Jordan-algebraic approach to convexity theorems for quadratic mappings

L. Faybusovich,
Department of Mathematics,
University of Notre Dame,
255 Hurley Hall, Notre Dame, IN, 46545

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Abstract

We describe a Jordan-algebraic version of results related to convexity of images of quadratic mappings as well as related results on exactness of symmetric relaxations of certain classes of nonconvex optimization problems. The exactness of relaxations is proved based on rank estimates. Our approach provides a unifying viewpoint on a large number of classical results related to cones of Hermitian matrices over real and complex numbers. We describe (apparently new) results related to cones of Hermitian matrices with quaternion entries and to the exceptional 27-dimensional Euclidean Jordan algebra.

1 Introduction

Starting from [F1, F2], Jordan-algebraic technique proved to be useful as a unifying tool for the description and analysis of interior-point algorithms. In the present paper we use this technique for similar goals but for studying the convexity images of quadratic mappings between finite-dimensional vector spaces. This circle of problems has numerous connections with optimization theory (see e.g. [Pol] for the discussion of various connections of this type and the bibliography). In particular, the questions like under what assumptions semidefinite relaxations of quadratically constrained quadratic programming problems are exact (see e.g. [YZ] and references therein) or when one can omit rank constraints in semidefinite programming problems (see [BM] and references therein) are important for modern optimization theory. Another very interesting connection is with the famous S-lemma (see [BN], pp. 300-314). Our approach is modelled on the work of A.Barvinok [B2, B3, B1] but is developed within a more general framework of Jordan algebras. The paper is organized as follows. In section 2 we briefly describe Jordan-algebraic concepts related to our discussion. In section 3 we give a complete description of the facial structure of a symmetric cone in the form somewhat different from one in [FK]. Section 4 is the central in the paper. It provides estimates on the rank of a feasible point in the intersection of an affine subspace and a symmetric cone. Our results are a direct (but not an immediate!) generalization of the results of A. Barvinok who considered the cones of Hermitian matrices over $\mathbf{R}$ and $\mathbf{C}$. In section 5 we derive from rank estimates some convexity results and results about exact convex relaxations of generally nonconvex optimization problems. It is done within the general Jordan-algebraic context. The central object here is the manifold of primitive idempotents in a simple Euclidean Jordan algebra (or its conic hull).

A very general Jordan-algebraic version of the well-known S-lemma is given. In section 6 we interpret results of Section 5 for concrete symmetric cones. The cases of symmetric cones corresponding to algebras of Hermitian matrices over **H** and exceptional 27-dimensional algebra seem to lead to new results.

Another type of convexity results (Jordan-algebraic version of Horn-Schur theorem) has been obtained in [LKF].

2 Jordan-algebraic concepts

We stick to the notation of an excellent book [FK]. We do not attempt to describe the Jordan-algebraic language here but instead provide detailed references to [FK]. Throughout this paper:
• V is a simple Euclidean Jordan algebra;
• $\mathrm{rank}(V)$ stands for the rank of V;
• $x \circ y$ is the Jordan algebraic multiplication for $x, y \in V$;
• $< x, y >= \mathrm{tr}(x \circ y)$ is the canonical scalar product in V; here tr is the trace operator on V;
• Ω is the cone of invertible squares in V;
• $\bar{\Omega}$ is the closure of Ω in V;
• An element $f \in V$ such that $f^2 = f$ and $\mathrm{tr}(f) = 1$ is called a primitive idempotent in V;
• The set $\mathcal{T}(V)$ of primitive idempotents is a smooth compact connected submanifold in V;
• Given $x \in V$, we denote by $L(x)$ the corresponding multiplication operator on V, i.e.

$$L(x)y = x \circ y, \quad y \in V;$$

• Given $x \in V$, we denote by $P(x)$ the so-called quadratic representation of x, i.e.

$$P(x) = 2L(x)^2 - L(x^2).$$

Given $x \in V$, there exist idempotents $f_1, \cdots, f_k$ in V such that $f_i \circ f_j = 0$ for $i \neq j$ and such that $f_1 + f_2 \cdots + f_k = e$, and distinct real numbers $\lambda_1, \cdots, \lambda_k$ with the following property:

$$x = \sum_{i=1}^{k} \lambda_i f_i \tag{1}$$

The numbers λ_i and idempotents f_i are uniquely defined by x. (see Theorem III. 1.1 in [FK]).

The representation (1) is called the spectral decomposition of x. Within the context of this paper the notion of rank of x is very important. By definition:

$$\mathrm{rank}(x) = \sum_{i:\lambda_i \neq 0} \mathrm{tr}(f_i). \tag{2}$$

Given $x \in V$, the operator $L(x)$ is symmetric with respect to the canonical scalar product. If f is an idempotent in V, it turns out that the spectrum of $L(f)$ belongs to $\{0, \frac{1}{2}, 1\}$. Following [FK], we denote by $V(1, f), V(\frac{1}{2}, f), V(0, f)$ corresponding eigenspaces.

It is clear that

$$V = V(0, f) \oplus V(1, f) \oplus V(\frac{1}{2}, f) \tag{3}$$

and the eigenspaces are pairwise orthogonal with respect to the scalar product $<,>$. This is the so-called Peirce decomposition of V with respect to an idempotent f. However, eigenspaces have more structure (see [FK], Proposition IV. 1.1). In particular, $V(0,f), V(1,f)$ are subalgebras in V. Let f_1, f_2 be two primitive orthogonal idempotents. It turns out that

$$\dim V(\frac{1}{2}, f_1) \cap V(\frac{1}{2}, f_2)$$

does not depend on the choice of the pair f_1, f_2 (see Corollary IV.2.6, p.71 in [FK]). It is called the degree of V (notation $\mathrm{d}(V)$).

If V is a simple Euclidean Jordan algebra, then

$$\dim V = \mathrm{rank}(V) + \frac{\mathrm{d}(V)}{2}\mathrm{rank}(V)(\mathrm{rank}(V) - 1).$$

Note that two simple Euclidean Jordan algebras are isomorphic if and only if their ranks and degrees coincide.
The next proposition will be frequently used in what follows.

Proposition 1 *Let $x, y \in \bar{\Omega}$. Then $< x, y > \geq 0$;*
$< x, y >= 0$ if and only if $x \circ y = 0$.

For a proof see e.g. [F2].

We summarize some of the properties of algebras $V(1,f)$.

Proposition 2 *Let f be an idempotent in a simple Euclidean Jordan algebra V. Then $V(1,f)$ is a simple Euclidean Jordan algebra with identity element f. Moreover,*

$$rank(V(1,f)) = rank(f)$$
$$d(V(1,f)) = d(V).$$

The trace operator on $V(1,f)$ coincides with the restriction of the trace operator on V. If $\tilde{\Omega}$ is the cone of invertible squares in $V(1,f)$ then $\bar{\tilde{\Omega}} = \bar{\Omega} \cap V(1,f)$.

Proposition 2 easily follows from the properties of Peirce decomposition on V (see section IV.2 in [FK]). Notice that if c is a primitive idempotent in $V(1,f)$, then c is primitive idempotent in V, i.e. $\mathcal{T}(V(1,f)) = \mathcal{T}(V) \cap V(1,f)$.

Indeed, let $c = c_1 + c_2$ where $c_1, c_2 \in V$ and $c_1^2 = c_1, c_2^2 = c_2$. Since $c \in V(1,f), c \circ (e-f) = 0$, i.e. $(e-f) \circ c_1 + (e-f) \circ c_2 = 0$.
Hence, $< e-f, c_1 > + < e-f, c_2 >= 0$. But $e-f, c_1, c_2 \in \bar{\Omega}$. Hence, $< e-f, c_1 > \geq 0, < e-f, c_2 > \geq 0$. We conclude that $< e-f, c_1 >=< e-f, c_2 >= 0$. By Proposition 1 $(e-f) \circ c_1 = (e-f) \circ c_2 = 0$, i.e. $c_1, c_2 \in V(1,f)$. But c is primitive in $V(1,f)$. Hence $c_1 = 0$ or $c_2 = 0$, which proves that c is primitive in V.

Let $f_1, \cdots, f_r$, where $r = \mathrm{rank}(V)$, be a system of primitive idempotents such that $f_i \circ f_j = 0$ for $i \neq j$ and $f_1 + \cdots f_r = e$. Such system is called a Jordan frame. Given $x \in V$, there exists a Jordan frame $f_1, \cdots, f_r$ and real numbers $\lambda_1, \cdots, \lambda_r$ such that

$$x = \sum_{i=1}^{r} \lambda_i f_i.$$

The numbers λ_i (with their multiplicities) are uniquely determined by x. (See Theorem III. 1.2 in [FK]).

It is clear that

$$\mathrm{tr}(x) = \sum_{i=1}^{r} \lambda_i, \mathrm{rank}(x) = \mathrm{card}\{i \in [1,r] : \lambda_i \neq 0\}.$$

Since primitive idempotents in $V(1,f)$ remain primitive in V, it easily follows that the rank of $x \in V(1,f)$ is the same as its rank in V.

