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Research Article

A New Sufficient Condition for Checking the Robust Stabilization of Uncertain Descriptor Fractional-Order Systems

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We consider the robust asymptotical stabilization of uncertain a class of descriptor fractional-order systems. In the state matrix, we require that the parameter uncertainties are time-invariant and norm-bounded. We derive a sufficient condition for the system with the fractional-order α satisfying $1 \leq \alpha < 2$ in terms of linear matrix inequalities (LMIs). The condition of the proposed stability criterion for fractional-order system is easy to be verified. An illustrative example is given to show that our result is effective.

1. Introduction

Descriptor systems arise naturally in many applications such as aerospace engineering, social economic systems, and network analysis. Sometimes we also call descriptor systems singular systems. Descriptor system theory is an important part in control systems theory. Since 1970s, descriptor systems have been wildly studied, for example, descriptor linear systems [1], descriptor nonlinear systems [2–4], and discrete descriptor systems [5–7]. In particular, Dai has systematically introduced the theoretical basis of descriptor systems in [8], which is the first monograph on this subject. A detailed discussion of descriptor systems and their applications can be found in [9, 10].

It is well known that fractional-order systems have been studied extensively in the last 20 years, since the fractional calculus has been found many applications in viscoelastic systems [11–14], robotics [15–18], finance system [19–21], and many others [22–26]. Studying on fractional-order calculus has become an active research field. To the best of our knowledge, although stability analysis is a basic problem in control theory, very few works existed for the stability analysis for descriptor fractional-order systems.

Many problems related to stability of descriptor fractional-order control systems are still challenging and unsolved. For the nominal stabilization case, N'Doye et al. [27] study the stabilization of one descriptor fractional-order system with the fractional-order α, $1 < \alpha < 2$, in terms of LMIs. N'Doye et al. [28] derive some sufficient conditions for the robust asymptotical stabilization of uncertain descriptor fractional-order systems with the fractional-order α satisfying $0 < \alpha < 2$. Furthermore, Ma et al. [29] study the robust stability and stabilization of fractional-order linear systems with positive real uncertainty. Note that, in Example 1, by applying Theorem 2 [27], it is harder to determine whether the uncertain descriptor fractional-order system (6) is asymptotically stable. Therefore, it is valuable to seek sufficient conditions, for checking the robust asymptotical stabilization of uncertain descriptor fractional-order systems.

In this paper, we study the stabilization of a class of descriptor fractional-order systems with the fractional-order α, $1 \leq \alpha < 2$, in terms of LMIs. We derive a new sufficient condition for checking the robust asymptotical stabilization of uncertain descriptor fractional-order systems with the fractional-order α satisfying $1 \leq \alpha < 2$, in terms of

LMIs. It should be mentioned that, compared with some prior works, our main contributions consist in the following: (1) we assume that the matrix of uncertain parameters in the uncertain descriptor fractional-order system is diagonal. Thus, compared with the results in [28], our conclusion, Theorem 8, is more feasible and effective and has wider applications; (2) compared with some stability criteria of fractional-order nonlinear systems, for example, in [9, 22], our method is easier to be used.

Notations: throughout this paper, $\mathbb{R}^{m\times n}$ stands for the set of m by n matrices with real entries, M^T stands for the transpose of M, $Sym\{X\}$ denotes the expression $X^T + X$, I_n denotes the identity matrix of order n, $\operatorname{diag}(a_1, a_2, \ldots, a_n)$ denotes the diagonal matrix, and $\bullet$ will be used in some matrix expressions to indicate a symmetric structure; i.e., if given matrices $H_1 = H_1^T \in \mathbb{R}^{m\times m}$ and $H_2 = H_2^T \in \mathbb{R}^{n\times n}$, then

$$\begin{pmatrix} H_1 & \bullet \\ L & H_2 \end{pmatrix} = \begin{pmatrix} H_1 & L^T \\ L & H_2 \end{pmatrix}. \quad (1)$$

2. Preliminary Results

Consider the following class of linear fractional-order systems:

$$\begin{aligned} {}_0^C D_t^\alpha x(t) &= Ax(t), \\ x(0) &= x_0, \end{aligned} \quad (2)$$

where $0 < \alpha < 2$ is the fractional-order, $x(t) \in \mathbb{R}^n$ is the state vector, $A \in \mathbb{R}^{n\times n}$ is a constant matrix, and ${}_0^C D_t^\alpha$ represent the fractional-order derivative, which can be expressed as

$${}_0^C D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\, d\tau, \quad (3)$$

where $\Gamma(\cdot)$ is the Euler Gamma function. For convenience, we use D^α to replace ${}_0^C D_t^\alpha$ in the rest of this paper. It is well known that system (2) is stable if [30–32]

$$\left|\arg(\operatorname{spec}(A))\right| > \alpha\frac{\pi}{2} \quad (4)$$

where $0 < \alpha < 2$ and $\operatorname{spec}(A)$ is the spectrum of all eigenvalues of A.

