Journal of Combinatorial Theory, Series A **98**, 60–73 (2002)
doi:10.1006/jcta.2001.3226, available online at http://www.idealibrary.com on IDEAL®

A Gleason Formula for Ozeki Polynomials

Y. Choie[1]

Dept. of Math., POSTECH, Pohang, Korea 790-784
E-mail: yjc@postech.ac.kr

and

P. Solé

CNRS-I3S, ESSI, 06 903 Sophia Antipolis, France
E-mail: ps@essi.fr

Received October 16, 2000; published online February 11, 2002

By using the structure theory of Jacobi forms we derive a simple expression for Ozeki polynomials of Type II self-dual binary codes.

1. INTRODUCTION

Ozeki polynomials of Type II self-dual codes were introduced and studied in [4, 9] under the name of Jacobi polynomials. They generalize split weight enumerators of [8] which correspond to the case of a translation vector of weight half the length. In view of the clash of notations with special function theory and of the contribution of Professor Ozeki in that area, we propose, following a suggestion of some Japanese colleagues to call henceforth Jacobi polynomials Ozeki polynomials. As the bivariate Molien series of [4] suggest the direct application of invariant theory to these polynomials does not yield a very simple expression for an arbitrary Ozeki polynomial of an arbitrary Type II code. In this work we derive a comparatively simpler expression for such an object as a polynomial into two simultaneous invariants (homogeneous polynomials in w, z, x, y) of bidegree (7, 1) and (11, 1) *with coefficients in* $\mathbf{Q}(\mathbf{x}, \mathbf{y})$. To this aim we use the

[1] This research has been partially supported by MSRC.

0097-3165/02 $35.00

Bannai–Ozeki map from Jacobi polynomials [2] into Jacobi forms combined with some classical results of Eichler–Zagier [6] on Jacobi forms.

2. JACOBI FORMS

Let M_k denote the complex vector space of modular forms of weight k. Recall its dimension [1, p. 119]:

$$\left[\frac{k}{12}\right] \qquad \text{if} \quad k \equiv 2 \pmod{12}$$

$$\left[\frac{k}{12}\right]+1 \qquad \text{otherwise.}$$

A standard result for classical modular forms is that the ring M_* of modular forms of even weight for the full modular group is a free ring in two generators E_4, E_6 the Eisenstein series of weight 4 and 6. The analogous result for Jacobi forms is Theorem 8.3 in [6].

THEOREM 1 (Eichler–Zagier). *Let $J_{2*,*}$ denote the ring of Jacobi forms of even weight and integral index, and let $E_{4,1}$ and $E_{6,1}$ denote the Jacobi Eisenstein series of weight* 4 *and* 6, *respectively, with index* 1. *Let $\Delta = (E_4^3 - E_6^2)/1728$. Then $J_{2*,*}$ is contained in $M_*[\frac{1}{\Delta}][E_{4,1}, E_{6,1}]$.*

For application to codes we shall require the following generalization.

THEOREM 2. *Let $J_{4*,*}$ denote the ring of Jacobi forms of weight a multiple of* 4 *and integral index, and let $\phi_{12,1}$ (in the notations of* [6, p. 39]) *denote the Jacobi cusp form of weight* 12 *and index* 1. *Then $J_{4*,*}$ is contained in $M_*[\frac{1}{\Delta}][E_{4,1}, \phi_{12,1}]$.*

Proof. First, we recall that Jacobi cusp forms $\phi_{10,1}$ and $\phi_{12,1}$ are algebraically independent (see the proof of Theorem 8.1 in [6]). We claim that $E_{4,1}(\tau, z)$ and $\phi_{12,1}(\tau, z)$ are also algebraically independent over M_*. But, from the well-known relation [6]

$$\phi_{10,1}(\tau, z) = \tfrac{1}{144}\,(E_6(\tau)\, E_{4,1}(\tau, z) - E_4(\tau)\, E_{6,1}(\tau, z)) \tag{1}$$

$$\phi_{12,1}(\tau, z) = \tfrac{1}{144}\,(E_4^2(\tau)\, E_{4,1}(\tau, z) - E_6(\tau)\, E_{6,1}(\tau, z)) \tag{2}$$

we have

$$E_{4,1}(\tau, z) = \frac{144}{E_4^3(\tau) - E_6^2(\tau)}\,(E_4(\tau)\,\phi_{12,1}(\tau, z) - E_6(\tau)\,\phi_{10,1}(\tau, z)). \tag{3}$$

If $E_{4,1}(\tau, z)$ and $\phi_{12,1}(\tau, z)$ are algebraically dependent over M_*, the relation (3) shows that $\phi_{10,1}(\tau, z)$ and $\phi_{12,1}$ are also algebraically dependent, which is a contradiction.