3 Facial structure of the cone of squares.

Throughout this paper we will use notation B_ϵ for an open ball in V with the center at 0 and of radius ϵ (with respect to the norm induced by the canonical Euclidean product). Given a subset $S \subset V$, We denote by Aff(S) the smallest affine subspace in V containing S (affine hull of S). The notation ri(S) is used for the relative interior of S:

$$\text{ri}(S) = \{x \in S : \exists \epsilon > 0, (x + B_\epsilon) \cap \text{Aff}(S) \subset S\}.$$

Let S be a convex subset of V. A face of S is a convex subset T of S such that whenever $\lambda x + \mu y \in T$, where $x, y \in S, \lambda, \mu > 0, \lambda + \mu = 1$, then $x, y \in T$. Recall the following theorem (Theorem 2.6.10 in [W]).

Theorem 1 *Let a be a point of convex set S in V. Let $\mathcal{F}_a$ be the intersection of all faces of S containing a. Then $\mathcal{F}_a$ is a face of S. Moreover, $a \in ri(\mathcal{F}_a)$ and the relative interiors of the faces of S form a partition of S.*

In this section we describe the facial structure of the cone of squares $\bar{\Omega}$ in V in the form somewhat different from one given in Proposition IV.3.1 of [FK].

Theorem 2 *Let $x \in \partial\Omega = \bar{\Omega} \backslash \Omega$ and*

$$x = \sum_{i=1}^{k+1} \lambda_i(x) f_i(x)$$

be the spectral decomposition of x, where $\lambda_i(x) > 0, i = 1, 2, \cdots, k, \lambda_{k+1}(x) = 0$ are pairwise distinct eigenvalues of x. Let

$$\mathcal{F}_x = \{z \in \bar{\Omega} : < f_{k+1}(x), z >= 0\}.$$

Then $\mathcal{F}_x$ is a face of $\bar{\Omega}$. Moreover, $x \in ri(\mathcal{F}_x), Aff(\mathcal{F}_x) = V(0, f_{k+1}(x)) = V(1, f_1(x) + \cdots + f_k(x)); ri(\mathcal{F}_x) = \tilde{\Omega}$, where $\tilde{\Omega}$ is the cone of invertible squares in $V(0, f_{k+1}(x)), \bar{\tilde{\Omega}} = \bar{\Omega} \cap V(0, f_{k+1}(x))$.

Corollary 1 *In particular,*

$$dim\mathcal{F}_x = dimV(1, f_1(x) + \cdots + f_k(x)) = \varphi_d(rank(x)),$$

$$\varphi_d(x) = x + \frac{dx(x-1)}{2}.$$

Here $d = d(V)$ is the degree of V.

Proof of theorem 2

Since $x \in \partial\Omega, f_{k+1}(x) \neq 0$. Let $H_x = \{z \in V; < z, f_{k+1}(x) >= 0\}$. If $z \in \bar{\Omega}$, then $< z, f_{k+1}(x) >\geq 0$ ($f_{k+1}^2(x) = f_{k+1}(x)$, hence, $f_{k+1}(x) \in \bar{\Omega}$). This implies that H_x is a supporting hyperplane to $\bar{\Omega}$ ($x \in H_x$). Thus, $\mathcal{F} = H_x \cap \bar{\Omega}$ is a face of $\bar{\Omega}$ and moreover, $x \in \mathcal{F}$. Furthermore , $y \in \mathcal{F}$ is equivalent to $y \in \bar{\Omega}$ and $< y, f_{k+1}(x) >= 0$. But then by Proposition 1 $y \circ f_{k+1}(x) = 0$ and consequently $y \in V(0, f_{k+1}(x)) = V(1, e - f_{k+1}(x))$. Thus, $\mathcal{F} \subset V(0, f_{k+1}(x))$, which implies Aff$(\mathcal{F}) \subset V(0, f_{k+1}(x))$. It is clear that $x \in \tilde{\Omega}$ (with inverse $\sum_{i=1}^{k} (1/\lambda_i) f_i(x)$ in $V(0, f_{k+1}(x))$).

Thus rank$V(0, f_{k+1}(x)) =$ rank(x). Since $\bar{\tilde{\Omega}} = \bar{\Omega} \cap V(0, f_{k+1}(x))$ by Proposition 2 we conclude that $\tilde{\Omega} \subset \mathcal{F}$ which implies that $V(0, f_{k+1}(x)) =$ Aff$(\tilde{\Omega}) \subset$ Aff$(\mathcal{F})$. Thus

$$V(0, f_{k+1}(x)) = \text{Aff}(\mathcal{F}).$$

Since $\tilde{\Omega}$ is relatively open in $V(0, f_{k+1}(x))$, we see that $\tilde{\Omega} \subset$ ri$(\mathcal{F})$. On the other hand, $\mathcal{F} = \mathcal{F} \cap \text{Aff}(\mathcal{F}) = \bar{\Omega} \cap V(0, f_{k+1}(x)) = \bar{\tilde{\Omega}}$. Hence, ri$(\mathcal{F}) =$ ri$(\bar{\tilde{\Omega}}) = \tilde{\Omega}$.

4 Rank estimates

We are now in position to generalize main results of [B2, B3] to arbitrary symmetric cones.

Theorem 3 *Let $\mathcal{A}$ be an affine subspace in V such that*

$$S = \bar{\Omega} \cap \mathcal{A} \neq \emptyset.$$

Then there exists $x \in S$ such that

$$\varphi_d(rank(x)) \leq codim_V \mathcal{A}.$$

Here d is the degree of V.

Proof Since S is closed, nonempty and does not contain straight lines, it contains an extreme point x (see e.g., [B2, B3], p. 53). Let rank$(x) = m$. There exists a unique face $\mathcal{F}_x$ of $\bar{\Omega}$ such that $x \in \text{ri}(\mathcal{F}_x)$. By Theorem 2, dim$\mathcal{F}_x = \varphi_d(m)$. It is clear that $\mathcal{F}_x \cap \mathcal{A}$ is a face of S and moreover, $x \in \text{ri}(\mathcal{F}_x \cap \mathcal{A})$. Indeed, $x \in \text{ri}(\mathcal{F}_x)$ implies that $\exists \epsilon > 0$ such that $(x + B_\epsilon) \cap \text{Aff}(\mathcal{F}_x) \subset \mathcal{F}_x$. But then $(x + B_\epsilon) \cap \text{Aff}(\mathcal{F}_x) \cap \mathcal{A} \subset \mathcal{F}_x \cap \mathcal{A}$. Since $\text{Aff}(\mathcal{F}_x \cap \mathcal{A}) \subset \text{Aff}(\mathcal{F}_x) \cap \mathcal{A}$ we have $x \in \text{ri}(\mathcal{F}_x \cap A)$. Since x is an extreme point of S, $x \in \text{ri}(\mathcal{F}_x \cap \mathcal{A})$ and $\mathcal{F}_x \cap \mathcal{A}$ is a face of S, we conclude that $\mathcal{F}_x \cap \mathcal{A} = \{x\}$ (by Theorem 1, there exists a unique face $\mathcal{F}$ of S such that $x \in \text{ri}(\mathcal{F})$).

Let $\text{Aff}(\mathcal{F}_x) = x + X, \mathcal{A} = x + Y$, where X, Y are vector subspace of V. We are going to show that $X \cap Y = 0$. We know that there exists $\epsilon > 0$ such that: $(x + B_\epsilon) \cap (x + X) \subset \mathcal{F}_x$. Hence, $(x + B_\epsilon) \cap (x + X) \cap (x + Y) = x + (B_\epsilon \cap X \cap Y) \subset \mathcal{F}_x \cap \mathcal{A} = \{x\}$. If $X \cap Y \neq 0$, we would arrive at a contradiction. Now, $X \cap Y = 0$ implies that $\dim(X + Y) = \dim X + \dim Y$. On the other hand, $\dim(X + Y) \leq dimV$. Hence, $\dim\mathcal{F}_x = \dim X \leq \dim V - \dim Y = \text{codim}_V \mathcal{A}$. We noticed before that $\dim\mathcal{F}_x = \varphi_d(\text{rank}(x))$. The result follows. $\diamond$

Remark Let $a_1, \cdots, a_k \in V, b_1, \cdots, b_k \in \mathbf{R}$ and

$$\mathcal{A} = \{z \in V :< a_i, z >= b_i, i = 1, \cdots, k\}.$$

It is clear that $\text{codim}_V \mathcal{A} \leq k$ provided $\mathcal{A} \neq \emptyset$. In this case Theorem 3 implies that $\varphi_d(rank(x)) \leq k$.

Remark In the case, where V is the Jordan algebra of real symmetric matrices, Theorem 3 coincides with the result on p. 194 of [B2]. See also [Pat]. In this case $\text{d}(V) = 1$.

Theorem 4 *Let $\mathcal{A}$ be an affine subspace in V such that*

$$S = \bar{\Omega} \cap \mathcal{A}$$

is nonempty and bounded. Suppose that there exists an integer $r \geq 1$ such that $codim_V(\mathcal{A}) \leq \varphi_d(r + 1)$, $rank(V) \geq r + 2$. Then there exists $x \in S$ such that $rank(x) \leq r$. Here $d = d(V)$ is the degree of V.

