Local Realizability Toposes and a Modal Logic for Computability (Extended Abstract)

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Abstract

This work is a step toward developing a logic for types and computation that includes both the usual spaces of mathematics and constructions and spaces from logic and domain theory. Using realizability, we investigate a configuration of three toposes, which we regard as describing a notion of relative computability. Attention is focussed on a certain local map of toposes, which we study first axiomatically, and then by deriving a modal calculus as its internal logic. The resulting framework is intended as a setting for the logical and categorical study of relative computability.

1 Introduction

We report here on the current status of research on the Logic of Types and Computation at Carnegie Mellon University [SAB$^+$]. The general goal of this research program is to develop a logical framework for the theories of types and computability that includes the standard mathematical spaces alongside the many constructions and spaces known from type theory and domain theory. One purpose of this goal is to facilitate the study of computable operations and maps on data that is not necessarily computable, such as the space of all real numbers.

Concretely, in the research described here we use the realizability topos over the graph model $P\mathbb{N}$ of the (untyped) lambda calculus, together with the sub-graph model given by the recursively enumerable subsets, to represent the classical and computable worlds, respectively. There results a certain configuration of toposes that can be regarded as describing a notion of *relative computability*.[1] We study this configuration axiomatically, and derive a higher-order, modal logic in which to reason about it. The logic can then be applied to the original model to formalize reasoning about computability in that setting. Moreover, the resulting logical framework provides a general, categorical semantics and logical syntax for reasoning in a formal way about abstract computability, which it is hoped could also be useful for formally similar concepts, such as logical definability.

In somewhat more detail, Section 2 begins by recalling the standard realizability toposes $\mathrm{RT}(A)$ and $\mathrm{RT}(A_\sharp)$ resulting from a partial combinatory algebra A and a subalgebra $A_\sharp$. We then identify a third category $\mathrm{RT}(A, A_\sharp)$ which plays a key role; very roughly speaking, it represents the world of all (continuous) objects, but with only computable maps between them. This category $\mathrm{RT}(A, A_\sharp)$ is a topos, the *relative realizability topos* on A with respect to the subalgebra $A_\sharp$.

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[1] Not to be confused with the standard notion of computability relative to an oracle.

The toposes $\mathrm{RT}(A)$ and $\mathrm{RT}(A_\sharp)$ are not particularly well-related by themselves; the purpose of the relative realizability topos $\mathrm{RT}(A, A_\sharp)$ is to remedy this defect. The *three* toposes are related to each other as indicated in the following diagram, in which the three functors on the left leg constitute a local geometric morphism, while the right leg is a logical morphism.

$$\begin{tikzcd}
& \mathrm{RT}(A, A_\sharp) \arrow[dl, shift left=2] \arrow[dr] & \\
\mathrm{RT}(A_\sharp) \arrow[ur, shift left=4] \arrow[ur] & & \mathrm{RT}(A)
\end{tikzcd}$$

The local geometric morphism on the left is our chief concern and the focus of Section 3, which also mentions some examples and properties of these fairly well-understood maps of toposes. When we first encountered it, we were pleased to recognize our situation as an instance of one that F.W. Lawvere has already called attention to and dubbed an *adjoint cylinder* or, more colorfully, a *unity and identity of opposites* [Law91, Law98].

In Section 4 we present four axioms for local maps of toposes and sketch the proof that they are sound and complete. Actually, since the situation we are mainly interested in—*i.e.*, realizability—forces the local map to be localic, we give the axioms in a form that implies this condition. We simply mention here that a modification of axiom 2 about generators will accomodate all (bounded) local maps. This axiomatization has been found useful in working with the particular situation we have in mind, but its general utility for local maps of toposes remains to be seen.

One application, of sorts, of the axioms for local maps is the investigation of their logical properties. These are given in Section 5 in the form of a logical calculus involving two propositional operations, written $\sharp\varphi$ and $\flat\varphi$, with $\sharp$ left adjoint to $\flat$. It turns out that $\sharp$ satisfies the S4 modal logic postulates for the box-operation. We here term the $\sharp$-calculus a *modal logic for computability*, since that is the interpretation we have in mind; but of course, this modal logic can be interpreted in any local topos. We intend to use it to investigate the logical relations that hold in the relative realizability topos; however, this aspect of our work is only just beginning.

Note that any local map also induces a closely related pair of adjoint operations on *logical types* (objects), in addition to the ones on formulas (subobjects) studied here, relating our work to [BMTS99, Ben95]. The idea of a modal "computability" operator $\sharp$ is due to the senior author (January 1998) and was the original impetus for this work, parts of which are from the second author's doctoral thesis [Bir98]. The final brief section of the paper spells out the intended interpretation of the $\sharp$-calculus in the relative realizability topos $\mathrm{RT}(A, A_\sharp)$.

2 Realizability toposes for computability

Let $(A, \cdot, \mathbf{K}, \mathbf{S})$ be a partial combinatory algebra (PCA); often we just denote it by its underlying set A. The binary operator $\cdot$ is the (partial) application and combinators $\mathbf{K}$ and $\mathbf{S}$ are taken to be part of the structure and not just required to exist.