Now, since the space $J_{4*,\ell}$ is a module of rank $\ell+1$ over M_*, we can write, for any $f(\tau, z) \in J_{4*,\ell}$,

$$f(\tau, z) = \sum_{j=0}^{\ell} g_j(\tau)\, E_{4,1}^{j}(\tau, z)\, E_{6,1}^{\ell-j}(\tau, z),$$

where $g_j(\tau)$ is a meromorphic modular form. From Theorem 1 and from the relation (2), we have the coefficient $g_j(\tau)$ in $M_*[\frac{1}{\Delta}]$. This means that $J_{4*,*} \subset M_*[\frac{1}{\Delta}][E_{4,1}, \phi_{12,1}(\tau, z)]$. ▮

We shall require the theory of weak Jacobi forms to control the exponent of Δ in these theorems. Let $\tilde{J}_{2*,*}$ denote the ring of so-called weak Jacobi forms of even weight.

THEOREM 3 (Eichler–Zagier). *The ring $\tilde{J}_{2*,*}$ is a polynomial algebra over M_* on two generators*

$$\tilde{\phi}_{0,1} = \frac{\phi_{10,1}}{\Delta} \in \tilde{J}_{-2,1}$$

$$\tilde{\phi}_{-2,1} = \frac{\phi_{12,1}}{\Delta} \in \tilde{J}_{0,1}$$

where $\phi_{10,1}$, $\phi_{12,1}$ are like in (1) *and* (2).

The following results improve on Theorem 3 for low m by removing the denominator Δ from the formulas. The proofs of the four theorems are all modelled on the proof of [6, Theorem 3.5]. First a map from modular to Jacobi forms is injective by algebraic independence reasons. Next the dimension of image and the relevant space of Jacobi forms coincide by the dimension formula of [6, p. 121]. Surjectivity of the said map follows.

First we deal with forms of weight multiple of 4.

THEOREM 4. *Let $l \geqslant 3$ be an integer. Every Jacobi form of $J_{4l,1}$ can be written as*

$$f_{4l-4}E_{4,1} + f_{4l-12}\phi_{12,1}$$

where $f_i \in M_i$ if $i > 0$ and zero otherwise.

Proof. Observe that, by the argument in the proof of Theorem 2 the forms $E_{4,1}$, $\phi_{12,1}$ are algebraically independent. It is known (see [6, p. 121]) that, for $k \geqslant m$, $k \equiv 0 \pmod 2$,

$$dim(J_{k,m}) = \sum_{v=0}^{m} \left(dim(M_{k+2v}) - \left\lceil \frac{v^2}{4m} \right\rceil \right).$$

Here, $\lceil x \rceil$ is the nearest integer $\geqslant x$. A computation combining the dimension formulas for the space of modular forms M_j yields

$$dim(J_{4l,1}) = dim(M_{4l}) + dim(M_{4l+2}) - 1 = dim(M_{4l-4}) + dim(M_{4l-12}),$$

so the result follows. ∎

THEOREM 5. *Let $l \geqslant 7$ be an integer. Every Jacobi form of $J_{4l,2}$ can be written as*

$$f_{4l-8}E_{4,1}^2 + f_{4l-16}E_{4,1}\phi_{12,1} + f_{4l-24}\phi_{12,1}^2 + \lambda E_4^{l-1}E_{4,2} + \mu f_{4l-20}\phi_{10,1}^2,$$

where λ, μ are arbitrary scalars and $f_i \in M_i$ if $i > 0$ and zero otherwise.

Proof. It is known (see [6, p. 121]) that, for $k \geqslant m$, $k \equiv 0 \pmod 2$,

$$dim(J_{k,m}) = \sum_{v=0}^{m} \left(dim(M_{k+2v}) - \left\lceil \frac{v^2}{4m} \right\rceil \right).$$

So, with dimension formular of the space of modular forms M_j, it follows from that

$$dim(M_{4l-8}) + dim(M_{4l-16}) + dim(M_{4l-24}) = dim(J_{4l,2}) - 2,$$

and the algebraic independence of $E_{4,2}$ and $E_{4,1}^2$, derived from Theorem 8.2 of [6, p. 96] where $A = E_{4,1}$ and $X = E_{4,2}$, ${}_{\Delta}z = \phi_{10,1}^2$. ∎

Next we treat forms of even weight.

THEOREM 6. *Let $l \geqslant 4$ be an integer. Every Jacobi form of $J_{2l,1}$ can be written as*

$$f_{2l-4}E_{4,1} + f_{2l-6}E_{6,1}$$

where $f_i \in M_i$ if $i > 0$ and zero otherwise.

Proof. This is essentially [6, Theorem 3.5]. ∎

THEOREM 7. *Let $l \geqslant 10$ be an integer. Every Jacobi form of $J_{2l,2}$ can be written as*

$$f_{2l-8}E_{4,1}^2 + f_{2l-10}E_{4,1}E_{6,1} + f_{2l-12}E_{6,1}^2 + \lambda f_{2l-20}\phi_{10,1}^2$$

where λ arbitrary scalar, $f_i \in M_i$ if $i > 0$ and zero otherwise.

