathcal{W}^h$, then there exists a constant c such that*

$$\|\mathbf{u}^h\|_{0,\gamma} \le ch^{-\eta}\|\mathbf{u}^h\|_{0,\gamma+\eta}, \tag{5.6}$$

for any real $\gamma \ge 0$ and any $\eta > 0$.

Proof. In [23], inequality 5.6 is shown in 2-dimension. The analogous 3-dimensional result can be proved in the same manner. □

Lemma 5.1 shows that there exists an approximation of the solution in the finite dimensional subspace $\mathcal{W}^h$ that yields an optimal weighted H^1-convergence. By using this, we can show the following error estimates.

THEOREM 5.3. *Let $\mathbf{u} \in \mathcal{U}$ be the solution of (2.3) and $\mathbf{u}^h \in \mathcal{W}^h$ be the solution of (5.1). Assume $\mathbf{u} \in H^{\alpha+\beta}_{\beta}(\Omega)^3$ and L satisfies theorem 3.4. Then,*

$$\|\mathbf{u}-\mathbf{u}^h\|_{l,\beta} \le c\ h^{\alpha+\beta-l}\|\mathbf{u}\|_{\alpha+\beta,\beta}, \quad \text{for } l = 0,1, \tag{5.7}$$

$$\|\mathbf{u}-\mathbf{u}^h\| \le c\ h^{\alpha}\|\mathbf{u}\|_{\alpha+\beta,\beta}, \tag{5.8}$$

where $3/2 < \alpha+\beta \le k+1$ and k is the degree of the piecewise polynomials in $\mathcal{W}^h$.

Proof. First, we prove (5.7) for $l = 1$. Since $\mathbf{u}^h$ is the minimizer of (5.1), by theorem 3.4, the triangle inequality, and lemma 5.1, we have

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}^h\|_{1,\beta} &\le c\|L(\mathbf{u}-\mathbf{u}^h)\|_{0,\beta} \le c\|L(\mathbf{u}-\mathcal{I}^h_0\mathbf{u})\|_{0,\beta} \le c\|\mathbf{u}-\mathcal{I}^h_0\mathbf{u}\|_{1,\beta}\\ &\le ch^{\alpha+\beta-1}\|\mathbf{u}\|_{\alpha+\beta,\beta}.\end{aligned} \tag{5.9}$$

In order to prove the result for $l = 0$, we consider the weighted L^2-norm on each element. We have

$$\|\mathbf{u}-\mathbf{u}^h\|^2_{0,\beta} = \sum_{\tau\in\mathcal{T}_h}\|\mathbf{u}-\mathbf{u}^h\|^2_{0,\beta,\tau}.$$

By Cauchy inequality, we have

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}^h\|^2_{0,\beta,\tau} &= \int_\tau r^{2\beta}|\mathbf{u}-\mathbf{u}^h|^2 d\tau \le \left(\int_\tau 1^{\frac{3}{2}}d\tau\right)^{\frac{2}{3}}\left(\int_\tau \left(r^{2\beta}|\mathbf{u}-\mathbf{u}^h|^2\right)^3 d\tau\right)^{\frac{1}{3}}\\ &\le c\ h^{3\cdot\frac{2}{3}}\left(\int_\tau \left(r^{\beta}|\mathbf{u}-\mathbf{u}^h|\right)^6 d\tau\right)^{\frac{1}{6}\cdot 2} = c\ h^2\|r^\beta(\mathbf{u}-\mathbf{u}^h)\|^2_{L^6(\tau)}.\end{aligned} \tag{5.10}$$

Since $H^1(\Omega)$ is continuously imbedded into $L^6(\Omega)$ ([11]) and $r^\beta\mathbf{u} \in H^1(\Omega)^3$,

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}^h\|^2_{0,\beta} &\le c\ h^2\sum_{\tau\in\mathcal{T}_h}\|r^\beta(\mathbf{u}-\mathbf{u}^h)\|^2_{L^6(\tau)} \le c\ h^2\|r^\beta(\mathbf{u}-\mathbf{u}^h)\|^2_{L^6(\Omega)}\\ &\le c\ h^2\|r^\beta(\mathbf{u}-\mathbf{u}^h)\|^2_1 \le c\ h^2\|\mathbf{u}-\mathbf{u}^h\|^2_{1,\beta}.\end{aligned}$$

