Generalized Hyers-Ulam-Stability of a New Generalized Mixed Type Cubic Quartic Functional Equation

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Abstract. In this paper, we obtain the general solution and investigate its generalized Hyers-Ulam-Rassias stability of a new generalized cubic and quartic functional equation

$$f(x+ny) + f(x-ny) = n^2 [f(x+y) + f(x-y)] - 2(n^2 - 1)f(x) + \frac{2}{n}(n+1)[f(ny) - n^3 f(y)]$$

for any real number n with $n \neq 0$ in Banach spaces.

Keywords: Cubic functional equation, Quartic functional equation, Mixed type functional equation, Hyers-Ulam-Rassias stability

1. Introduction

In 1940, S.M. Ulam proposed the following question concerning the stability of group homomorphism:

Let G_1 be a group and let G_2 be a metric group with the metric $d(.,.)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h: G_1 \rightarrow G_2$ satisfies the

inequality $d(h(x,y), h(x)*h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \rightarrow G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$? In other words, under what condition does there exists a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, D.H.Hyers gave a first affirmative answer to the question of Ulam for Banach spaces.

THEOREM 1.1.[9]. Let $f : X \rightarrow Y$ be a mapping from Banach spaces X , into Y for some $\epsilon > 0$, which satisfies

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon \quad (1.1)$$

for all $x, y \in X$. Then the limit

$$A(x) = \lim_{n \rightarrow \infty} \frac{f(2^n x)}{2^n} \quad (1.2)$$

exists for all $x \in X$ and $A : X \rightarrow Y$ is the unique mapping satisfying

$$\|f(x) - A(x)\| \leq \epsilon \quad (1.3)$$

for all $x \in X$. If $f(tx)$ is continuous in $t \in R$ for any fixed $x \in X$, then A is linear.

In 1978, Th.M.Rassias[19] gave the proof of stability of the linear mapping by permitting Cauchy difference to become unbounded. He proved the following theorem for a sum of powers of norms.

THEOREM 1.2.[19]. Let $f : E_1 \rightarrow E_2$ be a mapping from a normed vector space E_1 into Banach space E_2 such that $f(tx)$ is continuous in $t \in R$ and for each fixed $x \in E_1$ assume that there exist a constant $\epsilon > 0$ and $p \in [0, 1)$ with subject to the inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon (\|x\|^p + \|y\|^p) \quad (1.4)$$

for all $x, y \in E_1$, then there exists a unique linear mapping $T : E_1 \rightarrow E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{2\varepsilon}{2 - 2^p} \|x\|^p \quad (1.5)$$

for all $x \in E_1$.

In 1991, Gajda[5] answered the question for the case $p > 1$, which was raised by Th.M. Rassias. This new concept of stability phenomenon is known as Hyers-Ulam-Rassias stability of functional equations [1-3], [5-15], [22-23]. Later J.M.Rassias proved a similar result when the unbounded Cauchy difference is bounded by a product of powers of norms.

THEOREM 1.3.[16]. Let $f : E_1 \rightarrow E_2$ be a mapping from a normed vector space E_1 into Banach space E_2 subject to the inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon (\|x\|^p \|y\|^p) \quad (1.6)$$

for all $x, y \in E_1$, where ε and p are constants with $\varepsilon > 0$ and $0 \leq p \leq \frac{1}{2}$. Then there exists a limit

$$L(x) = \lim_{n \rightarrow \infty} \frac{f(2^n x)}{2^n} \quad (1.7)$$

for all $x \in E_1$ and $L : E_1 \rightarrow E_2$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \leq \frac{\varepsilon}{2 - 2^{2p}} \|x\|^{2p} \quad (1.8)$$

for all $x \in E_1$. If $p > \frac{1}{2}$ the inequality (1.6) holds for $x, y \in E_1$ and the limit

$$A(x) = \lim_{n \rightarrow \infty} 2^n f\left(\frac{x}{2^n}\right) \quad (1.9)$$

exist for all $x \in E_1$ and $A : E_1 \rightarrow E_2$ is a unique additive mapping which satisfies

$$\|f(x) - A(x)\| \leq \frac{\varepsilon}{2^{2p} - 2} \|x\|^{2p} \quad (1.10)$$

for all $x \in E_1$.

K.W.Jun and H.M.Kim[24] introduced the following cubic functional equation

$$f(2x+y)+f(2x-y)=2[f(x+y)+f(x-y)]+12f(x) \quad (1.11)$$

$x, y \in X$, where f is a mapping from a real vector space X into a real vector space Y . They established the general solution and the generalized Hyers-Ulam-Rassias stability for the functional equation (1.11). The function $f(x) = x^3$ satisfies the functional equation (1.11) which is thus called a cubic functional equation. Every solution of a cubic functional equation is said to be a cubic function. Jun and Kim proved that the mapping f from real vector space X and Y is solution of (1.11) if and only if there exists a unique function $C: X \times X \times X \rightarrow Y$ such that $f(x) = C(x, x, x)$ for all $x \in X$ and C is symmetric for fixed one variable and additive for fixed two variables.