Proof

We need to consider several cases.

(i) Let $\mathcal{A} \cap \Omega = \emptyset$. But $\mathcal{A} \cap \partial\Omega \neq \emptyset$. Let $y \in \mathcal{A} \cap \partial\Omega$. Since $\mathcal{A} \cap \Omega = \emptyset$, there exists a hyperplane H in V separating $\mathcal{A}$ and Ω. Since $\mathcal{A} \cap \partial\Omega \neq \emptyset$ we should have that H is a supporting hyperplane to $\bar{\Omega}$ and A (as an affine subspace) is a subset in H. Then $\mathcal{F} = H \cap \bar{\Omega}$ is a proper face of $\bar{\Omega}$. By Theorem 2 $H \cap \bar{\Omega}$ is the face of the form $\bar{\Omega}_0$ where $\bar{\Omega}_0$ is the cone of squares in the algebra $V(0, f)$ and f is a nonzero idempotent in V. Since $\mathcal{A} \subset H$, we have $S = \mathcal{A} \cap \bar{\Omega} \subset H \cap \bar{\Omega} = \bar{\Omega}_0$. Thus $S \subset \bar{\Omega}_0 \cap (V(0,f) \cap \mathcal{A})$. Since $\bar{\Omega}_0 = \bar{\Omega} \cap V(0,f) \subset \bar{\Omega}$, we have $\bar{\Omega}_0 \cap (V(0,f) \cap \mathcal{A}) \subset \bar{\Omega} \cap \mathcal{A}$. Thus

$$S = \bar{\Omega}_0 \cap (V(0,f) \cap \mathcal{A}).$$

Let us estimate $\text{codim}_{V(0,f)}\mathcal{A} \cap V(0,f)$.

Let $\mathcal{A} = y + Y$, where $y \in S, Y$ be a vector subspace in V. Denote $V(0,f)$ by W. We have $\dim(Y \cap W) = \dim(W) + \dim Y$ - $\dim(W+Y)$. Now $W + Y = (y+Y) + W \subset H$. Hence, $\dim(W+Y) \leq \dim H = \dim V - 1$. Consequently, $\dim W$ - $\dim(Y \cap W) = \dim(W+Y) - \dim Y \leq \dim V - \dim Y - 1 = \text{codim}_V \mathcal{A} - 1$. Thus $\text{codim}_W(Y \cap W) \leq \text{codim}_V \mathcal{A} - 1 \leq \varphi_d(r+1) - 1$.

The last inequality is due to the assumptions of the theorem. We can apply theorem 3 to $S = (\mathcal{A} \cap W) \cap \bar{\Omega}_0 \subset W$ to conclude that there is $x \in S$ such that $\varphi_d(\text{rank}(x)) \leq \varphi_d(r+1) - 1$. since φ_d is an increasing function on $[0, +\infty)$ (notice that $d \geq 1$), we conclude that $\text{rank}(x) \leq r$.

Suppose now that $\mathcal{A} \cap \Omega \neq \emptyset$

(ii) Consider first the case $\text{rank}(V) = r+2, r \geq 1, \mathcal{A} \subset V$ and $\text{codim}_V(\mathcal{A}) = \varphi_d(r+1)$. Since $\mathcal{A} \cap \Omega \neq \emptyset$, we have $\text{ri}(S) = \mathcal{A} \cap \Omega$, $\text{Aff}(S) = \mathcal{A}$. Hence, $\dim S = \dim \mathcal{A} = \dim V$ - $\text{codim}_V \mathcal{A} = \varphi_d(r+2) - \varphi_d(r+1) = 1 + d(r+1)$. It is also clear that $\partial_1 S = \text{rebd} S = \mathcal{A} \cap \partial\Omega$. If there exists $x \in \partial_1 S$ such that $\text{rank}(x) \leq r$, there is nothing to prove. For $x \in \text{ri}(S)$, $\text{rank}(x) = \text{rank}(V) = r+2$. Otherwise, notice that for $x \in \partial_1 S$ $\text{rank}(x) < \text{rank}(V) = r+2$. Thus we should have $\text{rank}(x) = r+1$ for all $x \in \partial_1 S$. Take $y \in \mathcal{A} \cap \Omega$. There exists $\epsilon > 0$ such that $(y + B_\epsilon) \cap \mathcal{A} \subset S$. Let $\mathcal{A} = y + Y, Y$ be a vector subspace in V. Consider $\mathbf{S}_\epsilon = \partial_1(B_\epsilon \cap Y) = \partial B_\epsilon \cap Y$. It is clear that $\mathbf{S}_\epsilon$ is homeomorphic to the $d(r+1)$ dimensional sphere. Given $z \in \mathbf{S}_\epsilon$, there exists a unique positive $t(z)$ such that $y + t(z)z \in \partial_1 S = \mathcal{A} \cap \partial\Omega$ (Recall that S is a convex compact set). The map $\psi : \mathbf{S}_\epsilon \to V, \psi(z) = y + t(z)z$ is clearly continuous. Since $\psi(\mathbf{S}_\epsilon) \subset \partial_1 S$, we have $\text{rank}\psi(z) = r+1$ for any $z \in \mathbf{S}_\epsilon$

We need the following Lemma.

Lemma 1 *Let V be a simple Euclidean Jordan algebra, rank$(V) = l$. Suppose that $0 < s < l$ and*

$$\bar{\Omega}_s = \{x \in \bar{\Omega} : \textit{rank}(x) = s\}.$$

For $x \in \bar{\Omega}_s$ consider the spectral decomposition

$$x = \sum_{j=1}^{k+1} \lambda_j(x) f_j(x),$$

where $\lambda_{k+1} = 0$. The map $\gamma_s = \bar{\Omega}_s \to V, \gamma_s(x) = f_{k+1}(x)$ is continuous.

We postpone the proof of the Lemma and continue with the proof of the Theorem.

Consider $\tilde{\psi} : \mathbf{S}_\epsilon \to V, \tilde{\psi}(z) = \gamma_{r+1}(\psi(z))$. Since $\psi(\mathbf{S}_\epsilon) \subset \bar{\Omega}_{r+1}$, the map $\tilde{\psi}$ is continuous. Notice that $\tilde{\psi}(\mathbf{S}_\epsilon) \subset \mathcal{T}(V)$ (the manifold of primitive idempotents in V). Indeed, let $\psi(z) = \sum_{j=1}^{k+1} \lambda_j f_j$ be spectral decomposition and $\lambda_{k+1} = 0$. Since $\text{rank}(\psi(z)) = r+1$, we have $\sum_{j=1}^{k} \text{tr}(f_j) = r+1$.

But $\sum_{j=1}^{k+1} f_j = e$. Hence, $\sum_{j=1}^{k+1} \text{tr}(f_j) = \text{tr}(e) = \text{rank}(V) = r+2$.

Thus, $\text{tr} f_{k+1} = 1$, i.e. $f_{k+1} \in \mathcal{T}(V)$. Notice that $\dim \mathcal{T}(V) = \text{d}(r+1)$ (see exercise 4a, p.78, in [FK]). Hence $\tilde{\psi} : \mathbf{S}_\epsilon \to \mathcal{T}(V)$ is a continuous map between two compact connected manifolds of the same dimension. But then $\tilde{\psi}$ cannot be injective. Indeed, if $\tilde{\psi}$ is injective, then $\tilde{\psi}$ should be a homeomorphism of $\mathbf{S}_\epsilon$ onto $\mathcal{T}(V)$ (see e.g. Corollary 28.4, p.172 in [Ha]). However, under out assumptions $\mathcal{T}(V)$ is not homeomorphic to a sphere. Indeed, we assume that $r \geq 1$ and rank$(V) =$ r+2, i.e. rank$(V) \geq 3$ (Notice that if rank$(V) = 2$, then $\mathcal{T}(V)$ is homeomorphic to a sphere). We need to consider two separate cases.

If $d = 1$, then $\mathcal{T}(V)$ is homeomorphic to $\mathbf{P}_{r+1}(\mathbf{R})$ (see exercise 5, p. 99 in [FK]), which is not homeomorphic to sphere for $r \geq 1$.

If $d > 1, r \geq 3$ (the only possible choices are $d = 2, d = 4, d = 8$), then according to [H], p.351, the following holds. Denote by $b_i(\mathcal{T}(V)), i = 0, 1, \cdots, d(r+1)$ the Betti numbers of $\mathcal{T}(V)$. Then

$$b_i(\mathcal{T}(V)) = \{ \begin{array}{cc} 1 & \text{when } i = 0(\text{mod } d) \\ 0 & \text{otherwise,} \end{array}$$

whereas $b_i(\mathbf{S}_\epsilon) = 1, i = 0, i = d(r+1)$ and $b_i(\mathbf{S}_\epsilon) = 0$ otherwise. It is then clear that, say, $\beta_d(\mathcal{T}(V)) = 1$ but $\beta_d(\mathbf{S}_\epsilon) = 0$ (recall that $r \geq 1$!). Thus under our assumption, there exist $z_1 \neq z_2$ in $\mathbf{S}_\epsilon$ such that $\tilde{\psi}(z_1) = \tilde{\psi}(z_2)$. Since $z_1 \neq z_2$, it is clear that $\psi(z_1) \neq \psi(z_2)$. Let $c = \tilde{\psi}(z_1) = \tilde{\psi}(z_2)$. According to our construction of c, we have that $\psi(z_1), \psi(z_2) \in V(0, c)$ and moreover, if Ω_0 is the cone of invertible squares in $V(0, c)$, then both $\psi(z_1)$ and $\psi(z_2) \in \Omega_0$ (see Theorem 2). Let L be a line passing through $\psi(z_1)$ and $\psi(z_2)$. It is clear that $L \subset \mathcal{A}$ and since Ω_0 does not contain lines, L hits its relative boundary at some point z_0. Then rank$(z_0) <$ rank$(\psi(z_1)) =$ rank$(\psi(z_2)) = r+1$. Clearly, $z_0 \in \bar{\Omega}_0 \cap \mathcal{A} \subset S$. Thus, rank$(z_0) \leq r$.