Proof. This follows (see [6, p. 121]) from

$$dim(M_{2l-8}) + dim(M_{2l-10}) + dim(M_{2l-12}) = dim(J_{2l,2}) - 1,$$

and the algebraic independence of $E_{4,1}$ and $E_{6,1}$ [6, Theorem 8.1, p. 90].) ▮

3. OZEKI POLYNOMIALS

3.1. *Weight Enumerators and Invariants*

A binary code of length n and dimension k is a $\mathbf{F}_2$-vector subspace of dimension k of $\mathbf{F}_2^n$. The dimension measures the size of a code since $|C| = 2^k$. Codes can be used to pack spheres for the Hamming metric $d_H(.,.)$, defined as

$$d_H(x, y) = \#\{i = 1, ..., n \mid x_i \neq y_i\}.$$

The Hamming weight $w(x)$ of a vector x is just its distance to zero $w(x) := d_H(x, 0)$. The generating function for the weight of codewords is the weight enumerator

$$W_C(x, y) := \sum_{c \in C} x^{n-w(c)} y^{w(c)}.$$

The dual of a code is understood w.r.t. duality of vector spaces

$$C^\perp := \{y \in \mathbf{F}_2^n \mid \forall x \in C,\ x.y = 0\}.$$

A code is *self-dual* if it is equal to its dual; it is said *doubly even* if, furthermore, the weight of everyone of its codewords is a multiple of 4. Note that a self-dual code satisfies $k = \frac{n}{2}$. The length of a self-dual code is therefore necessarily even and as invariant theory shows (cf. next theorem)

Let M, N denote the 2 by 2 matrices

$$M = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

and

$$N = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}.$$

Let G_{II} denote the matrix group generated by M and N. In general if a group G is acting on a ring R we denote by R^G the set of elements of R left invariant by G.

THEOREM 8. *The weight enumerator of a doubly even code of length n is an absolute invariant of degree n of the group* G_{II} *of order* 192.

Proof. Invariance by M follows from the MacWilliams formula and the self-duality of C. Invariance by N follows by the evenness of C. The order of the group can be checked by computer. ∎

Define the following invariants for G_{II} of degree 8 and 24, respectively:

$$\psi_8 = x^8 + 14x^4y^4 + y^8,$$

and

$$v_{24} = x^4y^4(x^4 - y^4)^4.$$

The following result incorporates the Gleason formula and the Broué–Enguehard map [3].

COROLLARY 1. (1) (Gleason) *The weight enumerator* W_C *of a doubly even code of length* $8n$ *is an isobaric polynomial in* ψ_8 *and* v_{24}. *There are scalars* a_j *such that*

$$W_C = \sum_{j=0}^{N} a_j \psi_8^{n-3j} v_{24}^j,$$

with $N := \lfloor \frac{n}{3} \rfloor$.

(2) (Broué–Enguehard) *The map* $O: f \to f(\theta_3, \theta_2)$ *is an algebra isomorphism from* $\mathbf{C}[\mathbf{x}, \mathbf{y}]^{G_{II}}$ *onto* $\mathbf{C}[E_4, \Delta]$.

Proof. This follows from the fact that $\mathbf{C}[\mathbf{x}, \mathbf{y}]^{G_{II}}$ is a free algebra on two generators ψ_8 and v_{24}. ∎

This has been extended by Ozeki [9] to the ring of even weight modular forms by considering relative invariants, i.e., homogeneous polynomials satisfying

$$\forall g \in G, g.r = \chi(g)\, r$$

for some character χ of G. In general, for given χ, relative invariants are not a ring but they are always a module on the absolute invariants. More specifically, consider the linear character χ defined on the generators as

$$\chi(M) = -1$$
$$\chi(N) = 1.$$

Let H_{II} denote the subgroup of $\langle M, N \rangle$, where χ is trivial. Its elements are called *formal* weight enumerators Then it can be shown [9] that

$$\mathbf{C}[\mathbf{x}, \mathbf{y}]^{\mathbf{H}_{\mathbf{II}}} = \mathbf{C}[\psi_8, \mathbf{k}_{12}]$$

where k_{12} is the degree 12 relative invariant of $\mathbf{C}[\mathbf{x}, \mathbf{y}]^{\mathbf{H}_{\mathbf{II}}}$ considered by Klein [7]

$$k_{12} := x^{12} - 33(x^8 y^4 + x^4 y^8) + y^{12}.$$

Ozeki's motivation was the following result.

THEOREM 9. *The map*

$$O\colon f \mapsto f(\theta_3, \theta_2)$$

is an algebra isomorphism from $\mathbf{C}[\psi_8, \mathbf{k}_{12}]$ *onto* $\mathbf{C}[E_4, E_6]$.