Let $A_\sharp$ be a sub-PCA of A, that is $A_\sharp$ is a subset of A containing $\mathbf{K}$ and $\mathbf{S}$ and closed under partial application. Intuitively, we are thinking of the realizers in A as "continuous" realizers and of theose in $A_\sharp$ as "computable" realizers. This intuition comes from the main example, where A is $P\mathbb{N}$, the graph model on the powerset of the natural numbers, and $A_\sharp$ is RE, the recursively enumerable sub-graph-model. Note that the model $P\mathbb{N}$ has a continuum of (countable) sub-PCA's. As another example, one may consider van Oosten's combinatory algebra $\mathcal{B}$ for sequential computation and its effective subalgebra $\mathcal{B}_{\mathit{eff}}$, see [Oos97, Lon98].

The PCA's $A_\sharp$ and A give rise to two realizability toposes $\mathrm{RT}(A_\sharp)$ and $\mathrm{RT}(A)$ in the standard way [HJP80]. One may think of $\mathrm{RT}(A)$ as a universe where all objects and all maps are realized by continuous realizers. Likewise, $\mathrm{RT}(A_\sharp)$ may be thought of as a universe where all objects and all maps are realized by computable realizers. Unfortunately, these two toposes are not very well related; in particular, it is not clear how to talk about computable maps operating on continuous objects, which is what one would like to do for the purposes of, *e.g.*, computable analysis [PER89]. Thus, one is led to introduce another realizability topos, $\mathrm{RT}(A, A_\sharp)$, where, intuitively, equality on all objects is realized by continuous realizers and all maps are realized by computable realizers.[2]

The topos $\mathrm{RT}(A, A_\sharp)$ is constructed by modifying the underlying tripos for $\mathrm{RT}(A)$ in the following way. The non-standard predicates φ, ψ on a set I are still functions $I \to PA$ into the powerset of A and the Heyting pre-algebra operations are the same as in the tripos underlying $\mathrm{RT}(A)$. The modification is in the definition of the entailment relation: we say $\varphi \vdash \psi$ over I iff there is a realizer a in $A_\sharp$ (not just in A) such that for all i in I, all $b \in \varphi(i)$, $a \cdot b$ is defined and $a \cdot b \in \psi(i)$. In the terminology of Pitts [Pit81], we have changed the "designated truth values" to be those subsets of A which have a non-empty intersection with $A_\sharp$. Denote this new tripos by $P : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Cat}$. Then $\mathrm{RT}(A, A_\sharp)$ is the topos $\mathbf{Set}[P]$ represented by P.

Explicitly, objects of $\mathrm{RT}(A, A_\sharp)$ are pairs $(X, \approx_X)$ with X a set and $\approx_X : X \times X \to PA$ a non-standard equality predicate with computable realizers for symmetry and transitivity. Morphisms from $(X, \approx_X)$ to $(Y, \approx_Y)$ are equivalence classes of functional relations $F\colon X \times Y \to PA$ with computable realizers proving that F is a functional relation. Two such functional relations F and G are equivalent iff there are computable realizers showing them equivalent. We now see that intuitively, it makes sense to think of objects of $\mathrm{RT}(A, A_\sharp)$ as objects with *continuous* realizers for existence and equality elements, and of morphisms $f = [F]$ as *computable* maps, since the realizers for the functionality of F are required to be computable.

3 Geometry of the realizability toposes for computability

Let $Q\colon \mathbf{Set}^{\mathrm{op}} \to \mathbf{Cat}$ be the standard realizability tripos on $A_\sharp$, *i.e.*, the tripos underlying $\mathrm{RT}(A_\sharp)$. We now define three $\mathbf{Set}$-indexed functors between Q and P:

$$\Delta : Q \to P \quad \text{and} \quad \Gamma\colon P \to Q \quad \text{and} \quad \nabla : Q \to P.$$

These are defined as follows. Over I, we have

$$\begin{aligned}
\Delta_I(\psi\colon I \to PA_\sharp)(i) &= \psi(i) \\
\Gamma_I(\varphi\colon I \to PA)(i) &= A_\sharp \cap \varphi(i) \\
\nabla_I(\psi\colon I \to PA_\sharp)(i) &= \bigcup_{\varphi \in PA} \Big(\varphi \wedge \big(A_\sharp \cap \varphi \supset \psi(i)\big)\Big),
\end{aligned}$$

where $\wedge$ and $\supset$ are calculated as in $P(1)$.

Theorem 3.1. *Under these definitions it follows that*

- (Δ, Γ) *is a geometric morphism of triposes from* P *to* Q.
- (Γ, ∇) *is a geometric morphism of triposes from* Q *to* P.
- *For all* $I \in \mathbf{Set}$, Δ_I *and* ∇_I *are both full and faithful.*

By Proposition 4.7 in [Pit81], these geometric morphisms lift to two geometric morphisms between the toposes, as in

$$\begin{tikzcd} \mathrm{RT}(A_\sharp) \arrow[r, bend left, "\Delta"] \arrow[r, bend right, "\nabla"'] & \mathrm{RT}(A, A_\sharp) \arrow[l, "\Gamma"'] \end{tikzcd}, \qquad \Delta \dashv \Gamma \dashv \nabla.$$

(Here we do not distinguish notationally between the functors at the tripos level and at the topos level). In particular, Δ preserves finite limits. Moreover, $\Delta, \nabla\colon \mathrm{RT}(A_\sharp) \to \mathrm{RT}(A, A_\sharp)$ are easily shown to also be both full and faithful. The geometric morphism $(\Delta, \Gamma)\colon \mathrm{RT}(A, A_\sharp) \to \mathrm{RT}(A)$ is therefore a (connected) surjection, while $(\Gamma, \nabla)\colon \mathrm{RT}(A_\sharp) \to \mathrm{RT}(A, A_\sharp)$ is an embedding. Note that $\Gamma \circ \nabla \cong 1 \cong \Gamma \circ \Delta$.