Applying (5.9) to the above implies

$$\|\mathbf{u}-\mathbf{u}^h\|_{0,\beta} \le ch\|\mathbf{u}-\mathbf{u}^h\|_{1,\beta} \le chh^{\alpha+\beta-1}\|\mathbf{u}\|_{\alpha+\beta,\beta} = ch^{\alpha+\beta}\|\mathbf{u}\|_{\alpha+\beta,\beta}.$$

To establish (5.8), we let $K = \{\tau \in \mathcal{T}_h : \tau \cap E \ne \emptyset\}$ and separate the norm $\|\mathbf{u}-\mathbf{u}^h\|$ into two parts

$$\|\mathbf{u}-\mathbf{u}^h\|^2 = \sum_{\tau\in K}\|\mathbf{u}-\mathbf{u}^h\|^2_{0,\tau} + \sum_{\tau\in\mathcal{T}_h\setminus K}\|\mathbf{u}-\mathbf{u}^h\|^2_{0,\tau}.$$

First, we consider the case $\beta < 1$. We have $r \le ch$ when $\tau \in K$, therefore,

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}^h\|^2 &= \sum_{\tau\in K}\|\mathbf{u}-\mathbf{u}^h\|^2_{0,\tau} + \sum_{\tau\in\mathcal{T}_h\setminus K}\|\mathbf{u}-\mathbf{u}^h\|^2_{0,\tau}\\ &\le \sum_{\tau\in K} h^{2(1-\beta)}\|\mathbf{u}-\mathbf{u}^h\|^2_{0,\beta-1,\tau} + \sum_{\tau\in\mathcal{T}_h\setminus K}\|\mathbf{u}-\mathbf{u}^h\|^2_{0,\tau}. \end{aligned} \tag{5.11}$$

When $\tau \in \mathcal{T}_h\setminus K$, similarly to (5.10), we have

$$\|\mathbf{u}-\mathbf{u}^h\|^2_{0,\tau} \le c\, h^2\|\mathbf{u}-\mathbf{u}^h\|^2_{L^6(\tau)}.$$

Then, the continuous imbedding from H^1 into L^6 and $h \le cr$ when $\tau \in \mathcal{T}_h\setminus K$ yield

$$\begin{aligned}&\sum_{\tau\in\mathcal{T}_h\setminus K}\|\mathbf{u}-\mathbf{u}^h\|^2_{0,\tau} \le c\, h^2 \sum_{\tau\in\mathcal{T}_h\setminus K}\|\mathbf{u}-\mathbf{u}^h\|^2_{L^6(\tau)} \le c\, h^2\|\mathbf{u}-\mathbf{u}^h\|^2_{L^6(\mathcal{T}_h\setminus K)}\\ &\le ch^2\|\mathbf{u}-\mathbf{u}^h\|^2_{H^1(\mathcal{T}_h\setminus K)} \le c\, h^2h^{-2\beta}\int_{\mathcal{T}_h\setminus K} r^{2\beta}\left(|\mathbf{u}-\mathbf{u}^h|^2 + |\nabla(\mathbf{u}-\mathbf{u}^h)|^2\right)d\Omega\\ &\le\ ch^{2(1-\beta)}\|\mathbf{u}-\mathbf{u}^h\|^2_{1,\beta}.\end{aligned}$$

Combining the above with (5.11) and using (5.9) give

$$\|\mathbf{u}-\mathbf{u}^h\| \le ch^{1-\beta}\|\mathbf{u}-\mathbf{u}^h\|_{1,\beta} \le ch^{\alpha}\|\mathbf{u}\|_{\alpha+\beta,\beta}.$$