We give the analogue of the Gleason formula for a formal weight enumerator.

PROPOSITION 1. *A formal weight enumerator W of degree* 4n *is an isobaric polynomial in ψ_8 and k_{12}. There are scalars b_j such that*

$$W = \sum_{j=0}^{N} b_j \psi_8^{\frac{n-3j}{2}} k_{12}^j,$$

with $N := \lfloor \frac{n}{3} \rfloor$.

3.2. *Ozeki Polynomials*

Let C be a binary code of length n. The Ozeki polynomial $J_{C,v}$ attached to C and an arbitrary binary vector v of length n is essentially the joint weight enumerator [8] of C with v. If $(a_1, b_1)(u)$ (resp. $(a_2, b_2)(u)$) denote the composition (i.e., (number of zeros, number of ones)) of u on the support of v (resp. the support of $\mathbf{1}+v$) then

$$J_{C,v}(w, z, x, y) := \sum_{u \in C} w^{a_1(u)} z^{b_1(u)} x^{a_2(u)} y^{b_2(u)}.$$

(Note that the order of the variables is different from [2, 9].) We see that $a_2 + b_2(u) = w(u)$ and $a_1 + b_1(u) = n - w(u)$ showing that this polynomial is homogeneous in each pair of variables (w, z) and (x, y). The space of homogeneous polynomials of degree t in w, z and total degree n will be denoted by

$$\mathbf{C}[\mathbf{w}, \mathbf{z}, \mathbf{x}, \mathbf{y}]_{\mathbf{t}, \mathbf{n}-\mathbf{t}}.$$

The subspace of such polynomials invariant under a group G will be denoted

$$\mathbf{C}[\mathbf{w}, \mathbf{z}, \mathbf{x}, \mathbf{y}]^{\mathbf{G}}_{\mathbf{t}, \mathbf{n}-\mathbf{t}}.$$

It is easy to show, using properties of joint weight enumerators that the Ozeki polynomial satisfies a MacWilliams relation. In fact if C is a doubly even self dual code it is invariant under $G_{II} \oplus G_{II} := \langle M_2, N_2 \rangle$ where

$$M_2 = \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix},$$

and similarly

$$N_2 = \begin{pmatrix} N & 0 \\ 0 & N \end{pmatrix}.$$

The bi-Molien series for this group can be found in [4], where the main motivation is to construct designs. Thus this group is the same abstract group as G_{II} but with a different representation the direct sum of the 2-dimensional representation with itself. More generally, if G is a matrix group we denote by $G \oplus G$ the group obtained as

$$G \oplus G := \left\{ \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} g \in G \right\}$$

If G acts on the ring R then $G \oplus G$ acts on the tensor product of R with itself. Invariants for $G \oplus G$ are called, following Schur, *simultaneous invariants* for the group G. The starting point of [2, 9] is that $J_{C,v}$ is a simultaneous invariant for G_{II}. An important tool in invariant theory is *polarization* which maps invariants into simultaneous invariants for the same group.

LEMMA 1. *Let G be a finite matrix group $\leqslant GL(2, \mathbf{C})$. If $f \in \mathbf{C}[\mathbf{x}, \mathbf{y}]^{\mathbf{G}}$ then $Af \in \mathbf{C}[\mathbf{w}, \mathbf{z}, \mathbf{x}, \mathbf{y}]^{\mathbf{G} \oplus \mathbf{G}}$, where the polarization operator $A := w\partial/\partial x + z\partial/\partial y$.*

We shall give special names to the two most important polarized polynomials. Let $\psi_{8,1} = A(\psi_8)$, $\psi_{8,2} = A^2(\psi_8)$ and $k_{12,1} = A(k_{12})$, as well as $v_{24,1} = A(v_{24})$.

3.3. *Bannai–Ozeki Map*

We recall two Jacobi theta functions we shall need.

$$\theta_3(\tau, z) := \sum_{n \in \mathbf{Z}} q^{n^2} \zeta^{2n}$$

and

$$\theta_3(\tau, z) := \sum_{n \in \mathbf{Z}} q^{(n+1/2)^2} \zeta^{2n+1},$$

where $q = \exp(\pi \sqrt{-1}\, \tau)$ and $\zeta = \exp(\pi \sqrt{-1}\, z)$.

We recall without proof the main result of [2].

THEOREM 10. *Let n be a multiple of 4. If $f \in \mathbf{C}[\mathbf{w}, \mathbf{z}, \mathbf{x}, \mathbf{y}]^{\mathbf{H_{II}} \oplus \mathbf{H_{II}}}_{\mathbf{m}, \mathbf{n}-\mathbf{m}}$ then*

$$BO(f) := f(\theta_3(2\tau, 2z), \theta_2(2\tau, 2z), \theta_3(2\tau, 0), \theta_2(2\tau, 0))$$

is a Jacobi form of weight $n/2$ and index m.