It follows by standard results that there is a Lawvere-Tierney topology j in $\mathrm{RT}(A, A_\sharp)$ such that $\mathrm{RT}(A_\sharp)$ is equivalent to the category $\mathrm{Sh}_j(\mathrm{RT}(A, A_\sharp))$ of sheaves.

The following theorem was known to Martin Hyland but apparently has never been published. We include a proof here.

[2] We first learned about the topos $\mathrm{RT}(A, A_\sharp)$ from Thomas Streicher in February 1998, but the construction has actually been known for a long time; see [Pit81, Page 15, item (ii)].

Theorem 3.2. *Let $\mathbb{C}$ be a finitely complete category and let P and Q be $\mathbb{C}$-triposes. Suppose $f = (f^*, f_*) : P \to Q$ is a geometric morphism of triposes. Then $\mathbb{C}[P]$ is localic over $\mathbb{C}[Q]$ via the induced geometric morphism $f = (f^*, f_*) \colon \mathbb{C}[P] \to \mathbb{C}[Q]$.*

Proof. We want to prove that $\mathbb{C}[P]$ is equivalent to the category of $\mathbb{C}[Q]$-valued sheaves on the internal locale $f_*(\Omega_{\mathbb{C}[P]})$ in $\mathbb{C}[Q]$. As usual [Joh77] it suffices to show that, for all $X \in \mathbb{C}[P]$, there exists a $Y \in \mathbb{C}[Q]$ and a diagram

$$\begin{tikzcd} S \arrow[r, tail] \arrow[d, two heads] & f^*Y \\ X & \end{tikzcd}$$

in $\mathbb{C}[P]$ presenting X as a subquotient of f^*Y for Y an object of $\mathbb{C}[Q]$. Write $\nabla_P \colon \mathbb{C} \to \mathbb{C}[P]$ for the functor $I \mapsto (I, \exists_{\delta_I}(T))$, where $\delta_I : I \mapsto I \times I$ is the diagonal map (the "constant objects functor" [Pit81]).

By a familiar property of realizability toposes, we have that for all $X \in \mathbb{C}[P]$, there exists an object $I \in \mathbb{C}$ and a diagram

$$\begin{tikzcd} S \arrow[r, tail] \arrow[d, two heads] & \nabla_P(I) = (I, \exists_{\delta_I}(T)) \\ X & \end{tikzcd} \tag{1}$$

in $\mathbb{C}[P]$ presenting X as a subquotient of a constant object $\nabla_P(I)$. Now since f^* is the inverse image of a geometric morphism of triposes, f^* preserves existential quantification (as an indexed left adjoint), so $f^*(\nabla_Q(I)) \cong \nabla_P(I)$, and the diagram in (1) is the required diagram. $\square$

We may conclude from Theorems 3.2 and 3.1 that $\mathrm{RT}(A, A_\sharp)$ is localic over $\mathrm{RT}(A_\sharp)$ via (Δ, Γ). Indeed, the geometric morphism $(\Delta, \Gamma) : \mathrm{RT}(A, A_\sharp) \to \mathrm{RT}(A_\sharp)$ is a *localic local map of toposes*, since Γ has a right adjoint ∇, for which $\Gamma \circ \nabla \cong 1$. Local maps have been studied by [JM89], and provide an instance of what Lawvere [Law91, Law98] has called *unity and identity of opposites.* The idea is that the full subcategories $\Delta(\mathrm{RT}(A, A_\sharp))$ and $\nabla(\mathrm{RT}(A, A_\sharp))$ are each equivalent to $\mathrm{RT}(A, A_\sharp)$, and yet are "opposite" in the sense of being coreflective and reflective, respectively, in $\mathrm{RT}(A, A_\sharp)$. We think of the objects in $\nabla(\mathrm{RT}(A, A_\sharp))$ as sheaves, and here we think of those in $\Delta(\mathrm{RT}(A, A_\sharp))$ as "computable".