W.G.Park and J.H.Bae[13] considered the following functional equation

$$f(x+2y)+f(x-2y)=4[f(x+y)+f(x-y)]+24f(y)-6f(x). \quad (1.12)$$

They proved that the function f between real vector space X and Y is a solution of (1.12) if and only if there exists a unique symmetric multi-additive function $B: X \times X \times X \times X \rightarrow Y$ such that $f(x) = B(x, x, x, x)$ for all $x \in X$ [3,4,20,21]. It is easy to show that the function $f(x) = x^4$ satisfies the functional equation (1.12), which is called a quartic functional equation and every solution of the quartic functional equation is said to be a quartic function.

In this paper, we deal with the new generalized functional equation deriving from cubic and quartic functions

$$f(x+ny)+f(x-ny)=n^2[f(x+y)+f(x-y)]-2(n^2-1)f(x)+\frac{2}{n}(n+1)[f(ny)-n^3f(y)] \quad (1.13)$$

for any real number n with $n \neq 0$ in Banach Spaces. This equation is called mixed type cubic and quartic functional equation because the function $f(x) = ax^3 + bx^4$ where a and b are arbitrary constants, becomes the solution of the equation (1.13). The general solution of the functional equation (1.13) is discussed in Section 2 and the generalized Hyers-Ulam-Rassias stability of the functional equation (1.13) is investigated in Section 3.

It may be observed that the above cubic and quartic functional equation (1.13) is not been dealt so far by any of the authors. So it is of great interest that the authors of this paper is much interested in finding the general solution and generalized Hyers-Ulam-Rassias stability of the generalized cubic and quartic functional equation (1.13) in Banach spaces.

In this paper, we consider the most general case of the functional equation (1.13) for any real number n with $n \neq 0$ and proceed to find its general solution.

2. The General Solution of the Functional Equation (1.13)

In this section, we establish the general solution of functional equation (1.13).

THEOREM 2.1. Let X, Y be vector spaces and let $f : X \rightarrow Y$ be function which satisfies

$$f(x+ny) + f(x-ny) = n^2[f(x+y) + f(x-y)] - 2(n^2-1)f(x) + \frac{2}{n}(n+1)[f(ny) - n^3f(y)]$$

for any real number n with $n \neq 0$ in Banach spaces then the following assertions hold

- a) If f is odd function then the function f is cubic functional equation.
- b) If f is even function then the function f is quartic functional equation.

Proof. a) Putting $x = y = 0$ in (1.13), we get $f(0) = 0$. Setting $x = 0$ in (1.13) and by oddness of f we obtain

$$f(ny) = n^3 f(y) \quad (2.1)$$

for all $y \in Y$. Replacing n by $\frac{1}{n}$ in (2.1), we get

$$f\left(\frac{1}{n}y\right) = \frac{1}{n^3} f(y) \quad (2.2)$$

for all $y \in Y$. Substituting (2.1) in (1.13), we get

$$f(x+ny) + f(x-ny) = n^2[f(x+y) + f(x-y)] - 2(n^2-1)f(x) \quad (2.3)$$

for all $x, y \in X$. Replacing x by nx in (2.3), we get

$$f(nx + ny) + f(nx - ny) = n^2 [f(nx + y) + f(nx - y)] - 2(n^2 - 1)f(nx) \quad (2.4)$$

for all $x, y \in X$. Again replacing y by x in (2.1), we get

$$f(nx) = n^3 f(x) \quad (2.5)$$

for all $x \in X$. Applying equation in (2.5) in (2.4), we get

$$f(nx + y) + f(nx - y) = n [f(x + y) + f(x - y)] + 2n(n^2 - 1)f(x) \quad (2.6)$$

for all $x, y \in X$. Replacing x by $(x + y)$ in (2.3), we get

$$f(x + (n+1)y) + f(x - (n-1)y) = n^2 [f(x + 2y) + f(x)] - 2(n^2 - 1)f(x + y) \quad (2.7)$$

for all $x, y \in X$. Replacing x by $(x - y)$ in (2.3)

$$f(x - (n+1)y) + f(x + (n-1)y) = n^2 [f(x - 2y) + f(x)] - 2(n^2 - 1)f(x - y) \quad (2.8)$$

for all $x, y \in X$. Adding (2.7) and (2.8), we get

$$\left[\begin{array}{l} f(x + (n+1)y) + f(x - (n-1)y) + \\ f(x + (n-1)y) + f(x - (n-1)y) \end{array} \right] = \left\{ \begin{array}{l} n^2 [f(x + 2y) + f(x - 2y)] + 2n^2 f(x) \\ -2(n^2 - 1)[f(x + y) + f(x - y)] \end{array} \right\} \quad (2.9)$$

for all $x, y \in X$. Further replacing y by $(x + y)$ in (2.3), we obtain

$$f((n+1)x + ny) + f((1-n)x - ny) = n^2 [f(2x + y) - f(y)] - 2(n^2 - 1)f(x) \quad (2.10)$$

for all $x, y \in X$ and replacing y by $(-x + y)$ in (2.3), we get

$$f((1-n)x + ny) + f((1+n)x - ny) = n^2 [f(y) + f(2x - y)] - 2(n^2 - 1)f(x) \quad (2.11)$$

for all $x, y \in X$. Adding (2.10) and (2.11), we get

$$\left[\begin{array}{l} f((n+1)x+ny) + f((n+1)x-ny) \\ + f((1-n)x+ny) + f((1-n)x-ny) \end{array} \right] = n^2 [f(2x+y) + f(2x-y)] - 4(n^2-1)f(x) \quad (2.12)$$

