

# Genericity results in linear conic programming – a tour d’horizon

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## Abstract

This paper is concerned with so-called generic properties of general linear cone programs. Many results have been obtained on this subject during the last two decades. It has, *e.g.*, been shown in [29] that uniqueness, strict complementarity and nondegeneracy of optimal solutions hold for almost all problem instances. Strong duality holds generically in a stronger sense: it holds for a generic subset of problem instances.

In this paper, we survey known results and present new ones. In particular we give an easy proof of the fact that Slater’s condition holds generically in linear cone programming. We further discuss the problem of stability of uniqueness, nondegeneracy and strict complementarity. We also comment on the fact that in general, cone programming cannot be treated as a smooth program and that techniques from nonsmooth geometric measure theory are needed.

**Keywords:** Conic programs; generic properties; Slater’s condition; strong duality; uniqueness; nondegeneracy; techniques from geometric measure theory.

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## 1 Introduction

Linear cone programs (CP) can be given in different equivalent forms. In this paper, we consider the pair of primal-dual linear conic programs

$$\max c^T x \quad \text{s. t.} \quad B - Ax \in \mathcal{K}, \quad (P)$$

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$$\min \langle B, Y \rangle \quad \text{s. t.} \quad A^T Y = c, \quad Y \in \mathcal{K}^*, \quad (D)$$

with given vectors  $c \in \mathbb{R}^n$ ,  $B \in \mathbb{R}^m$ , a matrix  $A \in \mathbb{R}^{m \times n}$  and variables  $x \in \mathbb{R}^n, Y \in \mathbb{R}^m$ . We assume that  $\mathcal{K} \subseteq \mathbb{R}^m$  is a pointed full-dimensional closed convex cone and  $\mathcal{K}^*$  is the dual cone of  $\mathcal{K}$  with respect to the Euclidean inner product  $\langle \cdot, \cdot \rangle$  in  $\mathbb{R}^m$ , *i.e.*,  $\mathcal{K}^* := \{Y \in \mathbb{R}^m \mid \langle Y, X \rangle \geq 0 \text{ for all } X \in \mathcal{K}\}$ .

Often, *e.g.*, in semidefinite and copositive programming the elements  $Y, B$  and the columns  $A_i$  of  $A$  are matrices from the set  $\mathcal{S}_k$  of (real) symmetric  $k \times k$ -matrices. We therefore write the vectors  $x, c \in \mathbb{R}^n$  in lower case but the vectors (matrices)  $Y, B, A_i \in \mathbb{R}^m$  in capital letters. Note that we can simply identify  $\mathcal{S}_k \equiv \mathbb{R}^m$  where  $m := \frac{1}{2}k(k+1)$ .

Linear conic programming represents an important class of convex problems with a multitude of applications. It contains linear programming (LP), semidefinite and copositive programming as special cases. We refer *e.g.*, to [26, 36, 28] for surveys on this topic.

In this paper, we study genericity results for such programs, *i.e.*, we wish to show that certain “nice” regularity conditions hold generically. Let  $\mathcal{P}$  be (a subset of) an Euclidean space  $\mathbb{R}^N$ . In what follows, we say that a subset  $\mathcal{P}_r$  of the set  $\mathcal{P}$  is a generic subset of  $\mathcal{P}$  if  $\mathcal{P}_r$  is open in  $\mathcal{P}$  and  $\mathcal{P} \setminus \mathcal{P}_r$  has Lebesgue measure zero. We call  $\mathcal{P}_r$  a weakly generic subset of  $\mathcal{P}$  if only it holds that  $\mathcal{P} \setminus \mathcal{P}_r$  has Lebesgue measure zero. A property is said to be (weakly) *generic* in the problem set  $\mathcal{P}$ , if it holds for a (weakly) generic subset  $\mathcal{P}_r$  of  $\mathcal{P}$ . So the weakly generic sets  $\mathcal{P}_r$  need not be open. Hence, genericity implies both density and stability of the nice problem instances, whereas weak genericity only assures density. Note that from a numerical viewpoint stability (*i.e.*, openness of  $\mathcal{P}_r$ ) is crucial, so genericity is the desirable property.

**Remark 1.1.** *Genericity can be defined in different ways. In [2, 4] the (weak) genericity results have been formulated with respect to (wrt.) the Lebesgue measure, in [29] wrt. the Hausdorff measure, and in [5] in terms of  $\sigma$ -porosity (cf., Lemma 3.9). It is well-known that in  $\mathbb{R}^N$  the  $N$ -dimensional Lebesgue- and Hausdorff measures coincide (see, *e.g.*, [25, Corollary 2.8]) and that a  $\sigma$ -porous set has Lebesgue measure zero (the converse does not hold). In [34] weak genericity is called metric genericity and some genericity results are given in terms of open and dense sets (see [34, Theorem 4.6, 4.7]). Note however, that openness and density of a set  $\mathcal{A} \subset \mathbb{R}^N$  does not imply that the complement  $\mathbb{R}^N \setminus \mathcal{A}$  has Lebesgue measure zero. So our concept of genericity is stronger and we think that for our purpose, (*i.e.*, for problem sets in  $\mathbb{R}^N$ ), our definition of genericity is very appropriate and meaningful.*

Genericity of properties like strong duality, nondegeneracy, strict complementarity and uniqueness of solutions of linear conic programs have been discussed before. Alizadeh, Haeberly, and Overton [2] as well as Shapiro [35] specifically discuss generic properties of semidefinite programs (SDP). Pataki and Tunçel [29] derive weak genericity results on strict complementarity, uniqueness, and nondegeneracy for general linear conic programs. Note however, that the results in [2] have been proven under the assumption that the Slater condition is satisfied, and in [29], the genericity results are restricted to so called gap-free problems (*i.e.*,

problems with finite optimal value and zero duality gap). The possibility that these assumptions generically fail has not been excluded, so that strictly speaking these genericity results were lacking some foundation. Merely for the SDP case, it is indicated in [35, p. 310] that the Slater condition (Mangasarian-Fromovitz condition) is generic. Recently in [4] Bolte, Daniilidis and Lewis gave special full genericity results wrt. uniqueness of solutions under the extra assumption that the cone  $\mathcal{K}$  is a semialgebraic set.

When we were working on an earlier version of this article, other results were brought to our attention, *e.g.*, the paper by Schurr et al. [34] and the one by Borwein et al. [5]. This has led to a complete revision of our earlier paper and resulted in the present article which has the following aims: To survey the known genericity results, to add new ones and to discuss the relations between the different genericity statements.

We start with some general remarks. Usually, genericity results in smooth optimization are proven by applying transversality theory from differential topology. We refer to [22] for such genericity results in smooth nonlinear finite programming, and to [2] for results in semidefinite programming. We also refer the reader to Section 5 for the special case of LP and SDP.

However, a general conic program is not a completely smooth problem. Indeed, a part of the problem is given by the general cone  $\mathcal{K}$  (or its dual  $\mathcal{K}^*$ ) and boundaries of convex cones are generally described by convex (thus Lipschitz) functions rather than by smooth functions. So to obtain genericity results in general linear cone programming we have to use techniques from nonsmooth convex analysis. Fortunately, in the field of geometric measure theory many results of differential geometry for  $C^1$ -functions have been generalized to similar results for Lipschitz functions. Founding work for this theory goes back to Federer and others (see [11, 25, 32] for an overview). The results in [29], [34], and [4] are based on this theory, and we also will use techniques from geometric measure theory.

In this paper, we try to prove our genericity results with techniques which are as basic as possible. Genericity of strong duality will be proven (based on Lemma 3.2) by purely topological arguments. For weak genericity of uniqueness more structure is needed. As we shall see, the classical result that Lipschitz-functions (convex functions) are differentiable almost everywhere will do the job. For weak genericity of nondegeneracy and strict complementarity (SC) more sophisticated techniques from geometric measure theory are still needed (see [29]).

The paper is organized as follows. Section 2 introduces some notation and presents two equivalent formulations for the cone programs ( $P$ ) and ( $D$ ). In Section 3 we show that the Slater condition holds generically in conic programming. By using well-known techniques this leads to genericity results for strong duality similar to the results in [34]. We compare the statements in [34] with our result and discuss related work. Section 4 deals with weak genericity results concerning uniqueness, nondegeneracy and strict complementarity in CP. In Section 4.1 we give an independent proof of the fact that uniqueness is weakly generic. This approach was brought to our attention by Alexander Shapiro (personal communication). The proof does not rely on deeper results from geometric measure theory as used in [29, Theorem 3]. Section 4.2 summarizes

the weak genericity results for nondegeneracy and strict complementarity from [29]. Section 4.3 comments on the fact that nondegeneracy implies Slater's condition. It further explains why most genericity results from linear semi-infinite optimisation (SIP) cannot be directly applied to CP.

In Section 5 we discuss the problem of stability for properties like uniqueness, nondegeneracy and strict complementarity. For some special classes of CP, such as LP and SDP, full genericity can be proven. For general cone programs it is still open whether the stability for these properties holds (fully) generically.

## 2 Preliminaries

We next discuss two other formulations for CP. Many authors (e.g., [29]) consider conic programs in **Self-dual formulation:**

$$(P_0) \quad \max \langle C, B \rangle - \langle C, X \rangle \quad \text{s. t.} \quad X \in (B + \mathcal{L}) \cap \mathcal{K}, \quad (2.1)$$

$$(D_0) \quad \min \langle B, Y \rangle \quad \text{s. t.} \quad Y \in (\mathcal{L}^\perp + C) \cap \mathcal{K}^*, \quad (2.2)$$

where  $C, B \in \mathbb{R}^m$ ,  $\mathcal{L} = \text{span}\{A_1, \dots, A_n\} \subset \mathbb{R}^m$  is the linear subspace spanned by  $A_i \in \mathbb{R}^m$ ,  $i = 1, \dots, n$ , and  $\mathcal{K}$  is a cone in  $\mathbb{R}^m$  (as above).

It is easy to see that the problems  $(P_0)$ ,  $(D_0)$  are equivalent to  $(P)$ ,  $(D)$ , respectively. Indeed, let  $A_i$  denote the columns of  $A$  and choose some  $C \in \mathbb{R}^m$  satisfying  $\langle A_i, C \rangle = c_i$  for  $i = 1, \dots, n$ . Then the feasible sets of  $(P_0)$  and  $(P)$  are directly related via the affine mapping  $X = B - Ax$  (in case the  $A_i$ 's are linearly independent the map is bijective). Also their objective function values are the same since for  $X = B - \sum_{i=1}^n x_i A_i$  we obtain

$$\langle C, B \rangle - \langle C, X \rangle = \langle C, B - X \rangle = \langle C, \sum_{i=1}^n x_i A_i \rangle = \sum_{i=1}^n x_i \langle C, A_i \rangle = c^T x.$$

The dual problems  $(D_0)$  and  $(D)$  have the same objective function, and in view of the relation

$$Y - C \in \mathcal{L}^\perp \quad \Leftrightarrow \quad \langle Y - C, A_i \rangle = 0 \text{ for all } i \quad \Leftrightarrow \quad \langle Y, A_i \rangle = c_i \text{ for all } i \quad \Leftrightarrow \quad A^T Y = c$$

the feasible sets coincide, so  $(D_0)$  and  $(D)$  are equivalent as well.

