Finite failure is and-compositional

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Abstract

We study some properties of SLD-trees related to finite failure. The main results are a theorem stating that the non-ground finite failure set is a correct and fully abstract semantics wrt finite failure and a second theorem stating that the complement of non ground finite failure is and-compositional, i.e. that the finite failure behaviour of conjunctive goals can be derived from the finite failure behaviour of atomic goals. The proofs are based on two new lemmata which generalize to infinite derivations theorems which are valid for successful and finitely failed derivations.

1 Introduction

The operational semantics of (positive) logic programs is usually based on SLD-trees. Several operational properties, useful for reasoning about programs, can be extracted from an SLD-tree. Examples are SLD-derivations, resultants, partial answers, computed answers, finite failures. All these properties, that we call *observables*, can be obtained as abstractions of the SLD-tree. The study of the observables may provide a new insight into the operational semantics and results that can be helpful in the theory of logic programs.

Sometimes it is useful to associate an observable with a denotation, which is a mathematical object which characterizes the behaviour of a program with respect to that observable. This is the approach chosen in the s-semantics for the computed answers observable [**?**] and in [6, 7, **?**] for other less abstract observables.

In the above mentioned examples it turns out that a lot of interesting theorems about the observable can more naturally be stated and proved if expressed in terms of the denotation, see, for example, [7] for the properties of resultants. The relation between the observable and the denotation can be understood in terms of the concepts of *correctness* and *full abstraction*.

Let $\circ$ be the observable, i.e. an abstraction of the SLD-trees. $\circ$ induces an observational equivalence $\approx_\circ$ on programs. Two programs are equivalent according to $\approx_\circ$, if they are observationally indistinguishable wrt $\circ$. Now, assume $D(P)$ is a denotation of program P. Then D is correct with respect to $\circ$, if $D(P_1) = D(P_2)$ implies $P_1 \approx_\circ P_2$. D is also fully abstract if $P_1 \approx_\circ P_2$ implies $D(P_1) = D(P_2)$. A denotation useful to discuss an observable must be correct with respect to it. In addition, if it is fully abstract, it enjoys a kind of minimality property.

In this paper we consider the properties of finite failures, following the above mentioned approach. After showing that the standard finite failure set is too weak as a denotation for characterizing finite failures, we define a denotation ($NGFF_P$ which we prove to be correct and fully abstract with respect to finite failures. Some possible applications of this result to program transformation and verification are discussed in section 10. A more technical result is and-compositionality. And-compositionality is one of the most important properties already proved for the above mentioned observables. It is related to the fact that the observable behaviour of conjunctive goals can be derived from the observable behaviors of atomic goals. And-compositionality allows us to build a denotation by considering atomic goals only. And-compositionality is not proved for $NGFF_P$, but for its " complement" $NGFF_P$ The result is technically quite complex and relies on a set of properties of (fair) SLD-trees, which are of some interest by themselves.

The paper is organized as follows. In section 3 we show that the Finite Failure set is not correct and we define the denotation $NGFF_P$ introduced in [10]. Section 4 contains some intermediate original results. Section 5 contains the proof of the correctness. We show that the denotation is fully abstract in section 6. An extension of the domain of our denotation is presented in section 8. Finally section 9 shows that the complement of $NGFF_P NGFF_P$ is and compositional.

2 Preliminaries

The reader is assumed to be familiar with the terminology of and the basic results in the semantics of logic programs [**?**, **?**]. Let the signature S of a program P consist of the set $\sum_P$ of *function symbols*, a finite set Π_P of *predicate symbols*, a denumerable set V of *variable symbols*. All the definitions in the following will assume a given signature S. Let T be the set of terms built on $\sum_P$ and V. Variable-free terms are called *ground*. A substitution is a mapping ϑ: $V \to T$ such that the set $Dom(\vartheta) = \{X | \; X\vartheta \neq X\}$ (*domain of* ϑ) is finite. If $W \subset V$, we denote by $\vartheta_{|W}$ the *restriction* of ϑ to the variables in W. ϵ denotes the empty substitution. The *composition* $\vartheta \cdot \sigma$ of the substitutions ϑ and σ is defined as functional composition. If $Dom(\vartheta) \cap Dom(\sigma) = \emptyset$ we can define the *union* of the substitutions ϑ and σ as $(\vartheta \cup \sigma)(X) = X\vartheta$ if $X \in Dom(\vartheta)$, $X\sigma$ otherwise.

A *renaming* is a substitution ρ for which there exists the inverse ρ^{-1} such that $\rho\rho^{-1} = \rho^{-1}\rho = \epsilon$. The pre-ordering $\leq$ (more general than) on substitutions is such that $\vartheta \leq \sigma$ iff there exists ϑ' such that $\vartheta \cdot \vartheta' = \sigma$. The result of the application of the substitution ϑ to a term t is an *instance* of t denoted by $t\vartheta$. Therefore t is an *anti-instance* of $t\vartheta$. We define $t \leq t'$ (t is more general than t') iff there exists ϑ such that $t\vartheta = t'$. A substitution ϑ is *grounding* for t if $t\vartheta$ is ground. The relation $\leq$ is a preorder. $\approx$ denotes the associated equivalence relation (*variance*). A substitution ϑ is a *unifier* of terms t and t' if $t\vartheta \equiv t'\vartheta$. The *most general unifier* of t_1 and t_2 is denoted by $mgu(t_1, t_2)$. All the above definitions can be extended to other syntactic expressions in the obvious way. An atom is an object of the form $p(t_1, \ldots, t_n)$, where $p \in \Pi_P$ and $t_1, \ldots, t_n$ belong to T. A *clause* is a formula of the form $H : -L_1, \ldots, L_n$, with $n \geq 0$, where H (the *head*) and $L_1, \ldots, L_n$ (the *body*) are atoms. ":-" and "," denote logic implication and conjunction respectively, and all the variables are universally quantified. If the body is empty the clause is a *unit clause*. A *program* is a finite set of clauses. A *goal* is a formula $\leftarrow L_1, \ldots, L_n$, where each L_i is an atom. By $Vars(E)$ we denote the set of variables occurring in the expression E. The *Herbrand base* B_P for P is the set of all the ground atoms belonging to the signature of the program P. A *Herbrand interpretation* is any subset of B_P. The *extended Herbrand base* B_V for P is the set of atoms built with the predicate symbols Π_P. If G is a goal, $G \overset{\vartheta}{\leadsto}_{P,R} B_1, \ldots, B_n$ denotes an SLD derivation with the selection rule R of $B_1, \ldots, B_n$ in the program P, where ϑ is the composition of the mgu's used in the derivation restricted to $Vars(G)$. $G \overset{\vartheta}{\leadsto}_{P,R} \Box$ denotes the refutation of G in the program P with computed answer substitution ϑ. We will denote by $\tilde{X}$ and $\tilde{t}$ a tuple of distinct variables and a tuple of terms respectively, while $\tilde{B}$ will denote a (possible empty) conjunction of atoms. We extend now the preorder on atoms by defining a preorder on conjunctions of atoms.

Definition 2.1 *$A_1, \ldots, A_k \leq \overline{A}_1, \ldots, \overline{A}_k$ if there exists a substitution ϑ such that for $i = 1, \ldots, k$ $\quad A_i\vartheta = \overline{A}_i$.*

Definition 2.2 *$A_1, \ldots, A_k < \overline{A}_1, \ldots, \overline{A}_k$ if there exist $J = \{j_1, \ldots, j_f\} \subseteq \{1, \ldots, k\}$, $J \neq \emptyset$, and a substitution ϑ such that $A_i < \overline{A}_i$ $\forall i \in J$ and $A_s\vartheta = \overline{A_s}$.*

3 Which semantics for finite failure

We first define the observational equivalence relation $\approx_{FF}$, induced on programs by the finite failures observable.

Definition 3.1 *Let P_1 and P_2 be programs, G be a goal and T_1 and T_2 be SLD-trees (defined by a fair selection rule) for G in P_1 and P_2 respectively. Then $P_1 \approx_{FF} P_2$ if, for every goal G, T_1 is finitely failed if and only if T_2 is finitely failed.*

As we will show now, the standard semantics, i.e. the ground finite failure set, is not able to model the behaviour of finite failure. Namely, the ground finite failure set cannot distinguish programs which have different sets of goals having a fair finitely failed SLD-tree.

The finite failure set FF_P is the set of all the ground atoms which finitely fail in P.

$$FF_P = \{\ A \mid A \text{ is a ground atom and } \leftarrow A \text{ has a fair finitely failed SLD-tree } \}.$$

It is easy to note that FF_P is not correct with respect to $\approx_{FF}$. Here is a counterexample.

Example 3.1

$P_1 :$

```
p(f(X)):-p(X)
s(a)
```

$P_2 :$

```
p(f(X)):-p(X),p(a)
s(a)
```

P_1 and P_2 have the same finite failure set.

$$FF_{P_1} = FF_{P_2} = \{ \quad p(a), p(f(a)), p(f(f(a))), \ldots \quad s(f(a)), s(f(f(a))), s(f(f(f(a)))), \ldots \}$$

However the goal $\leftarrow p(X)$ has a fair finitely failed SLD-tree in P_2 while $\leftarrow p(X)$ has only infinite fair SLD-trees in P_1.

The set FF_P is not adequate for modeling the finite failure of non-ground goals.

Our aim is to find a denotation which can distinguish programs which are not observationally equivalent wrt the finite failure behaviour. In other words we want to find a denotation σ which is correct wrt finite failure ($\approx_{FF}$), i.e. the following relation must hold:

$$\sigma(P_1) = \sigma(P_2) \Rightarrow P_1 \approx_{FF} P_2.$$

The denotation we consider is *the non-ground finite failure set*. It is the set of atomic goals which have at least a finitely failed SLD-tree. Namely,
$NGFF_P = \{A \mid \leftarrow A$ has a fair finitely failed SLD-tree $\}$.
$NGFF_P$ was first introduced in [10], where an equivalent bottom-up definition of $NGFF_P$ is also given.

We will show that $NGFF_P$ is correct wrt the equivalence $\approx_{FF}$, that is, given two programs P and Q, if $NGFF_P = NGFF_Q$ then if G finitely fails in P it also finitely fails in Q and vice versa, for any goal G. We will also show that $NGFF_P$ is a fully abstract denotation.

We will prove that the set of maximum elements belonging to the complement of the extension of $NGFF_P$ to the Complete Herbrand Base, is and-compositional wrt finite failure. Then the complement of $NGFF_P$ is and-compositional. Correctness wrt the equivalence relation $\approx_{FF}$, can be viewed as a consequence of and-compositionality. In fact, if a denotation is and-compositional, we have a method to characterize all the conjunctive goals which finitely fail in P, on the basis of the information contained in the denotation. Hence, if $NGFF_P = NGFF_Q$, then P and Q have exactly the same set of (conjunctive) goals which finitely fail.