for all $x, y \in X$. Interchanging x by y in (2.12), we arrive

$$\left[\begin{array}{l} f(nx+(n+1)y) + f(-nx+(n+1)y) \\ + f(nx+(1-n)y) + f(-nx+(1-n)y) \end{array} \right] = n^2 [f(x+2y) - f(x-2y)] - 4(n^2-1)f(y) \quad (2.13)$$

for all $x, y \in X$. Simplifying (2.13) and using oddness, we get

$$\left[\begin{array}{l} f(nx+(n+1)y) - f(nx-(n+1)y) \\ + f(nx-(n-1)y) - f(nx+(n-1)y) \end{array} \right] = n^2 [f(x+2y) - f(x-2y)] - 4(n^2-1)f(y) \quad (2.14)$$

for all $x, y \in X$. Subtracting (2.7) from (2.8), we get

$$\left[\begin{array}{l} f(x+(n+1)y) - f(x-(n+1)y) \\ + f(x-(n-1)y) - f(x+(n-1)y) \end{array} \right] = n^2 [f(x+2y) - f(x-2y)] - 2(n^2-1)[f(x+y) - f(x-y)] \quad (2.15)$$

for all $x, y \in X$. Replacing x by nx in (2.15), we get

$$\left[\begin{array}{l} f(nx+(n+1)y) - f(nx-(n+1)y) \\ + f(nx-(n-1)y) - f(nx+(n-1)y) \end{array} \right] = \left[\begin{array}{l} n^2 [f(nx+2y) - f(nx-2y)] \\ - 2(n^2-1)[f(nx+y) - f(nx-y)] \end{array} \right] \quad (2.16)$$

for all $x, y \in X$. By comparing (2.14) and (2.16), we get

$$n^2 [f(x+2y) - f(x-2y)] - 4(n^2-1)f(y) = \left[\begin{array}{l} n^2 [f(nx+2y) - f(nx-2y)] \\ - 2(n^2-1)[f(nx+y) - f(nx-y)] \end{array} \right] \quad (2.17)$$

for all $x, y \in X$. Now, by interchanging x and y in (2.17), we get

$$n^2 [f(2x+y) + f(2x-y)] - 4(n^2-1)f(x) = \left[\begin{array}{l} n^2 [f(2x+ny) + f(2x-ny)] \\ - 2(n^2-1)[f(x+ny) + f(x-ny)] \end{array} \right] \quad (2.18)$$

for all $x, y \in X$. Substituting equation (2.3) in (2.18), we get

$$f(2x+y)+f(2x-y)=\begin{bmatrix} [f(2x+ny)+f(2x-ny)]-2(n^2-1)[f(x+y)+f(x-y)] \\ +4(n^2-1)f(x) \end{bmatrix} \quad (2.19)$$

for all $x, y \in X$. Simplifying (2.19), we get

$$f(2x+ny)+f(2x-ny)=\begin{bmatrix} [f(2x+y)+f(2x-y)]+2(n^2-1)[f(x+y)+f(x-y)] \\ -4(n^2-1)f(x) \end{bmatrix} \quad (2.20)$$

for all $x, y \in X$. Replacing x by $2x$ in (2.3), we get

$$f(2x+ny)+f(2x-ny)=n^2[f(2x+y)+f(2x-y)]-16(n^2-1)f(x) \quad (2.21)$$

for all $x, y \in X$. Equating (2.20) and (2.21), we get

$$n^2[f(x+y)+f(x-y)]-2(n^2-1)f(x)=\begin{bmatrix} [f(2x+y)+f(2x-y)] \\ +2(n^2-1)[f(x+y)+f(x-y)] \\ -4(n^2-1)f(x) \end{bmatrix} \quad (2.22)$$

for all $x, y \in X$. Simplifying (2.22) we arrive a cubic functional equation

$$f(2x+y)+f(2x-y)=2[f(x+y)+f(x-y)]+12f(x),$$

which proves our first part of the theorem.

(b) putting $x = y = 0$ in (1.13), we get $f(0) = 0$, setting in $x = 0$ in (1.13) and using evenness of f , we get

$$f(ny) = n^4 f(y) \quad (2.23)$$

for all $y \in X$. Substitute (2.23) in (1.13), we get

$$f(x+ny)+f(x-ny)=n^2[f(x+y)+f(x-y)]-2(n^2-1)f(x)+2n^2(n^2-1)f(y) \quad (2.24)$$

for all $x, y \in X$. Replacing x by $2x$ in (2.24), we get

$$f(2x+ny)+f(2x-ny)=n^2[f(2x+y)+f(2x-y)]-32(n^2-1)f(x)+2n^2(n^2-1)f(y) \quad (2.25)$$

for all $x, y \in X$. Replacing x by $x + y$ in (2.24), we get

$$f(x + (n+1)y) + f(x + (1-n)y) = \begin{bmatrix} n^2[f(x+2y) + f(x)] - 2(n^2-1)f(x+y) \\ + 2n^2(n^2-1)f(y) \end{bmatrix} \quad (2.26)$$