For $\beta \ge 1$, by the triangle inequality, lemma 5.2, and (5.9), we have

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}^h\|^2 &\le c\|\mathbf{u}-\mathcal{I}_0^h\mathbf{u}\|^2 + c\|\mathcal{I}_0^h\mathbf{u}-\mathbf{u}^h\|^2 \le c\|\mathbf{u}-\mathcal{I}_0^h\mathbf{u}\|^2 + ch^{2(1-\beta)}\|\mathcal{I}_0^h\mathbf{u}-\mathbf{u}^h\|^2_{0,\beta-1}\\ &\le c\sum_{\tau\in\mathcal{T}_h}\|\mathbf{u}-\mathcal{I}_0^h\mathbf{u}\|^2_{0,\tau} + ch^{2(1-\beta)}\left(\|\mathbf{u}-\mathcal{I}_0^h\mathbf{u}\|^2_{0,\beta-1} + \|\mathbf{u}-\mathbf{u}^h\|^2_{0,\beta-1}\right)\\ &\le c\sum_{\tau\in\mathcal{T}_h}\|\mathbf{u}-\mathcal{I}_0^h\mathbf{u}\|^2_{0,\tau} + ch^{2(1-\beta)}\|\mathbf{u}-\mathcal{I}_0^h\mathbf{u}\|^2_{1,\beta}.\end{aligned} \tag{5.12}$$

If $\tau \in \mathcal{T}_h\setminus K$, then we simply use $h \le cr$ to obtain

$$\|\mathbf{u}-\mathcal{I}_0^h\mathbf{u}\|^2_{0,\tau} \le ch^{2(1-\beta)}\|\mathbf{u}-\mathcal{I}_0^h\mathbf{u}\|^2_{0,\beta-1,\tau}. \tag{5.13}$$

If $\tau \in K$, we recall the definition of the operator $\mathcal{I}_0^h$ and let δ be a smooth cut-off function defined in (5.4). By triangle inequality,

$$\|\mathbf{u}-\mathcal{I}_0^h\mathbf{u}\|^2_{0,\tau} \le c\|(1-\delta)\mathbf{u}-\mathcal{I}_0^h((1-\delta)\mathbf{u})\|^2_{0,\tau} + c\|\delta\mathbf{u}-\mathcal{I}_0^h(\delta\mathbf{u})\|^2_{0,\tau}.$$

Since $(1-\delta)\mathbf{u} \in H^{\alpha+\beta}(\tau)^3$ and $\mathcal{I}_0^h((1-\delta)\mathbf{u}) = \mathcal{I}^h((1-\delta)\mathbf{u})$ from the definition, the analogous calculation in the proof of lemma 5.1 provides

$$\begin{aligned}&\|(1-\delta)\mathbf{u}-\mathcal{I}_0^h((1-\delta)\mathbf{u})\|^2_{0,\tau} = \|(1-\delta)\mathbf{u}-\mathcal{I}^h((1-\delta)\mathbf{u})\|^2_{0,\tau}\\ &\le ch^{2(\alpha+\beta)}\int_\tau \left|D^{\alpha+\beta}((1-\delta)\mathbf{u})\right|^2 d\tau = ch^{2(\alpha+\beta)}\iiint_{\frac{h}{3k}}^{r(\theta)}\left|D^{\alpha+\beta}((1-\delta)\mathbf{u})\right|^2 d\tau\\ &\le ch^{2(\alpha+\beta)}h^{-2\beta}\iiint_{\frac{h}{3k}}^{r(\theta)} r^{2\beta}\left|D^{\alpha+\beta}((1-\delta)\mathbf{u})\right|^2 d\tau\\ &\le ch^{2\alpha}\sum_{l=0}^{\alpha+\beta}\int_\tau r^{2(\beta-(\alpha+\beta)+l)}\left|D^l\mathbf{u}\right|^2 d\tau = ch^{2\alpha}\|\mathbf{u}\|^2_{\alpha+\beta,\beta,\tau}.\end{aligned} \tag{5.14}$$

By the definition of $\mathcal{I}_0^h$ and $\|\mathbf{u}\|_{0,\beta-(\alpha+\beta)} \leq \|\mathbf{u}\|_{\alpha+\beta,\beta}$, we have

$$\begin{aligned}\|\delta\mathbf{u} - \mathcal{I}_0^h(\delta\mathbf{u})\|_{0,\tau}^2 &= \|\delta\mathbf{u}\|_{0,\tau}^2 \leq \int_\tau r^{2\alpha} r^{-2\alpha}|\mathbf{u}|^2 d\tau \leq ch^{2\alpha}\int_\tau r^{-2\alpha}|\mathbf{u}|^2 d\tau \\ &= ch^{2\alpha}\int_\tau r^{2(\beta-(\alpha+\beta))}|\mathbf{u}|^2 d\tau = ch^{2\alpha}\|\mathbf{u}\|_{0,\beta-(\alpha+\beta),\tau}^2 \leq ch^{2\alpha}\|\mathbf{u}\|_{\alpha+\beta,\beta,\tau}^2. \quad (5.15)\end{aligned}$$