The next lemma, given by Chilali et al. [33], contains the necessary and sufficient conditions of (4) in terms of LMI, when the fractional-order α belongs to $1 \le \alpha < 2$.

Lemma 1 (see [33]). *Let $A \in \mathbb{R}^{n\times n}$ be a real matrix and $1 \le \alpha < 2$. Then $|\arg(spec(A))| > (\pi/2)\alpha$ if and only if there exists $P > 0$ such that*

$$\begin{pmatrix} \left(AP + PA^T\right)\sin\theta & \bullet \\ \left(PA^T - AP\right)\cos\theta & \left(AP + PA^T\right)\sin\theta \end{pmatrix} < 0. \quad (5)$$

Consider the following uncertain descriptor fractional-order systems:

$$\begin{aligned} ED^\alpha x(t) &= \left(A + \Delta_A\right)x(t) + Bu(t) \\ x(0) &= x_0 \end{aligned} \quad (6)$$

where $1 \le \alpha < 2$, $x(t) \in \mathbb{R}^n$ is the semistate vector, $u(t) \in \mathbb{R}^m$ is the control input, $E \in \mathbb{R}^{n\times n}$ is singular, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are constant matrices, and the time-invariant matrix Δ_A corresponds to a norm-bounded parameter uncertainty, which is the following form:

$$\Delta_A = M_A \Delta N_A \quad (7)$$

where M_A and N_A are real constant matrices of appropriate sizes, and the uncertain matrix $\Delta = (\gamma_{ij})_{p\times q}$ satisfies

$$\Delta\Delta^T \le I_p. \quad (8)$$

Remark 2. Condition $\Delta\Delta^T \le I_p$ is rational because a lot of system uncertainties satisfy this inequality. Besides, this condition can also be used in many literatures, for example, in [9, 34–39].

It is well known that the following system

$$\begin{aligned} ED^\alpha x(t) &= Ax(t) + Bu(t) \\ x(0) &= x_0 \end{aligned} \quad (9)$$

is normalizable if and only if

$$\operatorname{rank}\begin{bmatrix} E & B \end{bmatrix} = n. \quad (10)$$

Further we have that the uncertain descriptor fractional-order systems (6) is normalizable if and only if the nominal descriptor fractional-order system (9) is normalizable.

Lemma 3 (see [28], Theorem 1). *System (6) is normalizable if and only if there exist a nonsingular matrix P and a matrix Y such that the following LMI*

$$EP + BY + P^T E^T + Y^T B^T < 0 \quad (11)$$

is satisfied. In this case, the gain matrix L is given by

$$L = YP^{-1}. \quad (12)$$

Assume that (6) is normalizable; by applying LMI (11), we obtain $L \in \mathbb{R}^{m\times n}$ such that $\operatorname{rank}(E + BL) = n$. Consider the feedback control for (6) in the following form:

$$u(t) = -LD^\alpha x(t) + Kx(t), \quad (13)$$

where $K \in \mathbb{R}^{m\times n}$ is one gain matrix such that the obtained normalized system is asymptotically stable. Then we have the closed-loop system:

$$(E + BL)D^\alpha x(t) = \left(A + \Delta_A + BK\right)x(t), \quad (14)$$

that is,

$$D^\alpha x(t) = \left(A_1 + B_1 K + E_1 \Delta_A\right)x(t) \quad (15)$$

where

$$\begin{aligned} E_1 &= (E + BL)^{-1}, \\ A_1 &= E_1 A, \\ B_1 &= E_1 B. \end{aligned} \quad (16)$$

To facilitate the description of our main results, we need the following results.

In [28], N'Doye et al. derive a sufficient condition for the robust asymptotical stabilization of uncertain descriptor fractional-order systems with the fractional-order α satisfying $1 \leq \alpha < 2$ in terms of LMIs.