for all $x, y \in X$. Replacing x by $x - y$ in (2.24), we get

$$f(x - (1+n)y) + f(x - (1-n)y) = \begin{bmatrix} n^2[f(x-2y) + f(x)] - 2(n^2-1)f(x-y) \\ + 2n^2(n^2-1)f(y) \end{bmatrix} \quad (2.27)$$

for all $x, y \in X$. Adding (2.26) and (2.27), we get

$$\begin{bmatrix} f(x + (n+1)y) + f(x + (1-n)y) \\ + f(x - (1+n)y) + f(x - (1-n)y) \end{bmatrix} = \begin{bmatrix} n^2[f(x+2y) + f(x-2y)] + 2n^2f(x) \\ - 2(n^2-1)[f(x+y) + f(x-y)] + 4n^2(n^2-1)f(y) \end{bmatrix} \quad (2.28)$$

for all $x, y \in X$. Replacing x by nx in (2.28), we get

$$\begin{bmatrix} f(nx + (n+1)y) + f(nx + (1-n)y) \\ + f(nx - (1+n)y) + f(nx - (1-n)y) \end{bmatrix} = \begin{bmatrix} n^2[f(nx+2y) + f(nx-2y)] + 2n^6f(x) \\ - 2(n^2-1)[f(nx+y) + f(nx-y)] \\ + 4n^2(n^2-1)f(y) \end{bmatrix} \quad (2.29)$$

for all $x, y \in X$. Replacing y by $x + y$ in (2.24)

$$f(x + n(x+y)) + f(x - n(x+y)) = \begin{bmatrix} n^2[f(2x+y) + f(y)] - 2(n^2-1)f(x) \\ + 2n^2(n^2-1)f(x+y) \end{bmatrix} \quad (2.30)$$

for all $x, y \in X$. Replacing y by $x - y$ in (2.24)

$$f((1+n)x - ny) + f((1-n)x + ny) = \begin{bmatrix} n^2[f(2x-y) + f(y)] - 2(n^2-1)f(x) \\ + 2n^2(n^2-1)f(x-y) \end{bmatrix} \quad (2.31)$$

for all $x, y \in X$. Adding (2.30) and (2.31), we get

$$\begin{bmatrix} f((1+n)x+ny) + f((1-n)x-ny) \\ + f((1+n)x-ny) + f((1-n)x+ny) \end{bmatrix} = \begin{bmatrix} n^2[f(2x+y) + f(2x-y)] + 2n^2 f(y) \\ -4(n^2-1)f(x) \\ + 2n^2(n^2-1)[f(x+y) + f(x-y)] \end{bmatrix} \quad (2.32)$$

for all $x, y \in X$. Interchanging x by y in (2.32)

$$\begin{bmatrix} f(nx+(1+n)y) + f(nx-(1-n)y) \\ + f(nx-(1+n)y) + f(nx+(1-n)y) \end{bmatrix} = \begin{bmatrix} n^2[f(x+2y) + f(x-2y)] + 2n^2 f(x) - 4(n^2-1)f(y) \\ + 2n^2(n^2-1)[f(x+y) + f(x-y)] \end{bmatrix} \quad (2.33)$$

for all $x, y \in X$. Equating (2.29) and (2.33)

$$\begin{bmatrix} n^2[f(nx+2y) + f(nx-2y)] + 2n^2 f(x) \\ - 2(n^2-1)[f(nx+y) + f(nx-y)] + 4n^2(n^2-1)f(y) \end{bmatrix} = \begin{bmatrix} n^2[f(x+2y) + f(x-2y)] + 2n^2 f(x) \\ - 4(n^2-1)f(y) + 2n^2(n^2-1)[f(x+y) + f(x-y)] \end{bmatrix} \quad (2.34)$$

for all $x, y \in X$. On simplifying (2.34), we get

$$\begin{bmatrix} f(nx+2y) + f(nx-2y) \\ - 4(n^2-1)[f(x+y) + f(x-y)] \\ - [f(x+2y) + f(x-2y)] \end{bmatrix} = \begin{bmatrix} 2(1-n^4) + 4(n^2-1)^2 \\ - 8(n^2-1) \end{bmatrix} f(x) - 8(n^2-1)f(y) \quad (2.35)$$

for all $x, y \in X$. Interchanging x and y in (2.25), we get

$$f(nx+2y) + f(nx-2y) = \begin{bmatrix} n^2[f(x+2y) + f(x-2y)] \\ - 32(n^2-1)f(y) + 2n^2(n^2-1)f(x) \end{bmatrix} \quad (2.36)$$

for all $x, y \in X$. Substituting (2.36) in (2.35), we get

$$\begin{bmatrix} n^2[f(x+2y) + f(x-2y)] - 32(n^2-1)f(y) \\ + 2n^2(n^2-1)f(x) - 4(n^2-1)[f(x+y) + f(x-y)] \\ - [f(x+2y) + f(x-2y)] \end{bmatrix} = \begin{bmatrix} 2(1-n^4) + 4(n^2-1)^2 \\ - 8(n^2-1) \end{bmatrix} f(x) - 8(n^2-1)f(y) \quad (2.37)$$

for all $x, y \in X$. Simplifying (2.37), we get

$$\begin{bmatrix} (n^2 - 1)[f(x + 2y) + f(x - 2y)] \\ -4(n^2 - 1)[f(x + y) + f(x - y)] \end{bmatrix} = -6(n^2 - 1)f(x) + 24(n^2 - 1)f(y) \quad (2.38)$$

for all $x, y \in X$. Dividing equation (2.38) by $(n^2 - 1)$, we get

$$f(x + 2y) + f(x - 2y) = 4[f(x + y) + f(x - y)] + 24f(y) - 6f(x) \quad (2.39)$$

which is quartic functional equation, this proves the theorem.