**Remark 2.1.** *Important special cases of CP are linear programs (LP) where  $\mathcal{K} = \mathcal{K}^* = \mathbb{R}_+^m$ , and semidefinite programs (SDP) where the columns  $A_i$  of  $A$  (i.e., the basis of  $\mathcal{L}$ ) as well as  $B, C$  are elements of the space  $\mathcal{S}_k$  of symmetric  $k \times k$ -matrices and  $\mathcal{K} = \mathcal{K}^*$  equals the set  $\mathcal{S}_k^+$  of positive semidefinite matrices in  $\mathcal{S}_k$ . Note that we can identify  $\mathcal{S}_k \equiv \mathbb{R}^m$  with  $m = \frac{1}{2}k(k+1)$ . Another example is given by the class of copositive programs (COP) where  $\mathcal{K}$  is the cone of copositive matrices with dual  $\mathcal{K}^*$ , the cone of completely positive*

matrices (see e.g., [6] for details).

In the sequel, the feasible sets and optimal values of these cone programs will be denoted by  $\mathcal{F}_{P_0}$ ,  $\mathcal{F}_{D_0}$  and  $v_{P_0}$ ,  $v_{D_0}$ , respectively. As usual we say that strong duality holds for a pair of primal, dual programs  $(P_0)$ ,  $(D_0)$  if the relation  $v_{P_0} = v_{D_0}$  holds.

**SIP formulation:** Linear cone programs can also be seen as special case of linear semi-infinite programs (SIP) of the form

$$(SIP_P) \quad \max_{x \in \mathbb{R}^n} c^T x \quad \text{s. t.} \quad b(Y) - a(Y)^T x \geq 0 \quad \text{for all } Y \in Z, \quad (2.3)$$

with a possibly infinite index set  $Z \subset \mathbb{R}^m$  and functions  $a : Z \rightarrow \mathbb{R}^n$  and  $b : Z \rightarrow \mathbb{R}$ . The (Haar-) dual reads:

$$(SIP_D) \quad \min \sum_{Y_j \in Z} y_j b(Y_j) \quad \text{s. t.} \quad \sum_{Y_j \in Z} y_j a(Y_j) = c, \quad y_j \geq 0, \quad (2.4)$$

where the min is taken over all finite sums. For an introduction to (linear) SIP we refer e.g., to [12]. Note that the condition  $X \in \mathcal{K}$  can be equivalently expressed as

$$\langle X, Y \rangle \geq 0 \quad \text{for all } Y \in \mathcal{K}^*.$$

In view of this relation the primal program  $(P)$  can be written as  $(SIP_P)$  with

$$a(Y) := A^T Y, \quad b(Y) := \langle B, Y \rangle, \quad \text{and} \quad Z := \mathcal{K}^* \quad (2.5)$$

The feasibility condition for  $(SIP_D)$  then becomes

$$c = \sum_j y_j A^T Y_j, \quad y_j \geq 0$$

and by putting  $Y := \sum_j y_j Y_j \in \mathcal{K}^*$ , this coincides with the feasibility condition  $c = A^T Y$  of  $(D)$ . Moreover, in view of  $\sum_j y_j b(Y_j) = \sum_j y_j \langle Y_j, B \rangle = \langle Y, B \rangle$ , the dual  $(SIP_D)$  is equivalent to  $(D)$ , and we simply denote both versions by  $(D)$ .

For the genericity results in this article we always assume that the cone  $\mathcal{K}$  (and thus  $\mathcal{K}^*$ ) and  $n, m$  are arbitrarily fixed. Then the set of problem instances of  $(P)$ ,  $(D)$  is given by

$$\mathcal{P} := \{(A, B, c) \in \mathbb{R}^{m \times n} \times \mathbb{R}^m \times \mathbb{R}^n\} \equiv \mathbb{R}^{m \cdot n + m + n}$$

endowed with some norm.

Often we prove results of the sort that for arbitrarily fixed  $A \in \mathbb{R}^{m \times n}$  a property holds for all  $(B, c)$

from a generic set  $S = S(A) \subset \mathbb{R}^{m+n}$ . We emphasize that this implies that the property holds for almost all problem instances in the whole space  $\mathcal{P} = \{(A, B, c)\}$ . Indeed, under this assumption for any fixed  $A \in \mathbb{R}^{m \times n}$  the property holds on the whole  $\mathbb{R}^{m+n}$  except for the set  $S(A)^C := \mathbb{R}^{m+n} \setminus S(A)$  of Lebesgue measure  $\mu(S(A)^C) = 0$  in  $\mathbb{R}^{m+n}$ . But then by Fubini's theorem the property holds for  $(A, B, c) \in \mathbb{R}^{m \times n} \times \mathbb{R}^m \times \mathbb{R}^n$  except for a set of measure  $\int_{\mathbb{R}^{m \times n}} \mu(S(A)^C) dA = 0$ .

Concerning openness, however, we have to be careful: If for any fixed  $A$  a property holds for any  $(B, c)$  from an open set  $S \subset \mathbb{R}^{m+n}$ , then this property need not hold for an open set in  $\mathcal{P}$ . [A counterexample is given by Example 3.8.](#)

Throughout the paper we assume that  $n \leq m$  holds. For the case  $n > m$  the genericity results can be summarized by the following statement: Generically for the case  $n > m$

- the dual ( $D$ ) is infeasible and
- the primal program ( $P$ ) is unbounded.

So in this case generically strong duality holds with  $v_P = v_D = +\infty$ . To prove this, we use the well-known fact that (see *e.g.*, [22, Ex. 7.3.23])

$$\text{a matrix } U \in \mathbb{R}^{N \times M} \text{ with } N \geq M \text{ generically has full rank } M. \quad (2.6)$$

We first show that generically wrt.  $(A, c)$  the system

$$c = A^T Y \quad \text{has no solution } Y \in \mathbb{R}^m. \quad (2.7)$$

Indeed, by (2.6) the matrix  $U := [A^T \ c] \in \mathbb{R}^{n \times (m+1)}$  generically has rank  $m + 1$  whence  $Uz = 0$  does not allow a nonzero solution. This means that generically the system in (2.7) is infeasible.

To show that ( $P$ ) is generically unbounded we consider the system  $Ax = B$ ,  $c^T x = \tau$ , any solution of which yields a primal feasible  $x$  with objective value  $\tau$ . Again, by using (2.6), generically, the matrix  $U := \begin{bmatrix} A \\ c^T \end{bmatrix} \in \mathbb{R}^{(m+1) \times n}$  has full rank  $m + 1$ , so  $Ax = B$ ,  $c^T x = \tau$  is solvable for any  $\tau$  (and  $B$ ).

### 3 Genericity of Slater's condition and strong duality

It is well-known that strong duality always holds in linear programming (unless both programs are infeasible), but strong duality need not hold in general cone programming. However, as we shall see, strong duality is a generic property.

In this section we give an independent easy proof of the fact that in cone programming the Slater condition holds generically. To do so we only make use of the result that the boundary of a convex set has measure zero. By applying well-known duality theorems this leads to an alternative proof of the genericity result

for strong duality in [34]. We also summarize other related results that were brought to our attention ([5] and [34]).

### 3.1 Genericity of the Slater condition

In this section we provide a purely topological proof of the genericity of the Slater condition.

**Definition 3.1.** *We say that Slater's condition holds for (P), if there exists a feasible  $x$  such that  $X := B - Ax \in \text{int } \mathcal{K}$ . Analogously, we say that Slater's condition holds for (D), if there exists a feasible  $Y$ , i.e.,  $A^T Y = c$ , such that  $Y \in \text{int } \mathcal{K}^*$ .*

Roughly speaking, Slater's condition says that the feasible set of the problem is not entirely contained in the boundary of the convex cone. For this reason, it is intuitive that the proof of a genericity result should be based on properties of this boundary. More specifically, we will use the fact that the boundary of a convex set has measure zero.

**Lemma 3.2.** *Let  $\mathcal{T}$  be a full-dimensional closed convex set in  $\mathbb{R}^s$ . Then the boundary of  $\mathcal{T}$  has  $s$ -dimensional Lebesgue measure zero.*

*Proof.* For the sake of completeness we repeat here the elegant proof of [24]. Consider an open ball  $\mathcal{B}_\varepsilon(p)$  with center  $p \in \text{bd } \mathcal{T}$  and radius  $\varepsilon > 0$ . Since there exists a hyperplane supporting the convex set  $\mathcal{T}$  at  $p$ , at least half of the ball does not contain points of  $\mathcal{T}$ . Therefore,

$$\limsup_{\varepsilon \downarrow 0} \frac{\mu(\mathcal{T} \cap \mathcal{B}_\varepsilon(p))}{\mu(\mathcal{B}_\varepsilon(p))} \leq \frac{1}{2}.$$

On the other hand, Lebesgue's density theorem (see e.g., [10]), says that for almost all points  $p$  of the Lebesgue measurable set  $\mathcal{T}$  we have that

$$\lim_{\varepsilon \downarrow 0} \frac{\mu(\mathcal{T} \cap \mathcal{B}_\varepsilon(p))}{\mu(\mathcal{B}_\varepsilon(p))} = 1.$$

This immediately implies that  $\text{bd } \mathcal{T}$  has measure zero. □

The next theorem shows that Slater's condition is indeed generic.

**Theorem 3.3.** *Let  $A \in \mathbb{R}^{m \times n}$  be given arbitrarily. Then there exist a generic subset  $S_1 \subset \mathbb{R}^n$  (open with complement of measure zero), such that for any  $c \in S_1$  precisely one of the following alternatives holds for the corresponding problem instance of (D):*

(1) *either the feasible set of (D) is empty, i.e.,  $\{Y \in \mathcal{K}^* \mid A^T Y = c\} = \emptyset$ , or*

(2) *Slater's condition holds for (D), i.e.,  $\{Y \in \text{int } \mathcal{K}^* \mid A^T Y = c\} \neq \emptyset$ ,*

An analogous result holds for the primal program (P), i.e., there is a generic subset  $\tilde{S}_1$  of  $\mathbb{R}^m$  such that for any  $B \in \tilde{S}_1$  either the corresponding program (P) is infeasible or (P) satisfies the Slater condition.

*Proof.* For the case of program (D), note that the set  $S := \{c = A^T Y \mid Y \in \mathcal{K}^*\} \subset \mathbb{R}^n$  is a convex set with  $\dim S =: k \leq n$ . We define  $S_1 := \text{int } S \cup (\mathbb{R}^n \setminus \text{cl } S)$ . As a union of two open sets,  $S_1$  is clearly open. Note that for  $c \in \mathbb{R}^n \setminus \text{cl } S$  the alternative (1) holds, i.e., the feasible set is empty. If  $k < n$  (i.e.,  $A$  does not have full rank  $n$ ) then the statement is true. So we can assume  $\dim S = n$ , and since by Lemma 3.2 the set  $\text{bd } S = \mathbb{R}^n \setminus S_1$  has measure zero, it is sufficient to show that for  $c \in \text{int } S$  the Slater condition holds (alternative (2)).

So let  $c \in \text{int } S$  be given. By assumption there exists some  $Y_0 \in \mathcal{K}^*$  for which  $A^T Y_0 = c$  holds. Consider the affine space  $Y_0 + \ker A^T$ . If  $Y_0 + \ker A^T \cap \text{int } \mathcal{K}^* \neq \emptyset$ , then Slater's condition holds and we are done.