Lemma 4 (see [28], Theorem 2). *Assume that (6) is normalizable; then there exists gain matrix K such that the uncertain descriptor fractional-order system (6) with fractional-order $1 \leq \alpha < 2$ controlled by the control (13) is asymptotically stable, if there exist matrices $X \in \mathbb{R}^{m\times n}$, $P_0 = P_0^T > 0 \in \mathbb{R}^{n\times n}$ and a real scalar $\delta > 0$, such that*

$$\begin{bmatrix} \Omega_{11} & \bullet & \bullet & \bullet \\ \Omega_{21} & \Omega_{22} & \bullet & \bullet \\ N_A P_0 & 0 & -\delta I & \bullet \\ 0 & N_A P_0 & 0 & -\delta I \end{bmatrix} < 0 \tag{17}$$

where

$$\begin{aligned} \Omega_{11} &= \Omega_{22} \\ &= \left(P_0 A_1^T + A_1 P_0 + B_1 X + X^T B_1^T\right)\sin\theta \\ &\quad + \delta E_1 M_A \left(E_1 M_A\right)^T, \\ \Omega_{21} &= \left(P_0 A_1^T - A_1 P_0 + X^T B_1^T - B_1 X\right)\cos\theta, \end{aligned} \tag{18}$$

with $\theta = \pi - \alpha(\pi/2)$ and matrices P and Y are given by LMI (11).

Moreover, the gain matrix K is given by

$$K = X P_0^{-1}. \tag{19}$$

Lemma 5 (see [40]). *For any matrices X and Y with appropriate sizes, we have*

$$X^T Y + Y^T X \leq \epsilon X^T X + \epsilon^{-1} Y^T Y, \tag{20}$$

for any $\epsilon > 0$.

Lemma 6 (see [41]). *Let X, Y, and Z be real matrices of appropriate sizes. Then, for any $x \in \mathbb{R}^n$,*

$$\begin{aligned} &\max\left\{\left(x^T X F Y x\right)^2 : F^T F \leq I\right\} \\ &\quad = \left(x^T X X^T x\right)\left(x^T Y^T Y x\right). \end{aligned} \tag{21}$$

3. Main Result

In this section, we present a new sufficient condition to design the gain matrix K. In the following theorem, Δ_M and Δ_N are given nonsingular matrices, such that

$$\Delta_M^{-1} \Delta \Delta_N^{-1} \left(\Delta_M^{-1} \Delta \Delta_N^{-1}\right)^T \leq I_p. \tag{22}$$

From now on, we denote $\widehat{\Delta} = \Delta_M^{-1} \Delta \Delta_N^{-1}$, $\widehat{\mathbf{M}} = E_1 M_A \Delta_M$, and $\widehat{\mathbf{N}} = \Delta_N N_A P$. It is obvious that $\widehat{\Delta}\widehat{\Delta}^T \leq I_p$. Thus, for any $\epsilon_1 > 0$ and $\epsilon_2 > 0$, by using Lemmas 5 and 6 and $\widehat{\Delta}\widehat{\Delta}^T \leq I_p$, we have

$$\begin{aligned} &\begin{bmatrix} \left(\widehat{\mathbf{M}}\widehat{\Delta}\widehat{\mathbf{N}} + \widehat{\mathbf{N}}^T \widehat{\Delta}^T \widehat{\mathbf{M}}^T\right)\sin\theta & 0 \\ 0 & \left(\widehat{\mathbf{M}}\widehat{\Delta}\widehat{\mathbf{N}} + \widehat{\mathbf{N}}^T \widehat{\Delta}^T \widehat{\mathbf{M}}^T\right)\sin\theta \end{bmatrix} \\ &= \begin{bmatrix} \widehat{\mathbf{M}}\left(\widehat{\Delta}\sin\theta\right) & 0 \\ 0 & \widehat{\mathbf{M}}\left(\widehat{\Delta}\sin\theta\right) \end{bmatrix} \begin{bmatrix} \widehat{\mathbf{N}} & 0 \\ 0 & \widehat{\mathbf{N}} \end{bmatrix} \\ &\quad + \begin{bmatrix} \widehat{\mathbf{N}} & 0 \\ 0 & \widehat{\mathbf{N}} \end{bmatrix}^T \begin{bmatrix} \widehat{\mathbf{M}}\left(\widehat{\Delta}\sin\theta\right) & 0 \\ 0 & \widehat{\mathbf{M}}\left(\widehat{\Delta}\sin\theta\right) \end{bmatrix}^T \\ &\leq \epsilon_1 \begin{bmatrix} \widehat{\mathbf{M}}\left(\widehat{\Delta}\sin\theta\right) & 0 \\ 0 & \widehat{\mathbf{M}}\left(\widehat{\Delta}\sin\theta\right) \end{bmatrix} \begin{bmatrix} \widehat{\mathbf{M}}\left(\widehat{\Delta}\sin\theta\right) & 0 \\ 0 & \widehat{\mathbf{M}}\left(\widehat{\Delta}\sin\theta\right) \end{bmatrix}^T \\ &\quad + \frac{1}{\epsilon_1} \begin{bmatrix} \widehat{\mathbf{N}} & 0 \\ 0 & \widehat{\mathbf{N}} \end{bmatrix}^T \begin{bmatrix} \widehat{\mathbf{N}} & 0 \\ 0 & \widehat{\mathbf{N}} \end{bmatrix} \\ &\leq \epsilon_1 \begin{bmatrix} \widehat{\mathbf{M}} & 0 \\ 0 & \widehat{\mathbf{M}} \end{bmatrix} \begin{bmatrix} \widehat{\mathbf{M}} & 0 \\ 0 & \widehat{\mathbf{M}} \end{bmatrix}^T + \frac{1}{\epsilon_1} \begin{bmatrix} \widehat{\mathbf{N}} & 0 \\ 0 & \widehat{\mathbf{N}} \end{bmatrix}^T \begin{bmatrix} \widehat{\mathbf{N}} & 0 \\ 0 & \widehat{\mathbf{N}} \end{bmatrix} \end{aligned} \tag{23}$$