Even if correctness is a consequence of and-compositionality, we have decided to give a separate direct proof of correctness, since it is much more insightful and does not require the completion construction of section 8. The two proofs are similar in same parts, yet are concerned with different domains (the Extended Herbrand Base and the Complete Herbrand Base).

We will often use the complement of $NGFF_P$, called $\overline{NGFF}_P$ and defined as $B_V/NGFF_P$, where B_V is the extended Herbrand base. An equivalent characterization can be given as
$\overline{NGFF}_P = \{ A \mid \exists\, \vartheta$ such that $\leftarrow A\vartheta$ has a successful SLD-tree or an infinite fair SLD-tree$\}$.

4 Some properties of infinite derivations

The proofs that will be presented in this paper are based on properties of infinite derivations, stated by lemmata 4.3 and 4.4. These lemmata generalize to two classes of infinite derivations the following two well known results about successful and finitely failed derivations.

Lemma 4.1 (Successful derivations) *[?] If the goal G has a successful SLD-tree in P, with computed answer substitution ϑ', then $G\vartheta$ has a successful SLD-tree in P, for every substitution ϑ such that $\vartheta \geq \vartheta'$.*

Lemma 4.2 (Finitely failed derivations) *[?] If the goal G has a finitely failed SLD-tree in P via R (where R is a fair selection rule), then also $G\vartheta$ has a finitely failed SLD-tree for every substitution ϑ.*

We want to find a similar property for infinite SLD-trees computed by fair selection rules. In other words, we want to be able to say something concerning the behaviour of the goal $G\vartheta$, once we know that the goal G has an infinite SLD-tree.

In order to achieve our aim we need to distinguish two kinds of infinite derivations for a given goal G.

- The substitutions computed during the infinite derivation of G keep instantiating the variables occurring in the goal G. This derivation is called *perpetual infinite derivation.*
- There exists k, such that all the substitutions computed at steps whose depth is $\geq k$ do not instantiate the variables occurring in the goal G and instantiate only variables introduced in the derivation. Such a derivation is called *non-perpetual infinite derivation.*

Let us give now the formal definition of perpetual derivation.

Definition 4.1 *Let d be an infinite derivation in the fair SLD-tree for the goal G. Let ϑ_i be the substitution computed at the i-th resolution step. Then d is a perpetual infinite derivation if $\forall i \ \exists n$*

$$G(\vartheta_0 \cdot \vartheta_1 \cdot \ldots \cdot \vartheta_i) < G(\vartheta_0 \cdot \vartheta_1 \cdot \ldots \cdot \vartheta_i \cdot \ldots \cdot \vartheta_{i+n}).$$

The following example is meant to help understanding the concept of perpetual infinite derivation.

Example 4.1

$P:$
```
p(f(X)):-p(X)
q(X):-t(X)
t(b):-t(X)
```

The goal $\leftarrow p(X)$ has an infinite perpetual derivation, while the goal $\leftarrow q(X)$ has an infinite SLD-tree with an infinite derivation which, on the contrary, is not perpetual.

It is worth noting that if d is a non-perpetual derivation for the goal G in P, then $\exists k$ such that $\forall i,\ i \geq k$

$$G(\vartheta_0 \cdot \vartheta_1 \cdot \ldots \cdot \vartheta_k) = G(\vartheta_0 \cdot \vartheta_1 \cdot \ldots \cdot \vartheta_k \cdot \ldots \cdot \vartheta_i).$$

$\vartheta = (\vartheta_0 \cdot \vartheta_1 \cdot \ldots \cdot \vartheta_k)_{|Vars(G)}$ is called the *definite answer* for the non-perpetual derivation d.

Now we have all the notions which are necessary to state the following lemma.

Lemma 4.3 (Non-perpetual infinite derivations) *Let G be a goal and R be a fair selection rule. Assume that G has an infinite SLD-tree via R, with at least one non-perpetual infinite derivation d, with definite answer ϑ'. Consider now the SLD-tree of $G\vartheta$, $\vartheta \geq \vartheta'$, via the selection rule R' which selects, at every resolution step, that atom in $G\vartheta$, which is in the same position of the one selected by R in G. Then $G\vartheta$ has an infinite SLD-tree via R' with a non-perpetual derivation which has ϵ as definite answer.*

The proof can be found in the appendix, section B.

Lemma 4.3. gives a property for fair SLD-trees of $G\vartheta$, $\vartheta \geq \vartheta'$, once we know that the goal G has an infinite fair SLD-tree with a definite answer ϑ' for an infinite non-perpetual derivation.

We want to find a similar result for a goal $G\vartheta$, in the case where G has an infinite fair SLD-tree with a perpetual derivation. To this aim, it is useful to distinguish which are the variables belonging to the goal G which are going to be instantiated infinitely many times in a given derivation and which are the ones that will not be instantiated anymore after a suitable resolution depth.

Definition 4.2 (Perpetual variables) *Let P be a program, G be a goal having at least one infinite perpetual derivation d. Let*

$$\vartheta_1, \vartheta_2, \ldots,$$

be the substitutions computed at each resolution step in the derivation d, restricted to the variables in $Vars(G)$. The set of perpetual variables of G, Per(G) is defined as follows. Per(G)=$\{X_1, \ldots, X_s\}$, where $\{X_1, \ldots, X_s\} \subseteq Vars(G)$ and $\forall X \in \{X_1, \ldots, X_s\}$ the term bound to X is instantiated infinitely many times by the substitutions $\vartheta_1, \vartheta_2, \ldots$. Namely, for every X there exists an infinite set of indexes of substitutions $\{s_1, s_2, \ldots\}$ such that

$$\vartheta_{s_1|X_i} < \vartheta_{s_2|X_i} < \ldots\ldots$$

The perpetual variables of a goal G, in a given derivation, are then the variables which are instantiated infinitely many times during an infinite perpetual derivation.

In the following we will sometimes use the complement of the set $Per(G)$ wrt to the set of all the variables occurring in G.

Definition 4.3 $\overline{Per}(G) = \{X_1, \ldots, X_n\}$ *such that for every* $X_i \in \{X_1, \ldots, X_n\}$, $X_i \in Vars(G)$ *and* $X_i \notin Per(G)$.

From the definitions of $\overline{Per}(G)$ and of perpetual derivation, we can easily prove that there exists a resolution depth k, such that the variables in $\overline{Per}(G)$ are not instantiated anymore by the substitutions computed by resolution steps of depth $> k$. Namely there exists k such that $\forall r,\ r > k$

$$\vartheta_{k_{|\overline{Per}(G)}} = \vartheta_{r_{|\overline{Per}(G)}}, \quad \text{where } \vartheta_r \text{ is the substitution computed at the } r\text{-th}$$

resolution step. $\vartheta_{k_{|\overline{Per}(G)}}$ is called the *partial perpetual answer* for the goal G and the perpetual infinite derivation we are analyzing. The following lemma characterizes the behaviour of $G\vartheta'$, if $\vartheta' \geq \overline{\vartheta}$ and $\overline{\vartheta}$ is the partial perpetual answer for the goal G, where G has at least one perpetual derivation.

Lemma 4.4 (Perpetual infinite derivations) *Let G be a goal and R be a fair selection rule. Assume that G has an infinite SLD-tree via R, with at least one perpetual infinite derivation d, with partial perpetual answer ϑ'. Consider now the SLD-tree of $G\vartheta$, $\vartheta \geq \vartheta'$ and $Dom(\vartheta) = Dom(\vartheta')$, via the selection rule R' which selects, at every resolution step, that atom in $G\vartheta$, which is in the same position as the one selected by R in G. Then $G\vartheta$ has an infinite SLD-tree via R' with at least one perpetual derivation.*

The proof can be found in the appendix, section B.

5 $NGFF_P$ is a correct semantics

We give now the proof that the set $NGFF_P$ is correct wrt finite failure.

Theorem 5.1 (Correctness) *Let P and Q be two programs such that $NGFF_P = NGFF_Q$. Let G be a goal then*

G has a fair finitely failed SLD-tree in P

if and only if

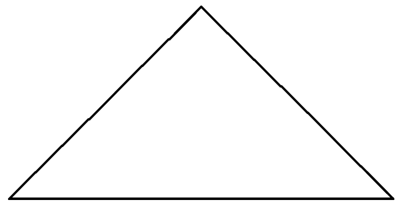
G has a fair finitely failed SLD-tree in Q.

Proof
Let us suppose that G has a finitely failed SLD-tree in P but that G does not have a finitely failed SLD-tree in Q.
Then the goal G can have

1. a successful tree in Q.
   If this is the case there exists a finite number of resolvents of the goal $G = \leftarrow A_1, \ldots, A_k$
   $$G_0 = G, \ldots, G_n \quad \text{such that} \quad G_n = \Box$$
   and the goal G has at least a computed answer ϑ.
   Then, by lemma 4.1., $G\vartheta$ has a successful tree in Q with empty answer substitution.
   Let us consider ϑ_a, which is grounding substitution for $G\vartheta$. Again by lemma 4.1, $G(\vartheta \cdot \vartheta_a)$ has a successful tree in Q, with an empty answer substitution.
   if a compound ground goal has a successful tree in Q then for each $A_i \in G$
   $\forall i: \ 1 \leq i \leq k \quad A_i(\vartheta \cdot \vartheta_a) \in \overline{NGFF_Q}$.
   But Q e P have the same set of goals which finitely fail, then
   $\forall i: \ 1 \leq i \leq k \quad A_i(\vartheta \cdot \vartheta_a) \in \overline{NGFF_P}$.
   If none of the n ground atoms finitely fail also $\leftarrow A_1(\vartheta \cdot \vartheta_a), \ldots, A_k(\vartheta \cdot \vartheta_a)$ does not fail in P.
   Then the compound goal $\leftarrow (A_1, \ldots, A_k)(\vartheta \cdot \vartheta_a)$ does not finitely fail in P.
   This contradicts the fact that $G = \leftarrow A_1, \ldots, A_k$ finitely fails in P since, if this were the case, by lemma 4.2 the goal $\leftarrow (A_1, \ldots, A_k)(\vartheta \cdot \vartheta_a)$ would have a finitely failed SLD-tree in P. This gives a contradiction.

2. The goal G has a fair infinite SLD-tree in Q.
   In this case
   $$G = \leftarrow A_1, \ldots, A_k \text{ has a finitely failed SLD-tree in } P$$

$G = \leftarrow A_1, \ldots, A_k$

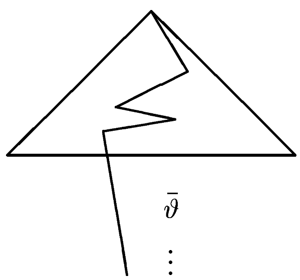


$\vartheta_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots \vartheta_m$

By $\vartheta_1, \ldots, \vartheta_m$ we denote the substitutions restricted to the variables of G computed before the failure of an atom belonging to G.
G has a fair infinite SLD-tree in Q,

(a) with at least a non perpetual derivation.

