Nondegeneracy concepts for zeros of piecewise smooth functions

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Abstract

A zero of a piecewise smooth function f is said to be nondegenerate if the function is Fréchet differentiable at that point. Using this concept, we describe the usual nondegeneracy notions in the settings of nonlinear (vertical, horizontal, mixed) complementarity problems and the variational inequality problem corresponding to a polyhedral convex set. Some properties of nondegenerate zeros of piecewise affine functions are described. We generalize a recent result of Ferris and Pang on the existence of a nondegenerate solution of an affine variational inequality problem which itself is a generalization of a theorem of Goldman and Tucker.

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1 Introduction

Various nondegeneracy concepts appear in the theory of linear programming, the linear complementarity problem and its generalizations, the variational inequalities, and in other areas of the mathematical programming [10]. They appear naturally in the investigations on the structure of the solution set, stability, convergence analysis of iterative algorithms, error bound analysis, etc.

For the linear complementarity problem LCP (M, q), see Section 3 for the description, a solution x^* is said to be nondegenerate (or strictly complementary) if $x^* + Mx^* + q > 0$. When M is skew-symmetric (the case corresponding to linear programming), a classical result due to Goldman and Tucker says that the LCP has a nondegenerate solution whenever it is solvable. There are many interesting and important results concerning nondegenerate solutions. For example, it has been shown in the context of the LCP that (a) nondegenerate solutions always lie in the relative interior of (some maximal convex component of) the solution set [19], (b) the solution set is finite if every solution is nondegenerate [19], [25], and (c) a nondegenerate solution is stable if and only if it is isolated [17]. Nondegeneracy ideas are present in various pivotal schemes [3], interior-point methods [16], [24], [37], error bound analysis [22], etc. In a recent paper, Ferris and Pang [8] extend this notion from LCP setting to that of an affine variational inequality (AVI) and prove important equivalence between the existence of a nondegenerate solution, weak sharp minima, minimum principle sufficiency, and error bounds. They also extend the above result of Goldman and Tucker to the context of AVI. For the generalized linear complementarity problem, the nondegeneracy concept was introduced by Szanc [33] who used it to analyze Lemke type pivotal algorithm for solving such problems. For the vertical linear complementarity problem (VLCP) and the horizontal linear complementarity problem (HLCP), Sznajder [34] introduced the nondegeneracy concept and discussed its connection to the structure, finiteness, and stability of the solution set. All of the above problems are piecewise affine in the sense that they can be formulated as problems of finding the zeros of piecewise affine functions. The nondegeneracy concept is equally important in the nonlinear setting such as nonlinear programs, variational inequalities, generalized equations, see e.g., [1], [2], [4], [9], [10], [29], [30], etc.

We show in this article that in the contexts of the nonlinear complementarity problem, the vertical (horizontal, mixed) nonlinear complementarity problem, and the variational inequality problem corresponding to a polyhedral convex set, the nondegeneracy concept is equivalent to the Fréchet differentiability of an appropriate piecewise smooth function at a zero. For example, for LCP (M, q), a solution x^* is nondegenerate if and only if there is a y^* such that at the point (x^*, y^*), the piecewise affine function

$$F(x, y) = \begin{bmatrix} y - Mx - q \\ x \wedge y \end{bmatrix}$$

vanishes and is Fréchet differentiable. The proof of this and other similar results depend on some simple observations such as: the function $g(x, y) = x \wedge y$ of two real variables x and y is Fréchet differentiable at a zero (x^*, y^*) if and only if $x^* + y^* > 0$.

Our results are described via the notion of nondegeneracy for zeros of piecewise smooth functions: a zero of a piecewise smooth function is *nondegenerate* if the function is Fréchet differentiable at this zero. We also introduce, for a composite piecewise smooth function of the form

$$f = g \circ h$$

where g is piecewise smooth and h is smooth, the notion of *g-nondegenerate* zero. Theorem 1 describes this concept in various equivalent ways.

In Section 6, we study the properties of nondegenerate zeros of piecewise affine functions. In Section 7, we prove some generalizations of a recent result of Ferris and Pang [8] on the existence of a nondegenerate solution of an affine variational inequality problem.

2 Preliminaries

A function $f : I\!R^n \to I\!R^m$ is said to be *piecewise smooth* (or PC^1) if it is a continuous selection of smooth (= continuously differentiable) functions: f is continuous and there exist continuously differentiable functions $f_j : I\!R^n \to I\!R^m$ such that

$$f(x) \in \{f_1(x), f_2(x), \ldots, f_J(x)\} \quad \text{for all} \quad x \in I\!R^n.$$

If each f_i is affine, we shall say that f is *piecewise affine.*

For an excellent introduction to piecewise smooth and piecewise affine functions, see [32]. We now briefly recall some basic properties of piecewise smooth functions. Let us fix a piecewise smooth function f and a point x^*. Then for any $d \in I\!R^n$, the directional derivative

$$f'(x^*; d) := \lim_{t \downarrow 0} \frac{f(x^* + td) - f(x^*)}{t}$$

exists and is positively homogeneous in d. Moreover, f is *B-differentiable* at x^*: the function $f(x^*) + f'(x^*; x - x^*)$ is a *first order approximation* of f at x^*, that is,

$$\lim_{x \to x^*} \frac{f(x) - [f(x^*) + f'(x^*; x - x^*)]}{||x - x^*||} = 0.$$

It turns out that the B-derivative $f'(x^*; \cdot)$ is a piecewise linear function given by

$$f'(x^*; d) \in \{f_i'(x^*)d : i \in I^e(f, x^*)\}$$

where f_i' is the Fréchet derivative of f_i and $I^e(f, x^*)$ is the collection of (so called essentially active) indexes i such that $x^* \in cl$ int $\{z : f(z) = f_i(z)\}$ (Prop. 4.1.3, [32]).

We note that f is Fréchet differentiable (F-differentiable, for short) at x^* precisely when $f'(x^*; d)$ is linear in d.

When f is given as $f = g \circ h$ where g is PC^1 and h is C^1, we have the chain rule (Thm. 3.1.1, [32])

$$f'(x^*; d) = g'(h(x^*), h'(x^*)d).$$

Also, when f is piecewise affine, the equality

$$f(x) = f(x^*) + f'(x^*; x - x^*) \tag{1}$$

holds for all x near x^*.

For natural numbers n and m, we write $\mathcal{PA}(I\!R^n, I\!R^m)$ for the set of all piecewise affine functions from $I\!R^n$ to $I\!R^m$. The zero set of a function f is denoted by $\mathcal{Z}(f)$, i.e.,

$$\mathcal{Z}(f) = \{x : f(x) = 0\}.$$

For any vector x, the components are denoted by $x_1, x_2, \ldots, x_n$. e denotes the vector of ones. $B(x^*, \varepsilon)$ denotes the closed ball (in the space under consideration) of radius ε around x^*. We define $x \wedge y$, $x \vee y$, and $\langle x, y\rangle\, (= x^t y)$ as, respectively, the componentwise minimum, componentwise maximum, and the usual inner product of vectors x and y. Also, $x^+ := x \vee 0$ and $x^- := (-x) \vee 0$. For a set X in $I\!R^n$, the dual is defined by $X^* := \{y \in I\!R^n : \langle y, x\rangle \geq 0 \text{ for all } x \in X\}$; the polar of X is defined as $-X^*$.