We shall need a few special instances of the preceding.

LEMMA 2. $BO(v_{24,1}) = 32\phi_{12,1}$

Proof. By the properties of the BO map and inspection of $A(v_{24})$ we see that $BO(A(v_{24}))$ is a Jacobi cusp form of weight 12 and index unity. By [6, p. 40] we see that it is a multiple of $\phi_{12,1}$. The factor 32 follows on comparing Taylor expansions of both sides using [6, p. 39]. ▮

The proof of the following three lemmata is analogous and omitted.

LEMMA 3. $BO(\psi_{8,1}) = 128E_{4,1}$ *and* $O(\psi_8) = 16E_4$

LEMMA 4. $BO(\psi_{8,2}) = 2^7 7E_{4,2}$

LEMMA 5. $BO(k_{12,1}) = -768E_{6,1}$ *and* $O(k_{12}) = -64E_6$

For our coding-theoretic application we shall require the following analogue and strengthening.

THEOREM 11. *Let n be a multiple of* 8.
If $f \in \mathbf{C}[\mathbf{w}, \mathbf{z}, \mathbf{x}, \mathbf{y}]^{\mathbf{G_{II}} \oplus \mathbf{G_{II}}}_{\mathbf{m}, \mathbf{n}-\mathbf{m}}$ *then*

$$BO(f) := f(\theta_3(2\tau, 2z), \theta_2(2\tau, 2z), \theta_3(2\tau, 0), \theta_2(2\tau, 0))$$

is a Jacobi form of weight $n/2$ *and index* m. *Furthermore the map* $f \mapsto BO(f)$ *of Theorem* 10 *is injective.*

Proof. The modularity follows by Theorem 10 since the map of Theorem 11 is a restriction of the map of Theorem 10.

Injectivity follows from [6, Theorem 8.5, p. 99] by observing that, by Lemmata 3 and 5 the image by BO of $\mathbf{C}[\mathbf{w}, \mathbf{z}, \mathbf{x}, \mathbf{y}]^{\mathbf{H_{II}} \oplus \mathbf{H_{II}}}$ contains $\mathbf{C}[E_4, E_6, E_{4,1}, E_{6,1}]$. ∎

Define $\psi_{20,1} := (-k_{12}\psi_{8,1} + \frac{3}{2}\psi_8 k_{12})$.

LEMMA 6. $BO(\psi_{20,1}) = 2^{17}3^2\phi_{10,1}$

3.4. *Gleason Formulas*

We shall prove the following three generalizations of the Gleason formula from weight enumerators to Ozeki polynomials.

The following result is the analogue of Theorem 2 for invariants.

THEOREM 12. *Every element of* $\mathbf{C}[\mathbf{w}, \mathbf{z}, \mathbf{x}, \mathbf{y}]^{\mathbf{G_{II}} \oplus \mathbf{G_{II}}}_{\mathbf{m}, \mathbf{n}-\mathbf{m}}$ *is an homogeneous polynomial of degree* m *in* $\psi_{8,1}$ *and* $\nu_{24,1}$ *with coefficients in* $\mathbf{C}[\mathbf{x}, \mathbf{y}]^{\mathbf{G_{II}}}[\mathbf{1}/\nu_{24}]$.

Proof. For any $f \in \mathbf{C}[\mathbf{w}, \mathbf{z}, \mathbf{x}, \mathbf{y}]^{\mathbf{G_{II}} \oplus \mathbf{G_{II}}}_{\mathbf{m}, \mathbf{n}-\mathbf{m}}$ the quantity $BO(f)$ is a Jacobi form of weight $\frac{n}{2}$ with index m. Since $BO(f)$ can be written uniquely as a polynomial in $E_{4,1}$ and $\phi_{12,1}$ over $M_*[\frac{1}{\Delta}]$ (see Theorem 2), the Bannai–Ozeki map BO is injective map from $\mathbf{C}[\mathbf{w}, \mathbf{z}, \mathbf{x}, \mathbf{y}]^{\mathbf{G_{II}} \oplus \mathbf{G_{II}}}_{\mathbf{m}, \mathbf{n}-\mathbf{m}}$ to $M_*[\frac{1}{\Delta}][E_{4,1}, \phi_{12,1}]$. Also, from the fact that $BO(\psi_{8,1}) = 128E_{4,1}$ and that $BO(\nu_{24,1}) = 32\phi_{12,1}$, the theorem follows. ∎

The following result, the analogue of Theorem 1 for invariants, extends the preceding to formal weight enumerators in the sense of Ozeki [9].

THEOREM 13. *Every element of* $\mathbf{C}[\mathbf{w}, \mathbf{z}, \mathbf{x}, \mathbf{y}]^{\mathbf{H_{II}} \oplus \mathbf{H_{II}}}_{\mathbf{m}, \mathbf{n}-\mathbf{m}}$ *is an homogeneous polynomial of degree* m *in* $\psi_{8,1}$ *and* $k_{12,1}$ *with coefficients in* $\mathbf{C}[\mathbf{x}, \mathbf{y}]^{\mathbf{H_{II}}}[1/v_{24}]$.