(iii) Consider now a general case, where codim$_V \mathcal{A} \leq \varphi_d(r+1)$, dim$V \geq r+2, r \geq 1$. (We still assume that $\mathcal{A} \cap \Omega \neq \emptyset$). If codim$_V \mathcal{A} < \varphi_d(r+1)$, then the result follows from Theorem 3. It suffices to consider the case codim$_V \mathcal{A} = \varphi_d(r+1)$. By Theorem 3 there exists $y \in S$ such that rank$(y) \leq r+1$. If rank$(y) < r+1$, we are done. Consider the case rank$(y) = r+1$. Let

$$y = \sum_{j+1}^{k+1} \lambda_j f_j(y)$$

be spectral decomposition of y and $\lambda_{k+1} = 0$. Let $f_{k+1}(y) = c_1 + \cdots + c_s$, where $c_1, \cdots, c_s$ are primitive pairwise orthogonal idempotents.

rank$(f_{k+1}(y)) =$ rank $(V)-$ rank $(y) =$ rank $(V) - (r+1) \geq 1$.

Thus $s =$ rank$(f_{k+1}(y)) \geq 1$. Let $W = V(0, c_1 + c_2 + \cdots + c_{s-1})$. Notice that $< y, f_{k+1}(y) >= 0$ implies $< y, c_i >= 0, i = 1, 2, \cdots, s$. Hence $y \in W$. Further, rank$(W) =$ rank$(y) + 1 = r+2$. Notice that codim$_W(\mathcal{A} \cap W) \leq$ codim$_V(\mathcal{A}) = \varphi_d(r+1)$ (as we saw in (i))

Let Ω_W be the cone of invertible squares in W. Since $y \in \bar{\Omega}_W \cap (\mathcal{A} \cap W) = S \cap W$, the result follows from Theorem 3 if codim$_W(\mathcal{A} \cap W) <$codim$_V(\mathcal{A})$ or we are in the case (ii) if codim$_W(\mathcal{A} \cap W) =$ codim$_V \mathcal{A} = \varphi_d(r+2)$. This completes the proof of the theorem. $\diamond$

<u>Proof of Lemma 1</u>

Since $\bar{\Omega}_s$ is an orbit of connected component of the group of automorphisms of Ω (see Proposition IV.3.1, (iii) in [FK]) it is a smooth connected submanifold in V.

Let $x \in \bar{\Omega}_s$ have a spectral decomposition

$$x = \sum_{j=1}^{k+1} \lambda_j f_j(x),$$

$\lambda_{k+1} = 0$. Using the Peirce decomposition associated with (complete) system of orthogonal idempotents $f_1(x), \cdots f_{k+1}(x)$ (see section IV.2 in [FK]), one can easily see that

$$\mathrm{Im}P(x) = V(1, f_1(x) + f_2(x) + \cdots + f_k(x)) = V(0, f_{k+1}(x)).$$

Recall that $P(x)$ is the quadratic representation of x. But dim $\mathrm{Im}P(x) = \mathrm{rank}(P(x))$ (rank of the $\mathbf{R}-$linear map $P(x) : V \to V$) and $\mathrm{rank}(P(x))$ is constant when x varies over $\bar{\Omega}_s$ (see Proposition IV.3.1, (IV) in [FK]). Since $\bar{\Omega}_s$ is connected and the map $x \to P(x)$ is continuous, we conclude that the map $x \to \mathrm{Im}P(x)$ is continuous. (see Proposition 13.6.1, p408 in [GLR]). Let $\pi(x) : V \to \mathrm{Im}P(x)$ be orthogonal projection (with respect to the canonical scalar product $<,>$). The continuity of the map $x \to \mathrm{Im}P(x)$ is equivalent to the continuity of the map $x \to \pi(x)$ (see [GLR], chapter 13). But $\pi(x) = P(f_1(x) + f_2(x) + \cdots + f_k(x))$. (see [FK], p.65). On the other hand, $P(f_1(x) + f_2(x) + \cdots + f_k(x))e = f_1(x) + \cdots + f_k(x) = e - f_{k+1}(x)$ which implies that the map $x \to f_{k+1}(x) = \gamma_s(x)$ is continuous.
Remark If V is the algebra of symmetric matrices with real entries, Theorem 4 coincides with theorem 1.2 in [B3].

5 Some applications

The natural question within the Jordan-algebraic approach developed here is the convexity of the image of the manifold $\mathcal{T}(V)$ of primitive idempotents (or its conic hull) under linear maps. We will show how to transform it to the setting of quadratic maps in the section 6.

Proposition 3 *Let V be a simple Euclidean Jordan algebra of degree d. Given $a_1, \cdots, a_k \in V$, consider the linear map*

$$N : V \to \mathbf{R}^k$$

$$N(x) = \begin{bmatrix} < a_1, x > \\ \vdots \\ < a_k, x > \end{bmatrix}. \tag{4}$$

If $\varphi_d^{-1}(k) < 2$, then

$$N(\bigcup_{\lambda \geq 0} \lambda \mathcal{T}(V)) = N(\bar{\Omega}).$$

In particular, $N(\bigcup_{\lambda \geq 0} \lambda \mathcal{T}(V))$ is a convex cone.

Proof Denote $\bigcup_{\lambda \geq 0} \lambda \mathcal{T}(V)$ by K. Since $K \subset \bar{\Omega}$, it is clear that $N(K) \subset N(\bar{\Omega})$. Let $b = (b_1, \cdots, b_k)^T \in N(\bar{\Omega})$. Then there exists $x \in \bar{\Omega}$ such that $N(x) = b$. Consider $S = \{y \in \bar{\Omega} : N(y) = b\}$. It is clear that $x \in S$, i.e. $S \neq \emptyset$. According to Theorem 3 there exists $z \in S$ such that $\varphi_d(\mathrm{rank}(z)) \leq k$ or (using monotonicity of φ_d) $\mathrm{rank}(z) \leq \varphi_d^{-1}(k) < 2$. Hence, $\mathrm{rank}(z) = 1$ or $\mathrm{rank}(z) = 0$. In both cases, it is clear that $z \in K$. Thus $b \in N(K)$. ⋄

Proposition 4 *Let V be a simple Euclidean Jordan algebra, $d(V) = d$, rank$(V) \geq 3$. Suppose that $k = \varphi_d(2), a_1, \cdots, a_k \in V$ are such that there exist $\tau_1, \cdots, \tau_k \in \mathbf{R}$ with the property*

$$\sum_{i=1}^{k} \tau_i a_i \in \Omega. \tag{5}$$

Then

$$N(\bigcup_{\lambda \geq 0} \lambda \mathcal{T}(V)) = N(\bar{\Omega}) \text{ is a closed convex cone.}$$

Proof In the notation of the proof of Proposition 3, it is clear that $N(K) \subset N(\bar{\Omega})$. Let $b \in N(\bar{\Omega})$, i.e. there exists $x \in \bar{\Omega}$ such that $N(x) = b$. The set $S = \{y \in \bar{\Omega} : N(y) = b\}$ is nonempty. Moreover, it is bounded. Indeed,

$$S \subset \{y \in \bar{\Omega} :< \sum_{i=1}^{k} \tau_i a_i, y >= \sum_{i=1}^{k} \tau_i b_i\} = T.$$

Since $\sum_{i=1}^{k} \tau_i a_i \in \Omega$, the set T is bounded (see Corollary I.1.6 in [FK]). Hence, S is bounded. By Theorem 4 (with $r = 1$) there exists $z \in \bar{\Omega}$ such that $\mathrm{rank}(z) \leq 1$. Hence, $z \in K$, i.e. $N(K) = N(\bar{\Omega})$. The closeness of $N(\bar{\Omega})$ immediately follows from the fact that $\mathrm{Ker}N \cap \bar{\Omega} = 0$ which in turn easily follows from (5). $\diamond$

Proposition 5 *Let $d(V) = d$, $rank(V) \geq 3$, N is defined as in (4) and $k = \varphi_d(2) - 1$. Then $N(\mathcal{T}(V))$ is convex.*

Proof Consider $\tilde{N} : V \to \mathbf{R}^{k+1}$,

$$\tilde{N}(x) = \begin{bmatrix} < a_1, x > \\ \vdots \\ < a_k, x > \\ \mathrm{tr}(x) =< e, x > \end{bmatrix}.$$