So assume by contradiction that  $Y_0 + \ker A^T \cap \text{int } \mathcal{K}^* = \emptyset$ . This implies in particular that  $Y_0 \in \text{bd } \mathcal{K}^*$ , and since  $\text{int } \mathcal{K}^* \neq \emptyset$ , there exists a separating hyperplane with normal vector  $N$  such that (see [30, Theorem 11.2])

$$\langle N, Y \rangle \geq \langle N, Y_0 \rangle \text{ for all } Y \in \mathcal{K}^* \quad \text{and} \quad N \perp \ker A^T. \quad (3.1)$$

Since  $c \in \text{int } S$ , there exists an open neighborhood  $\emptyset \neq \mathcal{N}_\varepsilon(c) \subset \text{int } S$  of  $c$  and by continuity of the mapping  $A^T Y$  there exists an open neighborhood  $\emptyset \neq \mathcal{N}_\delta(Y_0)$  of  $Y_0$  such that  $A^T \mathcal{N}_\delta(Y_0) \subset \mathcal{N}_\varepsilon(c)$ . The separating hyperplane divides  $\mathcal{N}_\delta(Y_0)$  into two parts. Take a point  $Y_1 \in \mathcal{N}_\delta(Y_0)$  such that  $\langle N, Y_1 \rangle < \langle N, Y_0 \rangle$ . By construction,  $c_1 := A^T Y_1 \in \mathcal{N}_\varepsilon(c) \subset \text{int } S$ . So there must exist a pre-image  $\tilde{Y}_1 \in \mathcal{K}^*$  with  $A^T \tilde{Y}_1 = c_1$ , i.e.,  $\tilde{Y}_1 = Y_1 + \tilde{Y}_0$  with  $\tilde{Y}_0 \in \ker A^T$ . Altogether using  $\langle N, \tilde{Y}_0 \rangle = 0$  and (3.1), we attain the contradiction

$$\langle N, Y_0 \rangle \leq \langle N, \tilde{Y}_1 \rangle = \langle N, Y_1 + \tilde{Y}_0 \rangle = \langle N, Y_1 \rangle < \langle N, Y_0 \rangle.$$

This concludes the proof for problem (D).

For the primal program we proceed as follows. We note that  $\mathbb{R}^m$  allows an orthogonal decomposition

$$\mathbb{R}^m = \text{im } A \oplus \ker A^T, \quad B = B_1 \oplus B_2 \quad \text{for } B \in \mathbb{R}^m$$

where  $B_2$  is the projection  $\text{proj}_{\ker A^T} B$  of  $B \in \mathbb{R}^m$  onto the linear space  $\ker A^T$ . Let  $Q \in \mathbb{R}^{m \times m}$  be the matrix representation of this projection, i.e.,  $B_2 = \text{proj}_{\ker A^T} B = QB$ . We now consider the convex cone  $R := Q\mathcal{K}$ . As before we have

$$QB \in \ker A^T \setminus \text{cl } R \quad \Rightarrow \quad \{B - Ax \mid x \in \mathbb{R}^n\} \cap \mathcal{K} = \emptyset$$

and we can show (with  $\text{int } R$  relative to  $\ker A^T$ )

$$QB \in \text{int } R \quad \Rightarrow \quad \{B - Ax \mid x \in \mathbb{R}^n\} \cap \text{int } \mathcal{K} \neq \emptyset.$$

Here again  $\text{bd } R$  has measure zero and thus  $R_1 := \text{int } R \cup (\ker A^T \setminus \text{cl } R)$  is relatively open in  $\ker A^T$  with  $\ker A^T \setminus R_1$  of measure zero in  $\ker A^T$ . Consequently, the set  $\tilde{S}_1 := \text{im } A \oplus R_1$  is open in  $\mathbb{R}^m$  with  $\mathbb{R}^m \setminus \tilde{S}_1$  of measure zero in  $\mathbb{R}^m$ . By construction, for  $B \in \tilde{S}_1$ , precisely one of the two alternatives holds.  $\square$

**Remark 3.4.** The Slater conditions for  $(P)$  and  $(P_0)$  are clearly equivalent. Also the genericity result for  $(D)$  in Theorem 3.3 wrt. parameter  $c$  can be translated to the following corresponding result for  $(D_0)$ : Let  $\mathcal{L}$  be given. Then there exists a generic subset  $Q_1 \subset \mathbb{R}^m$  such that for any  $C \in Q_1$  precisely one of the following alternatives holds for the corresponding problem instance of  $(D_0)$ :

(1') either the feasible set of  $(D_0)$  is empty, or

(2') Slater's condition holds for  $(D_0)$ , i.e.,  $\{Y \mid Y \in (\mathcal{L}^\perp + C) \cap \text{int } \mathcal{K}^*\} \neq \emptyset$ .

To see this, similar to the second part of the proof of Theorem 3.3, consider the orthogonal decomposition

$$\mathbb{R}^m = \mathcal{L}^\perp \oplus \mathcal{L}, \quad C = C_1 \oplus C_2 \quad \text{for } C \in \mathbb{R}^m.$$

Let  $P \in \mathbb{R}^{m \times m}$  be the matrix representation of the projection  $\text{proj}_{\mathcal{L}}$  onto  $\mathcal{L}$ , and let  $C_2 = PC = \text{proj}_{\mathcal{L}} C$ . Then as in the the proof of Theorem 3.3 above we consider the convex cone  $S := PK^* \subset \mathcal{L}$  and the set (relative to  $\mathcal{L}$ )

$$S_1 = \text{int } S \cup (\mathcal{L} \setminus \text{cl } S),$$

which is relatively open with  $\mathcal{L} \setminus S_1$  of measure zero. Note that for  $PC \in \text{int } S$  the alternative (2') holds and for  $PC \in \mathcal{L} \setminus \text{cl } S$  the condition (1') is true. So the set  $Q_1 = \mathcal{L}^\perp \oplus S_1$  is the required generic set in  $\mathbb{R}^m$ .

It is well-known (see [31], [34, Lemma 3.2], or [12, Theorem 8.1]) that if for some  $A, B$  the problem  $(P)$  satisfies the Slater condition, then for all  $c$  the strong duality relation  $v_P = v_D$  holds and, in case  $v_P = v_D$  is finite, the optimal value of  $(D)$  is attained. So the genericity of Slater's condition in Theorem 3.3 leads to the following genericity result for strong duality (similar to [34]):

**Corollary 3.5.** *Let  $A \in \mathbb{R}^{m \times n}$  be given arbitrarily. Then with the generic subset  $\tilde{S}_1 \subset \mathbb{R}^m$  from Theorem 3.3 the following holds for  $B \in \tilde{S}_1$ :*

- either the feasible set of  $(P)$  is empty,
- or  $(P)$  is strictly feasible and for any  $c \in \mathbb{R}^n$  we have  $v_P = v_D$ , meaning that if  $(D)$  is infeasible, then  $v_P = v_D = +\infty$ , and if  $(D)$  is feasible, then  $v_P = v_D$  is finite and the minimum value of  $(D)$  is attained.

An analogous result holds for the dual program  $(D)$  wrt.  $c \in S_1 \subset \mathbb{R}^n$  (with  $S_1$  from Theorem 3.3).

By combining the results for the primal and dual programs we obtain:

**Corollary 3.6.** *Let  $A \in \mathbb{R}^{m \times n}$  be given arbitrarily. Then with the generic subsets  $S_1 \subset \mathbb{R}^n$ ,  $\tilde{S}_1 \subset \mathbb{R}^m$  from Theorem 3.3, for any  $(B, c) \in \tilde{S}_1 \times S_1$  precisely one of the following alternatives holds:*

- (1) *Both feasible sets of  $(P)$  and  $(D)$  are empty.*
- (2) *Precisely one of the feasible sets of  $(P)$  or  $(D)$  is empty and  $v_P = v_D = \pm\infty$ .*
- (3) *Both  $(P)$  and  $(D)$  are feasible and for both problems the optimal value is attained with  $v_P = v_D$ .*

A corresponding result holds for  $(P_0)$ ,  $(D_0)$  wrt. to a generic set  $\tilde{S}_1 \times Q_1 \subset \mathbb{R}^m \times \mathbb{R}^m$  of parameters  $(B, C)$  (cf. Remark 3.4).

The statement in Corollary 3.6 could be called genericity of universal strong duality wrt. parameters  $(B, c)$  (for any fixed  $A$ ).

We next compare our result with that in [34] where the authors take  $A$  as a parameter, and they define: For given  $A$ , universal duality is said to hold (wrt.  $A$ ), if for any  $(B, c)$  the equality  $v_P = v_D$  holds for  $(P)$  and  $(D)$  (see also Section 3.2). They prove the following:

**Theorem 3.7.** [34, Theorem 4.5, Theorem 4.7]. *There is a generic subset  $S \subset \mathbb{R}^{m \times n}$  such that for any  $A \in S$  universal duality holds.*

The main difference between this statement and ours above is that by taking  $A$  as a parameter in the generic set  $S$  of Theorem 3.7, the case that both primal and dual are infeasible is excluded. In our approach, for fixed  $A$  we cannot exclude generically in  $(B, c)$  the infeasibility of both programs  $(P)$  and  $(D)$  simultaneously. We illustrate this difference between our result in Corollary 3.6 and the result from [34] as stated in Theorem 3.7 by an example.

**Example 3.8.** *Consider the LP:*

$$(P) \quad \max c^T x \quad \text{s.t.} \quad B - Ax \geq 0 \quad \text{with} \quad c = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

$$(D) \quad \min B^T Y \quad \text{s.t.} \quad A^T Y = c, \quad Y = (y_1, y_2, y_3) \geq 0.$$

*The primal resp. dual feasibility conditions are:*

$$x_2 \leq 0, \quad x_2 \geq 1, \quad x_1 \leq 0 \quad \text{resp.} \quad y_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + y_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix} + y_3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad y_i \geq 0.$$

Both programs are infeasible, and for fixed  $A$  this property is stable wrt. small perturbations of  $(B, c)$ . So in Corollary 3.6, the alternative (1) cannot be excluded generically. Recall however that according to the genericity concept in Theorem 3.7 (where  $A$  is the parameter) a generic perturbation of the matrix  $A$  above makes either  $(P)$  or  $(D)$  feasible.

Moreover, note that in Corollary 3.6 (in contrast to Theorem 3.7) also the existence of solutions is assured in case (3). We also emphasize that the proof of our genericity statement is more elementary than the proof in [34] which is based on a “deep” result (see [34, Lemma A.1]) from geometric measure theory.

The notion of universal duality goes back to Duffin [9]. His results allow another approach to genericity of strong duality which is shortly discussed in the next section.

### 3.2 Genericity results based on generic closedness of the image $M\mathcal{K}$

Recently, it was brought to our attention by Warren Moors that an approach from [5] allows another way to prove genericity of strong duality for cone programs: it is well-known that for  $M \in \mathbb{R}^{k \times m}$  the linear image  $M\mathcal{K} := \{MY \mid Y \in \mathcal{K}\}$  of a polyhedral closed convex cone  $\mathcal{K} \subset \mathbb{R}^m$  is closed. This is not generally true for non-polyhedral cones (see *e.g.*, [30, p.73,74] for a counterexample). In [5, Theorem 2] the following genericity statement has been shown.

**Lemma 3.9.** *Let  $k \in \mathbb{N}$  and let  $\mathcal{K} \subset \mathbb{R}^m$  be a closed convex cone. Then the set*

$$S_1 := \mathbb{R}^{k \times m} \setminus \text{int}\{M \in \mathbb{R}^{k \times m} \mid M\mathcal{K} \text{ is closed}\}$$

*is  $\sigma$ -porous.*

Note that  $\sigma$ -porosity of  $S_1$  implies that  $S_1$  has Lebesgue measure zero and is the countable union of nowhere dense sets (see [5]).

The following result for SIP, by Duffin et al. [9], provides the connection between strong duality and closedness of images  $M\mathcal{K}$ . We formulate these statements in terms of our problems  $(P)$  and  $(D)$ .

Under the assumption that  $(P)$  is feasible, in [9] the data  $(A, B) \in \mathbb{R}^{m \times n} \times \mathbb{R}^m$  are said to yield primal uniform LP duality for  $(P)$  and  $(D)$ , if for any  $c \in \mathbb{R}^n$  either  $\mathcal{F}_D = \emptyset$  and  $v_P = v_D = \infty$ ; or  $v_P = v_D$  is finite and a solution of  $(D)$  exists.

