Products, or How to Create Modal Logics of High Complexity

MAARTEN MARX, *Applied Logic Lab, ILLC, Universiteit van Amsterdam, Plantage Muidergracht 24, 1018TV, Amsterdam, The Netherlands. E-mail: marx@wins.uva.nl*

SZABOLCS MIKULÁS, *Department of Computer Science, Birkbeck College, Malet Street, London WC1H 7HX, UK. E-mail: szabolcs@dcs.bbk.ac.uk*

Abstract

The aim of this paper is to exemplify the complexity of the satisfiability problem of products of modal logics. Our main goal is to arouse interest for the main open problem in this area: a tight complexity bound for the satisfiability problem of the product $\mathbf{K}\times\mathbf{K}$. At present, only non-elementary decision procedures for this problem are known. Our modest contribution is two-fold. We show that the problem of deciding $\mathbf{K}\times\mathbf{K}$-satisfiability of formulas of modal depth two is already hard for nondeterministic exponential time, and provide a matching upper bound. For the full language, a new proof for decidability is given which combines filtration and selective generation techniques from modal logic. We put products of modal logics into an historic perspective and review the most important results. [1]

Keywords: modal logic, computational complexity

1 Introduction

Taking products of modal logics is one of the most straightforward ways to combine two or more modal logics. The construction is defined as follows for the basic modal logic $\mathbf{K}$. The product of two $\mathbf{K}$ logics is denoted by $\mathbf{K}^2$ or $\mathbf{K}\times\mathbf{K}$ (pronounced: K-square). The language consists of the propositional connectives $\wedge$ and $\neg$ and the modalities $\diamondminus$ and $\diamondvert$. We will also use the standard abbreviations, e.g., $\boxminus$ for $\neg\diamondminus\neg$.

Frames are defined in the following way. Let (U_0, R_0) and (U_1, R_1) be two $\mathbf{K}$-frames with universes U_0 and U_1 and accessibility relations R_0 and R_1, respectively. Then we form the (binary) *product* $(U_0 \times U_1, \mathsf{H}, \mathsf{V})$ of these frames by defining

$$\begin{aligned} (x,y)\mathsf{H}(u,v) &\iff x\mathsf{R}_0 u \;\&\; y = v \\ (x,y)\mathsf{V}(u,v) &\iff y\mathsf{R}_1 v \;\&\; x = u. \end{aligned}$$

That is, the accessibility relations are defined coordinate-wise. The class of $\mathbf{K}^2$-frames is defined as the class of all binary products of $\mathbf{K}$-frames.

A model $\mathfrak{M}$ is a frame $(U_0 \times U_1, \mathsf{H}, \mathsf{V})$ together with an evaluation $\mathfrak{k} : P \to \mathcal{P}(U_0 \times U_1)$ of the propositional variables. Truth is defined in the usual way — the non-

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propositional cases are: for every $x \in U_0 \times U_1$,

$$\begin{array}{lcl} \mathfrak{M}, x \Vdash \diamondminus\phi & \iff & \mathfrak{M}, y \Vdash \phi \text{ for some } (x,y) \in \mathsf{H} \\ \mathfrak{M}, x \Vdash \diamondbar\phi & \iff & \mathfrak{M}, y \Vdash \phi \text{ for some } (x,y) \in \mathsf{V}. \end{array}$$

We recall from [3] that the logic $\mathbf{K}^2$ is finitely axiomatizable by the standard **K**-axioms in both dimensions together with commutativity (Comm) and confluence (Conf):

PT: (enough) propositional tautologies,

DB: $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$ where $\Box \in \{\boxminus, \boxbar\}$,

Comm: $\diamondminus\diamondbar\phi \leftrightarrow \diamondbar\diamondminus\phi$,

Conf: $\diamondminus\boxbar\phi \to \boxbar\diamondminus\phi$,

and the usual rules of Modus Ponens, Universal Generalization and Substitution.

The first-order frame conditions corresponding to Comm and Conf are:

- *commutativity*, i.e., for every w, x and y with $w\mathsf{H}x\mathsf{V}y$, there is z such that $w\mathsf{V}z\mathsf{H}y$, and for every w, x and y with $w\mathsf{V}x\mathsf{H}y$, there is z such that $w\mathsf{H}z\mathsf{V}y$;
- *confluence*, i.e., for every w, x and y with $w\mathsf{H}x$ and $w\mathsf{V}y$, there is z such that $x\mathsf{V}z$ and $y\mathsf{H}z$.