Substituting (5.13), (5.14), and (5.15) into (5.12), and applying lemma 5.1 yield

$$\|\mathbf{u} - \mathbf{u}^h\|^2 \leq ch^{2\alpha}\sum_{\tau\in K}\|\mathbf{u}\|_{\alpha+\beta,\beta,\tau}^2 + ch^{2(1-\beta)}\|\mathbf{u} - \mathcal{I}_0^h\mathbf{u}\|_{1,\beta}^2 \leq ch^{2\alpha}\|\mathbf{u}\|_{\alpha+\beta,\beta}.$$

□

Minimizing the least-squares functional in the weighted norm using H^1-conforming elements and appropriate values of the weight allows the approximate solution to converge. The above theorem also implies optimal L^2-error convergence with H^1-conforming finite elements. In the next section, we consider error bounds using graded mesh refinements.

6. Graded Mesh Refinement. The paper [4] is one of the earliest papers to examine graded mesh refinement in the context of weighted Sobolev spaces. Following that, many have used similar mesh refinements when weighted Sobolev spaces were considered to overcome the difficulties associated with singularities (see [2], [3]). In this section we show some error estimate results using graded mesh refinements in our methodology.

Assume $\beta < 1$. For the global mesh parameter, h $(0 < h < 1)$, let $\mathcal{T}_h^m$ be a triangulation of the domain $\overline{\Omega} = \cup_{\tau\in\mathcal{T}_h^m}\tau$, where the diameter, $h_\tau = diam(\tau)$, of each tetrahedron, τ, is defined by

$$h_\tau \sim \begin{cases} h^{\frac{1}{1-\beta}} & \text{if} \quad \tau\cap E \neq \emptyset, \\ hr_\tau^\beta & \text{if} \quad \tau\cap E = \emptyset, \end{cases} \tag{6.1}$$

where $r_\tau = \min\{r = \sqrt{x^2+y^2} : (x,y,z)\in\tau\}$. Define $\mathcal{V}^h$ to be a space of $\mathcal{C}^0$ piecewise polynomials on each finite element satisfying $\mathbf{n}\times\mathbf{v}^h = \mathbf{0}$ on Γ_D and $\mathbf{n}\cdot\mathbf{v}^h = 0$ on Γ_N and let $\mathbf{u}^h$ satisfy

$$G_w(\mathbf{u}^h; f) = \min_{\mathbf{v}^h\in\mathcal{V}^h} G_w(\mathbf{v}^h; f). \tag{6.2}$$

THEOREM 6.1. *Let $\mathbf{u}\in H_\beta^{\alpha+\beta}(\Omega)^3\cap\mathcal{U}$ and $\mathbf{u}^h\in\mathcal{V}^h$ be the solutions of (2.3) and (6.2), respectively. Suppose that $\beta < 1$ and the hypotheses of lemma 3.1 are satisfied. Then,*

$$\|\mathbf{u} - \mathbf{u}^h\| \leq c\ h^{\alpha+\beta}\|\mathbf{u}\|_{\alpha+\beta,\beta}, \tag{6.3}$$

where $3/2 < \alpha+\beta \leq k+1$ and k is the degree of the piecewise polynomials in $\mathcal{V}^h$.