and

$$\begin{aligned} &\begin{bmatrix} 0 & \left(\widehat{\mathbf{M}}\widehat{\Delta}\widehat{\mathbf{N}} - \widehat{\mathbf{N}}^T \widehat{\Delta}^T \widehat{\mathbf{M}}^T\right)\cos\theta \\ \left(\widehat{\mathbf{N}}^T \widehat{\Delta}^T \widehat{\mathbf{M}}^T - \widehat{\mathbf{M}}\widehat{\Delta}\widehat{\mathbf{N}}\right)\cos\theta & 0 \end{bmatrix} \\ &\leq \epsilon_2 \begin{bmatrix} \widehat{\mathbf{M}} & 0 \\ 0 & \widehat{\mathbf{M}} \end{bmatrix} \begin{bmatrix} \widehat{\mathbf{M}} & 0 \\ 0 & \widehat{\mathbf{M}} \end{bmatrix}^T + \frac{1}{\epsilon_2} \begin{bmatrix} \widehat{\mathbf{N}} & 0 \\ 0 & \widehat{\mathbf{N}} \end{bmatrix}^T \begin{bmatrix} \widehat{\mathbf{N}} & 0 \\ 0 & \widehat{\mathbf{N}} \end{bmatrix}, \end{aligned} \tag{24}$$

that is,

$$\begin{aligned} &\begin{bmatrix} \left(\widehat{\mathbf{M}}\widehat{\Delta}\widehat{\mathbf{N}} + \widehat{\mathbf{N}}^T \widehat{\Delta}^T \widehat{\mathbf{M}}^T\right)\sin\theta & \left(\widehat{\mathbf{M}}\widehat{\Delta}\widehat{\mathbf{N}} - \widehat{\mathbf{N}}^T \widehat{\Delta}^T \widehat{\mathbf{M}}^T\right)\cos\theta \\ \left(\widehat{\mathbf{N}}^T \widehat{\Delta}^T \widehat{\mathbf{M}}^T - \widehat{\mathbf{M}}\widehat{\Delta}\widehat{\mathbf{N}}\right)\cos\theta & \left(\widehat{\mathbf{M}}\widehat{\Delta}\widehat{\mathbf{N}} + \widehat{\mathbf{N}}^T \widehat{\Delta}^T \widehat{\mathbf{M}}^T\right)\sin\theta \end{bmatrix} \\ &\leq \left(\epsilon_1 + \epsilon_2\right) \begin{bmatrix} \widehat{\mathbf{M}} & 0 \\ 0 & \widehat{\mathbf{M}} \end{bmatrix} \begin{bmatrix} \widehat{\mathbf{M}} & 0 \\ 0 & \widehat{\mathbf{M}} \end{bmatrix}^T \\ &\quad + \left(\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2}\right) \begin{bmatrix} \widehat{\mathbf{N}} & 0 \\ 0 & \widehat{\mathbf{N}} \end{bmatrix}^T \begin{bmatrix} \widehat{\mathbf{N}} & 0 \\ 0 & \widehat{\mathbf{N}} \end{bmatrix}. \end{aligned} \tag{25}$$