**Lemma 3.10.** *(see [9, Theorem 3.2] and [20, Theorem 6.14]) Let  $(A, B) \in \mathbb{R}^{m \times n} \times \mathbb{R}^m$  be such that  $(P)$  is feasible. Then the data  $(A, B)$  yield primal uniform LP duality if and only if the cone*

$$\mathcal{C} := \text{cone} \left( \left\{ \begin{pmatrix} A^T \\ B^T \end{pmatrix} Y \mid Y \in \mathcal{K}^* \right\} \cup e_{m+1} \right)$$

*is closed. Here,  $e_{m+1}$  denotes the unit vector  $e_{m+1} = (0, \dots, 0, 1) \in \mathbb{R}^{m+1}$ .*

Under the assumption that  $(P)$  is feasible, it is easy to show that

$$\text{if the cone } \mathcal{C}_1 := \left\{ \begin{pmatrix} A^T \\ B^T \end{pmatrix} Y \mid Y \in \mathcal{K}^* \right\} \text{ is closed, then } \mathcal{C} \text{ is closed.} \quad (3.2)$$

To see this, we note that the cone  $\mathcal{C}_2 := \text{cone}(e_{m+1})$  is closed, and apply a well-known result, *e.g.*, in the form [12, Theorem A4]:

Let  $\mathcal{C}_1, \mathcal{C}_2$  be closed cones with  $\mathcal{C}_1 \cap -\mathcal{C}_2 = \{0\}$ . Then  $\mathcal{C}_1 + \mathcal{C}_2$  is closed.

To show that under the assumption  $\mathcal{F}_P \neq \emptyset$  the relation  $\mathcal{C}_1 \cap -\mathcal{C}_2 = \{0\}$  holds, let us assume to the contrary that there exists an element  $0 \neq Z \in \mathcal{C}_1 \cap -\mathcal{C}_2$ . This means that there exists some  $\tilde{Y} \in \mathcal{K}^*$  such that  $Z := \begin{pmatrix} A^T \\ B^T \end{pmatrix} \tilde{Y} = -\alpha e_{m+1}$  with  $\alpha > 0$ , *i.e.*,  $A^T \tilde{Y} = 0$  and  $B^T \tilde{Y} = -\alpha < 0$ . But for any  $\bar{x} \in \mathbb{R}^n$  we then obtain  $(B - A\bar{x})^T \tilde{Y} = B^T \tilde{Y} < 0$ , *i.e.*,  $(P)$  is not feasible, a contradiction.

By combining Lemma 3.10 and (3.2) with Lemma 3.9 we obtain:

**Theorem 3.11.** *The set of parameters  $(A, B) \in \mathbb{R}^{m \times n} \times \mathbb{R}^m$  with nonempty primal feasible set  $\mathcal{F}_P$  where the (primal) uniform LP-duality fails is  $\sigma$ -porous in  $\mathbb{R}^{m \times n} \times \mathbb{R}^m$ . So, in particular, uniform LP duality (as defined above) is weakly generic in the space of parameters  $(A, B) \in \mathbb{R}^{m \times n} \times \mathbb{R}^m$ .*

A corresponding dual genericity result holds wrt. parameters  $(A^T, c)$ .

## 4 Genericity analysis for other properties

In this section we analyse the generic behavior of cone programs with respect to the uniqueness, nondegeneracy and strict complementarity (SC) of solutions. Note that even for linear programs (LP), these properties are not always fulfilled. But it appears that these properties hold for almost all instances of conic programs. We emphasize that the stability (openness of the “set of nice instances”) cannot be answered generally without extra assumptions on the cone. This aspect will be treated in Section 5.

Trying to derive the genericity results by using techniques which are as basic as possible, we show in the next section how the analysis of uniqueness can be based on the classical result that a convex function is differentiable almost everywhere. The weak genericity results for nondegeneracy and strict complementarity in Subsection 4.2 still require deeper results from geometric measure theory.

### 4.1 Analysis of uniqueness of solutions

We now study the uniqueness of solutions of cone programs. Weak genericity of uniqueness can be proven, as in [29], by using a result from geometric measure theory for convex bodies ([32, Theorem 2.2.9]). Alternatively, we will derive this result by using the fact that convex functions are differentiable almost everywhere.

This approach was brought to our attention by Alexander Shapiro (personal communication). It is based on a duality theory developed in Rockafellar [31]. Similar results have been proven for SIP programs in [13]. We will make use of these results, directly formulated in terms of cone programs.

In this section we consider for fixed  $A, B$  our primal problem  $P = P(c)$ , as the linear SIP (see Section 2)

$$P(c) \quad \max c^T x \quad \text{s. t.} \quad (B - Ax)^T Y \geq 0 \quad \text{for all } Y \in Z := \mathcal{K}^*$$

depending on  $c$  as a parameter, with optimal value function  $v_{P(c)}$ , feasible set  $\mathcal{F}_{P(c)}$  (not depending on  $c$ ) and the set  $\mathcal{S}_{P(c)}^* := \{x \in \mathcal{F}_{P(c)} \mid c^T x = v_{P(c)}\}$  of optimal solutions. The dual  $(D(c))$  has corresponding optimal values and feasible sets  $v_{D(c)}$ ,  $\mathcal{F}_{D(c)}$ , etc.

We introduce the cone  $\mathcal{M} := \{a(Y) = A^T Y \mid Y \in \mathcal{K}^*\}$  which will play a crucial role.

**Remark 4.1.** In semi-infinite optimization, the condition  $c \in \text{int } \mathcal{M}$  is just the standard Slater condition for  $(\text{SIP}_D)$  and it is not difficult to see that this condition is equivalent to the Slater condition for  $(D)$  in Definition 3.1 (see [1, Lemma 3.1]).

In the following,  $\mathcal{D}_P := \{c \in \mathbb{R}^n \mid v_{P(c)} < \infty\}$  denotes the effective domain of the function  $v_{P(c)}$  and  $\partial v_{P(c)}$  its subdifferential wrt.  $c$ .

**Lemma 4.2.** (see [13]) *Let  $B$  and  $A$  be such that  $\mathcal{F}_{P(c)} \neq \emptyset$ . Then the following holds:*

- (1)  $v_{P(c)}$  is a proper closed convex function of  $c$  on its effective domain  $\mathcal{D}_P$
- (2)  $\partial v_{P(c)} = \mathcal{S}_{P(c)}^*$ .
- (3)  $\mathcal{S}_{P(c)}^*$  is nonempty and compact if and only if  $c \in \text{int } \mathcal{M}$ .

*Proof.* See [13, page 262] for (1), and [13, Theorem 2.1] for (2) and (3). □

By using Lemma 4.2 and Rademacher's theorem for convex functions we can now prove the weak genericity of uniqueness in CP and obtain at the same time an alternative proof for the genericity of the Slater condition.

**Theorem 4.3.** *Let  $A$  and  $B$  be such that  $\mathcal{F}_{P(c)} \neq \emptyset$ . Then for almost all  $c \in \mathbb{R}^n$  the following alternative holds:*

- either  $\mathcal{F}_{D(c)} = \emptyset$ ,
- or the Slater condition holds for  $(D(c))$  and the solution of  $(P(c))$  is unique.

*A corresponding dual result holds wrt. parameter  $B$  (for fixed  $A, c$ ).*

*Proof.* Let  $A, B$  be such that  $\mathcal{F}_{P(c)} \neq \emptyset$ . Let  $\mathcal{D}_P$ , with boundary  $\text{bd } \mathcal{D}_P$ , be the (convex) effective domain of the convex function  $v_{P(c)}$  from Lemma 4.2. We distinguish the following three cases for  $c \in \mathbb{R}^n$ :

- (i)  $c \in \text{bd } \mathcal{D}_P$ ,      (ii)  $c \notin \text{cl } \mathcal{D}_P$ ,      (iii)  $c \in \text{int } \mathcal{D}_P$ .

By Lemma 3.2, case (i) occurs on a set of measure zero in  $\mathbb{R}^n$ . In case (ii), in view of the relation

$$\mathcal{F}_{D(c)} \neq \emptyset \quad \Rightarrow \quad c \in \mathcal{D}_P$$

we get  $\mathcal{F}_{D(c)} = \emptyset$  and the first alternative holds.

In case (iii), we use the fact that the convex function  $v_{P(c)}$  defined on the open set  $\text{int } \mathcal{D}_P$  is differentiable for almost all  $c \in \text{int } \mathcal{D}_P$  (cf., e.g., [30, Theorem 25.5]), i.e., for these  $c$  values the subgradient  $\partial v_{P(c)} = \mathcal{S}_{P(c)}^* = \{\nabla v_{P(c)}\}$  is a singleton by Lemma 4.2 (2). Moreover, in this case, by Lemma 4.2 (3) the Slater condition holds for  $\mathcal{F}_{D(c)}$ , (cf. Remark 4.1).

The proof of the dual statement is similar. □

A uniqueness result similar to the uniqueness statement in Theorem 4.3 can also be found in [4], even for more general convex programs.

By combining the statements of Theorem 4.3 for the primal and dual we obtain:

**Corollary 4.4.** *Let  $A \in \mathbb{R}^{m \times n}$  be given arbitrarily. Then for almost all  $(B, c) \in \mathbb{R}^m \times \mathbb{R}^n$  the following holds: If both  $(P)$  and  $(D)$  are feasible, then both satisfy the Slater condition and both have unique optimal solutions  $\bar{X}$  and  $\bar{Y}$ .*

*A corresponding result holds for  $(P_0), (D_0)$  wrt. almost all  $(B, C) \in \mathbb{R}^m \times \mathbb{R}^m$ .*

## 4.2 Nondegeneracy and strict complementarity

We now discuss nongeneracy and strict complementarity of optimal solutions in conic programming. It has been shown by Pataki and Tunçel in [29] that both properties hold for almost all problem instances. For completeness we summarize their results, which are formulated in terms of the cone programs in self-dual form  $(P_0), (D_0)$  (cf. Section 2). Note that in [29] these results have been proven under the assumption that the problems are gap-free. We emphasise that their arguments are completed by the results of Section 3 which assure (weak) genericity of gap-freeness.

We have to introduce some notation. Let us denote the minimal face of  $\mathcal{K}$  containing  $X$  and the minimal face of  $\mathcal{K}^*$  containing  $Y$ , respectively, by

$$J(X) = \text{face}(X, \mathcal{K}) \quad \text{and} \quad G(Y) = \text{face}(Y, \mathcal{K}^*).$$

Observe that for each feasible  $X$ , we have  $X \in \text{rint } J(X)$ . For a face  $F$  of  $\mathcal{K}$ , we define the complementary face as  $F^\Delta := \{Q \in \mathcal{K}^* \mid \langle Q, S \rangle = 0 \text{ for all } S \in F\}$ . Clearly,  $F^\Delta$  is a closed convex cone. Moreover, it is not difficult to see that if  $X \in \text{rint } F$ , then  $F^\Delta = \{Q \in \mathcal{K}^* \mid \langle Q, X \rangle = 0\}$ . This immediately implies that the complementary face of  $J(X)$  is equivalently given by

$$J^\Delta(X) = \{Q \in \mathcal{K}^* \mid \langle Q, X \rangle = 0\}. \quad (4.1)$$

Analogous definitions and results apply to  $G^\Delta(Y)$ , the complementary face of  $G(Y)$ .