Proof. Let $K = \{\tau\in\mathcal{T}_h^m : \tau\cap E\neq\emptyset\}$. By the Cauchy inequality and (6.1), we have

$$\begin{aligned}
\|\mathbf{u}-\mathbf{u}^h\|^2 &= \sum_{\tau\in K}\|\mathbf{u}-\mathbf{u}^h\|^2_{0,\tau} + \sum_{\tau\in\mathcal{T}_h^m\setminus K}\|\mathbf{u}-\mathbf{u}^h\|^2_{0,\tau} \\
&\le \sum_{\tau\in K} ch_\tau^{2(1-\beta)}\|\mathbf{u}-\mathbf{u}^h\|^2_{0,\beta-1,\tau} + \sum_{\tau\in\mathcal{T}_h^m\setminus K}\left(\int_\tau 1d\tau\right)^{\frac{2}{3}}\left(\int_\tau|\mathbf{u}-\mathbf{u}^h|^6 d\tau\right)^{\frac{1}{3}} \\
&\le \sum_{\tau\in K} ch^2\|\mathbf{u}-\mathbf{u}^h\|^2_{0,\beta-1,\tau} + \sum_{\tau\in\mathcal{T}_h^m\setminus K} ch_\tau^2||\mathbf{u}-\mathbf{u}^h||^2_{L^6(\tau)} \\
&\le \sum_{\tau\in K} ch^2\|\mathbf{u}-\mathbf{u}^h\|^2_{0,\beta-1,\tau} + \sum_{\tau\in\mathcal{T}_h^m\setminus K} ch^2||r^\beta(\mathbf{u}-\mathbf{u}^h)||^2_{L^6(\tau)}. \qquad (6.4)
\end{aligned}$$

The continuous imbedding of $H^1(\Omega)$ into $L^6(\Omega)$ and $r^\beta\mathbf{u}\in H^1(\Omega)^3$ lead to

$$\begin{aligned}
\sum_{\tau\in\mathcal{T}_h^m\setminus K} ch^2||r^\beta(\mathbf{u}-\mathbf{u}^h)||^2_{L^6(\tau)} &\le ch^2||r^\beta(\mathbf{u}-\mathbf{u}^h)||^2_{L^6(\Omega)} \le ch^2||r^\beta(\mathbf{u}-\mathbf{u}^h)||^2_1 \\
&\le ch^2||\mathbf{u}-\mathbf{u}^h||^2_{1,\beta}. \qquad (6.5)
\end{aligned}$$

We define $\mathcal{I}_0^h$ in the same manner as in section 5 on each element $\tau\in\mathcal{T}_h^m$. Then, a calculation analogous to the proof of lemma 5.1 yields

$$\begin{aligned}
\|\mathbf{u}-\mathcal{I}_0^h\mathbf{u}\|_{1,\beta} &\le ch^{\alpha+\beta-1}|\mathbf{u}|_{\alpha+\beta,(\alpha+\beta)\beta} + ch^{\frac{\alpha+\beta-1}{1-\beta}}\|\mathbf{u}\|_{\alpha+\beta,\beta} \\
&= ch^{\alpha+\beta-1}\left(|\mathbf{u}|_{\alpha+\beta,(\alpha+\beta)\beta} + ch^{\frac{\alpha+\beta-1}{1-\beta}\beta}\|\mathbf{u}\|_{\alpha+\beta,\beta}\right). \qquad (6.6)
\end{aligned}$$

In (6.6), since $r^{(\alpha+\beta)\beta} < r^\beta$ and $h^{\frac{\alpha+\beta-1}{1-\beta}\beta} < 1$, we have

$$\|\mathbf{u}-\mathcal{I}_0^h\mathbf{u}\|_{1,\beta} \le ch^{\alpha+\beta-1}\|\mathbf{u}\|_{\alpha+\beta,\beta}. \qquad (6.7)$$

Since $\mathbf{u}^h$ is the minimizer of (6.2), theorem 3.4 and triangle inequality yield

$$\|\mathbf{u}-\mathbf{u}^h\|_{1,\beta} \le c\|L(\mathbf{u}-\mathbf{u}^h)\|_{0,\beta} \le c\|L(\mathbf{u}-\mathcal{I}_0^h\mathbf{u})\|_{0,\beta} \le c\|\mathbf{u}-\mathcal{I}_0^h\mathbf{u}\|_{1,\beta}. \qquad (6.8)$$

Therefore,(6.4), (6.5), (6.8), and (6.7) imply the error bound (6.3). □

The above theorem shows that we obtain better L^2-error convergence if we use graded mesh refinement. However, the results in section 5 demonstrate optimal error convergence without doing the extra work of mesh refinement. In the following section, we present numerical tests on uniform grids.