Hence we will also consider frames $(W, \mathsf{H}, \mathsf{V})$ which are not necessarily products of two **K**-frames but which validate the axioms for $\mathbf{K}^2$ (abstract frames). Using the frame conditions, it is decidable whether a finite abstract model is in fact a model for $\mathbf{K}^2$.

In general, for **X**, **Y** two classes of modal frames, the logic $\mathbf{X}\times\mathbf{Y}$ consists of all products $(U_0 \times U_1, \mathsf{H}, \mathsf{V})$ of frames (U_0, R) in **X** and (U_1, R) in **Y**. For well known modal logics we abuse notation by writing for instance $\mathbf{S5}\times\mathbf{S5}$ for the logic consisting of all products of frames in which R is the universal relation, etc. It is easy to see that the commutativity and the confluence axioms hold for every product logic.

Note that both the commutativity and the confluence condition contain an existential quantifier in the consequent; a notoriously bad sign for the complexity of a modal logic. For example, the satisfiability problem of the uni-modal logic given by the class of frames satisfying the confluence property in one dimension $\forall xyz((Rxy \wedge Rxz) \to \exists w(Ryw \wedge Rzw))$ is complete for nondeterministic exponential time (Hemaspaandra, unpublished). These existential quantifiers also have a bad effect on the interpolation property. Every modal logic axiomatized by Sahlqvist axioms which correspond to universal Horn sentences has interpolation. But interpolation fails for every bi-modal logic defined by a class of product frames which contain at least the finite $\mathbf{S5}\times\mathbf{S5}$ products [10].

But let us first look at the history of products and some special cases. The most heavily studied products are the ones of the form $({}^nW, {}^nW \times {}^nW)$: i.e., n-dimensional powers of frames with the universal relation. These structures are the atom-structures of the full diagonal-free cylindric set algebras whose history goes back to the 40's [5]. Tarski introduced cylindric algebras as the algebraic analogue of first-order logic, just as boolean algebras are the algebraic counterpart of propositional logic. n-dimensional algebras correspond to first-order logic with n variables. For first-order logic with countably many variables we have to resort to ω-dimensional products. We refer to

[5] for more on these connections. Many results about this specific class are known and we review some here. The general trend is that logical properties switch from positive to negative when n becomes larger than two. This holds for decidability and finite Hilbert-style axiomatizability. Interpolation and Beth definability fail already in dimension two [5].

Note that in these products all modalities are *S5* and of course commutativity and confluence hold. In three dimensions also the following (Sahlqvist) formula is valid

$$\Diamond_0 p \wedge \Diamond_1 q \wedge \Diamond_2 r \to \Diamond_0\Diamond_1\Diamond_2[\Diamond_2(\Diamond_1 p \wedge \Diamond_0 q) \wedge \Diamond_1(\Diamond_2 p \wedge \Diamond_0 r) \wedge \Diamond_0(\Diamond_2 q \wedge \Diamond_1 r)]. \tag{1.1}$$

On the frame level the formula corresponds to the condition saying that every configuration $xR_0y_0 \wedge xR_1y_1 \wedge xR_2y_2$ can be extended to a cube. More information on this formula is found in Remark 1.5.22 in [5].

Let us denote with $S5^n$ the inference system defined by the standard modal axioms plus rules for all n diamonds, the *S5*-axioms for every diamond, and the commutativity axiom for every pair of diamonds (since we are in *S5*, the confluence axiom follows). The formula (1.1) does not follow from $S5^n$ for $n \geq 3$. (cf., [5] Construction 3.2.68). Table 1 shows how quickly the satisfiability problem becomes hard.

$S5^1$	$S5^2$	$S5^n$, $n \geq 3$
NP-complete [7]	NEXPTIME-complete [9]	undecidable [8]

TABLE 1. Complexity classes of $S5^n$.

We note that the undecidability result states something stronger: namely every axiomatic system extending $S5^n$ ($n \geq 3$) with axioms valid on the class of n-dimensional powers is undecidable as well.