3 Nondegenerate zeros

In classical differential topology, a point is said to be nondegenerate for a function from $I\!R^n$ into itself if its Fréchet derivative at this point is nonsingular [23]. Robinson's nondegeneracy concept [29], [30] deals with a differentiable function restricted to a closed convex set. The functions we are interested in this study, namely, piecewise smooth functions, are not in general Fréchet differentiable.

A slight weakening the nondegeneracy notion given in [6] for a piecewise affine function results in the following definition.

Definition 1 *Let f be piecewise smooth. We say that x^* is a nondegenerate zero of f if $f(x^*) = 0$ and f is Fréchet differentiable at x^*.*

The following elementary result is immediate from (1).

Proposition 1 *A piecewise affine function f is Fréchet differentiable at a point x^* if and only if f is affine in a neighborhood of x^*.*

To understand this nondegeneracy concept and to motivate our next definition, we consider the *linear complementarity problem* LCP (M,q) [3] which is to find a vector x such that

$$x \geq 0, \;\; Mx + q \geq 0, \;\; \text{and} \;\; \langle x, Mx + q \rangle = 0$$

where $M \in \mathbb{R}^{n \times n}$ and $q \in \mathbb{R}^n$. This LCP can be formulated as a piecewise affine equation in any number of ways. The 'min' function formulation is given by the equation

$$f(x) := x \wedge (Mx + q) = 0 \tag{2}$$

where we observe that f is piecewise affine. Now suppose that x^* is a *nondegenerate solution* of LCP (M,q), i.e., x^* is a solution of LCP (M,q) (equivalently, $f(x^*) = 0$) and $x^* + Mx^* + q > 0$ which means in particular that for each index i, either $x_i^* > 0$ and $(Mx^* + q)_i = 0$, or $x_i^* = 0$ and $(Mx^* + q)_i > 0$. An obvious observation is that f is affine in a neighborhood of x^*, i.e., f is F-differentiable at x^*. We conclude that every nondegenerate solution of LCP (M,q) is a nondegenerate zero of f. That the converse is false is seen by the following two (one-dimensional) examples.

$$(i) \quad \text{LCP}\,(\,[1]\,,\,(0)\,) \qquad\qquad \text{and} \qquad\qquad (ii) \quad \text{LCP}\,(\,[1]\,,\,(1)\,)\,.$$

Clearly, $x^* = 0$ is a solution for both the problems. Also, both problems can be formulated by the same piecewise affine function $f(x) = x \wedge (Mx + q) = x$. Thus $x^* = 0$ is a nondegenerate zero of f in both cases. However, x^* is a nondegenerate solution only for LCP in (ii). To explain this difference, we write f from (2) as

$$f = g \circ h$$

where $g(x,y) = x \wedge y$ and $h(x) = (x, Mx+q)$. It is easily seen that in example (i), $h(x^*) = (0,0)$ and g is not F-differentiable at $h(x^*)$ whereas in example (ii), $h(x^*) = (0,1)$ and g is F-differentiable at $h(x^*)$. In other words, it is the F-differentiability of g at $h(x^*)$ that shows the difference between these two examples.

With this motivation, we introduce the following

Definition 2 *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a composite PC^1-function given by*

$$f = g \circ h \tag{3}$$

where $g : \mathbb{R}^k \to \mathbb{R}^m$ is PC^1 and $h : \mathbb{R}^n \to \mathbb{R}^k$ is C^1. We say that x^* is a g-nondegenerate zero of f if $f(x^*) = 0$ and g is Fréchet differentiable at $h(x^*)$.

Clearly, x^* is a nondegenerate zero of f when it is a g-nondegenerate zero of f. We now give a characterization result for g-nondegeneracy.

Theorem 1 *Let f be a composite PC^1 function given by (3) where g is PC^1 and h is C^1. Let $x^* \in I\!R^n$. Then the following are equivalent.*

(a) x^* is a g-nondegenerate zero of f.

(b) $(x^*, h(x^*))$ is a nondegenerate zero of

$$\Phi(x, y) = \begin{bmatrix} y - h(x) \\ g(y) \end{bmatrix}. \tag{4}$$

(c) For every $H \in \mathcal{C}^0$, x^* is a nondegenerate zero of $F := g \circ H$ where

$$\mathcal{C}^0 := \{H : H \text{ is } C^1 \text{ and } \ H(x^*) = h(x^*)\}.$$

(d) For every $H \in \mathcal{L}^0$, x^* is a nondegenerate zero of $F := g \circ H$ where

$$\mathcal{L}^0 := \{h(x) + B(x - x^*) : B \text{ is a matrix}\}.$$

Moreover, when g is piecewise affine, these are further equivalent to

(e) $f(x^*) = 0$ and there exists an $\varepsilon > 0$ such that for every $H \in \mathcal{C}^\varepsilon$, $F := g \circ H$ is Fréchet differentiable at x^* where

$$\mathcal{C}^\varepsilon := \{H : H \text{ is } C^1 \text{ and } ||h(x^*) - H(x^*)|| \leq \varepsilon\}.$$

Proof. The equivalence of (a) and (b) is immediate. To see $(a) \Rightarrow (c)$, suppose (a) holds so that g is F-differentiable at $h(x^*)$. For any $H \in \mathcal{C}^0$, $H(x^*) = h(x^*)$ and so $F = g \circ H$ is Fréchet differentiable at x^* in view of the chain rule. Thus we have (c). Trivially, (c) implies (d). To see $(d) \Rightarrow (a)$, suppose (d) holds. We show that g is F-differentiable at $h(x^*)$ by showing that the B-derivative $g'(h(x^*); d)$ is linear in d. Fix scalars α and β, vectors d_1 and d_2; we can easily find a matrix B and vectors e_1 and e_2 such that $h'(x^*)e_1 + Be_1 = d_1$ and $h'(x^*)e_2 + Be_2 = d_2$. Clearly, the function $H(x) := h(x) + B(x - x^*) \in \mathcal{L}^0$. Since by assumption, $F = g \circ H$ is Fréchet differentiable at x^*, we have

$$\begin{aligned} g'\left(h(x^*), \alpha d_1 + \beta d_2\right) &= g'\left(H(x^*), (h'(x^*) + B)(\alpha e_1 + \beta e_2)\right) \\ &= g'\left(H(x^*), H'(x^*)(\alpha e_1 + \beta e_2)\right) \\ &= F'(x^*, \alpha e_1 + \beta e_2) \\ &= \alpha F'(x^*, e_1) + \beta F'(x^*, e_2) \\ &= \alpha g'\left(H(x^*), H'(x^*)e_1\right) + \beta g'\left(H(x^*), H'(x^*)e_2\right) \\ &= \alpha g'\left(h(x^*), d_1\right) + \beta g'\left(h(x^*), d_2\right) \end{aligned}$$

thus proving the linearity. Now assume that g is piecewise affine. Since (e) implies (c), we have $(e) \Rightarrow (a)$. To see the reverse implication, assume that g is F-differentiable at $h(x^*)$. By Proposition 1, g continues to be F-differentiable at points close to $h(x^*)$. Hence for a suitable $\varepsilon > 0$ and for every $H \in \mathcal{C}^\varepsilon$, $F = g \circ H$ is Fréchet differentiable proving (e). ∎

4 Complementarity problems

In this section, we specialize Theorem 1 to complementarity problems. We begin with an elementary result concerning the differentiability of the 'min' function.