Remark 7. Note that, when $\delta = 2$, we have $\epsilon_1 + \epsilon_2 \leq 2$ and

$$\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} \geq 2 > \frac{1}{\delta} = \frac{1}{2}. \tag{26}$$

That is, for any real scalar $\delta > 0$, and two matrices $X \in \mathbb{R}^{m\times n_1}$ and $Y \in \mathbb{R}^{n_2\times m}$, we cannot obtain real scalars $\epsilon_1 > 0$ and $\epsilon_2 > 0$ such that

$$\left(\epsilon_1 + \epsilon_2\right) X X^T + \left(\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2}\right) Y^T Y \leq \delta X X^T + \frac{1}{\delta} Y^T Y, \tag{27}$$

where

$$\begin{aligned} X &= \begin{bmatrix} E_1 M_A \Delta_M & 0 \\ 0 & E_1 M_A \Delta_M \end{bmatrix}, \\ Y &= \begin{bmatrix} \Delta_N N_A P & 0 \\ 0 & \Delta_N N_A P \end{bmatrix}. \end{aligned} \tag{28}$$

Theorem 8. *Assume that (6) is normalizable; then there exists a gain matrix K such that the uncertain descriptor fractional-order system (6) with fractional-order $1 \le \alpha < 2$ controlled by the controller (13) is asymptotically stable, if there exist matrices $X \in \mathbb{R}^{m\times n}$, $P = P^T > 0 \in \mathbb{R}^{n\times n}$ and two real scalars $\epsilon_1 > 0$ and $\epsilon_2 > 0$, such that*

$$\begin{bmatrix} \widehat{\Omega_{11}} & \bullet & \bullet & \bullet \\ \widehat{\Omega_{21}} & \widehat{\Omega_{22}} & \bullet & \bullet \\ \Delta_N N_A P & 0 & -\epsilon_1 I & \bullet \\ 0 & \Delta_N N_A P & 0 & -\epsilon_1 I \end{bmatrix} < 0, \tag{29}$$

where

$$\begin{aligned} \widehat{\Omega_{11}} &= \widehat{\Omega_{22}} \\ &= \left(PA_1^T + A_1P + B_1X + X^TB_1^T\right)\sin\theta \\ &\quad + (\epsilon_1 + \epsilon_2)\, E_1M_A\Delta_M \left(E_1M_A\Delta_M\right)^T, \\ \widehat{\Omega_{21}} &= \left(PA_1^T - A_1P + X^TB_1^T - B_1X\right)\cos\theta \end{aligned} \tag{30}$$

with $\theta = \pi - \alpha(\pi/2)$ and matrices P and Y are given by LMI (11).

Moreover, the gain matrix K is given by

$$K = XP^{-1}. \tag{31}$$

Proof. Suppose that there exist matrices $X \in \mathbb{R}^{m\times n}$, $P = P^T > 0 \in \mathbb{R}^{n\times n}$ and two real scalars $\epsilon_1 > 0$ and $\epsilon_2 > 0$ such that (29) holds. It is easy to derive that

$$\begin{bmatrix} \widehat{\Omega_{11}} & \bullet & \bullet & \bullet & \bullet & \bullet \\ \widehat{\Omega_{21}} & \widehat{\Omega_{22}} & \bullet & \bullet & \bullet & \bullet \\ \Delta_N N_A P & 0 & -\epsilon_1 I & \bullet & \bullet & \bullet \\ 0 & \Delta_N N_A P & 0 & -\epsilon_1 I & \bullet & \bullet \\ \Delta_N N_A P & 0 & 0 & 0 & -\epsilon_2 I & \bullet \\ 0 & \Delta_N N_A P & 0 & 0 & 0 & -\epsilon_2 I \end{bmatrix} < 0. \tag{32}$$