**Definition 4.5.** *The extreme points of  $\mathcal{F}_{P_0}$  (resp.  $\mathcal{F}_{D_0}$ ) are called primal (resp. dual) basic feasible solutions.*

The following characterization of basic solutions is given in [29, Theorem 1]:

**Lemma 4.6.** *Let  $X$  be feasible for  $(P_0)$ . Then  $X$  is a basic feasible solution if and only if*

$$\text{span}(J(X)) \cap \mathcal{L} = \{0\}. \quad (4.2)$$

A similar condition for the dual program leads to the concept of (primal) nondegeneracy:

**Definition 4.7.** *A primal feasible solution  $X$  is called nondegenerate, if*

$$\text{span}(J^\Delta(X)) \cap \mathcal{L}^\perp = \{0\}. \quad (4.3)$$

*Nondegeneracy of a dual feasible solution  $Y$  is defined analogously.*

**Definition 4.8.** *Optimal solutions  $\bar{X}$  of  $(P_0)$  and  $\bar{Y}$  of  $(D_0)$  are called complementary, if  $\langle \bar{X}, \bar{Y} \rangle = 0$ , i.e., if  $\bar{Y} \in J^\Delta(\bar{X})$ . The solutions  $\bar{X}$  and  $\bar{Y}$  are called strictly complementary, if we have*

$$\bar{Y} \in \text{rint } J^\Delta(\bar{X}). \quad (4.4)$$

*Recall that  $\bar{X} \in \text{rint } J(\bar{X})$  holds by definition.*

The following lemma shows some relations between nondegeneracy, strict complementarity, basic solutions and uniqueness.

**Lemma 4.9.** *(see [28], [29, Theorem 2]) Let  $X$  be an optimal solution of  $(P_0)$ . Then the following hold.*

- (a) *If  $X$  is a unique optimal solution, then  $X$  is a basic solution.*
- (b) *If  $X$  is nondegenerate, then any complementary solution  $Y$  of  $(D_0)$  must be basic. Moreover, if there is a complementary solution  $Y$ , it must be unique.*

(c) Suppose that  $Y$  is a dual feasible solution and  $X$  and  $Y$  are strictly complementary. Then  $Y$  is basic if and only if  $X$  is nondegenerate.

**Remark 4.10.** In [29], a slightly different definition of strict complementarity is given: the optimal solutions  $\bar{X}$  and  $\bar{Y}$  are called strictly complementary if

$$\bar{X} \in \text{rint } F \quad \text{and} \quad \bar{Y} \in \text{rint } F^\Delta \quad \text{holds for some face } F \text{ of } \mathcal{K}. \quad (4.5)$$

It is clear that (4.4) implies (4.5). Conversely, let (4.5) be satisfied. We always have  $\bar{X} \in \text{rint } J(\bar{X})$ . So  $\bar{X} \in \text{rint } F$  implies  $F^\Delta = J^\Delta(\bar{X})$  by (4.1). Therefore, (4.4) and (4.5) are equivalent.

In [28], strict complementarity for  $\bar{X}, \bar{Y}$  is defined by  $J^\Delta(\bar{X}) = G(\bar{Y})$ . It can be shown that this condition and (4.4) are equivalent, see the proof of [29, Theorem 2]. By considering the dual problem, strict complementarity can similarly be defined as (again  $\bar{Y} \in \text{rint } G(\bar{Y})$  holds by definition):

$$\bar{X} \in \text{rint } G^\Delta(\bar{Y}). \quad (4.6)$$

Neither of the conditions (4.4) or (4.6) implies the other unless  $\mathcal{K}$  or  $\mathcal{K}^*$  are facially exposed, as noted in [28, Remark 3.3.2]. For an illustrative example for these ‘‘asymmetric’’ definitions of strict complementarity we refer to [7, Example 1].

Note that not all cones appearing in conic programming are facially exposed: it is well known that the cone of semidefinite matrices is facially exposed, but the cone of copositive matrices is not, see [8, Theorem 8.22].

We now sketch the weak genericity result for nondegeneracy and strict complementarity of Pataki and Tunçel [29]. To prove their result, they consider for fixed  $\mathcal{L}$  the sets (see [29, p. 455 and Proposition 1])

$$\bar{\mathcal{D}}(\mathcal{L}) := \{(B, C) \mid \text{the corresponding problems } (P_0) \text{ and } (D_0) \text{ are feasible with } v_{P_0} = v_{D_0}\}$$

and

$$\mathcal{D}(\mathcal{L}) := \{(B, C) \in \bar{\mathcal{D}}(\mathcal{L}) \mid \text{some optimal solutions } \bar{X}, \bar{Y} \text{ of } (P_0), (D_0) \text{ are strictly complementary}\}.$$

Using a deep result from geometric measure theory [29, Theorem 3] they derive the following result.

**Lemma 4.11.** [29, Proposition 2] *For fixed  $\mathcal{L}$ , the set  $\bar{\mathcal{D}}(\mathcal{L}) \setminus \mathcal{D}(\mathcal{L})$  has  $\dim(\bar{\mathcal{D}}(\mathcal{L}))$ -dimensional Hausdorff measure zero.*

From this result Pataki and Tunçel [29] derive

**Theorem 4.12.** *Let  $\mathcal{L}$  be given arbitrarily. Then for almost all  $(B, C) \in \mathbb{R}^{2m}$  the following is true: If the corresponding programs  $(P_0), (D_0)$  are both feasible, then there exist unique optimal solutions  $\bar{X}$  of  $(P_0)$  and  $\bar{Y}$  of  $(D_0)$ . These solutions are nondegenerate and satisfy the strict complementarity condition.*

*Proof.* Similar to the arguments in [29, p.456], we combine several results. For fixed  $\mathcal{L}$  we consider the set  $\mathcal{P}^0$  of instances  $(B, C)$  such that the primal and dual are feasible. (Note that this set  $\mathcal{P}^0$  is of full dimension.) Corollary 4.4 together with Lemma 4.11 guaranty that for almost all instances in  $\mathcal{P}^0$  the primal and dual optimal solutions are unique and strictly complementary (as defined in (4.4)). Let  $\mathcal{P}_{sc}^0$  denote this weakly generic subset of  $\mathcal{P}^0$ . In view of Lemma 4.9(a) (also valid for the optimal solution  $Y$  of  $(D_0)$ ) the dual optimal solutions of instances in  $\mathcal{P}_{sc}^0$  are basic and by Lemma 4.9(c) the primal maximizers  $X$  are nondegenerate.

Note that Lemma 4.9(c) does not hold for  $X$  and  $Y$  interchanged unless  $\mathcal{K}$  is facially exposed (cf., Remark 4.10). However, if we define strict complementarity as in (4.6), then Lemma 4.9(c) holds for  $X$  and  $Y$  interchanged. Analogous to (4.4) following [29], one can show that (4.6) is a weakly generic property. Thus, using the same arguments, weakly generically at optimal solutions of  $(D_0)$  the nondegeneracy condition holds.

□

**Remark 4.13.** With the same projection trick as in Remark 3.4, the genericity result of Theorem 4.12 for  $(P_0), (D_0)$  can directly be translated to the following statement for the program in the form  $(P), (D)$ :

Let  $A \in \mathbb{R}^{m \times n}$  be arbitrary. Then for almost all  $(B, c) \in \mathbb{R}^n \times \mathbb{R}^m$  we have that if  $(P)$  and  $(D)$  are both feasible, then there exist unique optimal solutions  $\bar{X}$  of  $(P)$  and  $\bar{Y}$  of  $(D)$ . Moreover,  $\bar{X}$  and  $\bar{Y}$  are both nondegenerate and satisfy the strict complementarity condition.

Note that to assure uniqueness of the solution of  $(P)$  in terms of the variable  $x \in \mathbb{R}^n$  we have to assume that  $A$  has full rank  $n$ . Recall, however, that for  $m \geq n$  a matrix  $A \in \mathbb{R}^{m \times n}$  generically has full rank  $n$  (cf. (2.6)).

### 4.3 Connection between nondegeneracy and Slater's condition

We shortly comment on the fact that nondegeneracy implies the Slater condition. We again analyse this for conic programs of the form  $(P_0)$  (see (2.1)). The following is true.

**Lemma 4.14.** *Let  $X$  be a nondegenerate feasible solution of  $(P_0)$ . Then Slater's condition holds for  $(P_0)$ . An analogous result is true for the problems  $(D_0), (P)$ , and  $(D)$ .*

*Proof.* We first note that the nondegeneracy condition  $\mathcal{L}^\perp \cap \text{span}(J^\Delta(X)) = \{0\}$  is equivalent to  $\mathcal{L} + [\text{span}(J^\Delta(X))]^\perp = \mathbb{R}^m$ . So by choosing some  $X_0 \in \text{int } \mathcal{K}$  there is a representation

$$X_0 = L + Z \quad \text{with } L \in \mathcal{L} \text{ and } Z \in [\text{span}(J^\Delta(X))]^\perp.$$

By defining  $L := X_0 - Z$  and using  $\langle X_0, Y \rangle > 0 \forall Y \in \mathcal{K}^* \setminus \{0\}$  we find for any  $S \in J^\Delta(X) \setminus \{0\}$  the relation  $\langle S, Z \rangle = 0$  and then

$$\langle S, L \rangle = \langle S, X_0 \rangle - \langle S, Z \rangle = \langle S, X_0 \rangle > 0.$$

Let  $\mathcal{B}_1 := \{S \mid \|S\| = 1\}$  be the unit sphere in  $\mathbb{R}^m$ . By compactness of  $\mathcal{B}_1$  and the continuity of the linear function  $\langle L, \cdot \rangle$ , there exists some  $\varepsilon > 0$  such that

$$\langle L, S \rangle \geq 2\varepsilon \quad \text{for all } S \in J^\Delta(X) \cap \mathcal{B}_1. \quad (4.7)$$

We now will show that for  $\alpha > 0$  small enough we have  $(X + \alpha L) \in (B + \mathcal{L}) \cap \text{int } \mathcal{K}$ , i.e., Slater's condition holds for  $(P_0)$ . Clearly  $(X + \alpha L) \in B + \mathcal{L}$  since  $X \in B + \mathcal{L}$  and  $L \in \mathcal{L}$ . To prove  $(X + \alpha L) \in \text{int } \mathcal{K}$ , we have to show that

$$\langle X + \alpha L, S \rangle > 0 \quad \text{for all } S \in \mathcal{K}^* \cap \mathcal{B}_1. \quad (4.8)$$

To do so, in view of (4.7) by continuity there exists some  $\delta > 0$  such that

$$\langle L, S \rangle \geq \varepsilon \quad \text{for all } S \in J_\delta^\Delta(X) \cap \mathcal{B}_1, \quad (4.9)$$

where  $J_\delta^\Delta(X) := \{S \in \mathcal{K}^* \mid |S - \bar{S}| < \delta \text{ for some } \bar{S} \in J^\Delta(X)\}$ . Since  $X \in \mathcal{K}$ , we have  $\langle X, S \rangle \geq 0$  for all  $S \in \mathcal{K}^*$ , and by the definition of  $J^\Delta(X)$  in (4.1) we have that  $\langle X, S \rangle > 0$  for all  $S \in (\mathcal{K}^* \setminus J_\delta^\Delta(X)) \cap \mathcal{B}_1$ . By compactness of this set, there exists some  $T$  such that

$$\langle X, S \rangle \geq T > 0 \quad \text{for all } S \in (\mathcal{K}^* \setminus J_\delta^\Delta(X)) \cap \mathcal{B}_1. \quad (4.10)$$

Let  $M := \min\{\langle L, S \rangle \mid S \in (\mathcal{K}^* \setminus J_\delta^\Delta(X)) \cap \mathcal{B}_1\}$ . We claim that  $X + \alpha L \in \text{int } \mathcal{K}$  for all  $0 < \alpha < \frac{T}{|M|}$ . We have the following two cases:

$$\text{If } S \in (\mathcal{K}^* \setminus J_\delta^\Delta(X)) \cap \mathcal{B}_1: \quad \text{then } \langle X + \alpha L, S \rangle = \langle X, S \rangle + \langle \alpha L, S \rangle \geq T + \alpha M > 0.$$

$$\text{If } S \in J_\delta^\Delta(X) \cap \mathcal{B}_1: \quad \text{using } \langle X, S \rangle \geq 0 \text{ and (4.9), we have } \langle X + \alpha L, S \rangle = \langle X, S \rangle + \langle \alpha L, S \rangle \geq \alpha \varepsilon > 0.$$

By combining these two cases, we have shown that (4.8) holds, and the result follows.  $\square$

For the case of semidefinite programming, it has been shown implicitly in [2, Proof of Theorem 14] that given

$\mathcal{L}$ , for almost all  $B$  all feasible points of  $\mathcal{F}_P$  are nondegenerate. Note that in [2] a definition of nondegeneracy is used which is different but equivalent to (4.3): nondegeneracy is defined in terms of transversality conditions for certain tangent spaces. Hence, by applying Lemma 4.14, it follows for the SDP case that given  $\mathcal{L}$ , for almost all  $B$  we have: if  $\mathcal{F}_P \neq \emptyset$ , then  $\mathcal{F}_P$  has Slater points. This was also established in [35, page 310].