Proposition 2 *Consider the function*

$$g(y_1, y_2, \ldots, y_k) := y_1 \wedge y_2 \wedge \ldots \wedge y_k \tag{5}$$

where $y_l \in \mathbb{R}^n$ *for each index* l. *Then the vector* $y^* = (y_1^*, y_2^*, \ldots, y_k^*)$ *is a nondegenerate zero of* g *if and only if for each index* $j = 1, 2, \ldots, n$, *exactly one element in the set*

$$\{(y_1^*)_j, (y_2^*)_j, \ldots, (y_k^*)_j\}$$

is zero.

In particular, when $k = 2$, $y^* = (y_1^*, y_2^*)$ *is a nondegenerate zero of* g *if and only if* $g(y^*) = 0$ *and* $y_1^* + y_2^* > 0$.

Proof. We assume that $n = 1$, i.e., each variable $y_l \in R$; the general case can be handled by considering the component functions g_j. Fix a y^*. Then the B-derivative of g at y^* is given by ([27], Theorem 5)

$$g'(y^*, d) = \min\{E_l d : y_l^* = g(y^*)\}$$

where E_l is the F-derivative of the function $y = (y_1, y_2, \ldots, y_k) \mapsto y_l$. Since all the E_ls are different, the above minimum of linear functions can be linear if and only if all the linear functions coincide. That is, the above minimum can be linear with $g(y^*) = 0$ if and only if there is only one l such that $y_l^* = 0$. This proves the first part of the proposition. Specializing this to $k = 2$, we get the second part.

■

4.1 The nonlinear complementarity problem

Let ϕ be a C^1 function from $\mathbb{R}^n$ into itself. Then this problem, denoted by NCP (ϕ), is to find a solution of the equation

$$f(x) := x \wedge \phi(x) = 0. \tag{6}$$

(When $\phi(x) = Mx + q$ with $M \in \mathbb{R}^{n \times n}$ and $q \in \mathbb{R}^n$, the problem reduces to the linear complementarity problem LCP(M, q) described earlier.)

In this setting, a vector x^* is said to be a *nondegenerate solution* of NCP (ϕ) if $f(x^*) = 0$ and $x^* + \phi(x^*) > 0$. With

$$g(x, y) := x \wedge y \quad \text{and} \quad h(x) := (x, \phi(x)),$$

we can write f as $g \circ h$. By combining Theorem 1 and Proposition 2 and making minor modifications, we get the following

Theorem 2 *Let h, g, and f be as above and let $x^* \in I\!R^n$. Then the following are equivalent:*

(i) x^* is a nondegenerate solution of NCP (ϕ).

(ii) x^* is a g-nondegenerate zero of f.

(iii) $(x^*, \phi(x^*))$ is a nondegenerate zero of

$$\Phi(x, y) = \begin{bmatrix} y - \phi(x) \\ x \wedge y \end{bmatrix}. \tag{7}$$

(iv) For every matrix B, x^* is a nondegenerate zero of $F(x) := x \wedge (\phi(x) + B(x - x^*))$.

(v) x^* is a solution of NCP (ϕ) *and there exists an $\varepsilon > 0$ such that for every C^1-function ψ with $||\phi(x^*) - \psi(x^*)|| \le \varepsilon$, $F(x) := x \wedge \psi(x)$ is Fréchet differentiable at x^*.*

4.2 The horizontal nonlinear complementarity problem

Given a C^1-function $\phi(x, y) : I\!R^n \times I\!R^n \to I\!R^n$, we define HNCP (ϕ) as the problem of finding vectors x and y such that

$$f(x, y) := \begin{bmatrix} \phi(x, y) \\ x \wedge y \end{bmatrix} = 0.$$

When $\phi(x, y) = Ax - By - q$, with $A, B \in I\!R^{m \times n}$ and $q \in I\!R^m$, this problem reduces to the horizontal linear complementarity problem HLCP (A, B, q). Following [34], we shall say that (x^*, y^*) is a *nondegenerate solution* of HNCP (ϕ) if $f(x^*, y^*) = 0$ and $x^* + y^* > 0$. Proposition 2 gives the following result.

Theorem 3 *Consider ϕ and f as above and let $z^* = (x^*, y^*)$. Then the following are equivalent:*

(i) z^* is a nondegenerate solution of HNCP (ϕ);

(ii) z^* is a nondegenerate zero of f.

Remark. It is possible to state a result similar to Theorem 2 by specializing Theorem 1. We shall omit the details.

4.3 The vertical nonlinear complementarity problem

Given C^1-functions $\phi_1, \phi_2, \ldots, \phi_k$ from $\mathbb{R}^n$ into itself, this problem, denoted by VNCP $(\phi_1, \phi_2, \ldots, \phi_k)$, is to find a solution of the equation

$$f(x) = \phi_1(x) \wedge \phi_2(x) \wedge \cdots \wedge \phi_k(x) = 0\,.$$

When each $\phi_l(x)$ is affine, we get a vertical linear complementarity problem see, [14], [34], [35]. Following [34], a vector x^* is said to be a *nondegenerate solution* of VNCP $(\phi_1, \phi_2, \ldots, \phi_k)$ if for each index j, exactly one element in the set $\{(\phi_l(x^*))_j : l = 1, \ldots, k\}$ is zero. Defining $h(x) := (\phi_1(x), \phi_2(x), \ldots, \phi_k(x))$ and $g(y_1, y_2, \ldots, y_k) := y_1 \wedge y_2 \wedge \ldots \wedge y_k$, we can combine Proposition 2 and Theorem 1.

Theorem 4 *Let h, g, and f be as above and let $x^* \in \mathbb{R}^n$. Then the statement*

$$x^* \textit{ is a nondegenerate solution of } \mathrm{VNCP}\,(\phi_1, \phi_2, \ldots, \phi_k)$$

is equivalent to each of the statements $(a) - (e)$ in Theorem 1.

4.4 The mixed nonlinear complementarity problem

Given C^1-functions $\phi(x,y)$ and $\psi(x,y)$ from $\mathbb{R}^n \times \mathbb{R}^n$ into $\mathbb{R}^n$, the problem denoted by MNCP(ϕ, ψ) is to find a solution of the equation

$$f(x,y) = \begin{bmatrix} \phi(x,y) \\ y \wedge \psi(x,y) \end{bmatrix} = 0\,.$$

When ϕ and ψ are affine, we get a mixed linear complementarity problem [12]. Following the definition given in [8] for the affine case, we shall say that a solution (x^*, y^*) of this problem is *nondegenerate* if $y^* + \psi(x^*, y^*) > 0$.

Writing $f = g \circ h$ where

$$h(x,y) = \begin{bmatrix} \phi(x,y) \\ \psi(x,y) \\ y \end{bmatrix}$$

and

$$g(x,y,z) = \begin{bmatrix} x \\ y \wedge z \end{bmatrix},$$

we can apply Proposition 2 and Theorem 1 to describe the nondegeneracy of a solution of MNCP in terms of the F differentiability of appropriate functions. We omit the details.

5 The variational inequality problem

Given a continuous function ϕ from $\mathbb{R}^n$ into itself and a closed convex set $\mathcal{K}$ in $\mathbb{R}^n$, this problem, denoted by VI$(\phi, \mathcal{K})$ is to find a vector $x^* \in \mathcal{K}$ such that

$$\langle \phi(x^*), x - x^* \rangle \geq 0 \ \text{ for } \ \text{ all } \ x \in \mathcal{K}\,.$$

When $\phi(x) = Mx + q$ is affine and $\mathcal{K}$ is polyhedral, this problem is called an *affine variational inequality problem*, denoted by AVI$(M, q, \mathcal{K})$ [8], [12], [28]. Throughout this paper, we make the blanket assumption that

$$\phi \text{ is } C^1 \text{ and } \mathcal{K} \text{ is polyhedral.}$$

Denoting the Euclidean projection mapping onto $\mathcal{K}$ by $\Pi_{\mathcal{K}}$, it is easily seen that the solutions of VI$(\phi, \mathcal{K})$ are precisely the zeros of the piecewise smooth function

$$f(x) := x - \Pi_{\mathcal{K}}(x - \phi(x))$$

see [13] and [27] for details.