By using the Schur complement of (29), one obtains

$$\begin{aligned} &\begin{bmatrix} \left((A_1P + B_1X) + \left(PA_1^T + X^TB_1^T\right)\right)\sin\theta & \left((A_1P + B_1X) - \left(PA_1^T + X^TB_1^T\right)\right)\cos\theta \\ \left(\left(PA_1^T + X^TB_1^T\right) - (A_1P + B_1X)\right)\cos\theta & \left((A_1P + B_1X) + \left(PA_1^T + X^TB_1^T\right)\right)\sin\theta \end{bmatrix} + \epsilon_1 \begin{bmatrix} \widehat{\mathbf{M}} & 0 \\ 0 & \widehat{\mathbf{M}} \end{bmatrix} \begin{bmatrix} \widehat{\mathbf{M}} & 0 \\ 0 & \widehat{\mathbf{M}} \end{bmatrix}^T \\ &\quad + \frac{1}{\epsilon_1} \begin{bmatrix} \widehat{\mathbf{N}} & 0 \\ 0 & \widehat{\mathbf{N}} \end{bmatrix}^T \begin{bmatrix} \widehat{\mathbf{N}} & 0 \\ 0 & \widehat{\mathbf{N}} \end{bmatrix} + \epsilon_2 \begin{bmatrix} \widehat{\mathbf{M}} & 0 \\ 0 & \widehat{\mathbf{M}} \end{bmatrix} \begin{bmatrix} \widehat{\mathbf{M}} & 0 \\ 0 & \widehat{\mathbf{M}} \end{bmatrix}^T + \frac{1}{\epsilon_2} \begin{bmatrix} \widehat{\mathbf{N}} & 0 \\ 0 & \widehat{\mathbf{N}} \end{bmatrix}^T \begin{bmatrix} \widehat{\mathbf{N}} & 0 \\ 0 & \widehat{\mathbf{N}} \end{bmatrix} < 0. \end{aligned} \tag{33}$$

Write $K = XP^{-1}$. It follows from applying (25) that

$$\begin{aligned} &\begin{bmatrix} \left((A_1 + B_1K + E_1M_A\Delta N_A)P + P(A_1 + B_1K + E_1M_A\Delta N_A)^T\right)\sin\theta & \left((A_1 + B_1K + E_1M_A\Delta N_A)P - P(A_1 + B_1K + E_1M_A\Delta N_A)^T\right)\cos\theta \\ \left(-(A_1 + B_1K + E_1M_A\Delta N_A)P + P(A_1 + B_1K + E_1M_A\Delta N_A)^T\right)\cos\theta & \left((A_1 + B_1K + E_1M_A\Delta N_A)P + P(A_1 + B_1K + E_1M_A\Delta N_A)^T\right)\sin\theta \end{bmatrix} \\ &= \begin{bmatrix} \left((A_1P + B_1KP) + \left(PA_1^T + PK^TB_1^T\right)\right)\sin\theta & \left((A_1P + B_1KP) - \left(PA_1^T + PK^TB_1^T\right)\right)\cos\theta \\ \left(\left(PA_1^T + PK^TB_1^T\right) - (A_1P + B_1KP)\right)\cos\theta & \left((A_1P + B_1KP) + \left(PA_1^T + PK^TB_1^T\right)\right)\sin\theta \end{bmatrix} \\ &+ \begin{bmatrix} \left(\widehat{\mathbf{M}}\widehat{\Delta}\widehat{\mathbf{N}} + \widehat{\mathbf{N}}^T\widehat{\Delta}^T\widehat{\mathbf{M}}^T\right)\sin\theta & 0 \\ 0 & \left(\widehat{\mathbf{M}}\widehat{\Delta}\Delta_N N_A P + \widehat{\mathbf{N}}^T\widehat{\Delta}^T\widehat{\mathbf{M}}^T\right)\sin\theta \end{bmatrix} + \begin{bmatrix} 0 & \left(\widehat{\mathbf{M}}\widehat{\Delta}\widehat{\mathbf{N}} - \widehat{\mathbf{N}}^T\widehat{\Delta}^T\widehat{\mathbf{M}}^T\right)\cos\theta \\ \left(\widehat{\mathbf{N}}^T\widehat{\Delta}^T\widehat{\mathbf{M}}^T - \widehat{\mathbf{M}}\widehat{\Delta}\widehat{\mathbf{N}}\right)\cos\theta & 0 \end{bmatrix} < 0. \end{aligned} \tag{34}$$

By using the above inequality (34) and Lemma 1, we obtain

$$\left|\arg\left(spec\left(A_1 + B_1K + E_1M_A\Delta N_A\right)\right)\right| > \frac{\pi}{2}\alpha. \tag{35}$$