$G = \leftarrow A_1, \ldots, A_k$



Let $\overline{\vartheta}$ be the definite answer substitution for the non perpetual derivation we are considering. By lemma 4.3, $G\overline{\vartheta}$ has an infinite derivation in Q with computed answer equal to ϵ.
There exists a subset of substitutions (restricted to the variables in G) in the finitely failed SLD-tree of G in P which is related to $\overline{\vartheta}$. Namely,

$$\exists J = \{j_1, \ldots, j_n\} \subseteq \{1, \ldots, m\} \;\; J \neq \emptyset \;\; \text{such that}$$

$$\overline{\vartheta} < \vartheta_j \quad \forall j \in J \;\; .$$

This is because:

i. There does not exist i, $1 \leq i \leq m$ such that

$$\overline{\vartheta} \geq \vartheta_i,$$

because $\leftarrow (A_1,\ldots,A_k)\vartheta_i$ has a finitely failed SLD-tree in P. Hence there exists j such that $A_j\vartheta_i \in NGFF_P$. Therefore $A_j\vartheta_i \in NGFF_Q$.
Since we have supposed $\overline{\vartheta} \geq \vartheta_i$, by lemma 4.2

$$A_j\overline{\vartheta} \in NGFF_Q$$

holds. However

$$\leftarrow (A_1,\ldots,A_j,\ldots,A_k)\overline{\vartheta}$$

had an infinite fair SLD-tree in Q.

ii. There exists at least a $j \in J$ such that

$$\overline{\vartheta} < \vartheta_j.$$

If this were not the case let A_s be the first atom selected in the derivation of G in P via R. According to case i, $\overline{\vartheta} \geq \vartheta_i$ does not hold. Then

$$A_s\overline{\vartheta} \in NGFF_P \text{ and therefore}$$
$$A_s\overline{\vartheta} \in NGFF_Q$$

together with the fact that

$$\leftarrow (A_1,\ldots,A_s,\ldots,A_k)\overline{\vartheta}$$

has an infinite fair SLD-tree in Q, gives a contradiction.

Then there exists $J = \{j_1,\ldots,j_n\} \subseteq \{1,\ldots,m\}$ and $J \neq \emptyset$ such that

$$\forall j \in J \;\; \overline{\vartheta} < \vartheta_j$$

and $\forall h: \; h \in \{1,\ldots,m\} \backslash J$, $\overline{\vartheta}$ is not comparable with ϑ_h.
Let us choose $j_i \in J$, $\overline{\vartheta} < \vartheta_{j_i}$.

$$\leftarrow (A_1,\ldots,A_k)\overline{\vartheta}$$

has an infinite fair SLD-tree with a non perpetual derivation.

$$\leftarrow (A_1,\ldots,A_k)\vartheta_{j_i}$$

by lemma 4.3 has an infinite non perpetual derivation in Q with definite answer equal to ϵ. Hence

$$\forall\, i \;\; A_i\vartheta_{j_i} \in \overline{NGFF_Q}$$

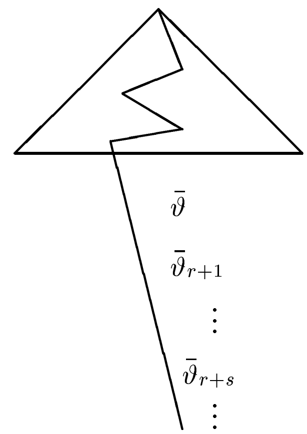
and therefore

$$\forall\, i \;\; A_i\vartheta_{j_i} \in \overline{NGFF_P}.$$

However we assumed that
$G = \leftarrow A_1,\ldots,A_k$ had a finite failure in P. Hence $\exists$ at least an index s with $1 \leq s \leq k$ such that $A_s\vartheta_{j_s} \in NGFF_P$.
This gives a contradiction.

(b) The infinite derivation is a perpetual derivation.

$$G = \leftarrow A_1, \ldots, A_k$$



Consider $\overline{\vartheta}_r$ the substitution computed at the r-th step of rewriting, restricted to the variables of the goal.
$\forall r$ there exists $J_r = \{j_1, \ldots, j_s\} \subseteq \{1, \ldots, m\}$ $J_r \neq \emptyset$ such that

$$\overline{\vartheta}_r < \vartheta_j \;\; \forall j \in J$$

This can be proved by using the same arguments used for the case of the non perpetual derivation. It holds also for a generic step of rewriting r. If we consider s such that $\overline{\vartheta}_{r+s} > \overline{\vartheta}_r$, such $\overline{\vartheta}_{r+s}$ exists by definition of perpetual answer substitution, also for $\overline{\vartheta}_{r+s}$ holds:

$$\exists J' \;:\; J' = \{j_1, \ldots, j_f\} \subseteq \{1, \ldots, m\}$$

defined above.
In general,

$$\forall s \;\; \exists\, J_s \subseteq J_{s-1} : \forall j \in J_s \quad \overline{\vartheta}_s < \vartheta_j$$

by the same arguments that ii, it's never the case that J_s becomes equal to $\emptyset$. Then there exists h such that

$$\forall s \quad \vartheta_h \in J_s,$$

$$\forall s \;\; \overline{\vartheta}_s < \vartheta_h.$$

By definition of perpetual derivation there exist infinitely many increasing indices of substitutions computed during the infinite derivation such that

$$\overline{\vartheta}_{s_1} < \overline{\vartheta}_{s_2} < \overline{\vartheta}_{s_3} < \ldots\ldots$$

Then for infinite indices

$$G\overline{\vartheta}_{s_i} = (A_1, \ldots, A_k)\overline{\vartheta}_{s_i} < (A_1, \ldots, A_k)\vartheta_h.$$

By definition of ordering on conjunctive atoms there exists A_p, with $1 \leq p \leq k$, such that $A_p\vartheta_h$ has infinitely many anti-instances.
By lemma A.1 in the appendix, this gives a contradiction. $\diamond$

6 $NGFF_P$ is fully abstract

We have just proved that $NGFF_P$ is correct wrt finite failures. The correctness wrt the observational behaviour we want to model is a necessary property for any sensible semantics.

There exists also another property we can be interested in, i.e. full abstraction. Full abstraction means that if two programs have the same behaviour, then they must have the same semantics, i.e.
$P_1 \approx_{FF} P_2 \Longrightarrow \sigma(P_1) = \sigma(P_2)$.
In the case of finite failures the full abstraction property guarantees that the semantics is "minimal" and there exists no better semantics.

The following theorem states that $NGFF_P$ is indeed fully abstract wrt $\approx_{FF}$.

Theorem 6.1 (Full abstraction) *Let P and Q be two programs.*
If $P \approx_{FF} Q$ then $NGFF_P = NGFF_Q$.

Proof
Trivial, since if $P \approx_{FF} Q$, P and Q have the same finite failure behaviour for atomic goals.

7 Towards And-compositionality

And-compositionality is not only relevant because it implies correctness, but it has a great relevance by itself. In fact, it is very useful to find a way to simulate the execution of a goal in the "semantics of a program P", in order to obtain the same result which one would obtain by computing the same goal in the program P.

The problem of determining whether a compound goal G finitely fails, by knowing whether $\leftarrow A$ finitely fails, for each A belonging to G, is not an easy problem, as shown by the following example.

Example 7.1

P_1 :
```
p(a)
q(b):-q(b)
s(a)
```

$\overline{NGFF}_{P_1} = \{p(a), p(X), q(b), q(X), s(a), s(X)\}$. *The goal* $G = \leftarrow p(X), q(X)$ *finitely fails in* P_1 *and both* $p(X)$ *and* $q(X)$ *belong to* $\overline{NGFF}_{P_1}$.

Example 7.2

P_2 :

```
p(X):-p(X)
q(b)
s(a)
```

$\overline{NGFF}_{P_2} = \{p(a), p(b), p(X), q(b), q(X), s(a), s(X)\}$. *The goal* $G = \leftarrow p(X), q(X)$ *has an infinite fair SLD-tree in* P_2 *and both* $p(X)$ *and* $q(X)$ *belong to* $\overline{NGFF}_{P_2}$.

The problem shown by the above examples is the following. Consider first the program P_1. Both $p(X)$ and $q(X)$ belong to $\overline{NGFF}_P$ because they have a successful SLD-tree and an infinite fair SLD-tree respectively. However, the SLD-trees instantiate the variables occurring in the goal computing incompatible substitutions for X. This is shown by the presence of $p(a)$ and $q(b)$ in $\overline{NGFF}_{P_1}$. Hence, for $p(X)$ both substitutions $\{X/a\}$ and $\{X/b\}$ are possible. In program P_2, on the contrary, $p(X)$ has an infinite SLD-tree, which does not instantiate the variables occurring in the goal. $q(X)$ has a successful tree which computes the substitution $\{X/b\}$. This is shown by the presence of $p(b)$ in $\overline{NGFF}_{P_2}$. One could try to solve this problem, by considering ground atoms only in $\overline{NGFF}_P$. This is not correct, as shown by the following example.

Example 7.3

P_3 :

```
q(f(X)):-q(X)
p(X)
s(a)
```

$$\overline{NGFF}_{P_3} = \{ \quad \begin{array}{l} q(X), q(f(X)), q(f(f(X))), \ldots \\ p(X), p(f(X)), p(f(f(X))), \ldots \\ p(a), p(f(a)), p(f(f(a))), \ldots \\ s(a), s(X) \end{array} \quad \}.$$

The goal $\leftarrow p(X), q(X)$ *does not finitely fail in* P_3. *Both* $p(X)$ *and* $q(X)$ *belong to* $\overline{NGFF}_{P_3}$ *but there exists no pair of ground atoms belonging to* $\overline{NGFF}_{P_3}$ *showing a compatible substitution for* $p(X)$ *and* $q(X)$.

8 Completion by ideals

In order to be able to formalize what we have understood from the previous examples, we need to define a Complete Herbrand Universe in order to admit atoms representative of limits of infinite computations, in our denotation. The same construction can be found in [10].

- Let $(P, \leq)$ be a preorder. A *directed* set in P is a subset D of P such that

$$\forall a, b \in D \;\; \exists c \in D \text{ such that } a \leq c \text{ and } b \leq c.$$

- An *ideal* S is a directed set which is downward closed, i.e. such that

$$\forall a \in S \quad b \leq a \Rightarrow b \in S.$$

The set of ideals of P, ordered by set inclusion, will be denoted by $(Id(P), \subseteq)$.