Proof. The same argument as Theorem 12 follows: combine the Bannai–Ozeki map BO of Theorem 10 with Theorem 1. Observe as per Lemmata 2 and 3 that $BO(\psi_{8,1}) = 128E_{4,1}$ and that $BO(k_{12,1}) = -768E_{6,1}$. ∎

We are now in a position to justify the title of the article.

THEOREM 14. *Every element of* $\mathbf{C}[\mathbf{w}, \mathbf{z}, \mathbf{x}, \mathbf{y}]^{\mathbf{H_{II}} \oplus \mathbf{H_{II}}}_{\mathbf{m}, \mathbf{n}-\mathbf{m}}$ *is an homogeneous polynomial of degree* m *in* $\psi_{20,1}/v_{24}$ *and* $v_{24,1}/v_{24}$, *namely*

$$\frac{1}{v_{24}^m} \sum_{j=0}^{m} d_j \psi_{20,1}^{m-j} v_{24,1}^j,$$

with $d_j \in \mathbf{C}[\mathbf{x}, \mathbf{y}]^{\mathbf{H_{II}}}_{\mathbf{n}+4\mathbf{m}-4\mathbf{j}}$.

Proof. The proof follows by combining Lemmata 2 and 6 with the properties of the Bannai–Ozeki map in Theorem 10 and Eichler and Zagier's theory of weak Jacobi forms [6, Theorem 3]. ∎

4. IMPROVEMENTS

We now apply Theorem 5 to 7 to invariants.

4.1. $m = 1$

THEOREM 15. *Let* $l \geqslant 4$ *be an integer. Every invariant of* $\mathbf{C}[\mathbf{w}, \mathbf{z}, \mathbf{x}, \mathbf{y}]^{\mathbf{G_{II}} \oplus \mathbf{G_{II}}}_{1, 8l-1}$ *can be written as*

$$f_{8l-8}\psi_{8,1} + f_{8l-24}v_{24,1},$$

where $f_i \in \mathbf{C}[\mathbf{x}, \mathbf{y}]^{\mathbf{G_{II}}}_{\mathbf{i}}$ *if* $i > 0$ *and zero otherwise.*

Proof. Theorem 12 implies that any $g \in \mathbf{C}[\mathbf{w}, \mathbf{z}, \mathbf{x}, \mathbf{y}]^{\mathbf{G_{II}} \oplus \mathbf{G_{II}}}_{1, 8l-1}$ can be written as

$$g(w, z, x, y) = \alpha\psi_{8,1} + \beta v_{24,1}$$

with $\alpha, \beta \in \mathbf{C}[\mathbf{x}, \mathbf{y}]^{\mathbf{G_{II}}}[1/v_{24}]$. By applying Bannai–Ozeki map BO, one has

$$BO(g) = 128 \cdot BO(\alpha)\, E_{4,1} + 32 \cdot BO(\beta)\, \phi_{12,1} \in J_{4l,1}.$$

Here, Theorem 4 implies that $128 \cdot BO(\alpha) = 128 \cdot O(\alpha) \in M_{4l-4}$, $32BO(\beta) = 128 \cdot O(\beta) \in M_{4l-12}$. Now, Corollary 1(2) implies that there exist unique $f_j \in \mathbf{C}[\mathbf{x}, \mathbf{y}]_{\mathbf{j}}^{\mathbf{G_{II}}}$ such that $f_{8l-8} = \alpha$ and $f_{8l-24} = \beta$. ∎

In the special case of Ozeki polynomials of Type II code this result can be checked by combining the polarization lemma of [4, Theorem 4] with the Assmus–Mattson theorem [4, Theorem 1].

The analogue for formal weight enumerators spells out.

THEOREM 16. *Let $l \geqslant 4$ be an integer. Every invariant of* $\mathbf{C}[\mathbf{w}, \mathbf{z}, \mathbf{x}, \mathbf{y}]_{\mathbf{1}, \mathbf{4l-1}}^{\mathbf{H_{II}} \oplus \mathbf{H_{II}}}$ *can be written as*

$$f_{4l-8}\psi_{8,1} + f_{4l-12}k_{12,1}$$

where $f_i \in \mathbf{C}[\mathbf{x}, \mathbf{y}]_{\mathbf{i}}^{\mathbf{H_{II}}}$ *if $i > 0$ and zero otherwise.*