We wish to mention that in geometric measure theory, transversality results have been proven which roughly speaking assert that weakly generically all intersection points of two convex sets are nondegenerate. For example, the following result has been shown in [21]:

**Lemma 4.15.** (see [21, Lemma 3.1]) *Let  $K, L \subset \mathbb{R}^m$  be compact convex sets with nonempty interiors. Then for almost all  $B \in \mathbb{R}^m$  (wrt. the Hausdorff measure) the sets  $K$  and  $L_B := B + L$  intersect almost transversally, i.e., for all  $X \in \text{bd } K \cap \text{bd } L_B$  we have*

$$N(K, X) \cap N(L_B, X) = \{0\} \quad \text{and} \quad N(K, X) \cap -N(L_B, X) = \{0\}$$

where  $N(K, X)$  denotes the normal cone of  $K$  at  $X$ .

A similar result has been given in [33, Theorem 2]. In combination with Lemma 4.14, also these results could be used to show that nondegeneracy and Slater's condition hold weakly generically in CP.

#### 4.4 Genericity results in linear semi-infinite optimization

In the preceding discussions we have made use of the fact that a conic program can be seen as a special case of a linear semi-infinite program (SIP) (cf. Sections 3.2 and 4.1). There are many papers dealing with generic properties (in the sense of density and stability) of semi-infinite problems in the form  $(\text{SIP}_P)$ ,  $(\text{SIP}_D)$  (see (2.3), (2.4)). We refer to [23] and [15, 16, 17, 18, 27]. In [14, Chapter 5] the interested reader finds an overview of stability and genericity results for linear semi-infinite problems.

One might expect that these genericity results for SIP can directly be transferred to CP, but unfortunately this is not the case. The reason is the following.

In the above articles, SIP programs are considered in the form (2.3) with infinite, compact index set  $Z \subset \mathbb{R}^m$ . In [23] the problem data  $(a(Y), b(Y), c)$  are elements of the space  $C^2(Z)^n \times C^2(Z) \times \mathbb{R}^n$ . In [15, 16, 17, 18] the data  $(a(Y), b(Y), c)$  are taken from  $C(Z)^n \times C(Z) \times \mathbb{R}^n$  endowed with the norm of uniform convergence

$$\|(a, b, c)\| = \max \left\{ \max_{Y \in Z} \|(a(Y), b(Y))\|_\infty, \|c\|_\infty \right\}.$$

But if we write CP in the form (2.3), (2.5), then the data  $(a(Y), b(Y))$  are of the special form

$$a(Y) = A^T Y, \quad b(Y) = \langle B, Y \rangle,$$

linear in  $Y$ . So this set of cone programs represent only a small subset of the set of SIP instances, *e.g.*, given by  $(a(Y), b(Y), c) \in C(Z)^n \times C(Z) \times \mathbb{R}^n$ . Thus this subset of CP programs allows much less freedom for perturbations, so that roughly speaking we can say:

- The density results cannot be transferred from the general SIP theory to the special case of CP.
- Openness results remain valid in the following sense: the sufficient conditions for stability in SIP remain valid for CP, but not the necessary conditions. Typically the conditions for stability in SIP are too strong in CP.

We just note that [17, Theorem 1] gives genericity results (density and openness) for the special case of (finite) linear programs.

## 5 Stability issues

The results so far do not present full genericity statements in the sense that stability is guaranteed wrt. perturbation of the whole set of parameters  $(A, B, c)$ . As we will show in a moment, in general CP, the Slater condition and strong duality are (fully) generic properties (density and openness). For the other desirable properties, namely, uniqueness, nondegeneracy and strict complementarity of solutions only weak genericity results (density without openness) have been established.

In smooth finite optimization (see [22]), the stability of such properties is typically proven by applying the (smooth) Implicit Function Theorem to an appropriate system of optimality conditions. As we shall see, this approach can be applied to the special case of LP and SDP. For the latter we make use of the fact (shown in [2]) that the set of positive semidefinite matrices of a given rank can locally be described by smooth manifolds. Similar techniques can be used if the cones  $\mathcal{K}, \mathcal{K}^*$  are so-called semi-algebraic sets: it is well-known that semi-algebraic sets allow a complete partition (stratification) of the set into smooth manifolds (see *e.g.*, [3, 2.5.1 Proposition]). For the sake of completeness we recall that a set  $\mathcal{A} \subset \mathbb{R}^N$  is called semi-algebraic if it is given by a finite union of sets

$$\{x \in \mathbb{R}^N \mid p_i(x) = 0, i = 1, \dots, k; \quad q_j(x) > 0, j = 1, \dots, s\}$$

with  $k, s \in \mathbb{N}$  and polynomial functions  $p_i, q_j \in \mathbb{R}[x_1, \dots, x_N]$ . The theory of semi-algebraic sets has been used in [4] to prove a genericity result for primal uniqueness. The stability is however shown only wrt. the objective vector  $c$  as parameter. We formulate one of their results in terms of our cone program:

[see [4, Theorem 5.1]] Let  $\mathcal{K}$  be a semi-algebraic cone, and let  $A, B$  be given such that  $\mathcal{F}_P$  is compact. Then there exists a generic set  $S \subset \mathbb{R}^n$  such that for all  $c \in S$  the corresponding program  $(P)$  has a unique maximizer.

It is not difficult to see that the cones of semidefinite, copositive and completely positive matrices are semi-algebraic.

However, general cones  $\mathcal{K}$  may have a much more complicated nonsmooth structure. So whether in general CP the properties of uniqueness, nondegeneracy and strict complementarity are stable (in a generic subset set of the problem set) remains an open problem.

We now establish some (full) genericity results. By using the stability of Slater's condition we firstly will prove that generically strong duality holds in general CP. To that end we restrict ourselves to the subset  $\mathcal{P}^1$  of CP instances (with fixed  $\mathcal{K}$ ,  $m \geq n$ ),

$$\mathcal{P}^1 = \{(A, B, c) \mid \text{the corresponding programs } (P), (D) \text{ are both feasible}\} .$$

Note that this set is of full dimension  $m \cdot n + m + n$ . By using results from Section 3 we can prove

**Theorem 5.1.** *There is a generic subset  $\mathcal{P}_{reg}^1$  of  $\mathcal{P}^1$  such that for any  $(A, B, c) \in \mathcal{P}_{reg}^1$  the Slater condition holds for  $(P)$  and  $(D)$  and both programs have optimal solutions with  $v_P = v_D$ .*

*Proof.* By Corollary 3.6 there is a weakly generic subset  $\mathcal{P}_1^1$  of  $\mathcal{P}^1$  such that for any  $(A, B, c) \in \mathcal{P}_1^1$  the Slater condition holds for the corresponding programs  $(P)$  and  $(D)$ . In view of (2.6) there also exists a generic subset  $\mathcal{P}_A$  of  $\mathbb{R}^{m \times n}$  such that for any  $A \in \mathcal{P}_A$  we have  $\text{rank } A = n$  (recall  $m \geq n$ ). We define the weakly generic subset  $\mathcal{P}_{reg}^1$  of  $\mathcal{P}^1$  by

$$\mathcal{P}_{reg}^1 = \mathcal{P}_1^1 \cap (\mathcal{P}_A \times \mathbb{R}^m \times \mathbb{R}^n) .$$

By definition, for any  $(\bar{A}, \bar{B}, \bar{c}) \in \mathcal{P}_{reg}^1$  the Slater condition holds for the corresponding programs  $(\bar{P}), (\bar{D})$ , i.e., there exist  $\bar{x} \in \mathcal{F}_{\bar{P}}, \bar{Y} \in \mathcal{F}_{\bar{D}}$ , such that

$$\bar{B} - \bar{A}\bar{x} \in \text{int } \mathcal{K} , \quad \bar{A}^T \bar{Y} = \bar{c}, \quad \bar{Y} \in \text{int } \mathcal{K}^* , \quad (5.1)$$

and  $\bar{A}$  has full rank  $n$ . Both Slater conditions in (5.1) are stable wrt. small perturbations of  $(\bar{A}, \bar{B}, \bar{c})$ . Indeed for  $(A, B, c)$  near  $(\bar{A}, \bar{B}, \bar{c})$  the point  $\bar{x}$  still satisfies the primal Slater condition. Moreover if we define  $Y = Y(A, c)$  as the (unique) solution of

$$\min \|Y - \bar{Y}\| \quad \text{s.t.} \quad A^T Y = c ,$$

by using  $\text{rank } \bar{A} = n$ , it is not difficult to show that  $Y(A, c)$  is continuously depending on  $A, c$  and satisfies  $Y(A, c) \rightarrow \bar{Y}$  for  $(A, c) \rightarrow (\bar{A}, \bar{c})$  and thus for  $(A, c)$  close to  $(\bar{A}, \bar{c})$  the vector  $Y(A, c)$  lies in the interior of  $\mathcal{K}^*$ . So the set  $\mathcal{P}_{reg}^1$  is an (open) generic subset of  $\mathcal{P}^1$ .

Moreover by the arguments before Corollary 3.5 for any  $(A, B, c) \in \mathcal{P}_{reg}^1$  both programs  $(P), (D)$  have optimal solutions and the strong duality relation  $v_P = v_D$  holds.

□

Before we give a (full) stability analysis for the case of SDP we consider linear programs as an illustrative example.