It is well know that it is a complete Partial Order (CPO) and it contains a sub-CPO isomorphic to $(P_{/\approx}, \leq)$, where $\approx$ is the equivalence relation induced by $\leq$.

Every atom $p(t)$ is now characterized by the ideal having $p(t)$ as the top element, i.e. the set $\{\, A \mid A \leq p(t)\}$. Infinite elements come also into the picture with sets such as $\{p(s^n(x)) \mid n \geq 0\}$, which represents an infinite element $p(s^\omega)$.

9 $Max(\overline{NGFF}_P)$ is and-compositional

We extend the Extended Herbrand Base with the usual order to the Complete Herbrand Base $(Id(B_v), \subseteq)$. Obviously also the concept of interpretation changes. The domain of our denotation will now be the Complete Herbrand Base.

$$\overline{NGFF}_P = \{\mathbf{A} \mid \;\; \mathbf{A} \in Id(B_V), \forall\, C \in \mathbf{A}, \exists \vartheta \;\; \leftarrow C\vartheta \text{ has a successful SLD-tree or an infinite fair SLD-tree}\}$$

Let Dsubst be the set of directed sets of substitutions, with typical elements $\Theta, \Psi \ldots$.

We define the application of Θ to an element $A \in B_v$ as

$$A\Theta = \{B \mid \exists \sigma \in \Theta : \; B \leq A\sigma\}.$$

It is possible to see [8] that any element of $Id(B_V)$ can be represented by a term t and an ideal of substitutions,
$\forall\, I \in Id(B_V)\; \exists t \in B_V, \exists \Theta \in Dsubst$ such that $I = t\Theta$.

Let us now introduce the Max operator which selects the maximal ideals of a subset of I.

Definition 9.1 *Let I be a subset of $Id(B_V)$.*

$$Max(I) = \{\mathbf{A} \in I \mid \;\; \nexists \mathbf{B} \;\; \mathbf{A} \subset \mathbf{B} \;\; \mathbf{B} \in I\}.$$

It is easy to show that Max selects the ideals representative of the more instantiated atoms of I. Max in fact captures the essential component of I, i.e. the component that was involved in the previous examples.

The characterization of a compound goal will be obtained by "computing" the goal in the finite failure semantics $NGFF_P$. To this aim we apply the Max operator to the interpretation $\overline{NGFF}_P$.

The following theorem states that a goal $G = \leftarrow A_1, \ldots, A_k$ has a finitely failed SLD-tree in P iff there does not exist an ideal of substitutions Θ such that for every atom A_i in G, $A_i\Theta$ belongs to the set $Max(\overline{NGFF}_P)$.

Theorem 9.1 (And-compositionality) *The goal* $G = \leftarrow A_1, \ldots, A_k$ *has a finitely failed SLD-tree in* P *if and only if*
$\nexists\Theta: \; A_i\Theta \in Max(\overline{NGFF}_P)$ *for i=1,...,k.*

Proof
$\Longrightarrow$
Assume that $\leftarrow A_1, \ldots, A_k$ has a finitely failed SLD-tree in P and that $\exists\Theta$
$A_i\Theta \in Max(\overline{NGFF}_P) \;\; for \; i = 1, \ldots, k$.
Let us call ϑ the possibly infinite substitution represented by the ideal of substitutions Θ.
Since $Max(\overline{NGFF}_P)$ is a subset of $\overline{NGFF}_P$, then no atom $A_i\vartheta$ has a finitely failed SLD-tree. Since $\forall i \;\; A_i\vartheta \in \overline{NGFF}_P$, we can have two cases related to the behaviour of the compound goal G.

1. Every goal $\leftarrow A_i\vartheta$ has a successful or infinite fair SLD-tree, which at each resolution step does not instantiate the variables in $A_i\vartheta$. Then the compound goal $\leftarrow A_1\vartheta, \ldots, A_k\vartheta$ has a successful fair SLD-tree or an infinite SLD-tree.
   Together with the assumption that the goal $\leftarrow A_1, \ldots, A_k$ had a finitely failed fair SLD-tree, this gives the contradiction.

2. There exists an A_h for which the above case 1 does not apply. $A_h\vartheta \in \overline{NGFF}_P$ and $A_h\vartheta$ has no successful or infinite fair SLD-tree, which at each resolution step does not instantiate the variables in $A\vartheta$. Then there exists a substitution ϑ' such that

$$A_h\vartheta \leq A_h\vartheta'$$

   and $\leftarrow A_h\vartheta'$ has a successful or an infinite fair SLD-tree.
   If this were the case, $A_h\vartheta$ would not be represented by an ideal in $Max(\overline{NGFF}_P)$, by definition of the Max operator.

$\Longleftarrow$
Assume that
$\nexists\,\Theta \;\; A_i\Theta \in Max(\overline{NGFF}_P)$, for $i = 1, \ldots, k$, and
the goal $\leftarrow A_1, \ldots, A_k$ does not finitely fail in P.
We have the following cases.

1. $\leftarrow A_1, \ldots, A_k$ has a successful fair SLD-tree. Then there exists at least a computed answer ϑ such that $\leftarrow (A_1, \ldots, A_k)\vartheta$ has a successful fair

SLD-tree. By lemma 4.1,

$$\forall\ \vartheta_a \text{ grounding substitution for } (A_1,\ldots,A_k)\vartheta$$

$$\leftarrow (A_1,\ldots,A_k)(\vartheta\cdot\vartheta_a) \text{ has a successful fair SLD-tree } \quad \Rightarrow$$

$$\forall\, i \quad A_i(\vartheta\cdot\vartheta_a) \text{ has a successful fair SLD-tree } \quad \Rightarrow$$

$$\forall\, i\ \ A_i(\vartheta\cdot\vartheta_a) \in \overline{NGFF}_P.$$

Since $\forall\, i\ \ A_i(\vartheta\cdot\vartheta_a)$ is ground, for all i there exists an ideal representative of $A_i(\vartheta\cdot\vartheta_a)$ which belongs to $Max(\overline{NGFF}_P)$, namely $\{B \mid B \leq A_i(\vartheta\cdot\vartheta_a)\}$. Then there exists an ideal of substitutions Θ which represents the grounding substitution $(\vartheta\cdot\vartheta_a)$. By hypothesis such an ideal of substitutions does not exist.

2. $\leftarrow A_1,\ldots,A_k$ has an infinite fair SLD-tree with at least one non-perpetual derivation. Let ϑ_a be a grounding substitution for $(A_1,\ldots,A_k)\overline{\vartheta}$, where $\overline{\vartheta}$ is the definite answer substitution. By lemma 4.3

   $(A_1,\ldots,A_k)(\overline{\vartheta}\cdot\vartheta_a)$ has an infinite fair SLD-tree with a non-perpetual

   infinite derivation, which has an empty definite answer. Since $\forall\, i\ \ A_i(\overline{\vartheta}\cdot\vartheta_a)$ is ground, for all i there exists an ideal representative of $A_i(\overline{\vartheta}\cdot\vartheta_a)$ which belong to $Max(\overline{NGFF}_P)$. Then there exist an ideal of substitutions Θ which represent the grounding substitution $(\overline{\vartheta}\cdot\vartheta_a)$. By hypothesis such an ideal of substitutions did not exist.

3. $\leftarrow A_1,\ldots,A_k$ has an infinite SLD-tree with at least one perpetual derivation. Then there exists a partial perpetual answer ϑ such that $Dom(\vartheta) \subseteq \overline{Per}(G)$ and, by lemma 4.4, $(A_1,\ldots,A_k)\vartheta$ has an infinite fair SLD-tree with a perpetual infinite derivation.
   Let ϑ_a be a grounding substitution for all the variables belonging to $\overline{Per}(G\vartheta)$. Namely

   $$Vars((A_1,\ldots,A_k)(\vartheta\cdot\vartheta_a)) \cap \overline{Per}(G) = \emptyset \text{ and } Dom(\vartheta_a) \cap Per(G) = \emptyset.$$

   Since $\leftarrow (A_1,\ldots,A_k)(\vartheta\cdot\vartheta_a)$ has a fair SLD-tree with at least one perpetual derivation, by lemma 4.4,

   $$\forall\, i\ \ \{\, B \mid B\ \leq\ A_i(\vartheta\cdot\vartheta_a)\} \in \overline{NGFF}_P.$$

   If, for some i, $Vars(A_i(\vartheta\cdot\vartheta_a)) = \emptyset$ then $A_i(\vartheta\cdot\vartheta_a)$ is ground and $\{\, B \mid B\ \leq A_i(\vartheta\cdot\vartheta_a)\} \in Max(\overline{NGFF}_P)$.
   There exists at least one j such that $A_j(\vartheta\cdot\vartheta_a)$ is not ground, since $Dom(\vartheta\cdot\vartheta_a) \cap Per(G) = \emptyset$.
   Since we are considering an infinite perpetual derivation of the SLD-tree for the goal $\leftarrow A_1,\ldots,A_k$ then $Per(A_1,\ldots,A_k) \neq \emptyset$.

Let us consider A_j such that $Vars(A_j(\vartheta \cdot \vartheta_a)) \neq \emptyset$.
This is due to our choice of ϑ_a and to the definition of partial perpetual answer.
Since $\leftarrow (A_1, \ldots, A_k)(\vartheta \cdot \vartheta_a)$ has a perpetual infinite derivation
there exist infinitely many substitutions ϑ_r, where ϑ_r is the substitution computed at the r-th resolution step, restricted to $Vars(G)$,

$$\forall\, r \;\; (A_1, \ldots, A_k)(\vartheta \cdot \vartheta_a \cdot \vartheta_r) \text{ does not fail } \quad \Rightarrow$$

$$\forall\, r \;\; A_j(\vartheta \cdot \vartheta_a \cdot \vartheta_r) \in \overline{NGFF}_P.\text{Hence}$$

$$A_j(\vartheta \cdot \vartheta_a \cdot \vartheta_r) \in \overline{NGFF}_P,$$

$$A_j(\vartheta \cdot \vartheta_a \cdot \vartheta_{r+1}) \in \overline{NGFF}_P,$$

$$A_j(\vartheta \cdot \vartheta_a \cdot \vartheta_{r+2}) \in \overline{NGFF}_P,$$

$$\vdots$$

Recalling that $Vars((A_j(\vartheta \cdot \vartheta_a)) \subseteq Per(G)$, we can always find indexes of substitutions, computed during the infinite perpetual derivation, such that

$$A_j(\vartheta \cdot \vartheta_a)\vartheta_{s_1} < A_j(\vartheta \cdot \vartheta_a)\vartheta_{s_2} < \ldots\ldots \text{ where } s_j < s_i \text{ if } j < i.$$

There exists an infinite descending chain of ideals, which belong to $\overline{NGFF}_P$, which are $\{\, A \mid A \leq A_j(\vartheta \cdot \vartheta_a)\vartheta_{s_1} \}, \{\, A \mid A \leq A_j(\vartheta \cdot \vartheta_a)\vartheta_{s_2} \}, \ldots$.
Then there exists an ideal representative of the limit of this chain, which still belongs to $\overline{NGFF}_P$.
This ideal representative of the limit element belongs also to the set $Max(\overline{NGFF}_P)$. This is because the chain is infinite and there exists no other ideal, not belonging to the chain, which contains all the ideals in an infinite descending chain.
Recalling that all the variables which are not perpetual are ground in the atoms represented by the infinite descending chain of ideals, it is worth noting that our argument for A_j holds for every atom belonging to the goal such that $Vars(A_j(\vartheta \cdot \vartheta_a)) \neq \emptyset$.
Since all the computed substitutions are the same for every A_j, then
there exists an ideal Θ of substitutions representative of the limit such that

$$\forall\, i \;\; A_i\Theta \in Max(\overline{NGFF}_P).$$

By hypothesis, such an ideal of substitutions does not exist. $\diamond$

We consider now examples 4.2 and 4.3 to show how the above result can be applied.