7. Computational results. In this section, we present some numerical examples. By minimizing the weighted-norm least-squares functional, we verify the optimal error convergence given in section 5. As mentioned in the introduction, if the internal angle of the edge, ω, is bigger than π or different types of boundary conditions meet at an edge with the internal angle bigger than $\pi/2$, then the boundary singularities occur. Thus, the solution of the Poisson equation is in H^{1+s}, where $s < \pi/\omega$ if the problem has Dirichlet or Neumann boundary conditions at a reentrant edge, or $s < \pi/(2\omega)$ if Dirichlet and Neumann boundary conditions meet at an edge with inner angle ω ([17]). Here, we construct our test problems based on the leading term of the singular part of the solution of Poisson's equation.

The software package FOSPACK (cf. [30]) was used to build the discrete system and to solve it by a conjugate gradient iterative method preconditioned by algebraic multigrid (AMG) using W(1,1)-cycles. The stopping criterion for the iteration was a residual reduction 10^{-8}. This unnecessarily large reduction was used to remove the algebraic error from the calculation of the convergence of the discrete solution. No graded mesh refinement is applied.

The domain of examples 7.1 and 7.2 is an L-shaped polyhedral cylinder :

$$\Omega = (-0.5, 0.5) \times (-0.5, 0.5) \times (0, 1) \backslash [0, 0.5) \times (-0.5, 0] \times (0, 1).$$

EXAMPLE 7.1. *We choose* $\mathbf{u} = \nabla p$*, where* p *has the form*

$$p = \delta(r) r^{\frac{2}{3}} \sin(2\theta/3) \sin(\pi z)$$

in the cylindrical coordinate system with $\delta(r)$ *a smooth cut-off function satisfying* $\delta(r) = 1$ *when* $r \le 0.25$ *and* $\delta(r) = 0$ *when* $r \ge 0.375$ *and choose* f *by* Δp*. Then,* p *satisfies*

$$\Delta p = f \ , \quad in\ \Omega \tag{7.1}$$

with $p = 0$ *on* $\partial\Omega$ *and* $\mathbf{u} \in H^{\frac{2}{3}}(\Omega)^3$ *satisfies*

$$\nabla \times \mathbf{u} = \mathbf{0}, \quad \nabla \cdot \mathbf{u} = f, \quad in\ \Omega \tag{7.2}$$

$$\mathbf{n} \times \mathbf{u} = \mathbf{0}, \quad on\ \partial\Omega. \tag{7.3}$$

	$\|\|\mathbf{u} - \mathbf{u}^h\|\|$		$\|\|\mathbf{u} - \mathbf{u}^h\|\|_{0,\beta}$		$\|\mathbf{u} - \mathbf{u}^h\|_{1,\beta}$	
1/8	2.398e-01	rates	3.734e-02	rates	7.775e-01	rates
1/16	1.074e-01	1.16	1.252e-02	1.58	2.223e-01	1.81
1/32	5.532e-02	0.96	3.116e-03	2.01	6.177e-02	1.85
1/64	3.350e-02	0.72	7.891e-04	1.98	1.875e-02	1.72
1/128	2.098e-02	0.68	1.978e-04	2.00	6.855e-03	1.45

TABLE 7.1
Example 1 with $\beta = 4/3$

It is easy to see that $\mathbf{u}$ in the above example is not in $H^1(\Omega)^3$. Since $\alpha = 2/3$ in this case, our theory holds for any β between 5/6 and 5/3. Here, we choose $\beta = 4/3$ so that $3/2 < \alpha + \beta = 2$ and observe the errors in L^2-, weighted L^2-, and weighted H^1-norms in table 7.1. The error estimates in theorem 5.3 predict the asymptotic L^2-error convergence rate to be approximately $O(h^{\frac{2}{3}})$ and the weighted L^2- and weighted H^1-rates at $O(h^2)$ and $O(h)$, respectively. We report the numbers in table 7.1 to support our theory. The convergence rate of the weighted H^1-seminorm in table 7.1 shows better convergence than expected at the resolution we are able to compute. However, based on two dimensional test results, it is likely that on increasingly fine meshes the asymptotic rate will approach to what theory predicted.