Examples of local maps in addition to the basic ones mentioned in [JM89] include the following:

(1) Let $\mathrm{RT}(A)$ be a realizability topos, and let $i \colon \mathbb{C}_A \to \mathrm{RT}(A)$ be a full subcategory of partitioned assemblies of suitably large, bounded cardinality, so that $\mathbb{C}_A$ is a small generating subcategory of projectives. The covering families in $\mathbb{C}_A$ are to be those which are epimorphic in $\mathrm{RT}(A)$. Then the Grothendieck topos $\mathrm{Sh}(\mathbb{C}_A)$ is local; let us write its structure maps as $\Delta' \dashv \Gamma' \dashv \nabla' \colon \mathbf{Set} \to \mathrm{Sh}(\mathbb{C}_A)$. There is a restricted Yoneda embedding,

$$Y = \mathrm{RT}(A)(i(\cdot), -) \colon \mathrm{RT}(A) \to \mathrm{Sh}(\mathbb{C}_A),$$

for which the diagram below commutes, in the sense that $\Gamma \cong \Gamma' \circ Y$, $\nabla' \cong Y \circ \nabla$.

$$\begin{tikzcd} \mathbf{Set} \arrow[r, shift right, "\nabla"'] \arrow[rr, bend right, "\nabla'"'] & \mathrm{RT}(A) \arrow[l, shift right, "\Gamma"'] \arrow[r, "Y"] & \mathrm{Sh}(\mathbb{C}_A) \arrow[ll, bend right, "\Gamma'"'] \end{tikzcd}$$

Thus we can regard $\Gamma \colon \mathrm{RT}(A) \to \mathbf{Set}$ as what *would be* the direct image of a local map, if only $\mathrm{RT}(A)$ had enough colimits.

(2) Let $\mathbb{C}$ be a small category with finite limits and $i \colon \mathbb{D} \hookrightarrow \mathbb{C}$ a full subcategory, closed under the same. The geometric morphism $\widehat{\mathbb{C}} \to \widehat{\mathbb{D}}$ with direct image the restriction i^* along i is then a local map. The image of $\widehat{\mathbb{D}}$ under the full embedding $i_!$ (where $i_! \dashv i^*$) then consists of those presheaves P on $\mathbb{C}$ for which

$$PC \cong \varinjlim_{D, h : C \to D} PD.$$

These are the objects that we are interested in as candidates for being "computable", when $\mathbb{D}$ represents the computable subcategory. They are the ones termed "discrete" in the sequel.

Regarding the choice of terminology: We use the term "discrete" by analogy to topological examples. We would have liked to call these objects "cosheaves" since they are the objects that are *co*orthogonal to the morphisms inverted by $\mathbf{a}$, and sheaves are those that are orthogonal. However, "cosheaf" has already been used to describe something else, namely a "covariant sheaf".

4 Axioms for localic local maps

In this section we present a set of axioms for localic local maps and sketch a proof that they are sound and complete, in the sense that whenever a given topos satisfies the axioms then it gives rise to a localic local map of toposes and, moreover, any localic local map of toposes satisfies the axioms. Later on we shall make use of the axiomatization in this section to describe a *modal logic for computability.* First we need a couple of definitions.

For the remainder of this section let $\mathcal{E}$ be an elementary topos and j a Lawvere-Tierney topology in $\mathcal{E}$. We write $V \mapsto \overline{V}$ for the associated closure operation on subobjects $V \rightarrowtail X$. We say that j is **principal** if, for all $X \in \mathcal{E}$, the closure operation on $\mathrm{Sub}(X)$ has a left adjoint $U \mapsto \mathring{U}$, called **interior**, that is,

$$\mathring{U} \leq V \iff U \leq \overline{V} \qquad \text{in } \mathrm{Sub}(X).$$

The interior operation is *not* assumed to be natural; that is, it is not assumed to commute with pullbacks. It follows that in general the interior operation is not induced by an internal map on the subobject classifier Ω in the $\mathcal{E}$, and in that sense is not a *logical operation* (in the internal logic of $\mathcal{E}$).

The interior operation extends to a functor $\mathcal{E} \to \mathcal{E}$, since, whenever $f\colon X \to Y$, we have $\mathring{X} \leq f^*(\mathring{Y})$. We say that an object $X \in \mathcal{E}$ is **open** if $\mathring{X} \cong X$. An object is open iff the interior of its diagonal equals its diagonal. An object $C \in \mathcal{E}$ is called **discrete** if it is **coorthogonal** to all morphisms inverted by the associated sheaf functor $\mathbf{a}$; that is, C is discrete if for all $e\colon X \to Y$ such that $\mathbf{a}(e)$ is an isomorphism, for all $f\colon C \to Y$, there exists a unique $f'\colon C \to X$ such that

$$\begin{tikzcd} & X \arrow[d, "e"] \\ C \arrow[ur, dashed, "f'"] \arrow[r, "f"'] & Y \end{tikzcd}$$

commutes.

Recall, *e.g.*, from [Joh77], that a sheaf can be characterized as an object which is orthogonal to all morphisms inverted by $\mathbf{a}$, and that it suffices to test orthogonality just with respect to the dense monomorphisms. For discrete objects there is a similar simplification: an object is discrete iff it is coorthogonal to all **codense epimorphisms**, where an epimorphism $e\colon X \twoheadrightarrow Y$ is codense iff the interior of its kernel is included in the diagonal of X. We write $\mathrm{D}_j\mathcal{E}$ for the full subcategory of $\mathcal{E}$ on the discrete objects.

Now we propose the following **axioms for a localic local map** on a topos $\mathcal{E}$ with topology j.

Axiom 1. The topology j is principal.