Example 9.1

$$P_1: \quad \texttt{p(a)} \quad \texttt{q(b):-q(b)}$$

$\overline{NGFF}_{P_1} = \{[p(X)p(a)], [p(X)], [q(X)q(b)], [q(X)]\}$
$Max(\overline{NGFF}_{P_1}) = \{[p(X)p(a)], [q(X)q(b)]\}$.
The goal $\leftarrow p(X), q(X)$ *has a finitely failed fair SLD-tree in* P_1 *since there exists no* Θ *such that* $(p(X), q(X))\Theta \in Max(\overline{NGFF}_{P_1})$.

Example 9.2

```
P2 :    p(X):-p(X)
        q(b)
        s(a)
```

$\overline{NGFF}_{P_2} = \{[p(X)p(a)], [p(X)p(b)], [p(X)], [q(X)q(b)], [q(X)][s(X), s(a)], [s(X)]\}$
$Max(\overline{NGFF}_{P_2}) = \{[p(X)p(a)], [p(X)p(b)], [q(X)q(b)], [s(X), s(a)]\}$
The goal $\leftarrow p(Y), q(Y)$ *does not have a finitely failed fair SLD-tree in* P_2 *since there exists* $\Theta = \{Y/X,\ Y/b\}$ *such that* $(p(Y), q(Y))\Theta \in Max(\overline{NGFF}_{P_2})$.

Example 9.3

```
P3 :    p(f(X),f(f(X))):-p(X,f(X))
        q(f(Y),f(Y)):-q(Y,Y)
```

$$\begin{aligned}\overline{NGFF}_{P_3} = \{ \quad & [p(X,Y)], [p(X,Y), p(X, f(X))],\\ & [p(X,Y), p(X, f(X)), p(f(X), f(f(X)))], \ldots\\ & [q(X,Y)], [q(X,Y)q(X,X)], [q(X,Y), q(X,X), q(f(X), f(X))], \ldots\}\end{aligned}$$

$$\begin{aligned}Max(\overline{NGFF}_{P_3}) = \{ \quad & [q(X,Y), q(X,X), q(f^n(X), q(f^n(X))|\ n \geq 1],\\ & [p(X,Y), p(X, f(X)), p(f^n(X), p(f^{n+1}(X))|\ n \geq 1]\ \}\end{aligned}$$

The goal $\leftarrow p(h,v), q(h,v)$ *has a finitely failed fair SLD-tree in* P_3 *because there exists no* Θ *such that* $(p(h,v), q(h,v))\Theta \in Max(\overline{NGFF}_{P_3})$.

It is worth noting that the set $Max(\overline{NGFF_P})$ does not in general contain ideals representative of ground atoms only, as shown by the last example. In general $Max(\overline{NGFF_P})$ contains ideals which represent ground atoms or limits of infinite chains.

Let us conclude with some remarks about the and-compositionality theorem. And-compositionality is a property of an observable rather then a property of a denotation. It tells us that we can determine the behaviours of all the (conjunctive) goals from the behaviours of the atomic goals. In particular, in the case of finite failures, the finite failure of any goal depends on the set of atomic goals which finitely fail ($NGFF_P$ and, therefore, $Max(\overline{NGFF_P})$). Theorem 10.1 obviously implies that $Max(\overline{NGFF_P})$ is correct and fully abstract. However,

$Max(\overline{NGFF_P})$ is undecidable (while $NGFF_P$ is recursively enumerable), and therefore it is not reasonable to take it as a denotation.

In the case of other observables, such as computed answers and resultants (see, for example, [7]), the and-compositional property has been very useful to simplify the proofs of several classic theorems in the theory of logic programs. We believe that our theorem should play a similar role in the theory of finite failure and negation as failure.

10 Conclusions

We have shown that $NGFF_P$ is correct and fully abstract. We also have shown that $Max(\overline{NGFF_P})$ is and-compositional wrt finite failures.

The problem of defining a denotation correct (and fully abstract) with respect to finite failures was an open problem in the theory of logic programs, interesting in the framework of program transformations. In fact, if we want a program transformation technique to preserve the semantics, we should require it to preserve computed answers and finite failures, i.e. the s-semantics (which models computed answers) and our finite failures semantics $NGFF_P$.

the usual theorems concerned with finite failures and the related theory of Negation as Failure.

It is worth noting that $NGFF_P$ is recursively enumerable and that the finite failure equivalence is not decidable. Note that this is also the case for other denotation, such as the success set and the computed answers semantics. The proof that some transformation rules do indeed preserve $NGFF_P$ can be done in analogy to proofs obtained for other denotations (see, for example [3]). To this end, it can be useful the least fixpoint characterization of $NGFF_P$ in [10]. The same fixpoint characterization can be used to extend to finite failure the theory of declarative diagnosis (or debugging), developed for the success set [12], for the set of atomic logical consequences (c-semantics) [5] and for the computed answers semantics (s-semantics) [4].

Other denotations were proved to be recursive for specific classes of programs, as for example the class of acceptable programs [2]. This was shown for the least Herbrand model in [1] and for the c-semantics and s-semantics in [11]. We are currently looking into the problem of finding similar decidability results for finite failure.

Let us finally discuss the compositionality results. We have not considered in this paper another property one can be interested in, i.e. compositionality wrt union of clauses, sometimes called or-compositionality. Or-compositionality is important if one wants to be able to reason about programs in a modular way. It is easy to realize that $NGFF_P$ is not or-compositional. An or-compositional semantics for finite failures can be found in [**?**]. It consists of more concrete abstractions of SLD-trees, namely sequences of resultants. The semantics is also and-compositional but not fully abstract.

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A Some preliminary results

We give first some results we used in the proofs of lemmas in section B. Let ID_{Π_P} and P^* be two sets of clauses, defined as follows:

$ID_{\Pi_P} = \{p(\widetilde{x}) : -p(\widetilde{x}) \mid p \in \Pi_P\}$.

$P^* = P \cup ID_{\Pi_P}$.

By these definitions we give a characterization of the set $\sigma_{\sim}(P)$:

$$\sigma_{\sim}(P) = \{A :- \; B_1, \ldots, B_m \in P \mid \exists\vartheta \textit{ such that } p(\widetilde{x}) \overset{\vartheta}{\leadsto}_{P,R} D_1, \ldots, D_k \overset{\gamma}{\leadsto}_{P^*,R} B_1, \ldots, B_m, \; m \geq 0, \; A = p(\widetilde{x})(\vartheta \cdot \gamma) \textit{ and } \widetilde{x} \textit{ are distinct new variables and } R \textit{ is fair} \}$$

$\sigma_{\sim}(P)$ is the set of all the resultants obtained in the derivation of $\leftarrow p(\tilde{x})$ in P by using any selection rule (the independence from the selection rule is obtained by using the set ID_{Π_P}). $\sigma_{\sim}(P)$ is the basic semantics of the semantic scheme in [?, 7] and is also the resultants semantics in [?]. The following proposition states the and-compositionality of $\sigma_{\sim}(P)$, i.e., it relates the existence of a derivation for the goal G in P, with the existence of resultants for the program P, obtained in a goal independent way for most general atomic goals.

Proposition A.1 *[?] Let P a program $G = \leftarrow A_1, \ldots, A_k$ a goal and R be a selection rule.*
Then there exists a derivation $G \overset{\vartheta}{\leadsto}_{P,R} G'$ iff the following conditions hold:

1. *$\exists\, J = \{j_1, \ldots, j_n\} \subseteq \{1, \ldots, k\}, J \neq \emptyset$ and $\forall j \in J$, there exists a variant, $c_j = H_j : -\widetilde{L}_j$, of a resultant belonging to $\sigma_{\sim}(P)$*
2. *$\exists\, \xi = mgu((A_{j_1}, \ldots, A_{j_n}), (H_{j_1}, \ldots, H_{j_n}))$, such that $\xi_{|Vars(G)} = \vartheta_{|Vars(G)}$,*
3. *the multisets $\widetilde{L}_1\xi, \ldots, \widetilde{L}_k\xi$ and G' are equal, where for $i = 1, \ldots, k$, $i \notin J$, $\widetilde{L}_i = A_i$.*

We introduce now a result that we will need in the following.

Lemma A.1 *[9] Any term t has only a finite number of anti-instances up to variable renaming.*

In the following we give some useful properties of unifiers.

Lemma A.2 *If $p(\widetilde{t_1}) \geq p(\widetilde{t_2})$ then there exists a substitution $\xi = mgu(p(\widetilde{t_1}), p(\widetilde{t_2}))$ and $\xi_{|Vars(p(\widetilde{t_1}))} = \epsilon$.*

The proof is straightforward.

Let us give now the last preliminary result of this section.

Lemma A.3 *Let W_1 and W_2 be two sets of variables such that they are a partition of the set of variables occurring in a given goal G.*
Consider $A\vartheta$ such that $Dom(\vartheta) \subseteq W_1$ and A is an atom belonging to the goal G.
If $\exists\, \xi = mgu(A, H)$ and $\vartheta \geq \xi_{|W_1}$, then $\exists\, \phi = mgu(A\vartheta, H)$ and $A(\vartheta \cdot \phi) \geq A\xi$.