THEOREM 2.2. A mapping $f : X \rightarrow Y$ satisfies the functional equation (1.13) for all $x, y \in X$ if there exist a Quartic mapping $Q : X \rightarrow Y$ and a cubic mapping $C : X \rightarrow Y$ such that $f(x) = Q(x) + C(x)$ for all $x \in X$.

Proof. Define the mappings $Q : X \rightarrow Y$ and $C : X \rightarrow Y$ by

$$Q(x) = \frac{1}{2}[f(x) + f(-x)] \quad (2.40)$$

$$\text{and } C(x) = \frac{1}{2}[f(x) - f(-x)] \quad (2.41)$$

for all $x \in X$. Then replacing x by $-x$ in (2.40) and (2.41), we get

$$Q(-x) = Q(x) \quad \text{and} \quad C(-x) = -C(x) \quad (2.42)$$

for all $x \in X$. Applying $Q(x)$ and $C(x)$ in (1.13), we arrive the following equations

$$Q(x + ny) + Q(x - ny) = n^2[Q(x + y) + Q(x - y)] - 2(n^2 - 1)Q(x) + \frac{2}{n}(n + 1)[Q(ny) - n^3Q(y)] \quad (2.43)$$

and

$$C(x + ny) + C(x - ny) = n^2[C(x + y) + C(x - y)] - 2(n^2 - 1)C(x) + \frac{2}{n}(n + 1)[C(ny) - n^3C(y)] \quad (2.44)$$

for all $x, y \in X$.

First we claim that Q is quadratic by substituting $x = y = 0$ in (2.40), we see that $Q(0) = 0$ and setting $(x, y) = (0, y)$ in (2.40), we obtain

$$Q(ny) = n^4 Q(y)$$

for all $y \in X$. Equation (2.40) is reduced to the form

$$Q(x+ny) + Q(x-ny) = n^2 [Q(x+y) + Q(x-y)] - 2(n^2 - 1)Q(x) + 2n^2(n^2 - 1)Q(y)$$

for all $x, y \in X$ and by Theorem 2.1 Q is quartic.

Secondly, we claim C is cubic. Substituting $x = y = 0$ in (2.40) we see that $C(0) = 0$ and by setting $(x, y) = (0, y)$ in (2.41), we obtain

$$C(ny) = n^3 C(y)$$

for all $y \in X$. Equation (2.41) takes the form

$$C(x+ny) + C(x-ny) = n^2 [C(x+y) + C(x-y)] - 2(n^2 - 1)C(x) \quad (2.45)$$

for all $x, y \in X$ and by Theorem 2.1, C is cubic.

If $f : X \rightarrow Y$ satisfies (1.13), then from (2.40) and (2.41) we have

$$f(x) = Q(x) + C(x)$$

for all $x \in X$.

3. Generalized Hyers-Ulam-Rassias Stability of (1.13)

Throughout this Section, X and Y be a real normed space and real Banach Space respectively. Let $f : X \rightarrow Y$ be a function then we define $D_f : X \times X \rightarrow Y$ by

$$D_f(x, y) = f(x+ny) + f(x-ny) - n^2 [f(x+y) + f(x-y)] + 2(n^2 - 1)f(x) - \frac{2}{n}(n+1) [f(ny) - n^3 f(y)] \quad (3.1)$$

for all $x, y \in X$ with $n \neq 0$.

THEOREM 3.1: Let $\varphi: X \times X \rightarrow [0, \infty)$ be a function satisfies $\sum_{i=0}^{\infty} \frac{\varphi(0, n^i x)}{n^{4i}} < \infty$ for all $x \in X$ and

$$\lim_{k \rightarrow \infty} \frac{\varphi(n^k x, n^k y)}{n^{4k}} = 0 \quad (3.2)$$

for all $x, y \in X$ with $n \neq 0$. If $f: X \rightarrow Y$ is a even function such that $f(0) = 0$, and

$$\|D_f(x, y)\| \leq \varphi(x, y) \quad (3.3)$$

for all $x, y \in X$, then there exists a unique Quartic function $Q: X \rightarrow Y$ satisfying equation (1.13) and

$$\|f(x) - Q(x)\| \leq \frac{1}{2n^3} \sum_{i=0}^{\infty} \frac{\varphi(0, n^i x)}{n^{4i}} \quad (3.4)$$

for all $x \in X$, where the function Q is given by

$$Q(x) = \lim_{k \rightarrow \infty} \frac{f(n^k x)}{n^{4k}} \quad (3.5)$$

for all $x \in X$.