In figure 7.1, we present the results with more β values when mesh size decreases from 1/64 to 1/128. It shows that the errors in L^2- and weighted H^1-norms behave as expected and better than expected for weighted L^2-norm error. The darker shaded region is the area where theorem 5.3 holds.

EXAMPLE 7.2. *In this example, we choose* $\mathbf{u} = \nabla p$ *and* $f = \Delta p$*, where*

$$p = \delta(r) r^{\frac{2}{3}} \cos(2\theta/3) \cos(\pi z)$$

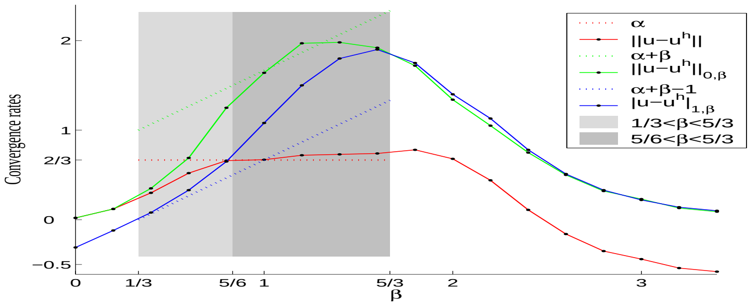

FIG. 7.1. *Finite element convergence rate as a function of β for example 7.1*

with the same smooth cut-off function, $\delta(r)$, defined in example 7.1. Then, p satisfies a Poisson problem with Neumann boundary condition, that is, (7.1) with $\mathbf{n}\cdot\nabla p = 0$ on $\partial\Omega = \Gamma_N$, and $\mathbf{u} \in H^{\frac{2}{3}}(\Omega)^3$ satisfies (7.2) but with $\mathbf{n}\cdot\mathbf{u} = 0$ on $\partial\Omega$.

The example 7.2 is similar to the example 7.1 but with a different type of boundary condition. The range of β satisfying theorem 5.3 is the interval $(5/6, 1)$, but numerical tests show that $\beta = 1$ gives the best convergence results. Here, we report the numerical results with $\beta = 1$ (see table 7.2). With $\beta = 1$, we expect the L^2-, weighted L^2-, and weighted H^1-error convergence rates to be approximately $O(h^{\frac{2}{3}})$, $O(h^{\frac{5}{3}})$, and $O(h^{\frac{2}{3}})$, respectively. Figure 7.2 shows similar behavior of figure 7.1 when the mesh h moves from $1/64$ to $1/128$. It shows that the convergence rates follow the theory for a while after β becomes bigger than 1, but soon the rates break down. As briefly mentioned in remark 3.2, the restriction on β is sufficient and figure 7.2 shows the possibility that the bounds hold beyond the restriction. However, again, we do not address that issue here.

	$\|\|\mathbf{u}-\mathbf{u}^h\|\|$		$\|\|\mathbf{u}-\mathbf{u}^h\|\|_{0,\beta}$		$\|\mathbf{u}-\mathbf{u}^h\|_{1,\beta}$	
1/8	2.192e-01	rates	5.197e-02	rates	1.099e+00	rates
1/16	1.122e-01	0.97	1.842e-02	1.50	3.288e-01	1.74
1/32	6.028e-02	0.90	4.986e-03	1.89	1.131e-01	1.54
1/64	3.683e-02	0.71	1.461e-03	1.77	5.425e-02	1.06
1/128	2.310e-02	0.67	4.733e-04	1.63	3.201e-02	0.76

TABLE 7.2
Example 2 with $\beta = 1$

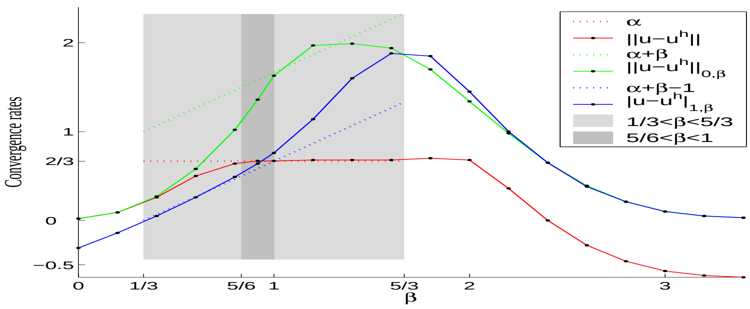

FIG. 7.2. *Finite element convergence rate as a function of β for example 7.2*

As the last example, we introduce mixed boundary conditions and the domain $\Omega = (-0.5, 0.5) \times (0, 0.5) \times (0, 0.5)$.