Axiom 2. For all $X \in \mathcal{E}$, there exists a discrete object C and a diagram

$$\begin{tikzcd} S \arrow[r, tail] \arrow[d, two heads] & C \\ X & \end{tikzcd}$$

in $\mathcal{E}$, presenting X as a subquotient of C.

Axiom 3. For all discrete $C \in \mathcal{E}$, if $X \rightarrowtail C$ is open, then X is also discrete.

Axiom 4. For all discrete $C, C' \in \mathcal{E}$, the product $C \times C'$ is again discrete.

Let $\mathcal{E}$ be a topos with a topology j satisfying the Axioms 1–4 for localic local maps. We can then prove:

Theorem 4.1. *The category of discrete objects* $\mathrm{D}_j\mathcal{E}$ *is coreflective in* $\mathcal{E}$, *that is, the inclusion* $\Delta\colon \mathrm{D}_j\mathcal{E} \hookrightarrow \mathcal{E}$ *has a right adjoint. Moreover,* $\mathrm{D}_j\mathcal{E}$ *is a topos,* Δ *is left exact, and* $(\Delta, \Gamma)\colon \mathcal{E} \to \mathrm{D}_j\mathcal{E}$ *is a localic local map.*

Proof Sketch. The associated discrete object of an object $X \in \mathcal{E}$ is obtained as follows. Present X as a subquotient of a discrete object C and consider the following diagram.

$$
\begin{tikzcd}
\mathring{K}_{\mathring{e}} \arrow[r, tail] & K_{\mathring{e}} \arrow[d, shift left] \arrow[d, shift right] & & \\
& \mathring{S} \arrow[r, tail, "m"] \arrow[d, "\mathring{e}"] \arrow[dl] & S \arrow[r, tail] \arrow[d, "e"] & C \\
\widetilde{\mathring{X}} \arrow[r, two heads, "h"'] & \mathring{X} \arrow[r, tail] & X &
\end{tikzcd}
$$

where $K_{\mathring{e}}$ is the kernel of $\mathring{e}$, and $\widetilde{\mathring{X}}$ is the coequalizer of its interior $\mathring{K}_{\mathring{e}}$. The object $\widetilde{\mathring{X}}$ can be shown to be the associated discrete object of X, in the sense that it is couniversal among all maps from discrete objects into X, so that $\widetilde{\mathring{X}} \twoheadrightarrow \mathring{X} \rightarrowtail X$ is the counit of the sought adjunction.

By results of Kelly and Lawvere [KL89, Propositions 2.1 and 2.4] it now follows that there is an equivalence of categories $\mathrm{D}_j\mathcal{E} \simeq \mathrm{Sh}_j\mathcal{E}$, under which $\Gamma \colon \mathcal{E} \to \mathrm{D}_j\mathcal{E}$ is sheafification. Thus Γ has a right adjoint. Moreover, Δ can be shown to be left exact. Since $\mathcal{E}$ is localic over $\mathrm{D}_j\mathcal{E}$ by Axiom 2, the geometric morphism $(\Delta, \Gamma) \colon \mathcal{E} \to \mathrm{D}_j\mathcal{E}$ is a localic local map of toposes. $\square$

Corollary 4.2. *For any discrete $C, C' \in \mathcal{E}$ and any $f \colon C' \to C$, and all open subobjects $U \rightarrowtail C$, the pullback $C' \times_C U$*

$$
\begin{tikzcd}
C' \times_C U \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & U \arrow[d, tail] \\
C' \arrow[r, "f"'] & C
\end{tikzcd}
$$

is open.

Theorem 4.3. *Every localic local map of toposes satisfies Axioms 1-4 for localic local maps.*

Proof Sketch. It suffices to consider localic local maps of the form $(L, \mathbf{a}) \colon \mathcal{E} \to \mathrm{Sh}_j\mathcal{E}$ with $\mathbf{a}$ the associated sheaf functor, $L \dashv \mathbf{a}$, and $\mathrm{D}_j\mathcal{E} \simeq \mathrm{Sh}_j\mathcal{E}$. The interior of an object X is obtained by taking the image of the counit $L\,\mathbf{a}\,X \to X$. The axioms are then easily verified. $\square$

5 A modal logic for computability

Let $\mathcal{E}$ be a topos with a topology j satisfying the axioms set out in the previous section. In this section our goal is to describe a *logic* with which one can reason about *both* of the two toposes $\mathcal{E}$ and $\mathrm{D}_j\mathcal{E}$. This will then apply to $\mathrm{RT}(A, A_\sharp)$ and $\mathrm{RT}(A_\sharp)$, see Section 6.

As a first attempt, one may consider the ordinary internal logic of $\mathcal{E}$ extended with a closure operator induced by the topology j and try to extend it futher with a logical operator corresponding to the interior operation. But since interior does not commute with pullback in general, it is not a logical operation on all subobjects of objects of $\mathcal{E}$. However, since the interior of an object X is the least dense subobject of X, one may instead add a new atomic predicate U_X for each type X and write down axioms expressing that it is the least dense subobject. This is straightforward. But, as yet, we do not have a convenient internal logical characterization of the discrete objects.