Proof. Theorem 13 implies that any $h \in \mathbf{C}[\mathbf{w}, \mathbf{z}, \mathbf{x}, \mathbf{y}]_{\mathbf{1}, \mathbf{4l-1}}^{\mathbf{H_{II}} \oplus \mathbf{H_{II}}}$ can be written as

$$h(w, z, x, y) = \alpha\psi_{8,1} + \beta k_{12,1}$$

with $\alpha, \beta \in \mathbf{C}[\mathbf{x}, \mathbf{y}]^{\mathbf{H_{II}}}[\mathbf{1}/v_{24}]$. By applying Bannai–Ozeki map BO, one has

$$BO(h) = 128 \cdot BO(\alpha)\, E_{4,1} - 768 \cdot BO(\beta)\, E_{6,1} \in J_{2l,1}.$$

Moreover, Theorem 6 implies that $128 \cdot BO(\alpha) = 128 \cdot O(\alpha) \in M_{2l-4}$, $-768 \cdot BO(\beta) = -768 \cdot O(\beta) \in M_{2l-6}$. Now, since $\mathbf{C}[\mathbf{x}, \mathbf{y}]^{\mathbf{H_{II}}}$ is isomorphic onto $\mathbf{C}[E_4, E_6]$ by Ozeki (see Theorem 9) one concludes that there exist unique $f_j \in \mathbf{C}[\mathbf{x}, \mathbf{y}]_{\mathbf{j}}^{\mathbf{H_{II}}}$ such that $f_{4l-8} = \alpha$ and $f_{4l-12} = \beta$. ∎

4.2. $m = 2$

THEOREM 17. *Let $l \geqslant 7$ be an integer. Every invariant of* $\mathbf{C}[\mathbf{w}, \mathbf{z}, \mathbf{x}, \mathbf{y}]_{\mathbf{2}, \mathbf{8l-2}}^{\mathbf{G_{II}} \oplus \mathbf{G_{II}}}$ *can be written as*

$$f_{8l-16}\psi_{8,1}^2 + f_{8l-32}\psi_{8,1}v_{24,1} + f_{8l-48}v_{24,1}^2$$

where $f_i \in \mathbf{C}[\mathbf{x}, \mathbf{y}]_{\mathbf{i}}^{\mathbf{G_{II}}}$ *if $i > 0$ and zero otherwise.*

Proof. Theorem 12 implies that any $g \in \mathbf{C}[\mathbf{w}, \mathbf{z}, \mathbf{x}, \mathbf{y}]_{\mathbf{2}, \mathbf{8l-2}}^{\mathbf{G_{II}} \oplus \mathbf{G_{II}}}$ can be written as

$$g(w, z, x, y) = a\psi_{8,1}^2 + b\psi_{8,1}v_{24,1} + cv_{24,1}^2$$

with $a, b, c \in \mathbf{C}[\mathbf{x}, \mathbf{y}]^{G_{II}}[1/v_{24}]$. So, by applying the Bannai–Ozeki map BO given in Theorem11, one has

$$BO(g) = 128^2 \cdot BO(a)\, E_{4,1}^2 + 128 \cdot 32 \cdot BO(b)\, E_{4,1}\phi_{12,1} + 32^2 \cdot BO(c)\, \phi_{12,1}^2 \in J_{4l,2}.$$

Here, Theorem 9 implies that $128^2 \cdot BO(a) = 128^2 \cdot O(a) \in M_{4l-8}$, $128 \cdot 32 \cdot BO(b) = 128 \cdot 32 \cdot O(b) \in M_{4l-16}$ and $32^2 \cdot BO(c) = 32^2 \cdot O(c) \in M_{4l-24}$. Therefore, Corollary 1 shows that there exist unique f_j in $\mathbf{C}[\mathbf{x}, \mathbf{y}]_{\mathbf{j}}^{G_{II}}$ such that $f_{8l-16} = a$, $f_{8l-32} = b$ and $f_{8l-48} = c$. ∎

For instance in agreement with [10] we find three degrees of freedom for $n = 40$, $m = 2$.

Analogue for formal weight enumerators is:

Theorem 18. *Let $l \geqslant 7$ be an integer. Every invariant of $\mathbf{C}[\mathbf{w}, \mathbf{z}, \mathbf{x}, \mathbf{y}]_{2, 4l-2}^{H_{II} \oplus H_{II}}$ can be written as*

$$f_{4l-16}\psi_{8,1}^2 + f_{4l-20}\psi_{8,1}k_{12,1} + f_{4l-24}k_{12,1}^2$$

where $f_i \in \mathbf{C}[\mathbf{x}, \mathbf{y}]_{\mathbf{i}}^{H_{II}}$ if $i > 0$ and zero otherwise.