It is well known that an element $x^* \in \mathcal{K}$ is a solution of VI$(\phi, \mathcal{K})$ if and only if

$$-\phi(x^*) \in \mathcal{N}_{\mathcal{K}}(x^*)$$

where $\mathcal{N}_{\mathcal{K}}(x^*)$ is the normal cone to $\mathcal{K}$ at x^* is defined by

$$\mathcal{N}_{\mathcal{K}}(x^*) := \{y : \langle y, x - x^* \rangle \leq 0 \quad \text{for all} \quad x \in \mathcal{K}\}\,.$$

A solution x^* of VI$(\phi, \mathcal{K})$ is said to be *nondegenerate* [8] if

$$-\phi(x^*) \in \text{ri}\, \mathcal{N}_{\mathcal{K}}(x^*)$$

where 'ri' refers to the relative interior. Our objective here is to describe this nondegeneracy by means of the differentiability of an appropriate piecewise piecewise smooth function.

We shall first look at the differentiability of the projection mapping onto $\mathcal{K}$. Consider a point $u^* \in \mathbb{R}^n$ and let $v^* = \Pi_{\mathcal{K}}(u^*)$. With $T(v^*) := T_{\mathcal{K}}(v^*)$ (the polar of $\mathcal{N}_{\mathcal{K}}(v^*)$) denoting the tangent cone of $\mathcal{K}$ at v^*, we define the critical cone to $\mathcal{K}$ at the point u^* by

$$C(u^*) := C_{\mathcal{K}}(u^*) := T(v^*) \cap (u^* - v^*)^-$$

where $(u^* - v^*)^-$ denotes the set of all vectors orthogonal to $u^* - v^*$. (When $\mathcal{K}$ is given as a set of linear inequalities $\{x : Ax \geq b\}$, the tangent cone at $v^* \in \mathcal{K}$ is given by $\{x : \overline{A}x \geq 0\}$ where $\overline{A}$ denotes the submatrix of A consisting of rows of A corresponding to the binding constraints at v^*. In this setting $C(u^*) = \{x : \overline{A}x \geq 0\} \cap (u^* - v^*)^-$.)

In [27], Pang proves the formula

$$\Pi_{\mathcal{K}}(u) = \Pi_{\mathcal{K}}(u^* + (u - u^*)) = \Pi_{\mathcal{K}}(u^*) + \Pi_{C(u^*)}(u - u^*)\,,$$

which is valid for all u near u^*. This formula shows that $\Pi_{\mathcal{K}}$ is B-differentiable at u^* and its B-derivative is the projection mapping onto the critical cone $C(u^*)$. Since the projection mapping onto a closed convex cone is linear if and only if the cone is a subspace, we have the following result.

Proposition 3 *Let $\mathcal{K}$ be a polyhedral set in $I\!R^n$ and fix $u^* \in I\!R^n$. Then the following are equivalent:*

(1) The mapping $u \to \Pi_{\mathcal{K}}(u)$ is F-differentiable at u^*.

(2) $C(u^*)$ is a linear subspace.

Remark: When $\mathcal{K} = \{x : Ax \geq b\}$, Pang [27] shows that $\Pi_{\mathcal{K}}$ is F-differentiable at u^* if and only if $\overline{A}\,\Pi_{C(u^*)}$ is identically zero where $\overline{A}$ is the submatrix of A consisting of rows corresponding to the binding constraints of the inequality system $Ax \geq b$ at $v^* = \Pi_{\mathcal{K}}(u^*)$.

In ([8], Prop.1) Ferris and Pang prove that for an AVI$(M, q, \mathcal{K})$ *a solution x^* is a nondegenerate solution if and only if the critical cone $C(x^* - Mx^* - q)$ is a linear subspace.* We use this to discuss the F-differentiability of

$$g(x, y) := x - \Pi_{\mathcal{K}}(x - y)\,. \tag{8}$$

(We note that when $\mathcal{K}$ is the nonnegative orthant, this reduces to the 'min' function introduced earlier in Section 3.)

Proposition 4 *Consider the function g defined above and a point (x^*, y^*). The following are equivalent:*

(a) g is F-differentiable at (x^*, y^*);

(b) The mapping $u \mapsto \Pi_{\mathcal{K}}(u)$ is F-differentiable at $u^* = x^* - y^*$;

(c) $C(x^* - y^*)$ is a linear subspace.

Moreover, when $x^* = \Pi_{\mathcal{K}}(x^* - y^*)$, the above conditions are equivalent to

(d) $-y^* \in \operatorname{ri} \mathcal{N}_{\mathcal{K}}(x^*)$.

Proof. The equivalence of the first three conditions follows from the previous proposition, the identity $\Pi_{\mathcal{K}}(u^* + u) = (x^* + u) - g(x^* + u, y^*)$, and the linearity of the mapping $(x, y) \mapsto x - y$. The equivalence of (c) and (d) follows from Proposition 1, [8] (applied to the constant mapping $x \to y^*$). ∎

We now go back to VI$(\phi, \mathcal{K})$ and observe that f defined by $f(x) := x - \Pi_{\mathcal{K}}(x - \phi(x))$ is nothing but $g \circ h$ where g is given by (8) and $h(x) := (x, \phi(x))$. By combining the previous proposition and Theorem 1, and making minor modifications, we arrive at the following

Theorem 5 *Consider the VI($\phi, \mathcal{K}$) and the associated functions f, g, and h.*

(i) x^* is a nondegenerate solution of VI($\phi, \mathcal{K}$).

(ii) x^* is a g-nondegenerate zero of f.

(iii) $(x^*, \phi(x^*))$ is a nondegenerate zero of

$$\Phi(x, y) = \begin{bmatrix} y - \phi(x) \\ g(x, y) \end{bmatrix}. \tag{9}$$

(iv) For every matrix B, x^* is a nondegenerate zero of $F(x) := x - \Pi_{\mathcal{K}}\left(x - \phi(x) - B(x - x^*)\right)$.

(v) x^* is a solution of VI($\phi, \mathcal{K}$) and there exists an $\varepsilon > 0$ such that for every C^1-function ψ with $||\phi(x^*) - \psi(x^*)|| \leq \varepsilon$, $F(x) := x - \Pi_{\mathcal{K}}(x - \psi(x))$ is Fréchet differentiable at x^*.

6 Properties of nondegenerate zeros of piecewise affine functions

In the Introduction, we stated three properties of nondegenerate solutions of LCPs. In this section, we present similar results for nondegenerate zeros of piecewise affine functions.