Instead we shall describe another approach where types and terms are objects and morphisms of $\mathrm{D}_j\mathcal{E}$ and predicates are all the predicates in $\mathcal{E}$ on objects from $\mathrm{D}_j\mathcal{E}$. More precisely, we consider the internal logic of the fibration $\begin{array}{c}\scriptstyle\mathrm{Pred}\\ \downarrow\\ \scriptstyle\mathrm{D}_j\mathcal{E}\end{array}$ obtained by change-of-base as indicated in:

$$
\begin{tikzcd}
\mathrm{Pred} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Sub}(\mathcal{E}) \arrow[d] \\
\mathrm{D}_j\mathcal{E} \arrow[r, "\Delta"'] & \mathcal{E}.
\end{tikzcd}
$$

Thus a predicate on an object $X \in \mathrm{D}_j\mathcal{E}$ is a subobject of ΔX in $\mathcal{E}$. Since

$$\mathrm{Sub}_{\mathcal{E}}(\Delta X) \cong \mathcal{E}(\Delta X, \Omega_{\mathcal{E}}) \cong \mathrm{D}_j\mathcal{E}(X, \Gamma\Omega_{\mathcal{E}}), \qquad (2)$$

the internal locale $\Gamma\Omega_{\mathcal{E}}$ is a generic object for the fibration $\begin{array}{c}\mathrm{Pred}\\ \downarrow\\ \mathrm{D}_j\mathcal{E}\end{array}$. Hence the internal logic is many-sorted higher-order intuitionistic logic.

Note that, since $\mathcal{E}$ is localic over $\mathrm{D}_j\mathcal{E}$, we can completely describe $\mathcal{E}$ in this internal logic in the standard way [FS79] as partial equivalence relations and functional relations between such.

By Corollary 4.2, the interior operation is a logical operation on predicates; we denote it by $\sharp$. Note that the ordinary logic of $\mathrm{D}_j\mathcal{E}$ is obtained by restricting attention to predicates of the form $\sharp\varphi$. The topology j in $\mathcal{E}$ induces a closure operator, which we are pleased to denote $\flat$, on predicates in this internal logic.

We now describe how the $\sharp$ and $\flat$ operations can be axiomatized. Logical entailment is written $\Gamma \mid \varphi \vdash \psi$, where Γ is a context of the form $x_1 : \sigma_1, \ldots, x_n : \sigma_n$ giving types σ_i to variables x_i, and where φ and ψ are formulas with free variables in Γ. We write $\emptyset$ for an empty list of assumptions. There are the usual rules of many-sorted higher-order intuitionistic logic plus the following axioms and rules:

$$\frac{}{\Gamma \mid \sharp\varphi \vdash \varphi}\,(3) \qquad \frac{}{\Gamma \mid \sharp\varphi \vdash \sharp\sharp\varphi}\,(4)$$

$$\frac{}{\Gamma \mid \emptyset \vdash \sharp(T)}\,(5) \qquad \frac{}{\Gamma \mid \sharp\varphi \wedge \sharp\psi \vdash \sharp(\varphi \wedge \psi)}\,(6)$$

$$\frac{\Gamma \mid \sharp\varphi \vdash \psi}{\Gamma \mid \varphi \vdash \flat\psi}\,(7) \qquad \frac{}{x : \sigma, y : \sigma \mid x = y \vdash \sharp(x = y)}\,(8)$$

One can then show that $\sharp$ has the formal properties of the box operator in the modal logic S4, *i.e.*, for formulas φ and ψ in context Γ:

$$\vdash \sharp\varphi \supset \varphi$$
$$\vdash \sharp(\varphi \supset \psi) \supset (\sharp\varphi \supset \sharp\psi)$$
$$\vdash \sharp\varphi \supset \sharp\sharp\varphi$$

and

$$\frac{\vdash \varphi}{\vdash \sharp\varphi}\,(9)$$

We therefore refer to this logic as a **modal logic for computability**.

We remark that the following principles of inference for quantifiers can be derived from (3)–(8):

$$\frac{\Gamma \mid \emptyset \vdash \sharp\forall x{:}X.\,\varphi}{\Gamma \mid \emptyset \vdash \forall x{:}X.\,\sharp\varphi}\,(10)$$

$$\frac{}{\Gamma \mid \sharp\exists x{:}X.\,\varphi \dashv\vdash \exists x{:}X.\,\sharp\varphi}\,(11)$$

Quite generally, the modal logic of any local map of toposes

$$\Gamma : \mathcal{E} \to \mathcal{F}, \qquad \Delta \dashv \Gamma \dashv \nabla$$

can be used to compare the internal logic of $\mathcal{E}$ with that of $\mathcal{F}$, since the types are then the discrete objects E in $\mathcal{E}$, for which $E \cong \Delta\Gamma E$ and

$$\mathrm{Sub}_{\mathcal{F}}(\Gamma E) \cong \mathrm{OpenSub}_{\mathcal{E}}(E),$$

where $\mathrm{OpenSub}_{\mathcal{E}}(E) \subseteq \mathrm{Sub}_{\mathcal{E}}(E)$ is the subposet of open subobjects of E in $\mathcal{E}$. Observe, *e.g.*, that the natural numbers object N is among the discrete objects, and that the identity relation on any discrete object is open.