By Proposition 4 $\tilde{N}(\bigcup_{\lambda \geq 0} \lambda \mathcal{T}(V)) = \tilde{N}(\bar{\Omega})$. Indeed, (5) is clearly satisfied if we take $\tau_1 = \tau_2 = \cdots = \tau_k = 0, \tau_{k+1} = 1$. Consider $H = \{(b_1, \cdots, b_{k+1})^T \in \mathbf{R}^{k+1} : b_{k+1} = 1\}$. H is a hyperplane in $\mathbf{R}^{k+1}$ and $H \cap \tilde{N}(\bar{\Omega})$ is convex. Denote by $\tilde{N}_1$ the restriction of $\tilde{N}$ on $(\bigcup_{\lambda \geq 0} \lambda \mathcal{T}(V))$. It is clear that

$$\tilde{N}_1^{-1}(H \cap \tilde{N}(\bar{\Omega})) = \{z \in \bigcup_{\lambda \geq 0} \lambda \mathcal{T}(V) : \mathrm{tr}(z) = 1\} = \mathcal{T}(V).$$

This means that

$$N(\mathcal{T}(V)) = H \cap \tilde{N}(\bar{\Omega}).$$

$\diamond$

Proposition 6 *Let $d(V) = d$, $rank(V) \geq 3$, $k < \varphi_d(2)$. Let $c, a_i, i = 1, 2, \cdots, k$ in V be such that there exist $\tau, \tau_i, i = 1, 2, \cdots, k$, in $\mathbf{R}$ with the property*

$$\tau c + \sum_{i=1}^{k} \tau_i a_i \in \Omega.$$

Let further, $b_i, i = 1, 2, \cdots, k$, be in $\mathbf{R}$.
Then

$$inf\{< c, x >:< a_i, x >= b_i, i = 1, 2, \cdots, k, \quad x \in \bigcup_{\lambda \geq 0} \lambda \mathcal{T}(V)\}$$

$$= inf\{< c, x >:< a_i, x >= b_i, i = 1, 2, \cdots, k, \quad x \in \bar{\Omega}\}$$

Proof We assume that the set $S = \{x \in \bar{\Omega} :< a_i, x >= b_i, i = 1, 2, \cdots, k\}$ is not empty. Otherwise, there is nothing to prove.

Let $y \in S, < c, y >= t$.

Consider the map $N : \bar{\Omega} \to \mathbf{R}^{k+1}$,

$$N(x) = \begin{bmatrix} < a_1, x > \\ \vdots \\ < a_k, x > \\ < c, x > \end{bmatrix}.$$

By Proposition 4 $N(\bar{\Omega}) = N(\bigcup_{\lambda \geq 0} \lambda \mathcal{T}(V))$.

Since $(b_1, \cdots, b_k, t)^T \in N(\bar{\Omega})$, we have $(b_1, \cdots, b_k, t)^T \in N(\bigcup_{\lambda \geq 0} \lambda \mathcal{T}(V))$. The result follows. $\diamond$

Proposition 7 *Let $d(V) = d$, $rank(V) \geq 3$. Let, further, $c, a_i, i = 1, 2, \cdots k, k < \varphi_d(2) - 1$. Suppose that the set*

$$S = \{x \in \bar{\Omega} :< a_i, x >= b_i, i = 1, 2, \cdots k, tr(x) = 1\} \text{ is not empty.}$$

Then

$$\min\{< c, x >: x \in \mathcal{T}(V), < a_i, x >= b_i, i = 1, 2, \cdots k\}$$
$$= \min\{< c, x >:< a_i, x >= b_i, i = 1, 2, \cdots k, tr(x) = 1, x \in \bar{\Omega}\}.$$

Proof Let $y \in S, < c, y >= t$. Consider the map $N : V \to \mathbf{R}^{k+2}$

$$N(z) = \begin{bmatrix} < a_1, z > \\ \vdots \\ < a_k, z > \\ < e, z >= \mathrm{tr}(z) \\ < c, z > \end{bmatrix}.$$

We have: $N(\bar{\Omega}) \cap \{(d_1, \cdots, d_{k+2})^T \in \mathbf{R}^{k+2} : d_{k+1} = 1\} = N(\mathcal{T}(V))$ by Proposition 5 (or more precisely its proof). It is clear that $(b_1, \cdots, b_k, 1, t) \in N(\bar{\Omega}) \cap \{(d_1, \cdots, d_{k+2})^T \in \mathbf{R}^{k+2} : d_{k+1} = 1\}$. The result follows. $\diamond$

Proposition 8 *Let $d(V) = d, r \geq 1$ and $1 \leq k < \varphi_d(r + 1)$ be such that $rank(V) \geq r + 2$. Let $a_1, \ldots, a_k \in V$. Consider the map N described as in (4). Then every point of convex hull $conv(N(\mathcal{T}(V)))$ can be represented as a convex combination of r (not necessarily distinct) points of $N(\mathcal{T}(V))$.*

Proof Let $b = (b_1, \cdots, b_k)^T \in \mathrm{conv}(N(\mathcal{T}(V)))$. Thus

$$b = \sum_{i=1}^{m} \lambda_i N(x_i)$$

for some $m \geq 1,\ x_i \in \mathcal{T}(V),\ \lambda_i \geq 0,\ \sum_{i=1}^{m} \lambda_i = 1$. Let $x = \sum_{i=1}^{m} \lambda_i x_i$. It is clear that $x \in \bar{\Omega},\ N(x) = b,\ \mathrm{tr}(x) = \sum_{i=1}^{m} \lambda_i \mathrm{tr}(x_i) = 1$.

It is clear that $S = \{z \in \bar{\Omega} : N(x) = b, \mathrm{tr}(x) = 1\}$ is nonempty and bounded. By theorem 4 there exist $z \in S$ such that $\mathrm{rank}(z) \leq r$. Let $\mu_1, \cdots, \mu_t, f_1, \cdots, f_t, t = \mathrm{rank}(V)$ be such that $f_1, \cdots, f_t$ is a Jordan frame and

$$z = \sum_{s=1}^{t} \mu_s f_s.$$

We notice earlier that $\mathrm{rank}(z) \leq r$ is equivalent to $\mathrm{card}\{s \in [1, t] : \mu_s > 0\} \leq r$.

Let $J = \{s \in [1, t] : \mu_s > 0\}$. We have:

$$z = \sum_{s \in J}^{t} \mu_s f_s.$$

since $\mathrm{tr}(z) = \sum_{s \in J} \mu_s = 1$, and $f_s \in \mathcal{T}(V)$ for all s, we conclude:

$$N(z) = b, \quad N(z) = \sum_{s \in J} \mu_s N(f_s).$$

The result follows. ◇

The next Proposition can be interpreted as an abstract version of the well-known S-lemma (see e.g. [BN]).

Proposition 9 *Let $c, a_i, i = 1, 2, \cdots, k$ in V be such that $N(\mathcal{T}(V))$ is a convex set. Here $N : V \to \mathbf{R}^{k+1}$,*

$$N(x) = \begin{bmatrix} < a_1, x > \\ \vdots \\ < a_k, x > \\ < c, x > \end{bmatrix}.$$

Suppose that there exists $x_0 \in \mathcal{T}(V)$ such that $\langle a_i, x_0 \rangle > 0$ for $i = 1, \cdots, k$. Let, further,

$$\Gamma = \{x \in \mathcal{T}(V) : \langle a_i, x \rangle \geq 0, i = 1, \cdots, k\}.$$

Then $\langle c, x \rangle \geq 0, \forall x \in \Gamma$ if and only if there exist nonnegative $\lambda_1, \cdots, \lambda_k$ such that

$$c - \sum_{i=1}^{k} \lambda_i a_i \in \bar{\Omega}.$$

Proof We prove (nontrivial) "only if" part. Let

$$Y = N(\bigcup_{\lambda \geq 0} \lambda \mathcal{T}(V))$$

and

$$Z = \{z \in \mathbf{R}^{k+1} : z_i \geq 0, i = 1, \cdots, k, z_{k+1} < 0\}.$$

Then by our assumptions $Y \bigcap Z = \emptyset$. Both Y and Z are convex. Hence, by separation theorem there exist real $\mu_1, \cdots, \mu_k, \lambda$ not all equal to zero and real a such that

$$\sum_{i=1}^{k} \mu_i y_i + \lambda y_{k+1} \geq a, \forall y \in Y,$$

$$\sum_{i=1}^{k} \mu_i z_i + \lambda z_{k+1} \leq a, \forall z \in Z.$$

The standard reasoning then shows that $\mu_i \leq 0$ for all i, $\lambda \geq 0$ and $a = 0$. Let us show that $\lambda > 0$. If $\lambda = 0$, then

$$\sum_{i=1}^{k} \mu_i y_i \geq 0$$

for all $y \in Y$, i.e.