Therefore, system (6) is asymptotically stable. This ends the proof. □

Remark 9. Write

$$\begin{aligned} T = &\begin{bmatrix} \widehat{\Omega_{11}} & \bullet & \bullet & \bullet \\ \widehat{\Omega_{21}} & \widehat{\Omega_{22}} & \bullet & \bullet \\ \Delta_N N_A P & 0 & -\epsilon_1 I & \bullet \\ 0 & \Delta_N N_A P & 0 & -\epsilon_1 I \end{bmatrix} \\ &- \begin{bmatrix} \Omega_{11} & \bullet & \bullet & \bullet \\ \Omega_{21} & \Omega_{22} & \bullet & \bullet \\ N_AP_0 & 0 & -\delta I & \bullet \\ 0 & N_AP_0 & 0 & -\delta I \end{bmatrix} \end{aligned} \tag{36}$$

Note that if we choose $\Delta_M = I_p$ and $\Delta_N = I_q$ in LMI (29),

$$T = \begin{bmatrix} (\epsilon_1 + \epsilon_2 - \delta) E_1 M_A (E_1 M_A)^T & \bullet & \bullet & \bullet \\ 0 & (\epsilon_1 + \epsilon_2 - \delta) E_1 M_A (E_1 M_A)^T & \bullet & \bullet \\ 0 & 0 & -(\epsilon_1 - \delta) I & \bullet \\ 0 & 0 & 0 & -(\epsilon_1 - \delta) I \end{bmatrix}. \quad (37)$$

It is easy to see the following:

(1) For given δ, when $\epsilon_1 - \delta > 0$, it is always true that $\epsilon_1 + \epsilon_2 - \delta > 0$; that is, there do not exist ϵ_1 and ϵ_2 such that $T < 0$. Therefore, Theorem 8 is not a special case of Lemma 4 [28, Theorem 2], when $\Delta_M = I_p$ and $\Delta_N = I_q$.

(2) For given ϵ_1 and ϵ_2, when $\epsilon_1 - \delta < 0$, there exists ϵ_2 such that T is positive definite; that is, there exists δ such that $T > 0$. Since conditions in Lemma 4 and Theorem 8 are both sufficient, we cannot derive Lemma 4 by applying Theorem 8; that is, Theorem 8 is not a generalization of Lemma 4 [28, Theorem 2].

4. A Numerical Example

In this section, we assume that the matrix of uncertain parameters Δ in the uncertain descriptor fractional-order system (6) is diagonal. We provide a numerical example to illustrate that Theorem 8 is feasible and effective with wider applications.

Example 1. Consider the uncertain descriptor fractional-order system described in (6) with parameters as follows:

$$E = \begin{bmatrix} 1 & 0 & 0.5 \\ 2 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$
$$A = \begin{bmatrix} 2.4 & 0.2 & 1.2 \\ 4 & 1.5 & 2 \\ 0 & 0 & 0 \end{bmatrix},$$
$$B = \begin{bmatrix} 4 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \quad (38)$$
$$M_A = \begin{bmatrix} 0.1 & 0 & 0 \\ 0.1 & 0.3 & 4.8 \\ 0 & 0.2 & 0 \end{bmatrix},$$
$$N_A = \begin{bmatrix} 0 & 0 & 0 \\ 0.2 & 0 & \frac{1}{30} \\ 0 & 0.1 & 0 \end{bmatrix},$$

where $\alpha = 1.23$.

It is easy to check that $\mathrm{rank}(E) = 2 < 3$; that is, E is singular. By applying the LMI (11), we obtain

$$P = 10^8 \times \begin{bmatrix} 0.2613 & -1.5587 & 1.0635 \\ -1.5587 & -1.2010 & -0.7272 \\ 1.0635 & -0.7272 & 0.4664 \end{bmatrix},$$
$$Y = 10^8 \times \begin{bmatrix} -1.4992 & 0.7485 & 0.3345 \\ 0.6535 & -0.2534 & -2.4425 \end{bmatrix}, \quad (39)$$

and the gain matrix L

$$L = YP^{-1} = \begin{bmatrix} 1.0224 & -0.5004 & -2.3944 \\ -2.8016 & 1.6203 & 3.6775 \end{bmatrix}. \quad (40)$$

It follows from (16) that

$$E_1 = \begin{bmatrix} -0.1025 & 0.3106 & -0.2545 \\ 0.2781 & 0.3221 & 0.1544 \\ -0.2483 & 0.1088 & -0.1188 \end{bmatrix},$$
$$A_1 = \begin{bmatrix} 0.9963 & 0.4454 & 0.4981 \\ 1.9559 & 0.5388 & 0.9780 \\ -0.1605 & 0.1136 & -0.0802 \end{bmatrix}, \quad (41)$$
$$B_1 = \begin{bmatrix} -0.3541 & -0.3010 \\ 1.5889 & 0.9091 \\ -1.0030 & -0.3770 \end{bmatrix}.$$