Proof
Without loss of generality, we can consider variants of the atoms A, $A\vartheta$ and H, such that the following relations hold:

$$Vars(A) \cap Vars(H) = \emptyset \text{ and } Vars(A\vartheta) \cap Vars(H) = \emptyset.$$

$\xi = mgu(A, H)$. Then, by definition of mgu, $A\xi = H\xi$.
In order to prove the thesis of this lemma, we need to relate the two instances of the atom A, $A\vartheta$ and $A\xi$. We recall that $\vartheta \geq \xi_{|W_1}$.
We decompose the substitution ϑ, by restricting his domain to the set of variables which belongs to the considered partition. Since $Dom(\vartheta) \subseteq W_1$ and $Dom(\xi_{|W_2}) \subseteq W_2$, it is possible to define the union of the two substitutions ϑ and $\xi_{|W_2}$.
Let us consider $\vartheta \cup \xi_{|W_2}$. By the previous observations, this is still a substitution and the following statements hold.

$$A(\vartheta \cup \xi_{|W_2}) \geq A\xi \text{ and}$$

$$A\xi = H\xi.$$

By this equality,

$$A(\vartheta \cup \xi_{|W_2}) \geq H\xi \geq H.$$

Then, by lemma A.2,

$$\exists\, \xi_1 = mgu(A(\vartheta \cup \xi_{|W_2}), H) \text{ and } \xi_{1|Vars(A(\vartheta \cup \xi_{|W_2}))} = \epsilon.$$

Since ξ_1 is an mgu

$$A(\vartheta \cup \xi_{|W_2}) = A(\vartheta \cup \xi_{|W_2})\xi_1 = H\xi_1 \text{ and}$$

$$A\vartheta \leq A(\vartheta \cup \xi_{|W_2}) = H\xi_1.$$

By lemma A.2,

$$\exists\, \xi_2 = mgu(A\vartheta, H\xi_1),\ \xi_{2|Vars(H\xi_1)} = \epsilon \text{ and}$$

$$A(\vartheta \cdot \xi_2) = H(\xi_1 \cdot \xi_2) = H\xi_1.$$

Our next step is to define the substitution $\xi_1 \cup \xi_2$. Such a union is still a substitution iff

$$Dom(\xi_1) \cap Dom(\xi_2) = \emptyset.$$

Let us verify the previous conditions in our case.
Since ξ_1 is a relevant mgu,

$$\xi_{1|Vars(A(\vartheta \cup \xi_{|W_2}))} = \epsilon \quad \Rightarrow$$

$$Dom(\xi_1) \subseteq Vars(H).$$

Analogously, ξ_2 is a relevant mgu and therefore

$$\xi_{2|Vars(H\xi_1)} = \epsilon \quad \Rightarrow$$

$$Dom(\xi_2) \subseteq Vars(A\vartheta).$$

We have assumed

$$Vars(A\vartheta) \cap Vars(H) = \emptyset.$$

Then $\alpha = \xi_1 \cup \xi_2$ is a substitution and

$$A(\vartheta \cdot \alpha) = H\alpha. \quad \text{Then there exists a unifier of } A\vartheta \text{ and } \; H \quad \Rightarrow$$

$$\exists \; \phi = mgu(A\vartheta, H).$$

In order to show $A\xi \leq A\vartheta\phi$ we prove that in general, it is always true that if

$$\exists \; \beta = mgu(A\vartheta, H) \text{ and } \exists \; \beta' = mgu(A, H) \; \text{ then}$$

$$A\beta' \leq A(\vartheta \cdot \beta).$$

Let us discuss why the previous conditions hold.
It is not the case that

- $A\beta'$ is not comparable with $A(\vartheta \cdot \beta)$. If this was the case than there would be a bind for a variable, let say X, $\{X \backslash t'\}$ belonging to β' and a binding $\{X \backslash t\}$ belonging to $(\vartheta \cdot \beta)$ such that t' and t are not comparable.
  Since β' is the most general unifier of A and H it means that the term in H corresponding to X in A is exactly t'.
  Consider now $H\beta$ and $A(\vartheta \cdot \beta)$, these two atom have to be equals, since β is an unifier of H and $A\vartheta$.
  We have $t'\beta$ in $H\beta$ in the position corresponding to X in A, while in $A(\vartheta\beta)$ we have t. Since t' and t are not comparable $t'\beta$ and t can not be equals. This gives a contradiction.

- $A\beta' > A(\vartheta \cdot \beta)$ holds. We can assume, without loss of generality,

$$Vars(A) \cap Vars(H) = \emptyset$$

$$Vars(A\vartheta) \cap Vars(H) = \emptyset, \text{ and}$$

$$Dom(\vartheta) \subseteq Vars(A).$$

We note that the substitution

$$(\vartheta \cdot \beta)_{|Vars(H)} = \beta_{|Vars(H)}.$$

Hence

$$A(\vartheta \cdot \beta) = H(\vartheta \cdot \beta), \quad \text{by definition of mgu, and}$$

$$A(\vartheta \cdot \beta) = H\beta.$$

By hypothesis, $A(\vartheta \cdot \beta) < A\beta'$. Then

$$H(\vartheta \cdot \beta) = A(\vartheta \cdot \beta) < A\beta' = H\beta' \quad \Rightarrow$$

$$(\vartheta \cdot \beta)_{|(Vars(A) \cup Vars(H))} < \beta'_{|(Vars(A) \cup Vars(H))}.$$

Since our mgu's are always relevant, it means that β' contains just the variables in A and H, then $(\vartheta \cdot \beta) < \beta'$. This means that there exists an mgu of A and H more general than β'.
We have assumed β' to be the most general unifier of A e H. This gives a contradiction.

$\diamond$

B Results on infinite derivations

Lemma B.1 (4.3) *[Non-perpetual infinite derivations] Let G be a goal and R be a fair selection rule. Assume that G has an infinite SLD-tree via R, with at least one non-perpetual infinite derivation d, with definite answer ϑ'. Consider now the SLD-tree of $G\vartheta$, $\vartheta \geq \vartheta'$, via the selection rule R' which selects, at every resolution step, that atom in $G\vartheta$, which is in the same position of the one selected by R in G. Then $G\vartheta$ has an infinite SLD-tree via R' with a non-perpetual derivation which has ϵ as definite answer.*

Proof
If the goal G has at least one non-perpetual infinite derivation then there exist

$$G_0 = G, \;\; G_1, \ldots\ldots, G_n, \ldots \quad \text{resolvents of the goal } G,$$

$$\vartheta_0, \vartheta_1, \ldots\ldots, \vartheta_n, \ldots \quad \text{substitutions computed at each resolution step.}$$

We are going to show that there exists a similar sequence of goals and substitutions

$$\overline{G}_0 = G\vartheta, \overline{G}_1, \ldots\ldots, \overline{G}_n \ldots \quad \text{resolvents of the goal } G\vartheta,$$

$$\overline{\vartheta}_0, \overline{\vartheta}_1, \ldots\ldots, \overline{\vartheta}_n, \ldots \quad \text{substitutions computed at each rewriting step,}$$

$$\text{such that } \; (\overline{\vartheta}_0 \cdot \overline{\vartheta}_1 \cdot \ldots\ldots \cdot \overline{\vartheta}_n \cdot \ldots)_{|Vars(G\vartheta)} = \epsilon.$$

The proof is obtained by induction on the length of the derivation.
Assume first $n = 1$, $G_0 = A_1, \ldots, A_k$, and A_h, $1 \leq h \leq k$, be the atom selected by a fair computation rule R.
Since $G \overset{\vartheta_0}{\leadsto}_{P,R} G_1$, by rewriting the atom A_h, by proposition A.1,

1. $\exists\, J = \{h\} \subseteq \{1, \dots, k\}$ and $\exists\, C_h = H_h : -\widetilde{L}_h \in \sigma_\sim(P)$,

2. $\exists\, \xi = mgu(A_h, H_h)$ such that $\xi_{|Vars(G)} = \vartheta_{0_{|Vars(G)}}$,

3. G_1 is equal to the conjunction $\widetilde{B}_1\xi, \dots, \widetilde{B}_k\xi$, where $\widetilde{B}_i = A_i \;\forall\, i \neq h$ and $\widetilde{B}_h = \widetilde{L}_h$ otherwise.

The above consequences of the derivation
$G \overset{\vartheta_0}{\leadsto}_{P,R} G_1$, allow us to apply proposition A.1 to the goal $\overline{G}_0 = (A_1, \dots, A_k)\vartheta$
$\overline{G}_0$ in order to show that there exists a derivation
$G\vartheta \overset{\overline{\vartheta}_0}{\leadsto}_{P,R'} \overline{G}_1$, such that

1. $\exists\, J = \{h\} \subseteq \{1, \dots, k\}$ and there exists $C_h \equiv H_h : -\widetilde{L}_h \in \sigma_\sim(P)$.
   This relation holds because it is the first consequence of proposition A.1 applied to the existence of the derivation $G \overset{\vartheta_0}{\leadsto}_{P,R} G_1$.

2. $\xi = mgu(A_h, H_h)$ and $\xi_{|Vars(G)} = \vartheta_{0_{|Vars(G)}}$. Obviously
   $Vars(A_h) \subseteq Vars(G)$, because A_h belongs to the conjunction of atoms of the goal G. Moreover $A_h\vartheta \geq A_h\vartheta_0$, because $\vartheta \geq \vartheta'$, where ϑ' is the definite answer for the non-perpetual infinite derivation. ϑ' is defined as $(\vartheta_0, \dots, \vartheta_k, \dots, \vartheta_s)_{|Vars(G)}$, where from the s-th step on, the substitution computed at each resolution step does not instantiate variables occurring in G.
   Thus
   $$A_h\vartheta \geq A_h\vartheta_0 = H_h\vartheta_0 \geq H_h.$$
   By lemma A.2, there exists $\overline{\xi} = mgu(A_h\vartheta, H_h)$ and $\overline{\xi}_{|Vars(A_h\vartheta)} = \epsilon$.
   Since $\overline{\xi}$ is an mgu computed by Robinson's algorithm, $\overline{\xi}$ is relevant. Namely, there does not exist any variable $v \in Dom(\overline{\xi})$ and
   $v \notin (Vars(A_h\vartheta) \cup Vars(H_h))$. Therefore $\overline{\xi}$ contains only pairs which bind variables occurring in H_h. Hence
   $$\overline{\xi}_{|Vars(G\vartheta)} = \epsilon \text{ and } \overline{\xi}_{|Vars(G\vartheta)} = \overline{\vartheta}_{0|Vars(G\vartheta)}.$$

3. Set $\overline{G}_1 = (A_1, \dots, A_{h-1}, \widetilde{L}_h, A_{h+1} \dots, A_k)(\vartheta \cdot \overline{\xi})$. $\overline{G}_1$ satisfies the third condition of proposition A.1.