PROOF: Setting $x = 0$ in (3.3) and dividing by $2n^3$, we get

$$\left\| \frac{f(ny)}{n^4} - f(y) \right\| \leq \frac{1}{2n^3} \varphi(0, y) \quad (3.6)$$

for all $x \in X$. Replacing y by x in the equation (3.6), we get

$$\left\| \frac{f(nx)}{n^4} - f(x) \right\| \leq \frac{1}{2n^3} \varphi(0, x) \quad (3.7)$$

for all $x \in X$. Replacing x by nx and dividing by n^4 in (3.7), we obtain

$$\left\| \frac{f(n^2x)}{(n^4)^2} - \frac{f(nx)}{n^4} \right\| \leq \frac{1}{2n^3} \frac{1}{n^4} \varphi(0, nx) \quad (3.8)$$

for all $x \in X$. Adding (3.7) and (3.8) we get

$$\begin{aligned} \left\| \frac{f(n^2x)}{(n^4)^2} - f(x) \right\| &\leq \frac{1}{2n^3} \left[\frac{1}{n^4} \varphi(0, nx) + \varphi(0, x) \right] \\ &\leq \frac{1}{2n^3} \sum_{i=0}^1 \frac{\varphi(0, n^i x)}{(n^4)^i} \end{aligned} \quad (3.9)$$

for all $x \in X$. Generalizing the above equation (3.9), we get

$$\left\| \frac{f(n^k x)}{(n^k)^4} - f(x) \right\| \leq \frac{1}{2n^3} \sum_{i=0}^{k-1} \frac{\varphi(0, n^i x)}{(n^{4i})} \quad (3.10)$$

for all $x \in X$.

$$\left\| \frac{f(n^k x)}{(n^k)^4} - f(x) \right\| \leq \frac{1}{2n^3} \sum_{i=0}^{\infty} \frac{\varphi(0, n^i x)}{(n^{4i})} \quad (3.11)$$

for all $x \in X$.

Now we prove that $\left\{ \frac{f(n^k x)}{n^{4k}} \right\}$ is a Cauchy Sequence for all $x \in X$.

For every positive integer k, s and for all $x \in X$, we have

$$\left\| \frac{f(n^k n^s x)}{n^{4(k+s)}} - \frac{f(n^k x)}{n^{4k}} \right\| = \frac{1}{n^{4k}} \left\| \frac{f(n^k n^s x)}{n^{4s}} - f(n^k x) \right\|$$

$$\leq \frac{1}{n^{4k}} \frac{1}{2n^3} \sum_{i=0}^{\infty} \frac{\varphi(0, n^{i+k}x)}{(n^{4i})}$$

$$\left\| \frac{f(n^k n^s x)}{n^{4(k+s)}} - \frac{f(n^k x)}{n^{4k}} \right\| \leq \frac{1}{2n^3} \sum_{i=0}^{\infty} \frac{\varphi(0, n^{i+k}x)}{(n^4)^{i+k}} \quad (3.12)$$

Taking the limit as $k \rightarrow \infty$ and using (3.2) the above equation (3.12) takes the form

$$\lim_{k \rightarrow \infty} \left\| \frac{f(n^{k+s}x)}{n^{4(k+s)}} - \frac{f(n^k x)}{n^{4k}} \right\| = 0 \quad (3.13)$$

for all $x \in X$. Thus the sequence $\left\{ \frac{f(n^k x)}{n^{4k}} \right\}$ is Cauchy sequence in Y . Since Y is a

Banach Space, then the sequence $\left\{ \frac{f(n^k x)}{n^{4k}} \right\}$ converges. Now we define $Q : X \rightarrow Y$

by $Q(x) = \lim_{k \rightarrow \infty} \frac{f(n^k x)}{n^{4k}}$ for all $x \in X$. Since f is even function, then Q is also even function. On the other hand we have

$$\|D_Q(x, y)\| = \lim_{k \rightarrow \infty} \frac{1}{n^{4k}} \|D_f(n^k x, n^k y)\|$$

$$\leq \lim_{k \rightarrow \infty} \frac{\varphi(n^k x, n^k y)}{n^{4k}} = 0$$

for all $x, y \in X$. Hence by theorem 2.1, Q is quartic function.

To do the completeness, take $(x, y) = (n^k x, n^k y)$ in (3.3) and divide the result by n^{4k} and taking the limit as $k \rightarrow \infty$ on both sides we get

$$\lim_{k \rightarrow \infty} \frac{1}{n^{4k}} \left\| \begin{aligned} &f(n^k(x+ny)) + f(n^k(x-ny)) - n^2[f(n^k(x+y)) + f(n^k(x-y))] \\ &+ 2(n^2-1)f(n^k(x)) + \frac{2}{n}(n+1)[f(n^k y) - n^3 f(n^k y)] \end{aligned} \right\| \leq \lim_{k \rightarrow \infty} \frac{\varphi(n^k x, n^k y)}{n^{4k}}$$

for all $x, y \in X$. Using the equations (3.2) and (3.5), we arrive

$$Q(x+ny)+Q(x-ny)=n^2[Q(x+y)+Q(x-y)]-2(n^2-1)Q(x)+\frac{2}{n}(n+1)[Q(ny)-n^3Q(y)]$$

Therefore function Q satisfies equation (1.13).