$$\langle \sum_{i=1}^{k} \mu_i a_i, x \rangle \geq 0$$

for any $x \in \mathcal{T}(V)$. This implies

$$\sum_{i=1}^{k} \mu_i a_i \in \bar{\Omega}.$$

By our assumptions there exists $x_0 \in \mathcal{T}(V)$ such that $\langle a_i, x_0 \rangle > 0$ for all i. We arrive at the contradiction, since all μ_i are nonpositive and not equal to zero simaltaneously . Hence, $\lambda > 0$. But then

$$c - \sum_{i=1}^{k} \lambda_i a_i \in \bar{\Omega}$$

for $\lambda_i = -\mu_i/\lambda$. ◇

6 Interpretation in terms of quadratic mappings.

To interpret the results of Section 5 in terms of quadratic mappings, we need to understand the structure of manifolds $\mathcal{T}(V)$ for various simple Euclidean Jordan algebras. If $\text{rank}(V) \geq 3$, every such algebra is of the type $\text{Herm}(m, A)$ where $A = \mathbf{R}, \mathbf{C}, \mathbf{H}$, or $\mathbf{O}$. Here $\mathbf{R}, \mathbf{C}, \mathbf{H}, \mathbf{O}$ are algebras of real, complex, quaternion, and octonion numbers respectively and $\text{Herm}(m, A)$ stands for the Jordan algebra of Hermitian matrices of size $m \times m$ with entries in A. Notice that if $A = \mathbf{O}, m \leq 3$. The Jordan-algebraic multiplication in all these cases is the same:

Given $C, D \in \text{Herm}(m, A)$,

$$C \circ D = \frac{CD + DC}{2},$$

where CD is the usual matrix multiplication. The list of corresponding manifolds $\mathcal{T}(V)$ is given on p.99 of [FK]. We now consider the situation for concrete series $\text{Herm}(m, A)$.

1. Let $A = \mathbf{R}$. In this case the Jordan-algebraic operator tr coincides with usual operator Tr of the matrix.

Thus

$$< C, D >= \text{Tr}(CD), \quad C, D \in \text{Herm}(m, \mathbf{R}).$$

and $\text{Herm}(m, \mathbf{R})$ is the algebra of m by m symmetric matrices with real entries.

$$\mathcal{T}(V) = \{C \in \text{Herm}(m, \mathbf{R}) : C^2 = C, \text{Tr}(C) = 1\}$$

i.e. $\mathcal{T}(V)$ is a manifold of one-dimensional orthogonal projections. Consider the map $\mu : \mathbf{R}^m \to \mathcal{T}(V)$, $\mu(x) = xx^T$. It is very well-known that $\mu(\mathbf{S}^{m-1}) = \mathcal{T}(V)$.

Proposition 10 *Let $q_i(x) = x^T C_i x, \ i = 1, 2$ be two quadratic forms on $\mathbf{R}^m$. Here $C_1, C_2 \in$ Herm$(m, \mathbf{R})$. Consider the map $\nu : \mathbf{R}^m \to \mathbf{R}^2$, $\nu(x) = (q_1(x), q_2(x))^T$. Then $\nu(\mathbf{R}^m)$ is a convex cone in $\mathbf{R}^2$*

Proof It is clear that $\mu(\mathbf{R}^n) = \bigcup_{\lambda \geq 0} \lambda \mathcal{T}(V)$. We are going to use Proposition 3. Notice that $d(\text{Herm}(m, \mathbf{R})) = 1$ for any $m \geq 2$. Hence, $\varphi_d(2) = 3$. Thus for $k = 2$ the image $N(\bigcup_{\lambda \geq 0} \lambda \mathcal{T}(V))$ is convex. In our case

$$N \circ \mu(x) = (x^T C_1 x, x^T C_2 x).$$

The result follows. ◇

Remark Proposition 10 is a classical theorem of Dines [D]. Similarly from Proposition 4 we obtain the following result.

Proposition 11 *Let* $q_i(x) = x^T C_i x, i = 1, 2, 3$ *be three quadratic forms on* $\mathbf{R}^m$. *Here* $C_1, C_2, C_3 \in$ *Herm*$(m, \mathbf{R})$ *are such that there exist real* τ_1, τ_2, τ_3 *with the property that*

$$\tau_1 C_1 + \tau_2 C_2 + \tau_3 C_3 > 0,$$

(i.e. the corresponding matrix is positive definite). Then for $m \geq 3$ *the image* $\nu(\mathbf{R}^m)$ *is convex closed cone.*

Remark Notice that rank(Herm$(m, \mathbf{R})) = m$. The result of Proposition 11 is central in [Pol].

Similarly, Proposition 5 yields classical Brickman's theorem [Br].

2. Let $A = \mathbf{C}$. Notice that rank(Herm$(m, \mathbf{C})) = m$, d(Herm$(m, \mathbf{C})) = 2$ In this case tr(C) =Tr(C), where Tr is the usual matrix trace. Hence, for the canonical scalar product we obtain:

$$< C, D >= \mathrm{Tr}(CD), \quad C, D \in \mathrm{Herm}(m, \mathbf{C}).$$

We have:

$$\mathcal{T}(\mathrm{Herm}(m, \mathbf{C})) = \{C \in \mathrm{Herm}(m, \mathbf{C}) : C^2 = C, \mathrm{Tr}(C) = 1\}.$$

Once again $\mathcal{T}(V)$ in this case is the manifold of orthogonal projections on (complex) one-dimension subspaces in $\mathbf{C}^m$. The map $\mu : \mathbf{C}^m \to \mathrm{Herm}(m, \mathbf{C}), \mu(x) = xx^*$, where $x^* = \bar{x}^T$ maps the unit sphere $\mathbf{S}^{2m-1}$ onto $\mathcal{T}(\mathrm{Herm}(m, \mathbf{C}))$. Notice that in this case $\varphi_d(2) = 4$ and all propositions from section 5 admit natural interpretation . For example, Proposition 3 leads to the following result.

Proposition 12 *Let* $q_i(x) = x^* C_i x, x \in \mathbf{C}^m, i = 1, 2, 3$ *be three Hermitian form. Consider the map* $\nu : \mathbf{C}^m \to \mathbf{R}^3$,

$$\nu(x) = (q_1(x), q_2(x), q_3(x)).$$

Then $\nu(\mathbf{C}^m)$ *is a convex cone.*

Remark This result is also known (see e.g. [Pol]).

Proposition 5 takes the following form.

Proposition 13 *Under the assumption of Proposition 12 let* $m \geq 3$. *Then*

$$\nu(\mathbf{S}^{2m-1}) \text{ is convex.}$$

Here $\mathbf{S}^{2m-1} = \{x \in \mathbf{C}^m : x^* x = 1\}$.

Remark This result is in [AP].

Here is an interpretation of Proposition 8.

Proposition 14 *Let* r, m, k *be such that* $r \geq 1, m \geq r + 2$ *and* $1 \leq k < \varphi_2(r + 1) = (r + 1)^2$. *Let* $C_1, \cdots, C_k \in$ *Herm*$(m, \mathbf{C})$. *Consider the map* $\nu : \mathbf{C}^m \to \mathbf{R}^k$, $\nu(x) = (x^* C_1 x, x^* C_2 x, \cdots, x^* C_k x)$. *Then every element of conv*$(\nu(\mathbf{S}^{2m-1}))$ *can be represented as a convex combination of* r *(not necessarily distinct) points of the form* $\nu(x), x \in \mathbf{S}^{2m-1}$.

Remark This result is essentially in [Poon].

The next Proposition immediately follows from Proposition 6.

Proposition 15 *Let $C_0, \cdots C_3 \in Herm(m, \mathbf{C})$ be such that*

$$\sum_{i=0}^{3} \tau_i C_i > 0$$

for some real τ_i. Let, further, $m \geq 3$. Consider the following quadratic optimization problem:

$$q_0(x) \rightarrow \min, q_i(x) = b_i, i = 1, 2, 3, x \in \mathbf{C^m}.$$

Here b_i are some real numbers. Consider, further, its semidefinite relaxation:

$$Tr(C_0 Y) \rightarrow \min, Tr(C_i Y) = b_i, i = 1, 2, 3, Y \geq 0, Y \in Herm(m, \mathbf{C}).$$

Then this semidefinite relaxation is exact.

3. Consider the case $A = \mathbf{H}$.

In principle, the same approach as in 1,2 works here. However, we prefer to work with complex Hermitian matrices. Notice that $d(\text{Herm}(m, \mathbf{H})) = 4$, $\text{rank}(\text{Herm}(m, \mathbf{H})) = m$.

Let $J = \begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix}$. Consider a subalgebra

$$V = \{C \in \text{Herm}(2m, \mathbf{C}) : JC = \bar{C}J\}.$$

It is shown in [FK] (see in particular, p.88 and exercise 1 of Chapter 3) that $\text{Herm}(m, \mathbf{H})$ is isomorphic (as a Jordan algebra) to subalgebra V of $\text{Herm}(2m, \mathbf{C})$.

Let $C = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix}$ be a partition of $2m \times 2m$ matrix with complex entries into four m by m blocks. Then $C \in V$ if and only if $C_1^* = C_1, C_4 = \bar{C}_1, C_2^T = -C_2$ and $C_3 = -\bar{C}_2$ (a direct computation). In other words, a typical element from V looks like this:

$$\begin{bmatrix} C_1 & C_2 \\ -\bar{C}_2 & \bar{C}_1 \end{bmatrix}, \tag{6}$$

where $C_1^* = C_1, C_2^T = -C_2$.