The study of cylindric algebras from a modal logical perspective started with the dissertation of Yde Venema [15] (Cf. also [11]). Up till very recently, other higher dimensional products than the one mentioned above have not been studied extensively. Recently R. Hirsch, I. Hodkinson and Á. Kurucz showed undecidability and non-finite axiomatizabilty for any class of product logics between $\mathbf{K}^3$ and $\mathbf{S5}^3$ [6].

Dimension two. In two dimensions there is a wide variety of systems which can be called products of modal logic. An early example is [14]. Many of the logics of knowledge and time for multi-agent systems from [2] are products of modal logics, notably with a temporal and an epistemic dimension. One variable fragments of modal predicate logics (with constant domains) can be viewed as products in which one dimension is **S5**. The **S5** diamond will be the existential quantifier. A good reference is the forthcoming book on products [4]. As an example of the kind of theorems in this area we mention a powerful result from [3] (Theorem 7.12) on axiomatizability of binary products. This will be used later in our decision procedures. We need one definition. A *pseudo-transitive* formula is one of the form $\nabla\Box_k p \to \triangle p$ where $\triangle$ is a sequence of (possibly different) boxes and ∇ a sequence of (possibly different) diamonds. A PTC formula is a pseudo-transitive or a closed (containing no propositional symbols) formula. A PTC logic is a logic axiomatized by PTC formulas. Note that PTC logics are canonical. Examples are the well-known logics **D**, **T**, **S4**, **S5** and **B**. The result of Gabbay and Shehtman says that every product of PTC logics is axiomatizable by adding the commutativity and the confluence axioms to the PTC axioms of the components. We stress that this result only holds in the case of two dimensions (see the discussion above).

Decidability and complexity. We now come to the main topic of this paper: decision algorithms for two-dimensional products of modal logics. We will be concerned with the following problem.

C-Satisfiability problem Given a class C of frames and a formula ϕ, is ϕ satisfiable on a model over a frame in C?

The grid-like nature of product frames (as exemplified by the commutativity and the confluence axioms) makes decision procedures for products in general much more expensive in terms of time and space than procedures for the uni-modal logics separately. This is very well documented for multi-dimensional logics with a temporal dimension in which case non-elementary and highly undecidable systems are abundant (cf. [2] and the references therein). Here we review some simpler examples. Both **K**×**S5**-satisfiability and **S5**×**S5**-satisfiability are complete for nondeterministic exponential time [9], while **K**-satisfiability is complete for polynomial space and **S5**-satisfiability even for nondeterministic polynomial time. The product $\mathbf{K4.3}^2$ is even undecidable [13], with **K4.3**-satisfiability again NP-complete. (**K4.3** stands for the class of transitive frames which do not branch towards the future.)

In a certain sense, binary products where one of the components is the class **S5** are very well behaved. At least we can often find decision procedures with a fixed upper bound. See the filtrations for products with an **S5** component in [3] and the mosaic/segment procedures in [12] and [9]. Another easy case are products in which one of the frame classes consists of a functional relation (**KAlt**): in many cases the complexity of the satisfiability problem of the product C×**KAlt** is equal to the complexity of the C-satisfiability problem [9].

Non-elementary procedures. Define $ex_m(k)$ inductively on m, by setting $ex_0(k) = k$ and $ex_{m+1}(k) = 2^{ex_m(k)}$. Intuitively $ex_m(k)$ looks like

$$2^{2^{\cdot^{\cdot^{2^k}}}}$$

where the length of the stack of 2's is m. We call a decision procedure for satisfiability *non-elementary* if for every m there is a formula ξ for which the procedure will take time at least $ex_m(|\xi|)$ for deciding ξ's satisfiability. Unfortunately for product logics in which one of the components is not **S5** or **KAlt** all procedures which are known are non-elementary. We now review what is known. [3] contains a non-elementary procedure for the product **K**×**K**. [16] contains a procedure for **K**× PDL with converse, from which we obtain for instance decidability of **K**×**S4**. We mention here that the decidability of the satisfiability problem for $\mathbf{S4}^2$ and $\mathbf{K4}^2$ is still open. [17] also contains non-elementary procedures for products of **K** with linear tense logics. Below we will provide a non-elementary procedure for $\mathbf{D}^2$. Here we briefly indicate how such a horrible upper bound is achieved.