We can finally say that there exists a derivation

$$G\vartheta \overset{\overline{\vartheta}_0}{\leadsto}_{P,R'} \overline{G}_1,$$

where $\overline{\vartheta}_{0|Vars(G\vartheta)} = \overline{\xi}_{|Vars(G\vartheta)} = \epsilon$.
By proposition A.1, we know that there exists some selection rule R'. However, in this case, we have obtained G_1 using a rule R', which selects the atom in the

same position as the one selected by R in G.
Next, let us prove it for a derivation of length n. We will show that

$$G\vartheta \overset{(\overline{\vartheta}_0\cdot\ldots\cdot\overline{\vartheta}_n)}{\rightsquigarrow}_{P,R'} \overline{G}_n, \text{ such that } (\overline{\vartheta}_0\cdot\ldots\cdot\overline{\vartheta}_n)_{|Vars(G\vartheta)} = \epsilon.$$

The first part of the proof is a direct application of proposition A.1, while the inductive hypothesis is used to show that the selection rule R' selects the atoms in $G\vartheta$ in a corresponding order than the ones selected by R in G.
By hypothesis, after n resolution steps, there exists a derivation
$G \overset{(\vartheta_0\cdot\ldots\cdot\vartheta_n)}{\rightsquigarrow}_{P,R} G_n$.
For the sake of simplicity, we can call α the composition of substitutions $(\vartheta_0\cdot\ldots\cdot\vartheta_n)$. By applying proposition A.1 to the derivation $G \overset{\alpha}{\rightsquigarrow}_{P,R} G_n$, we obtain the following results:

1. $\exists\ J = \{j_1,\ldots,j_s\} \subseteq \{1,\ldots,k\}$, where j_i is the index of an atom belonging to the goal, such that A_{j_i} was selected once and some atoms in the derivation of A_{j_i} were selected arbitrarily many times, during the first n steps of derivation,
   $J \neq \emptyset$ and $\forall j \in J\ \ \exists\, C_j = H_j : -\widetilde{L}_j \in \sigma_\sim(P)$.
2. $\exists\ \xi = mgu((A_{j_1},\ldots,A_{j_s}),(H_{j_1},\ldots,H_{j_s}))$ such that $\xi_{|Vars(G)} = \alpha_{|Vars(G)}$,
3. G_n is a conjunction $(\widetilde{B}_1,\ldots,\widetilde{B}_k)\xi$, where $\widetilde{B}_i = \widetilde{L}_i$ if $i \in J$ and $\widetilde{B}_i = A_i$ otherwise.

As for the case $n = 1$, the previous relations allow us to apply proposition A.1 to the goal $\overline{G}_0 = \leftarrow (A_1,\ldots,A_n)\vartheta$ to obtain the proof of the existence of the derivation $G\vartheta \rightsquigarrow_{P,R'} \overline{G}_n$. We know that,

1. There exists the previous set $J = \{j_1,\ldots,j_s\} \subseteq \{1,\ldots,k\}$. For this set, we know that the following relation holds, $\forall j \in J\ \ \exists\, C_j = H_j : -\widetilde{L}_j \in \sigma_\sim(P)$.
2. We know from the point 2 that there exists $\xi = mgu((A_{j_1},\ldots,A_{j_s}),(H_{j_1},\ldots,H_{j_s}))$. We want now to find a mgu of $(A_{j_1},\ldots,A_{j_s})\vartheta$ and $(H_{j_1},\ldots,H_{j_s})$ with the properties required in the hypothesis of the lemma. Since

   $$(A_{j_1},\ldots,A_{j_s})\vartheta \geq (A_{j_1},\ldots,A_{j_s})\vartheta' \geq (A_{j_1},\ldots,A_{j_s})(\vartheta_0\cdot\ldots\cdot\vartheta_n) =$$

   $$= (A_{j_1},\ldots,A_{j_s})\alpha = (A_{j_1},\ldots,A_{j_s})\xi = (H_{j_1},\ldots,H_{j_s})\xi \geq (H_{j_1},\ldots,H_{j_s}),$$

   by lemma A.2, $\exists\overline{\xi} = mgu((A_{j_1},\ldots,A_{j_s})\vartheta,(H_{j_1},\ldots,H_{j_s}))$ and

   $$\overline{\xi}_{|Vars((A_{j_1})\vartheta\cup\ldots\cup Vars(A_{j_s})\vartheta)} = \epsilon.$$

   Since $\overline{\xi}$ is a relevant mgu $\overline{\xi}_{|Vars(G\vartheta)} = \epsilon$ and
   $\overline{\xi}_{|Vars(G\vartheta)} = (\overline{\vartheta}_0\cdot\ldots\cdot\overline{\vartheta}_n)_{|Vars(G\vartheta)}$.

3. Set $\overline{G}_n = (\widetilde{C}_1, \ldots \widetilde{C}_k)\vartheta\overline{\xi}$, where $\widetilde{C}_i = \widetilde{L}_i$ if $i \in J$ and $\widetilde{C}_i = A_i$ otherwise.

By applying proposition A.1, we know that there exists a derivation

$$G\vartheta \overset{(\overline{\vartheta}_0 \cdot \ldots \cdot \overline{\vartheta}_n)}{\leadsto}_{P,R'} \overline{G}_n, \text{ such that } (\overline{\vartheta}_0 \cdot \ldots \cdot \overline{\vartheta}_n)_{|Vars(G\vartheta)} = \epsilon.$$

We still have to prove that R' is a rule which selects the atoms in $G\vartheta$ in the same order used by R to select them in G.
It is worth noting that the goal G_n was obtained from the goal G_{n-1}, by one resolution step from an atom selected by rule R.
The rule R in such a resolution step can have selected:

a) An atom A_f, belonging to the initial goal G, which was never selected in the first $n-1$ resolution steps, at the n-th resolution step R chooses A_f.
In this case, at the n-th step, the set J is enriched with the index f and $C_f = H_f : -\widetilde{L}_f$ will be of depth 1.
When we apply proposition A.1 to the derivation of $G\vartheta$ to obtain $\overline{G}_n$, we use the same set of indexes and the same resultants which were used in the derivation $G_0 \to G_n$. Since $\overline{G}_{n-1}$ was obtained, by inductive hypothesis, by selecting in the derivation $\overline{G}_0 \to \overline{G}_{n-1}$ atoms according to the same order in which atoms were selected by R in the derivation $G \to G_{n-1}$, R' has selected in $\overline{G}_{n-1}$ at the n-th step the atom in the f-th position.

b) An atom which did not belong to the initial goal, but was introduced by the rewriting process of some atom belonging to the initial goal, say A_f.
In this case $C_f \equiv H_f : -\widetilde{L}_f$ is a resultant of depth $r+1$, if r was the depth of the resultant C_f, used in the derivation of the goal G_{n-1}, in correspondence with the atom A_f.
Since $\overline{G}_0 = G\vartheta$ and there exists a derivation $\overline{G}_0 \to \ldots \to \overline{G}_{n-1} \to \overline{G}_n$, we have already proved that at the n-th step, $\overline{G}_0$ is "computed" by the same resultants used to derive G at the n-th step. Then $\overline{G}_n$ is obtained by using the same resultant, $H_f : -\widetilde{L_f}$, for rewriting $A_f\vartheta$. This resultant is of depth $r+1$. By hypothesis, $\overline{G}_{n-1}$ is obtained by selecting the atoms in the derivation $\overline{G}_0 \to \overline{G}_{n-1}$ according to the same order in which the atoms were selected by R in the derivation $G_0 \to G_{n-1}$. Therefore, at the n-th step, the goal $\overline{G}_0$ will be obtained by rewriting $A_f\vartheta \in \overline{G}_0$ with a resultant of depth $r+1 \in \sigma_{\sim}(P)$. At the n-th step, R' has selected, in the derivation of $\overline{G}_0$, an atom in the derivation of $A_f\vartheta$.
In order to obtain $\overline{G}_n$, we are using exactly the same resultants used to obtain G_n. This implies that there exists R' which selects the atom exactly in the same position as the one selected by R in the derivation of G.

The following implication holds,

$$G \overset{(\vartheta_0 \cdot \ldots \cdot \vartheta_n)}{\leadsto}_{P,R} G_n \quad \Rightarrow$$

$$G\vartheta \overset{(\overline{\vartheta}_0\cdot\ldots\cdot\overline{\vartheta}_n)}{\leadsto}_{P,R'} \overline{G}_n \quad \text{with } (\overline{\vartheta}_0 \cdot \ldots \cdot \overline{\vartheta}_n)_{|Vars(G\vartheta)} = \epsilon.$$

R and R' select the atoms respectively in the derivation of G and in the derivation of $G\vartheta$, exactly in the same order. $\diamond$

Lemma B.2 (4.4) *[Perpetual infinite derivations] Let G be a goal and R be a fair selection rule. Assume that G has an infinite SLD-tree via R, with at least one perpetual infinite derivation d, with partial perpetual answer ϑ'. Consider now the SLD-tree of $G\vartheta$, $\vartheta \geq \vartheta'$ and $Dom(\vartheta) = Dom(\vartheta')$, via the selection rule R' which selects, at every resolution step, that atom in $G\vartheta$, which is in the same position as the one selected by R in G. Then $G\vartheta$ has an infinite SLD-tree via R' with at least one perpetual derivation.*

Proof

If the goal G has a perpetual infinite derivation, then there exists a sequence of goals and a sequence of substitutions:

$$G_0 = G, \;\; G_1, \ldots\ldots G_n, \ldots \quad \text{resolvents of the goal } G,$$

$$\vartheta_0, \vartheta_1, \ldots\ldots \vartheta_n, \ldots \quad \text{substitutions computed at each resolution step.}$$

We want to show that there exists a corresponding sequence of goals and substitutions for the goal $G\vartheta$:

$$\overline{G}_0 = G\vartheta, \overline{G}_1, \ldots\ldots \overline{G}_n \ldots \quad \text{resolvents of the goal } G\vartheta,$$

$$\overline{\vartheta}_0, \overline{\vartheta}_1, \ldots\ldots \overline{\vartheta}_n, \ldots \quad \text{substitutions computed at each resolution step,}$$

$$\text{such that} \quad G(\vartheta \cdot \overline{\vartheta}_0 \cdot \ldots \cdot \overline{\vartheta}_i) \geq G(\vartheta_0 \cdot \ldots \cdot \vartheta_i) \quad \forall\, i.$$

We want to prove this last formula in order to show that there exists a derivation for $G\vartheta$, which is a perpetual infinite derivation.