Theorem 6 *Assume that $f \in \mathcal{PA}(\mathbb{R}^n, \mathbb{R}^m)$ and $x^* \in \mathcal{Z}(f)$ is nondegenerate. Then x^* is an isolated zero of f if and only if f is one-to-one in a neighborhood of x^* (i.e., the Fréchet derivative of f at x^* is one-to-one).*

Proof. Suppose $f(x) = Ax + a$ in a neighborhood U of x^*. Since the linear equation $Ax = -a$ has a unique solution x^* in U if and only if A is one-to-one, the result follows. ■

As pointed out in the Introduction, in the context of LCP, isolated nondegenerate solutions have stability properties. We now show that a similar result holds for piecewise affine functions. We recall that a zero x^* of a piecewise affine function $f \in \mathcal{PA}(\mathbb{R}^n, \mathbb{R}^m)$ is *stable* if x^* is an isolated zero of f and for each $\varepsilon > 0$, there exists a $\delta > 0$ such that for any continuous function $g : B(x^*, \varepsilon) \to \mathbb{R}^m$ with $\sup_{x \in B(x^*, \varepsilon)} ||f(x) - g(x)|| < \delta$ the set $\mathcal{Z}(g) \cap B(x^*, \varepsilon)$ is nonempty. The proof of the following result relies on degree theory [20], [26].

Theorem 7 *Suppose that x^* is an isolated and nondegenerate zero of $f \in \mathcal{PA}(\mathbb{R}^n, \mathbb{R}^n)$. Then f is stable at x^*.*

Proof. It follows from the previous proposition that in a neighborhood of x^*, $f(x) = Ax + a$ where A is one-to-one. Since A is square, it is nonsingular. Thus the index of f at x^* (which is the sign of the determinant of A) is nonzero. A routine application of the nearness property of the degree (Thm. 2.1.2, [20]) completes the proof. ■

One can now specialize the above result to various situations. For example, if x^* is an isolated and nondegenerate solution of AVI $(M, q, \mathcal{K})$, then x^* is a stable solution (where stability refers to perturbations with respect to M and q). We omit the details.

Our next result gives sufficient conditions for the set $\mathcal{Z}(f)$ to be finite.

Theorem 8 *Suppose that* $f \in \mathcal{PA}(I\!R^n, I\!R^m)$,
1. $\mathcal{Z}(f)$ *contains no lines, and*
2. Every $x^* \in \mathcal{Z}(f)$ *is nondegenerate.*
Then $\mathcal{Z}(f)$ *is finite.*

Proof. Since f is piecewise affine, $\mathcal{Z}(f)$ is a finite union of polyhedral sets. If every element in $\mathcal{Z}(f)$ is isolated, then each of these polyhedral sets is a singleton set proving the finiteness of $\mathcal{Z}(f)$. Suppose there is an $x^* \in \mathcal{Z}(f)$ which is not isolated. We claim that for some nonzero vector u, $x^* \pm \varepsilon u \in \mathcal{Z}(f)$ for all small positive ε. To see this we write $f(x) = Ax + a$ for all x in a neighborhood U of x^*. We pick y^* in $\mathcal{Z}(f) \cap U$ different from x^* and let $u := x^* - y^*$. We have $Au = 0$ and hence

$$f(x^* \pm \varepsilon u) = A\,(x^* \pm \varepsilon u) + a = A\,x^* + a \pm \varepsilon Au = 0$$

for all small $\varepsilon > 0$ proving the claim. The assumption that $\mathcal{Z}(f)$ contains no lines proves that the extended numbers

$$\lambda^* := \inf\{\,\lambda \le 0 : [x^* + \lambda u, x^*] \subseteq \mathcal{Z}(f)\,\} \quad \text{and } \mu^* := \sup\{\,\mu \ge 0 : [x^*, x^* + \mu u] \subseteq \mathcal{Z}(f)\,\}\,.$$

cannot both be infinite; here $[x, y]$ denotes the line segment joining vectors x and y. Assume without loss of generality that $0 > \lambda^* > -\infty$. Since f is continuous and $\mathcal{Z}(f)$ is closed, $z^* := x^* + \lambda^* u \in \mathcal{Z}(f)$ and $[z^*, x^*] \subseteq \mathcal{Z}(f)$. Clearly, z^* is non-isolated, and nondegenerate (by assumption). With u defined as before, we can easily show that $z^* \pm \delta u \in \mathcal{Z}(f)$ for all small positive δ. This contradicts the definition of λ^*. Hence every element of $\mathcal{Z}(f)$ is isolated proving the finiteness of $\mathcal{Z}(f)$. ∎

We note that the above result is applicable when $\mathcal{Z}(f)$ is bounded or when $\mathcal{Z}(f)$ is required to lie in a set without lines. As an illustration, consider AVI $(M, q, \mathcal{K})$ where $\mathcal{K}$ does not contain any line. If each solution of this problem is nondegenerate, then the solution set of this problem is finite. We omit the details.

Theorem 9 *Suppose f is piecewise affine. Then each nondegenerate zero of f is contained in the relative interior of some maximal convex component of $\mathcal{Z}(f)$.*

Proof. Let x^* be a nondegenerate zero of f so that in a neighborhood U of x^*, $f(x) = Ax + a$. It follows that the set $\mathcal{Z}(f) \cap \overline{U}$ is convex and hence contained in a convex component, say, X of $\mathcal{Z}(f)$. We claim that x^* is in the relative interior to X. Let $x^\circ \in X$. Since $x^* + \varepsilon(x^\circ - x^*) \in$

$U \cap X$, $f(x^* + \varepsilon(x^\circ - x^*)) = 0$ for small positive ε. It follows that $A(x^0 - x^*) = 0$, and as in the previous proposition,

$$[x^* - \varepsilon(x^\circ - x^*)\,,\, x^* + \varepsilon(x^\circ - x^*)] \subseteq \mathcal{Z}(f) \cap \overline{U} \subseteq X\,.$$

Thus (cf. Thm. 6.4 [31]), x^* is in the relative interior of X. ■

Theorem 10 *Assume that the range of a piecewise affine function $f \in \mathcal{PA}(\mathbb{R}^n, \mathbb{R}^n)$ contains an open set G. Then there exists an open and dense set $G_0 \subseteq G$ such that for all $q \in G_0$ the set $\mathcal{Z}(f-q) := \{x \in \mathbb{R}^n : f(x) = q\}$ is nonempty, finite, and every element in $\mathcal{Z}(f-q)$ is nondegenerate.*

Proof. Consider the description of f as given in Section 2. By the assumption on the range, there is at least one A_i which is nonsingular. Without loss of generality assume that the matrices $A_1, \ldots, A_l$ are singular $A_{l+1}, \ldots, A_K$ are nonsingular. Since each Ω_i has nonempty interior, the set $\cup_{l+1}^K (A_i(\Omega_i) + a_i)$ has nonempty interior. At the same time, $\left(\bigcup_1^l (A_i(\Omega_i) + a_i)\right)$ is a union of lower dimensional polyhedral sets and so its complement is dense and open in $\mathbb{R}^n$. Put

$$G_0 := G \cap \left[\left(\bigcup_{i=1}^{l} (A_i(\Omega_i) + a_i) \right)^c \cap \bigcup_{i=l+1}^{K} (A_i(\text{int } \Omega_i) + a_i) \right].$$

Then $\forall q \in G_0 \;\; \emptyset \neq \mathcal{Z}(f-q) \subseteq \cup_{i=l+1}^K \text{int } \Omega_i$ and each zero of $f - q$ is nondegenerate. ■

Remark. For a given piecewise affine function f, let us say that a vector $q \in ran f$ is *nondegenerate* if every element of $\mathcal{Z}(f-q)$ is nondegenerate (for f). Theorem 10 says that within every open set in the *range* (f), there is an open and dense subset consisting of nondegenerate vectors. If all the matrices of f have the same nonzero determinantal sign (in which case f is said to be *coherently oriented*), then for all nondegenerate vectors $q \in range\,(f)$, $f^{-1}(q)$ has the same (finite) number of elements (see Proposition 2.3.8 of [32]).