To give a sample application, call a formula ϑ **stable** if it is built up from atomic predicates (including equations) and first-order logic and if for every subformula of the form $\varphi \supset \psi$, the formula φ has no $\forall$ or $\supset$. For any sentence ϑ, we write $\mathcal{F} \vDash \vartheta$ to mean that ϑ holds in the standard internal logic of $\mathcal{F}$ with basic types σ interpreted by objects X_σ of $\mathcal{F}$ and atomic formulas R on type σ interpreted as subobjects $S_R \rightarrowtail X_\sigma$. We then write $\mathcal{E} \vDash \vartheta$ to mean that ϑ holds in the standard logic of $\mathcal{E}$ with basic types σ interpreted by objects ΔX_σ and atomic formulas R interpreted by $\Delta S_R \rightarrowtail \Delta X_\sigma$.

Proposition 5.1. *For any stable sentence* ϑ*,*

$$\mathcal{F} \vDash \vartheta \qquad \textit{iff} \qquad \mathcal{E} \vDash \vartheta.$$

Proof Sketch. There are the interpretations $[\![-]\!]_{\mathcal{F}}$ and $[\![-]\!]_{\mathcal{E}}$ for which one shows by induction that for any stable formula ϑ

$$\Delta[\![\vartheta]\!]_{\mathcal{F}} = [\![\sharp\vartheta]\!]_{\mathcal{E}},$$

using the fact that Γ preserves $\forall$ along maps between discrete objects. Thus for any stable sentence ϑ:

$$\begin{aligned}\mathcal{F} \vDash \vartheta \quad &\text{iff} \quad 1 = [\![\vartheta]\!]_{\mathcal{F}} \\ &\text{iff} \quad 1 = \Delta 1 = \Delta[\![\vartheta]\!]_{\mathcal{F}} = [\![\sharp\vartheta]\!]_{\mathcal{E}} \\ &\text{iff} \quad \mathcal{E} \vDash \sharp\vartheta \\ &\text{iff} \quad \mathcal{E} \vDash \vartheta.\end{aligned}$$

The proposition can be used to show that, *e.g.*, if $\mathcal{F}$ has choice for functions from N to N in the external sense, then so does $\mathcal{E}$. Indeed, let R by any relation (not necessarily open) on N in $\mathcal{E}$ and suppose that

$$\mathcal{E} \vDash \forall n{:}N.\, \exists m{:}N.\, R(n,m).$$

Then we reason informally as follows

$$\begin{array}{ll}\mathcal{E} \vDash \forall n{:}N.\, \exists m{:}N.\, R(n,m) & \\ \mathcal{E} \vDash \sharp\forall n{:}N.\, \exists m{:}N.\, R(n,m) & \text{by (9)} \\ \mathcal{E} \vDash \forall n{:}N\sharp.\, \exists m{:}N.\, R(n,m) & \text{by (10)} \\ \mathcal{E} \vDash \forall n{:}N.\, \exists m{:}N.\, \sharp R(n,m) & \text{by (11)} \\ \mathcal{F} \vDash \forall n{:}N.\, \exists m{:}N.\, \sharp R(n,m) & \text{by stability} \\ \mathcal{F} \vDash \forall n{:}N.\, \sharp R(n,f(n)) & \text{for some } f\colon N \to N \\ & \text{by AC in } \mathcal{F} \\ \mathcal{E} \vDash \forall n{:}N.\, \sharp R(n,f(n)) & \text{by stability} \\ \mathcal{E} \vDash \forall n{:}N.\, R(n,f(n)) & \text{by (3)}\end{array}$$

The model of the modal logic for computability given by the fibration $\begin{smallmatrix}\text{Pred}\\ \downarrow \\ \mathrm{D}_j\mathcal{E}\end{smallmatrix}$ defined above is in fact a tripos, namely the standard tripos on the internal locale $\Gamma\Omega_{\mathcal{E}}$, see (2). Indeed, we can give the following general definition and theorem.

Definition 5.2. A canonically presented tripos $P = \mathcal{E}(-,\Sigma)$ on a topos $\mathcal{E}$ is **local** if there is a topology $\flat\colon \Sigma \to \Sigma$ on P (see [Pit81, Page 59]) and an **interior** map $\sharp\colon \Sigma \to \Sigma$ such that

- $\vdash \sharp T$
- $\sharp p \wedge \sharp q \vdash \sharp(p \wedge q)$
- $\sharp p \vdash q$ iff $p \vdash \flat q$.

Theorem 5.3. *A local tripos induces a localic local map of toposes and, moreover, every localic local map of toposes arises from a local tripos.*

The tripos P underlying the topos $\mathrm{RT}(A, A_\sharp)$ is local: $\sharp\colon PA \to PA$ is induced by the functor $\Delta \circ \Gamma$,

$$\sharp(\varphi) = \varphi \cap A_\sharp$$

and $\flat\colon PA \to PA$ is induced by the functor $\nabla \circ \Gamma$,

$$\flat(\varphi) = \bigcup_{\psi \in PA} (\psi \wedge (\psi \cap A_\sharp \supset \varphi \cap A_\sharp)).$$

6 Interpretation of the modal logic in $\mathrm{RT}(A_\sharp)$ and $\mathrm{RT}(A, A_\sharp)$

Finally, we briefly describe in concrete terms how the modal logic for computability is interpreted in $\mathrm{RT}(A_\sharp)$ and $\mathrm{RT}(A, A_\sharp)$.