Proof. Theorem 13 implies that, any $h \in \mathbf{C}[\mathbf{w}, \mathbf{z}, \mathbf{x}, \mathbf{y}]_{2, 4l-2}^{H_{II} \oplus H_{II}}$ can be written as

$$h(w, z, x, y) = a\psi_{8,1}^2 + b\psi_{8,1}k_{12,1} + ck_{12,1}^2$$

with $a, b, c \in \mathbf{C}[\mathbf{x}, \mathbf{y}]^{H_{II}}[1/v_{24}]$. So, by applying Bannai–Ozeki map BO given in Theorem 10, one has

$$BO(h) = 128^2 \cdot BO(a)\, E_{4,1}^2 - 128 \cdot 768 \cdot BO(b)\, E_{4,1}E_{6,1} + 768^2 E_{6,1}^2 \in J_{2l,2}.$$

But Theorem 7 implies that $128 \cdot BO(a) = 128 \cdot O(a) \in M_{2l-8}$, $128 \cdot 768 \cdot BO(b) = 128 \cdot 768 \cdot O(b) \in M_{2l-10}$ and $768^2 \cdot BO(c) = 768^2 \cdot O(c) \in M_{2l-12}$. Now, since $\mathbf{C}[\mathbf{x}, \mathbf{y}]^{H_{II}}$ is isomorphic onto $\mathbf{C}[E_4, E_6]$ by Ozeki (see Theorem 9), one concludes that there exist unique f_j in $\mathbf{C}[\mathbf{x}, \mathbf{y}]_{\mathbf{j}}^{H_{II}}$ such that $f_{4l-16} = a$, $f_{4l-20} = b$ and $f_{4l-24} = c$. ∎

5. CONCLUSION

The main open problem is to find a direct, invariant-theoretic proof of our results.

The generalization of the results of Theorems 4–7 to $m > 2$ is of arithmetic interest.

The following diagram commutes:

$$\begin{array}{ccc} \mathbf{C}[\mathbf{x},\mathbf{y}]^{\mathbf{H}_{\mathrm{II}}} & \xrightarrow[A]{} & \mathbf{C}[\mathbf{w},\mathbf{z},\mathbf{x},\mathbf{y}]^{\mathbf{H}_{\mathrm{II}}\oplus\mathbf{H}_{\mathrm{II}}} \\ {\scriptstyle O}\downarrow & & \downarrow{\scriptstyle BO} \\ \mathbf{C}[\mathbf{E}_4,\mathbf{E}_6] & \xrightarrow[p]{} & \mathbf{C}[\mathbf{E}_4,\mathbf{E}_6,\mathbf{E}_{4,1},\mathbf{E}_{6,1}] \end{array}$$

Here the maps O and BO denote the maps in, respectively, Theorems 9 and 10. We believe it of interest to derive an intrinsic expression for the pull back p of A. This would constitute a modular form analogue of the polarization operator. A natural candidate would be the Kuznetsov–Cohen lifting KC of [6]. Observe that a necessary condition for commutation is $p(E_4) = E_{4,1}$ and $p(E_6) = E_{6,1}$. Indeed it can be checked that $KC(E_4) = E_{4,1}$ and $KC(E_6) = E_{6,1}$.

ACKNOWLEDGMENTS

The authors thank Don Zagier for helpful discussions. The second named author thanks POSTECH Math Dept for its kind hospitality.

REFERENCES

1. T. A. Apostol, “Modular Functions and Dirichlet Series in Number Theory,” Springer-Verlag, Berlin/New York, 1976.
2. E. Bannai and M. Ozeki, Construction of Jacobi forms from certain combinatorial polynomials, *Proc. Jpn. Acad. Ser. A Math. Sci.* **72**, No. 1 (1996), 12–15.
3. M. Broué and M. Enguehard, Polynômes des poids de certains codes et fonctions theta de certains réseaux, *Ann. Sci. Ecole Norm. Sup.* **5** (1972), 157–181.
4. A. Bonnecaze, B. Mourrain, and P. Solé, Jacobi polynomials, type II codes and designs, *Desig. Codes Cryptogr.* **16** (1999), 215–234.
5. J. H. Conway and N. J. A. Sloane, “Sphere Packings, Lattices and Groups,” 3rd ed., Springer-Verlag, Berlin/New York, 1998.
6. M. Eichler and D. Zagier, “The Theory of Jacobi Forms,” Birkhäuser, Basel, 1985.
7. F. Klein, “Vorlesungen über das Ikosaheder” Birkhäuser, Basel, 1993.
8. F. J. MacWilliams and N. J. A. Sloane, “The Theory of Error Correcting Codes,” North-Holland, Amsterdam, 1981.
9. M. Ozeki, On the notion of Jacobi polynomials for codes, *Math. Proc. Cambridge Philos. Soc.* **121** (1997), 15–30.
10. M. Ozeki, On covering radii and coset weight distributions of extremal binary self-dual codes of length 40, *Theoret. Comput. Science* **235** (2000), 283–308.
11. N.-P. Skoruppa and D. Zagier, A trace formula for Jacobi forms, *J. Reine Angew. Math.* **393** (1989), 168–198.
12. N.-P. Skoruppa and D. Zagier, Jacobi forms and a certain space of modular forms, *Invent. Math.* **94** (1988), 113–146.