7 Existence of nondegenerate solutions in affine variational inequalities

A result of Goldman and Tucker [15] says that every solvable LCP corresponding to a skew symmetric matrix has a nondegenerate solution. In [7], Ferris and Mangasarian show that the primal-dual linear complementarity formulation of a convex quadratic program has a nondegenerate solution if and only if a certain "minimum principle sufficiency" holds. Recently, Ferris and Pang [8] extended the above result of Goldman and Tucker to the setting of an AVI by replacing the skew symmetric matrix by a positive semidefinite matrix whose corresponding quadratic form vanishes on the feasible set of the given AVI. They deduce this result as a consequence of a theorem describing the equivalence of the error bound property, the minimum principle sufficiency property, and the existence of a nondegenerate solution. In this section we show that the above result of Ferris and Pang is valid when the matrix under consideration is copositive on the given polyhedral set and whose corresponding quadratic form vanishes on a certain subset of the feasible set of the given AVI. Our proof is based on Motzkin's theorem of the alternative and perhaps similar in spirit to the proof of the Goldman-Tucker result.

We now consider AVI $(M, q, \mathcal{K})$. When $\mathcal{K} = \{x : Ax \geq b\}$, this AVI can be formulated as a mixed LCP [12]:

$$\begin{aligned} Mx - A^t y + q &= 0\,, \\ y \wedge (Ax - b) &= 0\,. \end{aligned}$$

In [8], Ferris and Pang show that a vector x^* is a nondegenerate solution of AVI (M, q, K), if and only if there exists a vector y^* such that

$$\begin{aligned} Mx^* - A^t y^* + q &= 0\,, \\ y^* \wedge (Ax^* - b) &= 0\,, \\ y^* + (Ax^* - b) &> 0\,. \end{aligned} \tag{10}$$

Corresponding to the AVI $(M, q, \mathcal{K})$, let SOL $(M, q, \mathcal{K})$ denote the solution set and let Feas $(M, q, \mathcal{K}) := \{x : x \in \mathcal{K}, Mx + q \in (0^+\mathcal{K})^*\}$ denote the *feasible set* where $0^+\mathcal{K}$ denotes the recession cone of $\mathcal{K}$ and $(0^+\mathcal{K})^*$ is the dual of the cone $0^+\mathcal{K}$. For $\mathcal{K} = \{x : Ax \geq b\}$, we have $0^+\mathcal{K} = \{x : Ax \geq 0\}$, $(0^+\mathcal{K})^* = \{A^t y : y \geq 0\}$,

$$\text{Feas}\,(M, q, \mathcal{K}) = \{\, x : \exists\, y \geq 0 \;\text{ such that }\; Ax \geq\; b,\, Mx - A^t y + q = 0 \,\},$$

and

$$\text{SOL}\,(M, q, \mathcal{K}) = \{\, x : \exists\, y \geq 0 \;\text{ such that }\; Ax \geq b,\, Mx - A^t y + q = 0,\, \langle y, Ax - b\rangle = 0 \,\}.$$

We introduce a subset of the feasible set as follows:

$$\text{E}\,(M, q, \mathcal{K}) = \{\, x : \exists\, y \geq 0 \;\text{ such that }\; Ax \geq b,\, Mx - A^t y + q = 0,\; \text{and}\; \langle q, x\rangle - \langle b, y\rangle \leq 0 \,\}\,.$$

We recall that a matrix M is said to be *copositive* on a set if its quadratic form $Q(x) := \langle Mx, x\rangle$ is nonnegative on that set. The inequality

$$0 \leq \langle y, Ax - b\rangle = \langle Mx, x\rangle + \langle q, x\rangle - \langle b, y\rangle$$

for x and y with $Ax \geq b$, $y \geq 0$, and $Mx - A^t y + q = 0$, shows that when M is copositive on $\mathcal{K}$,

$$\text{SOL}\,(M, q, \mathcal{K}) \subseteq \text{E}\,(M, q, \mathcal{K}). \tag{11}$$

Our main result in this section is the following.

Theorem 11 *Suppose that* $\mathrm{E}(M,q,\mathcal{K}) \neq \emptyset$, *$M$ is copositive on $\mathcal{K}$, and*

$$\langle Mx, x\rangle = 0 \quad \textit{for all} \quad x \in \mathrm{E}(M,q,\mathcal{K})\,.$$

Then

(i) $\mathrm{SOL}(M,q,\mathcal{K}) \neq \emptyset$.

(ii) $\mathrm{AVI}(M,q,\mathcal{K})$ *has a nondegenerate solution.*

(iii) $\mathrm{SOL}(M,q,\mathcal{K}) = \mathrm{E}(M,q,\mathcal{K})$.

Our proof consists in first considering the homogeneous problem $\mathrm{AVI}(M,0,0^+\mathcal{K})$ and proving the general result by using augmented matrices. We begin with a proposition.

Proposition 5 *Assume that M is copositive on $0^+\mathcal{K}$. Then there exists a pair (x^*,y^*) such that* $Mx^* = A^t y^*,\ y^* \geq 0,\ Ax^* \geq 0,$ *and* $y^* + Ax^* > 0\,.$

Proof. Motzkin's theorem of the alternative (see [21]) shows that exactly one of the following two systems has a solution:

$$(I)\quad \left\{\ [\,I \quad A\,]\begin{pmatrix} y^* \\ x^* \end{pmatrix} > 0, \quad \begin{bmatrix} I & 0 \\ 0 & A \end{bmatrix}\begin{pmatrix} y^* \\ x^* \end{pmatrix} \geq 0, \quad \begin{bmatrix} -A^t & M \end{bmatrix}\begin{pmatrix} y^* \\ x^* \end{pmatrix} = 0 \right.$$

and

$$(II)\quad \left\{ \begin{array}{l} \begin{bmatrix} I \\ A^t \end{bmatrix} w + \begin{bmatrix} I & 0 \\ 0 & A^t \end{bmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{bmatrix} -A \\ M^t \end{bmatrix} z = 0 \\ \qquad\qquad w \gneqq 0, \quad u,\, v \geq 0\,, \end{array} \right.$$

where $w \gneqq 0$ means $w \geq 0$, $w \neq 0$. We will show that (II) cannot hold. Suppose it does, so there exists (w,u,v,z) such that

$$w + u = Az\,,$$
$$A^t(w+v) + M^t z = 0\,.$$

Now, $Az = w+u \geq 0$ implies $z \in 0^+\mathcal{K}$ and by the copositivity of M on $0^+\mathcal{K}$ we get $\langle z, -A^t(w+v)\rangle \geq 0$, that is, $\langle w+v, Az\rangle \leq 0$, which in turn, implies that $\langle w+v, w+u\rangle \leq 0$, contradicting the inequalities $w \gneqq 0$, u, $v \geq 0$. Thus (I) holds and the result follows: ■

Corollary 1 *Suppose that M is copositive on $0^+\mathcal{K}$ and*

$$Ax \geq 0,\ Mx = A^t y,\ y \geq 0 \Longrightarrow \langle Mx, x\rangle = 0\,.$$

Then $\mathrm{AVI}(M,0,0^+\mathcal{K})$ *has a nondegenerate solution.*

Proof. Let x^* and y^* be as in the previous proposition. Since $x^* \in 0^+\mathcal{K}$, $Mx^* \in (0^+\mathcal{K})^*$, $\langle y^*, Ax^*\rangle = \langle Mx^*, x^*\rangle = 0$ and $y^* + Ax^* > 0$, we see that x^* is a nondegenerate solution of AVI$\,(M, 0, 0^+\mathcal{K})$.