Every conjunction of atoms in the sequence $\overline{G}_0\overline{\vartheta}_0, \overline{G}_0(\overline{\vartheta}_0 \cdot \overline{\vartheta}_1), \;\; \overline{G}_0(\overline{\vartheta}_0 \cdot \overline{\vartheta}_1 \cdot \overline{\vartheta}_2), \ldots$ is more instantiated then the corresponding conjunction of atoms in the sequence $G_0\vartheta_0, G_0(\vartheta_0 \cdot \vartheta_1), G_0(\vartheta_0 \cdot \vartheta_1 \cdot \vartheta_2), \ldots$ which is an infinite descending chain. Moreover

$$\forall \;\; i \quad \overline{G}_0(\overline{\vartheta}_0 \cdot \ldots \cdot \overline{\vartheta}_i) \leq \; \overline{G}_0(\overline{\vartheta}_0 \cdot \ldots \cdot \overline{\vartheta}_i \cdot \overline{\vartheta}_{i+1}).$$

Then the sequence $\overline{G}_0\overline{\vartheta}_0, \overline{G}_0(\overline{\vartheta}_0 \cdot \overline{\vartheta}_1), \overline{G}_0(\overline{\vartheta}_0 \cdot \overline{\vartheta}_1 \cdot \overline{\vartheta}_2), \ldots$ is an infinite descending chain.

This proof is essentially similar to the proof of lemma B.1

Namely it is proved by induction on the length of the derivation and by applying proposition A.1.

Assume first $n = 1$, $G_0 = \leftarrow A_1, \ldots, A_k$ and let A_h, $1 \leq h \leq k$, be the atom selected by R.

Since $G_0 \overset{\vartheta_0}{\leadsto}_{P,R} G_1$ by rewriting the atom A_h, by proposition A.1, the following facts hold:

1. $\exists\, J = \{h\} \subseteq \{1, \ldots, k\}$ and $\exists\, C_h = H_h : -\widetilde{L}_h \in \sigma_{\sim}(P)$,
2. $\exists\, \xi = mgu(A_h, H_h)$ such that $\xi_{|Vars(G)} = \vartheta_{0|Vars(G)}$,
3. G_1 and $\widetilde{B}_1\xi, \ldots, \widetilde{B}_k\xi$ are equal, where $\widetilde{B}_i = A_i \ \forall\ i \neq h$ and $\widetilde{B}_h = \widetilde{L}_h$ otherwise.

We want to apply proposition A.1 to the goal $\overline{G}_0 = \leftarrow (A_1, \ldots, A_k)\vartheta$. We know that

1,2 and 3 are true because there exists the derivation $G_0 \overset{\vartheta_0}{\leadsto}_{P,R} G_1$.

Let us verify the hypothesis of proposition A.1, in order to show that the derivation $\overline{G}_0 = G\vartheta \overset{\overline{\vartheta}_0}{\leadsto}_{P,R'} \overline{G}_1$ does exist:

1. $\exists\, J = \{h\} \subseteq \{1, \ldots, k\}$ and there exists $C_h \equiv H_h : -\widetilde{L}_h \in \sigma_{\sim}(P)$. This is directly due to the first consequence of the existence of $G_0 \overset{\vartheta_0}{\leadsto}_{P,R} G_1$.
2. If there exists $\xi = mgu(A_h, H_h)$, then let $I_1 = \overline{Per}(G)$ and $I_2 = Per(G)$. I_1 and I_2 are a partition of the variables in G, $Dom(\vartheta) \subseteq \overline{Per}(G)$ and

   $$\vartheta \geq \vartheta' = (\vartheta_0 \cdot \ldots \cdot \vartheta_s)_{|I_1} \geq \vartheta_{0_{|I_1}}.$$

   Then, by lemma A.3,

   $$\exists\ \overline{\xi} = mgu(A_h\vartheta, H_h) \ \text{ and } \ A_h(\vartheta \cdot \overline{\xi}) \geq A_h\xi.$$

   Since $\vartheta_{0|Vars(G)} = \xi_{|Vars(G)}$. Let

   $$\overline{\vartheta}_{0_{|Vars(G)}} = \overline{\xi}_{|Vars(G)}.$$

   $$\text{Then } A_h(\vartheta \cdot \overline{\vartheta}_0) \geq A_h\vartheta_0 \quad \Rightarrow$$
   $$(\vartheta \cdot \overline{\vartheta}_0)_{|Vars(A_h)} \geq \vartheta_{0_{|Vars(A_h)}},$$

   since ϑ_0 is a relevant mgu, the domain is a subset of the variables occurring in A_h and in H_h and $Vars(A_h) \cap Vars(H_h) = \emptyset$ then

   $$G(\vartheta \cdot \overline{\vartheta}_0) \geq G\vartheta_0.$$

3. Let $\overline{G}_1 = (A_1, \ldots, A_{(h-1)}, \widetilde{L}_h, \ldots, A_k)\vartheta\overline{\xi}$. $\overline{G}_1$ satisfies the third condition of proposition A.1.

By proposition A.1, there exists a derivation

$$G\vartheta \overset{\overline{\vartheta}_0}{\leadsto}_{P,R'} \overline{G}_1,$$

where $G(\vartheta \cdot \overline{\vartheta}_0) \geq G\vartheta_0$.

We note that even if proposition A.1 asserts that there exists some rule R', R'

is a rule which has selected the atom in the same position as the one selected by R in the derivation of G.
Next, let us prove it for a derivation of length n.
The first part of the proof is, as in the previous lemma, a direct application of proposition A.1, while the inductive hypothesis is used to show that the selection rule R' selects the atoms in $G\vartheta$ in a corresponding order than the ones selected by R in G. Assume now that the result holds for a derivation of length $n-1$ and consider a derivation of length n.
After n resolution steps, by hypothesis, there exists a derivation
$G \overset{(\vartheta_0 \cdot \ldots \cdot \vartheta_n)}{\leadsto}_{P,R} G_n$.
For the sake of simplicity, we call α the substitution $(\vartheta_0 \cdot \ldots \cdot \vartheta_n) = \alpha$. Therefore $G \overset{\alpha}{\leadsto}_{P,R} G_n$.
This implies:

1. $\exists\ J = \{j_1, \ldots, j_s\} \subseteq \{1, \ldots, k\}$, where j_i is an index of an atom belonging to the goal, such that A_{j_i} was selected once and atoms in the derivation of A_j were selected arbitrarily many times, in the first n steps of the derivation. $J \neq \emptyset$ and $\forall j \in J\ \ \exists\ C_j = H_j : -\widetilde{L}_j \in \sigma_{\sim}(P)$,
2. $\exists\ \xi = mgu((A_{j_1}, \ldots, A_{j_s}), (H_{j_1}, \ldots, H_{j_s}))$ such that $\xi_{|Vars(G)} = \alpha_{|Vars(G)}$,
3. G_n is equal to the conjunction $(\widetilde{B}_1, \ldots, \widetilde{B}_k)\xi$, where $\widetilde{B}_i = \widetilde{L}_i$ if $i \in J$ and $\widetilde{B}_i = A_i$ otherwise.

We want now to verify the conditions of proposition A.1, which will allow us to prove the existence of the derivation $\overline{G}_0 = \leftarrow (A_1, \ldots, A_n)\vartheta \leadsto_{P,R'} \overline{G}_n$.

1. Let us consider the set of indexes $J = \{j_1, \ldots, j_s\}$. This set is the same set J, which we know to exist as a consequence of proposition A.1, applied to the existence of the derivation $G \overset{(\vartheta_0 \cdot \ldots \cdot \vartheta_n)}{\leadsto}_{P,R} G_n$.
For this set the following fact holds
$$\forall j \in J\ \ \exists\ C_j = H_j : -\widetilde{L}_j \in \sigma_{\sim}(P).$$
2. $\exists \xi = mgu(A_{j_1}, \ldots, A_{j_s}, H_{j_1}, \ldots, H_{j_s})$. The hypotheses of lemma B.2 are verified.
Let $I_1 = \overline{Per}(G)$ and $I_2 = Per(G)$. I_1 and I_2 are a partition of the set of variables occurring in G. Assume ϑ' to be the partial perpetual answer for the infinite partial derivation of G. Then $Dom(\vartheta') \subseteq I_1$.
There exists f, a suitable resolution step, such that after that step the variables belonging to $\overline{Per}(G)$ are not instantiated by the following resolution steps, $\vartheta' = (\vartheta_0 \cdot \ldots \cdot \vartheta_f)_{|I_1}$.
We have assumed $\vartheta \geq \vartheta'$. Hence the following facts hold:
$$\text{if}\quad n \geq f \quad \vartheta \geq \vartheta' = (\vartheta_0 \cdot \ldots \cdot \vartheta_f)_{|I_1} = (\vartheta_0 \cdot \ldots \cdot \vartheta_f \cdot \ldots \cdot \vartheta_n)_{|I_1} = \alpha_{|I_1}.$$

if $\quad n < f \quad \vartheta \geq \vartheta' = (\vartheta_0 \cdot \ldots \cdot \vartheta_f)_{|I_1} > (\vartheta_0 \cdot \ldots \cdot \vartheta_n)_{|I_1} = \alpha_{|I_1}.$

In both cases the following fact is true,

$$\vartheta \geq \vartheta' = (\vartheta_0 \cdot \ldots \cdot \vartheta_f)_{|I_1} \geq (\vartheta_0 \cdot \ldots \cdot \vartheta_n)_{|I_1} = \alpha_{|I_1}.$$

By lemma A.3

$$\exists\, \overline{\xi} = mgu(A_{j_1}\vartheta, \ldots, A_{j_s}\vartheta, H_{j_1}, \ldots, H_{j_s})$$

and

$$(A_{j_1}, \ldots, A_{j_s})\vartheta\overline{\xi} \geq (A_{j_1}, \ldots, A_{j_s})\xi.$$

Since $\overline{\xi}_{|Vars(G\vartheta)} = (\overline{\vartheta}_0 \cdot \ldots \cdot \overline{\vartheta}_n)_{|Vars(G\vartheta)}$ and $\xi_{|Vars(G)} = \alpha_{|Vars(G)} = (\vartheta_0 \cdot \ldots \cdot \vartheta_n)_{|Vars(G)}$ then

$$G(\vartheta \cdot \overline{\vartheta}_0 \cdot \ldots \cdot \overline{\vartheta}_n) \geq G(\vartheta_0 \cdot \ldots \cdot \vartheta_n),$$

since all the involved substitutions are relevant.

3. Let $\overline{G}_n = (\widetilde{C}_1, \ldots \widetilde{C}_k)(\vartheta \cdot \overline{\xi})$, where $\widetilde{C}_i = \widetilde{L}_i$, if $i \in J$ and $\widetilde{C}_i = A_i$ otherwise.

By proposition A.1,

$$G\vartheta \overset{(\overline{\vartheta}_0 \cdot \ldots \cdot \overline{\vartheta}_n)}{\rightarrow}_{P,R'} \overline{G}_n$$

The proof that R' selects the atoms in $G\vartheta$ in the same order used by R to select them in G, is the same that the one in the theorem B.1. $\diamond$