**Stability analysis for LP:** Consider the pair of primal-dual LP's

$$\begin{aligned} (P) \quad & \max c^T x \quad \text{s. t.} \quad X := B - Ax \in \mathbb{R}_+^m, \\ (D) \quad & \min \langle B, Y \rangle \quad \text{s. t.} \quad A^T Y = c, \quad Y \in \mathbb{R}_+^m, \end{aligned}$$

for instances  $Q := (A, B, c)$  (with  $A$  of full rank  $n$ ). Let again  $\mathcal{P}^1$  denote the set of LP instances  $Q$  such that the corresponding programs  $(P), (D)$  are both feasible. In view of Theorem 4.12 and Remark 4.13 there exists a weakly generic subset  $\mathcal{P}_{reg}^1 \subset \mathcal{P}^1$  of instances  $Q$  such that the pair of primal, dual optimal solutions  $\bar{X}, \bar{Y}$  of  $(P), (D)$  are both unique, nondegenerate, and strictly complementary. Now let  $\bar{Q} := (\bar{A}, \bar{B}, \bar{c})$  be an element of  $\mathcal{P}_{reg}^1$  with solutions  $\bar{X}, \bar{Y}$ . Let us denote the active index set  $\bar{I} = \{i \in \{1, \dots, m\} \mid \bar{X}_i = 0\}$ , its complement  $\bar{I}^C = \{i \in \{1, \dots, m\} \mid \bar{X}_i > 0\}$ , and  $\bar{\mathcal{L}} := \text{span}\{\bar{A}_j \mid j = 1, \dots, n\}$ , where  $\bar{A}_j$  is the  $j$ th column of  $\bar{A}$ . It follows that

$$J(\bar{X}) = \text{cone}\{e_i \mid i \in \bar{I}^C\} = G^\Delta(\bar{Y}) \quad , \quad G(\bar{Y}) = \text{cone}\{e_i \mid i \in \bar{I}\} = J^\Delta(\bar{X}). \quad (5.2)$$

The nondegeneracy condition for  $\bar{X}$  resp.  $\bar{Y}$  reads

$$\bar{\mathcal{L}}^\perp \cap \text{lin } J^\Delta(\bar{X}) = \{0\} \quad \text{resp.} \quad \bar{\mathcal{L}} \cap \text{lin } G^\Delta(\bar{Y}) = \{0\}. \quad (5.3)$$

The strict complementarity condition means that  $\bar{Y}_i = 0$  holds if and only if  $i \in \bar{I}^C$ . From (5.3) we deduce  $|\bar{I}| \leq n$ , resp.  $|\bar{I}^C| \leq m - n$  and thus, using  $m = |\bar{I}| + |\bar{I}^C| \leq m - n + n = m$ , we find  $|\bar{I}| = n$ . Moreover, the condition  $\bar{\mathcal{L}} \cap \text{lin } G^\Delta(\bar{Y}) = \bar{\mathcal{L}} \cap \text{lin}\{e_i \mid i \in \bar{I}^C\} = \{0\}$  implies that the matrix

$$\begin{pmatrix} \bar{A}^T \\ e_i^T, i \in \bar{I}^C \end{pmatrix} \quad \text{and thus the } n \times n\text{-matrix} \quad \bar{A}_{\bar{I}} := ([\bar{A}_1]_{\bar{I}}, \dots, [\bar{A}_n]_{\bar{I}})$$

is nonsingular (where  $[\bar{A}_1]_{\bar{I}} := ([\bar{A}_1]_j, j \in \bar{I})^T$ ). It finally follows that for  $(A, B, c)$  near  $(\bar{A}, \bar{B}, \bar{c})$  the solutions  $x$  (resp.  $X$ ) of  $(P)$  and  $Y$  of  $(D)$  are given as the solutions of the systems

$$B_{\bar{I}} - A_{\bar{I}} x = 0 \quad \text{and} \quad A_{\bar{I}}^T Y_{\bar{I}} - c = 0, \quad (5.4)$$

with  $Y$  defined by  $Y_i = [Y_{\bar{I}}]_i$  for  $i \in \bar{I}$  and  $Y_i = 0$  otherwise. These solutions yield unique, nondegenerate

and strictly complementary optimal solutions  $X, Y$  of  $(P), (D)$ . So we obtain the (well-known) result.

**Theorem 5.2.** *There is a generic subset  $\mathcal{P}_{reg}^1 \subset \mathcal{P}^1$  such that for all instances  $Q = (A, B, c)$  in  $\mathcal{P}_{reg}^1$  the primal, dual optimal solutions are unique, nondegenerate, and strictly complementary.*

*Moreover for any  $\bar{Q} \in \mathcal{P}_{reg}^1$  with primal solution  $\bar{X}$  and corresponding active index set  $\bar{I}$  ( $|\bar{I}| = n$ ) there exists a neighborhood  $\mathcal{N}$  of  $\bar{Q}$  such that for any  $Q = (A, B, c) \in \mathcal{N}$  the optimal solutions  $X, Y$  of the corresponding LP is given as the solution of the system (5.4).*

**Stability analysis for SDP:** We now study the stability of uniqueness, nondegeneracy and strict complementarity for SDP, *i.e.*, for the case  $\mathcal{K} = \mathcal{S}_k^+ = \{X \in \mathcal{S}_k \mid X \text{ is positive semidefinite}\}$  and  $A_i \in \mathcal{S}_k \equiv \mathbb{R}^m$  with  $m = \frac{1}{2}k(k+1)$ . Since we will make use of results in [2], we consider SDP in the form

$$(P_0) \quad \max \langle C, B \rangle - \langle C, X \rangle \quad \text{s. t.} \quad X := B - \sum_{i=1}^n x_i A_i \in \mathcal{S}_k^+$$

$$(D_0) \quad \min \langle B, Y \rangle \quad \text{s. t.} \quad Y := \sum_{j=1}^{m-n} y_j A_j^\perp + C \in \mathcal{S}_k^+.$$

as programs depending on the parameter  $Q := (\{A_i\}_{i=1}^n, B, C) \in (\mathcal{S}_k)^{n+2}$  (with  $m \geq n$ ). We again can assume that the matrices  $A_i, i = 1, \dots, n$ , are linearly independent (generic condition according to (2.6)) and that  $A_j^\perp, j = 1, \dots, m-n$ , is a basis of the orthogonal complement of  $\text{span}\{A_i\}_{i=1}^n$ .

For completeness we sketch the proof of the weak genericity results in [2]. We however present the arguments in a more explicit form which will enable us to apply the Implicit Function Theorem to establish stability, *i.e.*, full genericity.

We start by collecting some well-known facts in differential geometry.

- (1) Let be given a function  $f \in C^1(\mathbb{R}^q, \mathbb{R}^s)$ . Then  $0 \in \mathbb{R}^s$  is called a regular value of  $f$  if

$$\nabla f(x) \quad \text{has (full) rank } s \text{ for all } x \text{ such that } f(x) = 0. \quad (5.5)$$

- (2) (See *e.g.*, [22, Remark 3.1.5].) A set  $M \subset \mathbb{R}^s$  is called a  $C^r$ -manifold of codimension  $c_d, 0 \leq c_d \leq s$ , (dimension  $s - c_d$ ) if for any  $\bar{x} \in M$  there exist a neighborhood  $\mathcal{N}_{\bar{x}}$  and a  $C^r$  vector function  $h : \mathcal{N}_{\bar{x}} \rightarrow \mathbb{R}^{c_d}$  such that  $\nabla h(x)$  has rank  $c_d$  for all  $x \in \mathcal{N}_{\bar{x}}$  and

$$x \in \mathcal{N}_{\bar{x}} \text{ is in } M \text{ if and only if } h(x) = 0.$$

- (3) Let  $f : \mathbb{R}^q \rightarrow \mathbb{R}^s$  be a  $C^1$ -function and  $M \subset \mathbb{R}^s$  a manifold of codimension  $c_d$ , locally (in  $\mathcal{N} \subset \mathbb{R}^q$ ) defined by  $h(y) = 0$  with a  $C^1$ -function  $h : \mathcal{N} \rightarrow \mathbb{R}^{c_d}$ . Then we say that  $f$  is transversal to  $M$  if (*cf.*,

[22, Theorem 7.3.4])

$$\nabla f(x)[\mathbb{R}^s] + T_{f(x)}M = \mathbb{R}^s \quad \text{holds for all } x \text{ with } f(x) \in M, \quad (5.6)$$

where  $T_{f(x)}M$  is the tangent space to  $M$  at  $f(x)$ . By [22, Remark 7.3.5] an equivalent formulation of (5.6) is (with the defining equations  $h(y) = 0$  for  $M$ ):

$$\nabla h(f(x)) \text{ has full rank } c_d \text{ for all } x \text{ with } f(x) \in M. \quad (5.7)$$

The following is a useful generalisation of the Sard Theorem (see *e.g.*, [37, Prop. 78.10] for a proof).

**Theorem 5.3.** [*Parametric Sard Theorem*]

Let  $h : Q \times P \subset \mathbb{R}^q \times \mathbb{R}^p \rightarrow \mathbb{R}^s$ ,  $(x, y) \mapsto h(x, y)$ , be a  $C^r$ -mapping with  $r > \max\{0, q - s\}$  and open sets  $Q \subset \mathbb{R}^q, P \subset \mathbb{R}^p$ . If  $0 \in \mathbb{R}^s$  is a regular value of  $h$  then for almost all  $y \in P$  the value 0 is a regular value of the function  $h_x(x) := h(x, y)$ .

We now introduce the relevant functions and manifolds for the genericity results. It is well-known (see *e.g.*, [22, Example 7.3.24]) that for any  $r$ ,  $0 \leq r \leq k$  the set

$$W_r := \{X \in \mathcal{S}_k \mid \text{rank } X = r\} \quad \text{is a } C^\infty\text{-manifold in } \mathcal{S}_k \text{ of codimension } c_d = \frac{(k+1-r)(k-r)}{2}.$$

Let this manifold locally be defined by the system  $K(X) = 0$ .

In [2, Lemma 22] it has been proven that for any  $r, s$ ,  $0 \leq r, s$  and  $0 \leq r + s \leq k$  the set

$$W_{r,s} := \{(X, Y) \in \mathcal{S}_k \times \mathcal{S}_k \mid \text{rank } X = s, \text{rank } Y = r, \langle X, Y \rangle = 0\}$$

is a smooth  $C^\infty$ -submanifold of  $\mathcal{S}_k \times \mathcal{S}_k$  with  $\dim W_{r,s} = m - \frac{(k+1-r-s)(k-r-s)}{2}$  and thus with codimension  $c_d = m + \frac{(k+1-r-s)(k-r-s)}{2}$ . Given a pair  $(\bar{X}, \bar{Y}) \in W_{r,s}$  such that  $\bar{X} \in \mathcal{S}_s^+, \bar{Y} \in \mathcal{S}_r^+$  (positive semidefinite). By continuity of the eigenvalues for  $(X, Y)$  close to  $(\bar{X}, \bar{Y})$  the pair  $(X, Y)$  is in  $W_{r,s}$  if and only if  $(X, Y) \in W_{r,s}^+$  where

$$W_{r,s}^+ := \{(X, Y) \in \mathcal{S}_k^+ \times \mathcal{S}_k^+ \mid \text{rank } X = s, \text{rank } Y = r, \langle X, Y \rangle = 0\}.$$

So also the set  $W_{r,s}^+$  is a manifold of the same codimension  $c_d$ . This means that with locally defined smooth functions  $H$  (with  $H(X, Y) \in \mathbb{R}^{c_d}$ ) we have  $(X, Y) \in W_{r,s}^+$  if and only if  $H(X, Y) = 0$ . Note also that for  $(X, Y) \in W_{r,s}^+$  the relation  $\langle X, Y \rangle = 0$  implies  $X \cdot Y = 0$ . So the condition  $r + s \leq k$  must hold.

Now for  $x \in \mathbb{R}^n, y \in \mathbb{R}^{m-n}$  and SDP instance  $Q := (\{A_i\}_{i=1}^n, B, C)$  we define the mappings:

$$F(x, Q) := B - \sum_{i=1}^n x_i A_i, \quad G(y, Q) := C + \sum_{j=1}^{m-n} y_j A_j^\perp \quad (5.8)$$

For parameters  $Q = (\{A_i\}_{i=1}^n, B, C)$  in a sufficiently small neighborhood of  $\bar{Q} = (\{\bar{A}_i\}_{i=1}^n, \bar{B}, \bar{C})$  we can assume that the orthogonal complement  $\{A_j^\perp\}_{j=1, \dots, m-n}$  depends at least  $C^1$ -smoothly on the parameters  $\{A_i\}_{i=1}^n$ . Indeed, we obtain the  $\{A_j^\perp\}$ 's by a smooth Gram-Schmidt orthogonalization process (to compute  $\{\bar{A}_j^\perp\}_{j=1, \dots, m-n}$ ). So these functions  $F(x, Q), G(y, Q)$  can be seen as smooth functions of all parameters.