∎

Before we give a proof of Theorem 11, we make two observations. The first one says that *if M is copositive on a polyhedral set X, then it is copositive on the recession cone 0^+X.* This is easily seen by expanding the left side of the inequality $\langle M(x+\lambda u), x+\lambda u\rangle \geq 0$ which holds for all $x \in X$, $u \in 0^+X$ and $\lambda \geq 0$. The second observation is that $(a) \Longrightarrow (b)$ where

(a) $\langle Mx, x\rangle = 0 \quad \text{for all} \quad x \in \mathrm{E}\,(M, q, \mathcal{K})\,.$

(b) $Ax \geq 0,\ Mx = A^t y,\ \langle q, x\rangle - \langle b, y\rangle \leq 0,\ y \geq 0 \Longrightarrow \langle Mx, x\rangle = 0\,.$

This follows from the first observation since the set defined by the inequalities on the left side of implication in (b) is nothing but the recession cone of the set $\mathrm{E}\,(M, q, \mathcal{K})$.

Proof of Theorem 11. We put

$$B := \begin{bmatrix} A & -b \\ 0 & 1 \end{bmatrix}, \qquad N := \begin{bmatrix} M & q \\ -q^t & 0 \end{bmatrix}, \qquad \text{and} \qquad z := \begin{pmatrix} x \\ s \end{pmatrix}.$$

Consider the cone $\mathcal{L} := \{\, z : Bz \geq 0 \,\}$. In order to apply Corollary 1 to AVI$\,(N, 0, 0^+\mathcal{L})$ we first show that N is copositive on $0^+\mathcal{L} = \mathcal{L}$. Obviously, $\langle Nz, z\rangle = \langle Mx, x\rangle$, and

$$Bz \geq 0 \quad \text{if and only if} \quad Ax - bs \geq 0 \quad \text{and} \quad s \geq 0\,. \tag{12}$$

Case (1): $s = 0$ in (12). Then $Ax \geq 0$, so $x \in 0^+\mathcal{K}$, and since M is copositive on $0^+\mathcal{K}$ (M is already copositive on $\mathcal{K}$), $\langle Mx, x\rangle \geq 0$.
Case (2): $s > 0$. By the copositivity of M on $\mathcal{K}$,

$$\frac{x}{s} \in \mathcal{K},\ \left\langle M\left(\frac{x}{s}\right), \left(\frac{x}{s}\right)\right\rangle \geq 0 \Longrightarrow \langle Mx, x\rangle \geq 0\,.$$

Hence, N is copositive on $\mathcal{L}$.

Now, we will show that the conditions $Bz \geq 0$, $Nz = B^t w$ and $w \geq 0$ imply $\langle Nz, z\rangle = 0$. When expressed in terms of M, q, A and b the above conditions become

$$\left.\begin{array}{r} Ax - bs \geq 0 \\ s \geq 0 \end{array}\right\}, \qquad \left.\begin{array}{rcl} Mx + qs &=& A^t y \\ -\langle q, x\rangle &=& -\langle b, y\rangle + r \end{array}\right\}, \qquad \text{and} \qquad \begin{pmatrix} y \\ r \end{pmatrix} \geq 0\,. \tag{13}$$

Again, we consider two cases.

Case (1): $s=0$. That is, $Ax \geq 0$, $Mx = A^t y$, $\langle q,x\rangle - \langle b,y\rangle \leq 0$, $y \geq 0$. This, by the above observations, implies $\langle Mx,x\rangle = 0$, which in turn, gives $\langle Nz,z\rangle = 0$.

Case (2): $s > 0$. In this case, (13) shows that $x/s \in \mathrm{E}(M,q,\mathcal{K})$; whence by the hypothesis of the theorem, $\langle M(x/s),(x/s)\rangle = 0$, that is, $\langle Nz,z\rangle = 0$.

Now Corollary 1 is applicable to $\mathrm{AVI}(N,0,\mathcal{L})$ and so we have two elements

$$z^* = \begin{pmatrix} x^* \\ s^* \end{pmatrix}, \quad \text{and} \quad w^* = \begin{pmatrix} y^* \\ r^* \end{pmatrix}$$

such that $Bz^* \geq 0$, $Nz^* = B^t w^*$, $w^* \geq 0$, $\langle Nz^*,z^*\rangle = 0$, and $w^* + Bz^* > 0$. This is equivalent to the set of inequalities

$$\begin{aligned} Ax^* - bs^* \geq 0\,, \qquad & Mx^* + qs^* = A^t y^*, \\ s^* \geq 0\,, \qquad & -\langle q,x^*\rangle = -\langle b,y^*\rangle + r^*, \end{aligned} \qquad \text{and} \qquad \begin{pmatrix} y^* \\ r^* \end{pmatrix} \geq 0\,.$$

Note that $0 = \langle Nz^*,z^*\rangle = \langle B^t w^*,z^*\rangle = \langle w^*,Bz^*\rangle$, so $\langle Ax^* - bs^*, y^*\rangle = 0 = \langle r^*,s^*\rangle$ as well as $y^* + Ax^* - bs^* > 0$ and $r^* + s^* > 0$.

Case (1): $s^* > 0$. Then with $\overline{x} = x^*/s^*$ and $\overline{y} = y^*/s^*$ one has

$$\overline{x} \in \mathcal{K}, \quad M\overline{x} + q = A^t\overline{y} \in (0^+\mathcal{K})^*, \quad \langle \overline{y}, A\overline{x} - b\rangle = 0, \quad \overline{y} + A\overline{x} - b > 0, \quad \overline{y} \geq 0, \text{ and } \quad A\overline{x} - b \geq 0\,.$$

This means that $\overline{x}$ is a nondegenerate solution to $\mathrm{AVI}(M,q,\mathcal{K})$.

Case (2): $s^* = 0$. Then

$$\begin{aligned} Mx^* = A^t y^*,\ Ax^* \geq 0, \\ -\langle q,x^*\rangle = -\langle b,y^*\rangle + r^*,\ y^* \geq 0, \end{aligned} \quad y^* + Ax^* > 0\,,\ \langle y^*, Ax^*\rangle = 0\,. \tag{14}$$

We show that the above set of inequalities is inconsistent. Let $u \in \mathrm{E}(M,q,\mathcal{K})$ so $Au \geq b$, $Mu+q = A^t v$, $\langle q,u\rangle - \langle b,v\rangle \leq 0$, for some $v \geq 0$. It follows from (14) that for $\lambda \geq 0$, $u + \lambda x^* \in \mathrm{E}(M,q,\mathcal{K})$, and by the assumption of the theorem, $\langle M(u+\lambda x^*), u+\lambda x^*\rangle = 0$, that is, $\langle A^t(v+\lambda y^*) - q, u+\lambda x^*\rangle = 0$, so

$$-\langle q, u+\lambda x^*\rangle + \langle v+\lambda y^*, A(u+\lambda x^*)\rangle = 0\,.$$

Thus, $0 \geq -\langle q,u+\lambda x^*\rangle + \langle v+\lambda y^*, b\rangle$ which leads to $0 \geq -\langle q,x^*\rangle + \langle y^*,b\rangle = r^*$, contradicting $r^* > 0$. Thus case (2) cannot happen so that case (1) must hold in which case, we have a nondegenerate solution to $\mathrm{AVI}(M,q,\mathcal{K})$. Thus we have proved statements (i) and (ii) in the Theorem. To see (iii), we recall from (11) that $\mathrm{SOL}(M,q,\mathcal{K}) \subseteq \mathrm{E}(M,q,\mathcal{K})$. When $\langle Mx,x\rangle = 0$ on $\mathrm{E}(M,q,\mathcal{K})$, we have the reverse inclusion proving the equality of the two sets under consideration. ∎

The following example shows the copositivity condition on M is essential in Theorem 11.