Firstly, we compute P_0, X, δ, and K by using Lemma 4 [28, Theorem 2]. A feasible solution of LMI (11) is as follows:

$$P_0 = 10^{-7} \times \begin{bmatrix} 0.0560 & -0.0060 & -0.1056 \\ -0.0060 & 0.0489 & -0.0407 \\ -0.1056 & -0.0407 & 0.2552 \end{bmatrix},$$
$$X = \begin{bmatrix} -0.0022 & -0.0010 & 0.0056 \\ 0.0043 & 0.0001 & -0.0087 \end{bmatrix}, \quad (42)$$
$$\delta = 10^{-15} \times 1.0293,$$
$$K = XP_0^{-1} = 10^8 \times \begin{bmatrix} -2.0350 & -1.0959 & -1.0150 \\ 1.7432 & 0.9354 & 0.8674 \end{bmatrix}.$$

We choose

$$\Delta = \begin{bmatrix} \cos(0.8) & 0 & 0 \\ 0 & e^{-0.8} & 0 \\ 0 & 0 & \sin(0.1) \end{bmatrix}. \quad (43)$$

It follows from (15) that

$$S = A_1 + B_1K + E_1M_A\Delta N_A = 10^8 \times \begin{bmatrix} 0.1959 & 0.1065 & 0.0983 \\ -1.6487 & -0.8909 & -0.8242 \\ 1.3839 & 0.7465 & 0.6910 \end{bmatrix} \quad (44)$$

and the arguments of all eigenvalues of S are

$$3.1416, \quad 3.1416, \quad 0. \quad (45)$$

Based on those results, it is debatable whether or not system (6) is stable.

In the second way, we compute P_0, X, ϵ_1, ϵ_2, and K by using Theorem 8; we choose

$$\Delta_M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{12} \end{bmatrix}, \quad \Delta_N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 12 \end{bmatrix}. \quad (46)$$

It is easy to check that

$$\Delta_M^{-1}\Delta\Delta_N^{-1} = \begin{bmatrix} \cos(0.8) & 0 & 0 \\ 0 & e^{-0.8} & 0 \\ 0 & 0 & \sin(0.1) \end{bmatrix}, \quad M_A\Delta_M = \begin{bmatrix} 0.1 & 0 & 0 \\ 0.1 & 0.3 & 0.4 \\ 0 & 0.2 & 0 \end{bmatrix}, \quad \Delta_N N_A = \begin{bmatrix} 0 & 0 & 0 \\ 0.2 & 0 & 0.4 \\ 0 & 0.1 & 0 \end{bmatrix}. \quad (47)$$

It follows that a feasible solution of LMI (11) is

$$P_0 = \begin{bmatrix} 3074 & -5431 & -1331 \\ -5431 & 11221 & 885 \\ -1331 & 885 & 1912 \end{bmatrix}, \quad X = \begin{bmatrix} -6379 & 7189 & 4104 \\ 12249 & -13814 & -6269 \end{bmatrix}, \quad \epsilon_1 = 1055.1, \quad \epsilon_2 = 173.0328, \quad (48)$$

asymptotically stabilizing state-feedback gain is

$$K = XP_0^{-1} = \begin{bmatrix} -324.0313 & -143.8217 & -156.8345 \\ 840.6966 & 373.4044 & 409.0759 \end{bmatrix}, \quad (49)$$

$$\widehat{S} = A_1 + B_1K + E_1\Delta_A = \begin{bmatrix} -137.3101 & -61.0208 & -67.0911 \\ 251.3914 & 111.4837 & 123.6975 \\ 7.9010 & 3.5938 & 3.0047 \end{bmatrix}, \quad (50)$$

and the arguments of all eigenvalues of $\widehat{S}$ are

$$3.1416, \quad 3.1416, \quad 3.1416. \quad (51)$$

Therefore, system (6) is stable.

5. Conclusion

In this paper, the robust asymptotical stability of uncertain descriptor fractional-order systems (6) with the fractional-order α belonging to $1 \le \alpha < 2$ has been studied. We derive a new sufficient condition for checking the robust asymptotical stabilization of (6) in terms of LMIs. Out results can be seen as a generalization of [28, Theorem 2]. By adding appropriate parameters into LMIs, our result has wider applications. One special numerical example has shown that our results are feasible and easy to be used.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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