We need to describe $\mathcal{T}(V)$.

Lemma 2 *Let $\xi \in \mathbf{C}^{2m}$ be such that $\xi^* \xi = 1$. Consider $f(\xi) = \xi\xi^* + (J\bar{\xi})(J\bar{\xi})^*$. Then $f(\xi) \in V$, $f(\xi)^2 = f(\xi)$.*

<u>Proof</u> A direct computation.

Lemma 3 *Let $\xi_1, \cdots \xi_m \in \mathbf{C}^{2m}$ be such that $\xi_i^* \xi_i = 1, i = 1, \cdots, m,\ \xi_i^* \xi_j = 0,\ \xi_i^* J\bar{\xi}_j = 0$ for $i \neq j$. Then*

$$f(\xi_i) \circ f(\xi_j) = \delta_{ij} f(\xi_i), i, j = 1, 2, \cdots, m,$$

and

$$f(\xi_1) + \cdots + f(\xi_m) = I_{2m}.$$

<u>Proof</u> Notice that under our assumptions $\xi_1, \ldots, \xi_m, J\bar{\xi}_1, \cdots, J\bar{xi}_m$ form an orthonormal basis in $\mathbf{C}^{2m}$. The result follows by a direct computation.

Since $\text{rank}(V) = m$, we see that $f(\xi_1), \cdots, f(\xi_m)$ form a Jordan frame in V. In particular, $\text{tr}(f(\xi)) = 1$ if $\xi^* \xi = 1$. Since $\text{Tr}(f(\xi)) = 2$, we conclude that

$$\text{tr}(X) = \frac{1}{2}\text{Tr}(X),\ X \in V,$$

where Tr is the usual matrix trace.

Lemma 4

$$\mathcal{T}(V) = \{f(\xi) : \xi \in \mathbf{S}^{4m-1}\}.$$

<u>Proof</u> By lemma 3 we have : $f(\xi) \in \mathcal{T}(V)$ if $\xi^*\xi = 1.$. The connected component of the identity of the group $O(V)$ of (Jordan algebra) automorphisms of V acts transitively on $\mathcal{T}(V)$. In our case

$$O(V) = \{C \in \mathrm{Mat}(2m, \mathbf{C}) : C^* = C^{-1}, JC = \bar{C}J\}.$$

See p. 98 in [FK]. Let $C \in O(V)$. Then

$$C \cdot f(\xi) = C\xi\xi^*C^* + C(J\bar{\xi})(J\bar{\xi})^*C^* = (C\xi)(C\xi)^* + (J\overline{C\xi})(J\overline{C\xi})^* = f(C\xi)).$$

Notice that $C\xi \in \mathbf{S}^{4m-1}$. We see that $O(V)$ maps $\{f(\xi) : \xi \in \mathbf{S}^{4m-1}\}$ onto itself.

Hence the result. ◇

Let us compute

$$\Delta = \mathrm{tr}(C \circ f(\xi)) =< C, f(\xi) > \quad \text{for } C \in V.$$

We have

$$\Delta = \frac{1}{2}\mathrm{Tr}(Cf(\xi)) = \frac{1}{2}\mathrm{Tr}(\xi^*C\xi + (J\bar{\xi})^*C(J\bar{\xi})).$$

Now, $(J\bar{\xi})^*C(J\bar{\xi}) = -\bar{\xi}^*JCJ\bar{\xi} = -\bar{\xi}^*\bar{C}J^2\bar{\xi} = \bar{\xi}^*\bar{C}\bar{\xi} = \overline{\xi^*C\xi}$.

But $\xi^*C\xi$ is real, since C is Hermitian.

Hence, $< C, f(\xi) >= \xi^*C\xi$.

We summarize our results in the following proposition.

Proposition 16 *Consider a realization of Herm*$(m, \mathbf{H})$ *in the form:*

$$V = \{C \in \mathit{Herm}(2m, \mathbf{C}) : JC = \bar{C}J\}.$$

Then

$$\mathcal{T}(V) = \{f(\xi) = \xi^*\xi + (J\bar{\xi})(J\bar{\xi})^* : \xi \in \mathbf{S}^{2m-1}\}.$$

given $C \in V, < C, f(\xi) >= \xi^*C\xi$.

In particular, we see that $\mu : \mathbf{C}^{2m} \to V$, $\mu(\xi) = \xi\xi^* + (J\bar{\xi})(J\bar{\xi})^*$ is such that $\mu(\mathbf{S^{2m-1}}) = \mathcal{T}(V)$.

We see now that Propositions 3 - 9 admit a natural interpretation in terms of convexity of images of families of quadratic forms. Notice that $\varphi_4(2) = 6$.

As an example, consider the reformulation of Proposition 5.

Proposition 17 *Let* $D_1, \cdots, D_5$ *be matrices of the form (6). Let* $q_i(x) = x^*D_ix, x \in \mathbf{C^{2m}}$, $m \geq 3$, *and*

$\mu : \mathbf{C}^{2m} \to \mathbf{R}^5$ *is defined as*

$$\mu(x) = (q_1(x), \cdots q_5(x))^T.$$

Then $\mu(\mathbf{S}^{4m-1})$ *is convex.*

Here $\mathbf{S}^{4m-1} = \{x \in \mathbf{C}^{2m} : x^*x = 1\}$.

Let now $D_0, \cdots, D_5$ be matrices of the form (6) and such that

$$\sum_{i=0}^{5} \tau_i D_i > 0$$

for some real τ_i. The next two propositions immediately follow from Propositions 6,9.

Proposition 18 *Let $m \geq 3$. Consider the following quadratic optimization problem:*

$$q_0(x) \rightarrow \min, q_i(x) = b_i, i = 1, \cdots, 5, x \in \mathbf{C}^{2m}.$$

Here b_i are some real numbers and $q_i(x) = x^* D_i x$. Consider, further, its semidefinite relaxation:

$$Tr(D_0 Y) \rightarrow \min, Tr(D_i Y) = b_i, i = 1, \cdots, 5,$$

$$Y \geq 0, Y \in Herm(m, \mathbf{H}),$$

(i.e. Y is of the form (6)). Then this semidefinite relaxation is exact.

Proposition 19 *Let $m \geq 3$,*

$$\Gamma = \{x \in \mathbf{C^{2m}} : q_i(x) \geq 0, i = 1, \cdots, 5\}.$$

Suppose that there exists $x_0 \in \mathbf{C^{2m}}$ such that $q_i(x) > 0, i = 1, \cdots, 5$.Let, further, $q_0(x) \geq 0, \forall x \in \Gamma$. Then there exist real nonnegative $\lambda_1, \cdots, \lambda_5$ such that

$$D_0 - \sum_{i=1}^{5} \lambda_i D_i \geq 0.$$

Remark Notice that all results obtained from Propositions 3 - 9 for $A = \mathbf{H}$ seem to be new.

4. Consider the last case $A = \mathbf{O}$, $V = \text{Herm}(3, \mathbf{O})$. In this case $\text{rank}(V) = 3, d(V) = 8, \dim V = 27$. Notice that $\varphi_8(2) = 10$.

Let

$$C = \begin{bmatrix} \xi_1 & x_3 & \bar{x_2} \\ \bar{x_3} & \xi_2 & x_1 \\ x_2 & \bar{x_1} & \xi_3 \end{bmatrix}, \quad D = \begin{bmatrix} \eta_1 & y_3 & \bar{y_2} \\ \bar{y_3} & \eta_2 & y_1 \\ y_2 & \bar{y_1} & \eta_3 \end{bmatrix} \in \text{Herm}(3, \mathbf{O}) \tag{7}$$

Here $\xi_i, \eta_i \in \mathbf{R}, \quad x_i, y_i \in \mathbf{O}, i = 1, 2, 3$.

Recall (see e.g. [E]) that octonions can be identified with the pair of quaternions: $z \in \mathbf{O} \Leftrightarrow z = (z_1, z_2), z_1, z_2 \in \mathbf{H}$.

Moreover, if $t = (t_1, t_2) \in \mathbf{O}$, then

$$zt = (z_1 t_1 - \bar{z}_2 t_2, z_2 \bar{t}_1 + t_2 z_1),$$

$\bar{z} = (\bar{z}_1, -z_2)$. The trace operator tr coincides with the matrix Tr ([FK]), p.88-90).

Hence,

$$< C, D >= \text{Tr } (C \circ D) = \text{Tr } \frac{(CD + DC)}{2}.$$

A short computation with C, D in the form (7) yields:

$$< C, D >= \sum_{i=1}^{3} \xi_i \eta_i + \text{Re}(\sum_{i=1}^{3} \bar{x}_i y_i + x_i \bar{y}_i) = \sum_{i=1}^{3} \xi_i \eta_i + 2 \sum_{i=1}^{3} < x_i, y_i >_{\mathbf{O}}$$

.