Types and terms are interpreted by objects and morphisms of $\mathrm{RT}(A_\sharp)$ in the standard way. A predicate φ on an type $(X, \approx_X) \in \mathrm{RT}(A_\sharp)$ is an equivalence class of a strict, extensional relation in $P(X \times X)$ (recall P is the tripos underlying $\mathrm{RT}(A, A_\sharp)$), that is, φ is an equivalence class of set-theoretic functions $X \times X \to PA$ which are strict and extensional via computable realizers, two such functions being equivalent iff they are isomorphic as objects of $P(X \times X)$.

On such a predicate φ on an object $(X, \approx)$, $\sharp\varphi$ is just $x \mapsto \varphi(x) \cap A_\sharp$ and $\flat\varphi$ is

$$x \mapsto \bigcup_{\psi \in PA} (\psi \wedge (\psi \cap A_\sharp \supset \varphi(x) \cap A_\sharp)).$$

Thus we can think of $\sharp\varphi$ as φ being computably true, *i.e.*, realized via computable realizers.

Objects of $\mathrm{RT}(A, A_\sharp)$ are then described as pairs $((X, \approx), \varphi)$ with $(X, \approx) \in \mathrm{RT}(A_\sharp)$ and φ a partial equivalence in Pred on $(X, \approx) \times (X, \approx)$. Likewise, morphisms are described as functional relations in the standard way.

In this realizability model, we have the following further principle for $\sharp$:

$$\Gamma \mid \neg\varphi \dashv\vdash \sharp\neg\varphi$$

because the types are the objects of $\mathrm{RT}(A_\sharp)$. From this it follows that

$$\Gamma \mid \neg\neg\varphi \dashv\vdash \sharp\neg\neg\varphi$$

which accords with the intuition that double-negation closed formulas have no computational content.

Note also that since $\Gamma\colon \mathrm{RT}(A_\sharp) \to \mathbf{Set}$ has a right adjoint, the same is true for the global sections functor $\Gamma\colon \mathrm{RT}(A, A_\sharp) \to \mathbf{Set}$. Thus in $\mathrm{RT}(A, A_\sharp)$ too, 1 is indecomposable and projective; so $\mathrm{RT}(A, A_\sharp)$ has the logical disjunction and existence properties.

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References

[Ben95] P.N. Benton. A mixed linear and non-linear logic: Proofs, terms and models (preliminary report). Technical report, University of Cambridge, 1995.

[Bir98] L. Birkedal. Developing theories of types and computability. Thesis Proposal, April 1998.

[BMTS99] Z.E-L. Benaissa, E. Moggi, W. Taha, and T. Sheard. A categorical analysis of multi-level languages. Manuscript, January 1999.

[FS79] M.P. Fourman and D.S. Scott. Sheaves and logic. In M.P. Fourman, C.J. Mulvey, and D.S. Scott, editors, *Applications of Sheaves*, pages 302–401. Springer-Verlag, 1979.

[HJP80] J.M.E. Hyland, P.T. Johnstone, and A.M. Pitts. Tripos theory. *Mathematical Proceedings of the Cambridge Philosophical Society 88*, 1980.

[JM89] P.T. Johnstone and I. Moerdijk. Local maps of toposes. *Proc. London Math. Soc.*, 3(58):281–305, 1989.

[Joh77] P.T. Johnstone. *Topos Theory*. Number 10 in L.M.S. Monographs. Academic Press, 1977.

[KL89] G.M. Kelly and F.W. Lawvere. On the complete lattice of essential localizations. *Bull. Soc. Math. Belg. Ser. A*, XLI(2):289–319, 1989.

[Law91] F.W. Lawvere. Some thoughts on the future of category theory. In A. Carboni, M.C. Pedicchio, and G. Rosolini, editors, *Category Theory. Proceedings of the International Conference held in Como, Italy, July 22–28, 1990*, volume 1488 of *Lecture Notes in Mathematics*, pages 1–13. Springer-Verlag, 1991.

[Law98] F.W. Lawvere. Toposes generated by codiscrete objects in combinatorial topology and functional analysis. Notes for Colloquium lectures given at North Ryde, New South Wales, Australia on April 18, 1998 and at Madison, USA, on December 1, 1989, 1998.

[Lon98] J. Longley. The sequentially realizable functionals. Technical Report ECS–LFCS–98–402, University of Edinburgh, 1998.

[Oos97] J. van Oosten. A combinatory algebra for sequential functionals of finite type. Technical Report 996, University of Utrecht, 1997.

[PER89] M.B. Pour-El and J.I. Richards. *Computability in Analysis and Physics*. Springer-Verlag, 1989.

[Pit81] A.M. Pitts. *The Theory of Triposes*. PhD thesis, University of Cambridge, 1981.

[SAB+] D.S. Scott, S. Awodey, A. Bauer, L. Birkedal, and J. Hughes. Logics of Types and Computation at Carnegie Mellon University.
`http://www.cs.cmu.edu/Groups/LTC/`.