To show that Q is unique, if there exists another quartic function $Q' : X \rightarrow Y$ which satisfies equation (1.13) and (3.4). We have $Q(n^k x) = n^{4k}Q(x)$, $Q'(n^k x) = n^{4k}Q'(x)$ for all $x \in X$. It follows that

$$\begin{aligned}\|Q'(x)-Q(x)\| &= \frac{1}{n^{4k}}\|Q'(n^k x)-Q(n^k x)\| \\ &\leq \frac{1}{n^{4k}}\left[\|Q'(n^k x)-f(n^k x)\|+\|f(n^k x)-Q(n^k x)\|\right] \\ &\leq \frac{1}{n^3}\sum_{i=0}^{\infty}\frac{\varphi(0,n^{i+k}x)}{(n^4)^{i+k}}\end{aligned}$$

for all $x \in X$. By taking the limit $k \rightarrow \infty$ on both sides of the above equation and by equation (3.2), we get

$$Q'(x) = Q(x)$$

that is Q is unique.

THEOREM 3.2: Let $\varphi : X \times X \rightarrow [0, \infty)$ be a function satisfies

$$\sum_{i=0}^{\infty} n^{4i} \varphi(0, n^{-i-1}x) < \infty$$

for all $x \in X$, and

$$\lim_{k \rightarrow \infty} n^{4k} \varphi(n^{-k}x, n^{-k}y) = 0$$

for all $x, y \in X$ with $n \neq 0$. Suppose that there is an even function $f : X \rightarrow Y$ satisfies $f(0) = 0$ and (3.1), then the limit

$$Q(x) = \lim_{k \rightarrow \infty} n^{4k} f(n^{-k}x) \text{ exists}$$

for all $x \in X$ and $Q : X \rightarrow Y$ is unique quartic function satisfies (1.13) and

$$\|f(x) - Q(x)\| \leq \frac{n}{2} \sum_{i=0}^{\infty} n^{4i} \varphi(0, n^{-i-1}x) \quad (3.14)$$

for all $x, y \in X$.

PROOF: By Putting $x = 0$ in (3.3), we get

$$\left\| \frac{2}{n} f(ny) - 2n^3 f(y) \right\| \leq \varphi(0, y) \quad (3.15)$$

for all $y \in X$. Replacing y by $n^{-1}x$ in (3.15), and dividing by $\frac{2}{n}$, we get

$$\|n^4 f(n^{-1}x) - f(x)\| \leq \frac{n}{2} \varphi(0, n^{-1}x) \quad (3.16)$$

for all $x \in X$. Replacing x by $n^{-1}x$ in (3.16), we get

$$\|n^4 f(n^{-2}x) - f(n^{-1}x)\| \leq \frac{n}{2} \varphi(0, n^{-2}x) \quad (3.17)$$

for all $x \in X$. Multiplying the above equation by n^4 , we get

$$\|(n^4)^2 f(n^{-2}x) - n^4 f(n^{-1}x)\| \leq \frac{n}{2} n^4 \varphi(0, n^{-2}x) \quad (3.18)$$

for all $x \in X$. Adding the equations (3.16) and (3.18), we get

$$\|(n^4)^2 f(n^{-2}x) - f(x)\| \leq \frac{n}{2} [n^4 \varphi(0, n^{-2}x) + \varphi(0, n^{-1}x)] \quad (3.19)$$

for all $x \in X$.

By induction, we get

$$\|(n^4)^k f(n^{-k}x) - f(x)\| \leq \frac{n}{2} \sum_{i=0}^{k-1} n^{4i} \varphi(0, n^{-i-1}x)$$

$$\leq \frac{n}{2} \sum_{i=0}^{\infty} n^{4i} \varphi(0, n^{-i-1}x) \quad (3.20)$$

for all $x \in X$.

To show that the sequence $\{n^{4k} f(n^{-k}x)\}$ is a cauchy sequence in Y . For every positive integer k, s and for all $x \in X$, we have

$$\begin{aligned} \left\| n^{4(k+s)} f(n^{-(k+s)}x) - n^{4k} f(n^{-k}x) \right\| &= n^{4k} \left\| n^{4s} f(n^{-(k+s)}x) - f(n^{-k}x) \right\| \\ &\leq \frac{n}{2} \sum_{i=0}^{\infty} n^{4(i+k)} \varphi(0, n^{-(i+k)-1}x) \end{aligned}$$

for all $x \in X$. By taking the limit $k \rightarrow \infty$, it follows that $\{n^{4k} f(n^{-k}x)\}$ is a cauchy sequence in Y and sequence $\{n^{4k} f(n^{-k}x)\}$ converges since Y is a Banach space.

Now we define $Q: X \rightarrow Y$ by

$$Q(x) = \lim_{s \rightarrow \infty} n^{4s} f(n^{-s}x)$$

for all $x \in X$. The rest of the proof is similar to the proof of Theorem 3.1.