Here $< x, y >_{\mathbf{O}} = \text{Re}(x\bar{y}), \ x, y \in \mathbf{O}$ Notice that the last equality follows from $\text{Re}(xy) = \text{Re}(yx), x, y \in \mathbf{O}$. See Proposition V.1.2 in [FK].)

As usual,

$$\mathcal{T}(V) = \{C \in V : C^2 = C, \text{Tr}(C) = 1\}.$$

A direct computation (see e.g. [BP]) yields.

Proposition 20 *Let C be parameterized as in (7). Then*
$\mathcal{T}(Herm(3,\mathbf{O})) = \{(x_1,x_2,x_3,\xi_1,\xi_2,\xi_3) \in \mathbf{O}^3 \times \mathbf{R}^3 : \xi_1+\xi_2+\xi_3 = 1,\ \xi_1 = \xi_1^2 + \|x_2\|^2 + \|x_3\|^2,\ \xi_2 = \xi_2^2 + \|x_1\|^2 + \|x_3\|^2,\ \xi_3 = \xi_3^2 + \|x_1\|^2 + \|x_2\|^2,\ \xi_1\bar{x}_1 = x_2x_3,\ \xi_2\bar{x}_2 = x_3x_1,\ \xi_3\bar{x}_3 = x_1x_2\}$
Here $\|x\|^2 = x\bar{x} = \bar{x}x$.

Propositions 3 - 8 are the statements about the convexity of linear images of the manifold $\mathcal{T}(V)$ or its conic hull. Notice that $\dim\mathcal{T}(V) = d(V)(\mathrm{rank}(V)-1) = 16$.

It is not so easy, however, to translate these results to ones about images of quadratic forms. Acting in analogy with the cases of $A = \mathbf{R},\mathbf{C},\mathbf{H}$, we need to consider the map: $\mu : \mathbf{O}^3 \to \mathrm{Herm}(3,\mathbf{O})$,

$$\mu(d_1,d_2,d_3) = (d_i\bar{d}_j), i,j = 1,2,3.$$

Let $\mathbf{S}^{23} = \{(d_1,d_2,d_3) \in \mathbf{O}^3 : \|d_1\|^2 + \|d_2\|^2 + \|d_3\|^2 = 1\}$.

Unfortunately, $\mu(\mathbf{S}^{23}) \neq \mathcal{T}(\mathrm{Herm}(3,\mathbf{O}))$. More precisely $\mathcal{T}(\mathrm{Herm}(3,\mathbf{O})) \subset \mu(\mathbf{S}^{23})$ but the inclusion is strict. Similar construction for $A = \mathbf{R},\mathbf{C},\mathbf{H}$ yields coincidence of corresponding sets.

The problem is due to the fact that multiplication in $\mathbf{O}$ is not associative (see[BP] for details). Nevertheless, we have the following Proposition.

Proposition 21 *Let $\tilde{\mu}$ bee the restriction of μ to $\mathbf{O}\times\mathbf{O}\times\mathbf{R}$. Then $\tilde{\mu}(\mathbf{S}^{16}) = \mathcal{T}(V)$.*
Here $\mathbf{S}^{16} = \{(d_1,d_2,\zeta) \in \mathbf{O}\times\mathbf{O}\times\mathbf{R} : \|d_1\|^2 + \|d_2\|^2 + \zeta^2 = 1\}$, $\|d\| = \sqrt{d\bar{d}}, d \in \mathbf{O}$.

Proof Let us show first that $\tilde{\mu}(\mathbf{S}^{16}) \subset \mathcal{T}(V)$. We simply need to check all conditions of Proposition 20.

We have $\mu(d_1,d_2,\zeta) = (x_1,x_2,x_3,\xi_1,\xi_2,\xi_3)$ is equivalent to:

$\xi_i = \|d_i\|^2, i = 1,2,3,\ x_3 = d_1\bar{d}_2,\ \bar{x}_2 = \zeta d_1, x_1 = \zeta d_2$. Let us check, for example, that $\xi_1 = \xi_1^2 + \|x_2\|^2 + \|x_3\|^2$. We have $\Delta = \xi_1^2 + \|x_2\|^2 + \|x_3\|^2 = \|d_1\|^4 + \zeta^2\|d_1\|^2 + \|d_1\|^2\|d_2\|^2$. Here we used $\|d_1d_2\| = \|d_1\|\|d_2\|$. Hence, $\Delta = \|d_1\|^2(\|d_1\|^2 + \zeta^2 + \|d_2\|^2) = \|d_1\|^2 = \xi_1$.

The other conditions are verified similarly. Let us show that $\tilde{\mu}(\mathbf{S}^{16}) \supset \mathcal{T}(V)$. Let $(\xi_1,\xi_2,\xi_3,x_1,x_2,x_3) \in \mathcal{T}(V)$. Consider, first, the case where $\xi_3 > 0$.

Take $d_1 = \frac{\bar{x}_2}{\sqrt{\xi_3}}, d_2 = \frac{x_1}{\sqrt{\xi_3}}, \zeta = \sqrt{\xi_3}$.

Notice that $\|d_1\|^2 + \|d_2\|^2 + \zeta^2 = \frac{\|x_2\|^2 + \|x_1\|^2 + \xi_3^2}{\xi_3} = 1$ because of the one of the defining relations for $\mathcal{T}(V)$. We can easily check that $\mu(d_1,d_2,\zeta) = (\xi_1,\xi_2,\xi_3,x_1,x_2,x_3)$.

Consider now the case $\xi_3 = 0$. Due to the condition $\xi_3 = \xi_3^2 + \|x_1\|^2 + \|x_2\|^2$, we obtain $x_1 = x_2 = 0$. Hence, we have:

$$\xi_1 = \xi_1^2 + \|x_3\|^2, \quad \xi_2 = \xi_2^2 + \|x_3\|^2, \quad \xi_1 + \xi_2 = 1.$$

This system has two solutions:

$$\xi_1 = \frac{1}{2} \pm \frac{\sqrt{1-4\|x_3\|^2}}{2}$$
$$\xi_2 = \frac{1}{2} \mp \frac{\sqrt{1-4\|x_3\|^2}}{2},$$

provided $\|x_3\| \leq \frac{1}{2}$. Take $d_1 = \frac{\sqrt{\xi_1}x_3}{\|x_3\|}, d_2 = \sqrt{\xi_2}, \zeta = 0$ if $x_3 \neq 0$. It $x_3 = 0$, then $\xi_1 = 0, \xi_2 = 1$ or $\xi_1 = 1, \xi_2 = 0$. In both cases take $d_1 = \sqrt{\xi_1}, d_2 = \sqrt{\xi_2}, \zeta = 0$. We easily check that $\mu(d_1,d_2,\eta) = (\xi_1,\xi_2,\xi_3,x_1,x_2,x_3)$. $\diamond$

We identify $\mathbf{O} \times \mathbf{O} \times \mathbf{R}$ with $\mathbf{R}^{17}$. Consider on $\mathbf{R}^{17}$ quadratic forms of the following type:

$$f_{y,\eta}(d_1, d_2, \zeta) = \eta_1 \|d_1\|^2 + \eta_2 \|d_2\|^2 + \eta_3 \zeta^2$$

$$+2 < y_1, d_2 >_{\mathbf{O}} \zeta + 2 < y_2, \bar{d}_1 >_{\mathbf{O}} \zeta + 2 < y_3, d_1 \bar{d}_2 >_{\mathbf{O}} \zeta. \quad (8)$$

Here $y_1, y_2, y_3 \in \mathbf{O}, \eta_1, \eta_2, \eta_3 \in \mathbf{R}$.

Notice that if D is constructed from $(y_1, y_2, y_3, \eta_1, \eta_2, \eta_3)$ as in (7), then

$$f_{y,\eta}(d_1, d_2, \zeta) = \mathrm{Tr}(D \circ \tilde{\mu}(d_1, d_2, \zeta)).$$

We can now easily reformulate Propositions 3 - 8 in terms of quadratic forms $f_{y,\eta}$ on $\mathbf{R}^{17}$. For example, Proposition 3 leads to the following result. Notice that $\varphi_8(2) = 10$.

Proposition 22 *Let $q_i, i = 1, 2, \cdots 9$, be nine quadratic forms of type (8) on $\mathbf{R}^{17}$ identified with $\mathbf{O} \times \mathbf{O} \times \mathbf{R}$. Consider the map:*

$$\nu(d_1, d_2, \zeta) = (q_1(d_1, d_2, \zeta), \cdots, q_9(d_1, d_2, \zeta)).$$

Then $\nu(\mathbf{R}^{17})$ is a convex cone.

7 Concluding remarks

In the present paper we have considered a large number of classical results related to the convexity of image of quadratic mappings in a general context of Euclidean Jordan algebras. The technique used is a generalization of semidefinite relaxation technique which has been used by A. Barvinok for similar purposes. Our context is more general and allows one to obtain convexity result corresponding the series Herm$(m, \mathbf{H})$ of Euclidean Jordan algebra and exceptional 27-dimensional algebra which seem to be new. The present paper does not exhaust all possibilities offered by Jordan-algebraic technique for the analysis for this circle of questions. We plan to address the remaining issues elsewhere. This research was supported in part by NSF grant DMS-042740.

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