Example.

$$A := \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad b := \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \quad M := \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}, \quad q := \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and

$$\mathcal{K} := \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \ : \ A \begin{pmatrix} u \\ v \end{pmatrix} \geq b \right\}, \qquad \text{so} \qquad 0^{+}\mathcal{K} := \left\{ \begin{pmatrix} u \\ v \end{pmatrix} : u = 0, \, v \geq 0 \right\}.$$

It is easily seen that M is copositive on $0^{+}\mathcal{K}$ but not on $\mathcal{K}$, and

$$x \in \mathcal{K}, \ \ Mx + q \in (0^{+}\mathcal{K})^{*} \Longrightarrow \langle Mx, x \rangle = 0\,.$$

Simple algebra shows that AVI $(M, q, \mathcal{K})$ has no nondegenerate solutions.

Remark. It should be noted that AVI $(M, q, \mathcal{K})$ may not have a nondegenerate solution when $\langle Mx, x \rangle = 0$ on SOL $(M, q, \mathcal{K})$. A simple example is $M = I$, $\mathcal{K} = R^{n}_{+}$, $q = 0$.

We now consider the question of existence of a nondegenerate solution when the quadratic form is not identically zero on the set E $(M, q, \mathcal{K})$. The following results are proved in the spirit of Ferris-Pang theorem (Thm. 4.1, [8]).

Theorem 12 *Suppose that* M *is copositive on* $\mathcal{K}$ *and*

$$\emptyset \neq \mathrm{SOL}\,(M, q, \mathcal{K}) = \mathrm{E}\,(M, q, \mathcal{K}).$$

Then AVI $(M, q, \mathcal{K})$ *has a nondegenerate solution.*

Proof. We proceed as in the proof of Lemma 4.6 of [8]. We assume that AVI $(M, q, \mathcal{K})$ has no nondegenerate solution and consider the linear program

$$\begin{array}{ll} \text{minimize} & -\varepsilon \\ \text{subject to} & Ax \geq b, \ y \geq 0, \\ & Mx - A^{t}y + q = 0, \\ & \langle q, x \rangle - \langle b, y \rangle \leq 0, \\ & y + Ax - b \geq \varepsilon e. \end{array}$$

In view of our assumptions, the feasible set is nonempty and the optimal objective value is zero. By letting (u, v, ζ, w) be an optimal dual solution, we have

$$\begin{array}{l} M^{t}u + A^{t}(v + w) - \zeta q = 0, \\ -Au + \zeta b + w \leq 0, \\ \langle e, w \rangle = 1, \\ v, \zeta, w \geq 0, \\ -\langle q, u \rangle + \langle b, v + w \rangle = 0. \end{array}$$

Premultiplying the first equation by u, the second inequality by $v+w$, and the last equation by $-\zeta$, adding the resulting constraints and simplifying, we get

$$\langle Mu, u\rangle + \langle v+w, w\rangle \leq 0. \tag{15}$$

From the second constraint we see that $\frac{1}{\zeta}u \in \mathcal{K}$ (when $\zeta > 0$) or $u \in 0^+\mathcal{K}$ (when $\zeta = 0$) and hence $\langle Mu, u\rangle \geq 0$. Since both v and w are nonnegative, (15) implies that $w = 0$, contradicting the equation $\langle e, w\rangle = 1$. Hence we conclude that the AVI $(M, q, \mathcal{K})$ has a nondegenerate solution. ■

By modifying the proof of the previous result for $\mathcal{K} = \{x : Ax \geq 0\}$, we get the following.

Theorem 13 *Suppose that M is copositive on $\mathcal{K}$ and*

$$\emptyset \neq \mathrm{SOL}\,(M, q, \mathcal{K}) = \{x : \exists y \geq 0,\ Ax \geq 0,\ Mx - A^t y + q = 0,\ \langle q, x\rangle = 0\}.$$

Then AVI $(M, q, \mathcal{K})$ *has a nondegenerate solution.*

Finally, the following LCP result is obtained by specializing Theorems 11, 12, and 13 to $\mathcal{K} := \mathbb{R}^n_+$:

Theorem 14 *Suppose that M is a copositive matrix, i.e., M is copositive on $\mathbb{R}^n_+$. Then* LCP (M, q) *has a nondegenerate solution when any one of the following three conditions is satisfied:*

- $\langle Mx, x\rangle = 0$ *on* $\mathrm{E}\,(M, q) := \{x : x \geq 0,\ Mx + q \geq 0,\ \langle q, x\rangle \leq 0\} \neq \emptyset$.
- $\emptyset \neq \mathrm{SOL}\,(M, q) = \{x : x \geq 0,\ Mx + q \geq 0,\ \langle q, x\rangle \leq 0\}$.
- $\emptyset \neq \mathrm{SOL}\,(M, q) = \{x : x \geq 0,\ Mx + q \geq 0,\ \langle q, x\rangle = 0\}$.

8 Miscellaneous remarks

As mentioned in the Introduction, nondegeneracy concepts are intimately related to error bound properties in various complementarity problems and the variational inequality problem. Now that the nondegeneracy concept is related to Fréchet differentiability, can one deduce the error bound results directly from the differentiability properties? We do not have an answer but offer the following as an illustration. Consider a piecewise affine function F having a nondegenerate zero, say x^*, and whose zero set is convex. Let A be the derivative of F at x^*. By considering the line segment joining $\overline{x} \in \mathcal{Z}(F)$ and x^*, we can easily show that $A\overline{x} = Ax^*$, proving the inclusion $\mathcal{Z}(F) \subseteq \{x : Ax = Ax^*\}$. Suppose we have a polyhedral set Ω in $\mathbb{R}^n$ such that

$$\mathcal{Z}(F) = \Omega \cap \{x : Ax = Ax^*\}.$$

By describing Ω in terms of linear inequalities, and by applying Hoffman's Lemma [18], we can prove the existence of a positive constant μ such that

$$\mathrm{dist}\,(x, \mathcal{Z}(F)) \leq \mu||Ax - Ax^*|| \quad \text{for all} \quad x \in \Omega\,.$$

In the case of LCP (M, q) with a positive semidefinite matrix M, by taking the feasible set of this LCP as Ω, we can show that in the presence of a nondegenerate solution, the above error bound result yields the following estimation due to Mangasarian [22]:

$$\mathrm{dist}\,(x, \mathrm{SOL}\,(M, q)) \leq \mu\langle x, Mx + q\rangle \quad \text{for all} \quad x \in \mathrm{Feas}\,(M, q)\,.$$

Ferris and Pang [8] extend this result to the context of an AVI and prove even the converse that error bound of this type implies the existence of a nondegenerate solution. It would be interesting if their results can be deduced via Theorem 5.

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