THEOREM 3.3: Let $\varphi: X \times X \rightarrow [0, \infty)$ be a function satisfies $\sum_{i=0}^{\infty} \frac{\varphi(0, n^i x)}{n^{3i}} < \infty$ for all $x \in X$ and

$$\lim_{k \rightarrow \infty} \frac{\varphi(n^k x, n^k y)}{n^{3i}} = 0 \quad (3.21)$$

for all $x, y \in X$ with $n \neq 0$. If $f: X \rightarrow Y$ is a odd function such that $f(0) = 0$, and

$$\|D_f(x, y)\| \leq \varphi(x, y) \quad (3.22)$$

for all $x, y \in X$, then there exists a unique cubic function $C: X \rightarrow Y$ satisfying equation (1.13) and

$$\|f(x) - C(x)\| \leq \frac{1}{2n^2(n+1)} \sum_{i=0}^{\infty} \frac{\varphi(0, n^i x)}{n^{3i}} \quad (3.23)$$

for all $x \in X$, where the function C is given by

$$C(x) = \lim_{k \rightarrow \infty} \frac{f(n^k x)}{n^{3k}} \quad (3.24)$$

for all $x \in X$.

PROOF: The Proof is as similar as to that of Theorem 3.1.

THEOREM 3.4 : Let $\varphi : X \times X \rightarrow Y$ be a function such that

$$\sum_{i=0}^{\infty} \frac{\varphi(0, n^i x)}{n^{3i}} \leq \infty \text{ and } \lim_n \frac{\varphi(n^k x, n^k x)}{n^{3k}} = 0$$

for all $x \in X$. Suppose that a function $f : X \rightarrow Y$ satisfies the inequality

$$\|D_f(x, y)\| \leq \varphi(x, y) \quad (3.25)$$

for all $x, y \in X$, and $f(0) = 0$. Then there exists a unique cubic function $C : X \rightarrow Y$ and a unique Quartic function $Q : X \rightarrow Y$ satisfying equation (1.13) and

$$\|f(x) - Q(x) - C(x)\| \leq \left[\sum_{i=0}^{\infty} \left[\frac{1}{2n^3} \left(\frac{\varphi(0, n^i x) + \varphi(0, -n^i x)}{2n^{4i}} \right) + \frac{1}{2n^2(n+1)} \left(\frac{\varphi(0, n^i x) + \varphi(0, -n^i x)}{2n^{3i}} \right) \right] \right] \quad (3.26)$$

for all $x, y \in X$.

PROOF: We have

$$\|D_{f_e}(x, y)\| \leq \frac{1}{2} [\varphi(x, y) + \varphi(-x, -y)] \quad (3.27)$$

for all $x, y \in X$. Since $f_e(0) = 0$ and f_e is an even function, then by Theorem 3.1, there exists a unique Quartic function $Q : X \rightarrow Y$ satisfying

$$\|f(x) - Q(x)\| \leq \frac{1}{2n^3} \left[\sum_{i=0}^{\infty} \left(\frac{\varphi(0, n^i x) + \varphi(0, -n^i x)}{2 \times n^{4i}} \right) \right] \quad (3.28)$$

for all $x \in X$. On the other hand f_o is odd function and

$$\|D_{f_o}(x, y)\| \leq \frac{1}{2} [\varphi(x, y) + \varphi(-x, -y)] \quad (3.29)$$

for all $x, y \in X$. Then by Theorem 3.3, there exists a unique cubic function $C : X \rightarrow Y$ such that

$$\|f(x) - C(x)\| \leq \frac{1}{2n^2(n+1)} \left[\sum_{i=0}^{\infty} \left(\frac{\varphi(0, n^i x) + \varphi(0, -n^i x)}{2 \times n^{3i}} \right) \right] \quad (3.30)$$

for all $x \in X$. Combining the equations (3.28) and (3.30), we obtain (3.31)

$$\|f(x) - Q(x) - C(x)\| \leq \left[\sum_{i=0}^{\infty} \left[\frac{1}{2n^3} \left(\frac{\varphi(0, n^i x) + \varphi(0, -n^i x)}{2n^{4i}} \right) + \frac{1}{2n^2(n+1)} \left(\frac{\varphi(0, n^i x) + \varphi(0, -n^i x)}{2n^{3i}} \right) \right] \right] \quad (3.31)$$

This completes the proof of the theorem.

The following corollary is the immediate consequence of the above theorem which gives generalized Hyers-Ulam stability of functional equation (1.13).

COROLLARY 3.5 Let X and Y be a real normed space and Banach Space respectively, and let ε, p, q be real numbers such that $\varepsilon \geq 0, q > 0$ and either $p, q < 4$ or $p, q > 4$ suppose that a function $f : X \rightarrow Y$ satisfies

$$\|D_f(x, y)\| \leq \varepsilon (\|x\|^p + \|y\|^q) \quad (3.32)$$

for all $x, y \in X$ then there exists a unique Quartic function $Q : X \rightarrow Y$ which satisfies equation (1.13) and the inequality

$$\|f(x) - Q(x)\| \leq \frac{\varepsilon}{2|n^4 - 2^p|} \|x\|^p \quad (3.33)$$

for all $x \in X$ and for all $x \in X - \{0\}$ if $p > 0$. The function Q is given by

$$Q(x) = \lim_{k \rightarrow \infty} \frac{f(n^k x)}{n^{4k}} \quad \text{if } p, q < 4 \quad (3.34)$$

$$\left(Q(x) = \lim_{k \rightarrow \infty} n^{4k} f\left(\frac{x}{n^k}\right) \quad \text{if } p, q > 4 \right)$$

for all $x \in X$. Also, if for each fixed $x \in X$, the mapping $t \rightarrow f(tx)$ for R to Y is continuous, then $Q(rx) = r^4 Q(x)$ for all $r \in R$.

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