With these preparations we can prove the following full genericity result for SDP.

**Theorem 5.4.** *There is a generic subset  $\mathcal{P}_{reg}^1$  of the set*

$$\mathcal{P}^1 = \{(\{A_i\}_{i=1}^n, B, C) \mid \text{the corresponding problems } (P_0), (D_0) \text{ are both feasible}\} \subset (\mathcal{S}_k)^{n+2}$$

of SDP instances such that the following holds. For any  $Q \in \mathcal{P}_{reg}^1$  there exist unique, nondegenerate, and strictly complementary solutions  $x, y$  (or  $X, Y$ ) of  $(P_0), (D_0)$ . Moreover if  $\bar{Q} \in \mathcal{P}_{reg}^1$  is such that the corresponding (unique, nondegenerate, strictly complementary) solutions  $\bar{x}, \bar{y}$  (or  $\bar{X}, \bar{Y}$ ) of  $(P_0), (D_0)$  have  $\text{rank } \bar{X} = s, \text{rank } \bar{Y} = r$  with  $r + s = k$ , then there exists a (nonempty open) neighborhood  $\mathcal{N}$  of  $\bar{Q}$  such that for any  $Q \in \mathcal{N}$  the corresponding SDP programs  $(P_0)$  and  $(D_0)$  have (unique, nondegenerate, strictly complementary) solutions  $x(Q) \approx \bar{x}, y(Q) \approx \bar{y}$  (or  $(X(Q), Y(Q)) \approx (\bar{X}, \bar{Y})$ ) with the same ranks,  $\text{rank } X(Q) = s$  and  $\text{rank } Y(Q) = r$ .

*Proof.* We first sketch the proof of the weak genericity result as in [2]. Let  $\mathcal{P}_0^1$  denote the weakly generic subset of  $\mathcal{P}^1$  such that for all  $Q \in \mathcal{P}_0^1$  optimal solutions  $X, Y$  of  $(P_0), (D_0)$  exist with  $\langle X, Y \rangle = 0$  (see Corollary 3.6).

For fixed  $r, s$  ( $0 \leq r + s \leq k$ ) we now consider the system of  $c_d$  equation  $H(X, Y) = 0$  which (locally) define the manifold  $W_{r,s}^+$  of codimension  $c_d = m + \frac{(k+1-r-s)(k-r-s)}{2}$ . We further introduce with  $F, G$  in (5.8) the equations

$$\tilde{H}(x, y, Q) := H(F(x, Q), G(y, Q)) = 0$$

Let in the sequel  $\nabla_z f(z, y)$  denote the (partial) derivative of  $f$  with respect to the variable  $z$ . Since the derivative  $\nabla_{B,C}(F(x, Q), G(y, Q))$  has full rank  $2m$ , the derivative

$$\nabla \tilde{H}(x, y, Q) = \nabla H(F(x, Q), G(y, Q)) \cdot \nabla(F(x, Q), G(y, Q))$$

has full rank  $c_d$  for all  $x, y, Q$  with  $(F(x, Q), G(y, Q)) \in W_{r,s}^+$ . By the parametric Sard Theorem for almost all  $Q$  also for the function  $\tilde{H}_{x,y}(x, y) := \tilde{H}(x, y, Q)$ :

$$\nabla \tilde{H}_{x,y}(x, y) = \nabla_{(x,y)}[H(F(x, Q), G(y, Q))] \text{ has full rank } c_d \forall x, y \text{ with } (F(x, Q), G(y, Q)) \in W_{r,s}^+. \quad (5.9)$$

For  $r + s < k$  this means that for almost all  $Q$  there is no  $(x, y) \in \mathbb{R}^m$  such that  $(F(x, Q), G(y, Q)) \in W_{r,s}^+$ . For  $r + s = k$ , strict complementarity holds for all feasible pairs  $(X, Y) \in W_{r,s}^+$ . Taking all finitely many

combinations  $r, s$  with  $r + s \leq k$  into account we have proven that there is a weakly generic subset  $\mathcal{P}_1^1$  of  $\mathcal{P}^1$  such that for all  $Q \in \mathcal{P}_1^1$  any complementary solutions  $X, Y$  of  $(P_0), (D_0)$  are strictly complementary.

For the weak genericity of primal nondegeneracy we proceed also similar to [2, Proof of Th. 14]. Given  $s, 0 \leq s \leq k$  we consider the set  $W_s$  above and instances  $Q$  with primal feasible  $X = F(x, Q)$  in  $W_s$ . With the (linear independent) system of  $c_d$  equations  $K(X) = 0$ , which (locally) define the manifold  $W_s$  of codimension  $c_d = \frac{(k+1-s)(k-s)}{2}$ , we thus consider  $x, Q$  such that

$$\tilde{K}(x, Q) := K(F(x, Q)) = 0 .$$

Again since  $\nabla_B F(x, Q)$  has full rank  $m$ , the derivative  $\nabla F(x, Q)$  has full rank  $m$  for all  $x, Q$  and thus (in view of the definition of a manifold)  $\nabla K(X)$  has full rank  $c_d = \frac{(k-s+1)(k-s)}{2}$  for  $X \in W_s$ . So we find

$$\nabla \tilde{K}(x, Q) = \nabla K(F(x, Q)) \cdot \nabla F(x, Q) \quad \text{has full rank } c_d \text{ for all } x, Q \text{ with } F(x, Q) \in W_s .$$

The parametric Sard Theorem implies that for almost all  $Q$  for the function  $\tilde{K}_x(x) := \tilde{K}(x, Q)$  we have

$$\nabla \tilde{K}_x(x) = \nabla_x [K(F(x, Q))] \quad \text{has full rank } c_d \text{ for all } x \text{ with } F(x, Q) \in W_s . \quad (5.10)$$

This means, (see (5.6),(5.7) in fact (3) above) that for almost all  $Q$  the function  $F(x, Q)$  is transversal to the manifold  $W_s$ , so that for almost all  $Q$ :

$$\nabla_x F(x, Q)[\mathbb{R}^n] + T_{F(x, Q)} W_s = \mathcal{S}_k \quad \text{for all } x \text{ with } F(x, Q) \in W_s . \quad (5.11)$$

Since  $\nabla_x F(x, Q)[\mathbb{R}^n] = \text{span}\{\{A_i\}_{i=1}^n\}$  this condition is just the primal nondegeneracy condition [2, (18)]. (Note that our primal is the dual in [2] and the nondegeneracy condition in [2] is different but equivalent to the nondegeneracy relation in our paper.) Again by considering all possible  $s, 0 \leq s \leq k$ , we obtain a weakly generic subset  $\mathcal{P}_2^1$  of  $\mathcal{P}$  such that for all  $Q \in \mathcal{P}_2^1$  all primal feasible solutions are nondegenerate. The same can be done for the dual to obtain a set  $\mathcal{P}_3^1$  of instances such that for all  $Q \in \mathcal{P}_3^1$  all dual feasible solutions are nondegenerate. Note that if the primal and dual solutions are nondegenerate by Lemma 4.9(b) the optimal solutions must be unique. So by intersecting the weakly generic sets,  $\mathcal{P}_{reg}^1 := \bigcap_{i=0,1,2,3} \mathcal{P}_i^1$ , we have constructed a weakly generic subset  $\mathcal{P}_{reg}^1$  of  $\mathcal{P}^1$  such that for any  $Q \in \mathcal{P}_{reg}^1$  there exist unique, nondegenerate and strictly complementary solutions  $x, y$  (or  $X, Y$ ) of  $(P_0), (D_0)$ .

We now show the stability of these nice properties, *i.e.*, openness of  $\mathcal{P}_{reg}^1$ . This will be done by applying the Implicit Function Theorem to the system of equations above.

To do so, let  $\bar{Q} := (\{\bar{A}_i\}_{i=1}^n, \bar{B}, \bar{C})$  be a given instance in  $\mathcal{P}_{reg}^1$ . So  $\bar{x}, \bar{y}$  (or  $\bar{X}, \bar{Y}$ ) are unique, nondegenerate, strictly complementary solutions of the corresponding SDP pair  $(P_0)$  and  $(D_0)$  with  $\text{rank } \bar{X} = s$ ,  $\text{rank } \bar{Y} = r$ ,  $r + s = k$  and  $(\bar{X}, \bar{Y}) \in W_{r,s}^+$ , where  $\bar{X} = F(\bar{x}, \bar{Q}) = \bar{B} - \sum_{i=1}^n \bar{x}_i \bar{A}_i$  and  $\bar{Y} = G(\bar{y}, \bar{Q}) =$

$\sum_{j=1}^{m-n} \bar{y}_j \bar{A}_j^\perp + \bar{C}$ . By the discussion above (see (5.9)) the derivative

$$\nabla_{(x,y)}[H(F(\bar{x}, \bar{Q}), G(\bar{y}, \bar{Q}))] \quad \text{has full rank } m \quad (5.12)$$

at  $(F(\bar{x}, \bar{Q}), G(\bar{y}, \bar{Q}))$  satisfying  $H(F(\bar{x}, \bar{Q}), G(\bar{y}, \bar{Q})) = 0$ , a system of  $m$  equations. Locally near  $(\bar{x}, \bar{y}, \bar{Q})$  we consider the system

$$\tilde{H}(x, y, Q) := H(F(x, Q), G(y, Q)) = 0 \quad (5.13)$$

in the variables  $(x, y, Q)$ . By applying the Implicit Function Theorem to (5.13), and taking into account (5.12), we see that for  $Q \approx \bar{Q}$  there exists a unique  $C^\infty$ -solution function  $x(Q), y(Q)$  of the system

$$\tilde{H}(x(Q), y(Q), Q) = 0.$$

By construction, these solutions  $x(Q), y(Q)$  define strictly complementary optimal solutions of the programs  $(P_0), (D_0)$  wrt. the data  $Q \approx \bar{Q}$  with  $\text{rank } F(x(Q), Q) = s$ ,  $\text{rank } G(y(Q), Q) = r$ . So we have proven the stability of strict complementarity.

To see that also nondegeneracy of the solutions is stable we take for the instance  $\bar{Q} \in \mathcal{P}_{reg}^1$  the primal solution  $\bar{X} = X(\bar{Q})$  (see above). By the previous discussion (see (5.10)) with the defining equation  $K(X) = 0$  for the manifold  $W_s$  of codimension  $c_d = \frac{(k-s+1)(k-s)}{2}$  we have

$$\nabla_x[K(F(\bar{x}, \bar{Q}))] \quad \text{has full rank } c_d .$$

But then by continuity for  $Q \approx \bar{Q}$  and  $x(Q) \approx \bar{x}$  also  $\nabla_x[K(F(x(Q), Q))]$  has full rank  $c_d$  and (see (5.10), (5.11) above) the primal maximizers  $x(Q)$  ( $X(Q)$ ) are nondegenerate.

The same can be done for the dual. So finally we have established a full genericity result for SDP.  $\square$

## 6 Conclusion

In this paper we survey and complete genericity results for general conic programs. The results show that Slater's condition and strong duality are fully generic properties of CP, *i.e.*, they hold for almost all problem instances and are stable wrt. small perturbations of all problem data. Other nice properties such as uniqueness, nondegeneracy and strict complementarity are weakly generic, *i.e.*, they hold for almost all problem instances. For the special cases of SDP these properties are also stable. Whether this stability holds in general CP is still an open question.

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