EXAMPLE 7.3. *Let* $\mathbf{u} = \nabla p$ *and* $f = \Delta p$, *where* p *has the form*

$$p = \delta(r) r^{\frac{1}{2}} \sin(\theta/2) \sin(2\pi z).$$

Then, $\mathbf{u} \in H^{\frac{1}{2}}(\Omega)^3$ *satisfies (7.2) and boundary conditions*

$$\begin{aligned} \mathbf{n} \cdot \mathbf{u} = 0 \quad &\text{on } \Gamma_N = \{(r, \theta, z) \in \partial\Omega : \theta = \pi\}, \\ \mathbf{n} \times \mathbf{u} = \mathbf{0} \quad &\text{on } \Gamma_D = \partial\Omega \backslash \Gamma_N. \end{aligned}$$

	$\|\|\mathbf{u} - \mathbf{u}^h\|\|$		$\|\|\mathbf{u} - \mathbf{u}^h\|\|_{0,\beta}$		$\|\mathbf{u} - \mathbf{u}^h\|_{1,\beta}$	
1/6	3.702E-01	rates	3.332E-02	rates	6.694E-01	rates
1/12	1.168E-01	1.66	1.106E-02	1.59	2.608E-01	1.36
1/24	6.542E-02	0.84	3.553E-03	1.64	7.157E-02	1.87
1/48	4.447E-02	0.56	8.975E-04	1.99	2.213E-02	1.69
1/96	3.125E-02	0.51	2.301E-04	1.96	8.405E-03	1.40
1/192	2.204E-02	0.50	6.179E-05	1.90	3.975E-03	1.08

TABLE 7.3
Example 3 with $\beta = 1.4$

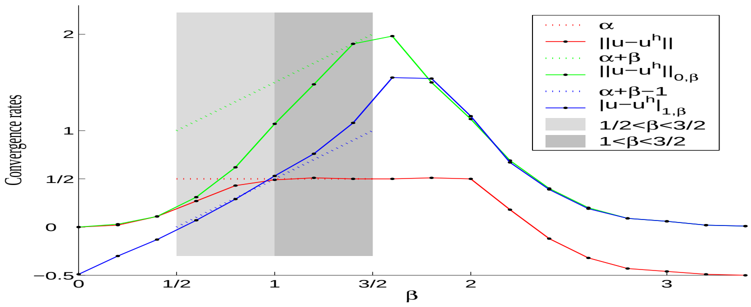

FIG. 7.3. *Finite element convergence rate as a function of* β *for example 7.3*

In example 7.3, different types of boundary conditions meet at an edge with inner angle π. Still, boundary singularity occurs along this edge, since the internal angle is bigger than $\pi/2$. In this case, the condition $1 < \beta < 3/2$ guarantees that theorem 5.3 holds. According to theorem 5.3, asymptotic convergence rates are $O(h^{\frac{1}{2}})$ for L^2-error, $O(h^{\frac{1}{2}+\beta})$ for weighted L^2-error, and $O(h^{\beta-\frac{1}{2}})$ for weighted H^1-error. Table 7.3 shows that the error convergence in L^2-, weighted L^2-, weighted H^1-norms with $\beta = 1.4$, where $3/2 < 1/2 + \beta$. In figure 7.3, the results with more β values when mesh size decreases from 1/96 to 1/192 are presented.

8. Conclusion. Low regularity of the solution caused by singularities on the boundary usually precludes the use of H^1-conforming finite elements in standard L^2 FOSLS functional. To overcome these difficulties, we use weighted norms in the least-squares functional. This is based on the idea of unweighting the area where the singularity occurs by multiplying by the weight functions. In this paper, we showed that minimization of least-squares functional in the weighted norm allows to approximately solve the div/curl system in H^1-conforming finite element spaces with optimal error convergence in L^2-, weighted L^2-, and weighted H^1-norms without using graded mesh refinement.

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