One way to prove decidability of the satisfiability problem of a modal logic, say modal logic **K**, is to show that it has the strong finite model property. This means that any satisfiable formula ϕ is satisfiable on a finite model whose size is bounded by some recursive function in the length of the input formula ϕ. In modal logic, the technique of filtration is widely used for such proofs. The key ingredient is the following. One takes a model $\mathfrak{M} = (W, R, v)$ which satisfies ϕ. One creates the set $Sub(\phi)$ consisting of all subformulas of ϕ and then one creates a filtrated model $\mathfrak{M}^*$ by factoring W through $Sub(\phi)$ as follows. An equivalence relation $\equiv_{Sub(\phi)}$ on W is defined as

$$x \equiv_{Sub(\phi)} y \text{ iff } x \text{ and } y \text{ satisfy the same formulas in } Sub(\phi) \text{ in model } \mathfrak{M}.$$

The domain of the filtrated model then consists of the set of all $\equiv_{Sub(\phi)}$-equivalence classes. How large is it? There are at most $2^{|Sub(\phi)|}$ equivalence classes, whence the model is bounded by an exponential in the length of ϕ. Of course one still has to define the accessibility relation and the valuation on the filtrated model, and show that it satisfies ϕ and is a model of the logic. Obviously this gets more complicated once we ask for models in which the relations satisfy certain conditions. As we will see below, in the case of products this is highly non–trivial. The only way we can do it at present is mathematically simple and elegant, but a nightmare for resource conscious researchers. Instead of factoring through $Sub(\phi)$, we factor through the set $D_m(\phi)$ defined as follows. Let $\{p_1, \dots, p_n\}$ be the propositional variables in ϕ. Let $d(\phi) = m$ denote the modal depth[2] of ϕ. $D_m(\phi)$ contains all formulas generated from $\{p_1, \dots, p_n\}$ of modal depth less than or equal to m. Clearly $D_m(\phi)$ is infinite, but it is finite up to logical equivalence by an application of the Fraïssé–Hintikka theorem (cf. [1] for a direct proof in modal logic). In fact this exercise gives us an upper bound on the set of all possible satisfiable subsets of $D_m(\phi)$, namely $ex_{m+1}(|\{p_1, \dots, p_n\}|)$. Thus a filtration through $D_m(\phi)$ will lead to a model whose size is bounded by a non-elementary function in $|\phi|$.

Organization of the paper. The paper is organized as follows. We first show how a mix of filtration and selective generation techniques from modal logic leads to decidability by means of the finite (Kripke) model property. The bound on the size of the models is non-elementary. We then show that in $\mathbf{K}^2$ we can create big models with rather simple formulas: we provide a satisfiable formula of modal depth 2 which can only be satisfied on a model whose size is exponential in the length of the formula. We then define a notion measuring the interaction between modalities in a formula: its switching depth. We finish by showing that the $\mathbf{K}^2$-satisfiability problem for formulas of switching depth at most one is complete for NEXPTIME (the class of non deterministic exponential time solvable problems). To conclude we formulate our conjecture about the complexity of the $\mathbf{K}^2$ satisfiability problem.

2 Deciding satisfiability for the whole language

We establish the bounded model property for the product $\mathbf{D}^2$. The proof is a combination of filtration and selective generation techniques used in modal logic (cf., e.g., [1]). Recall that $\mathbf{D}^2$ is defined as $\mathbf{K}^2$ plus the *seriality* condition saying that every world has a successor. We will show that every satisfiable formula can be satisfied in

[2] The modal depth is defined inductively as follows: $d(p) = 0$, for propositional variables p, $d(\neg\phi) = d(\phi)$, $d(\phi \wedge \psi) = \max(d(\phi), d(\psi))$ and $d(\Diamond\phi) = d(\phi) + 1$, for any $\Diamond$.

a finite model for $\mathbf{D}^2$ whose size is computable from the formula. The finite model will be a Kripke model satisfying the $\mathbf{D}^2$ axioms. Since $\mathbf{D}^2$ is finitely axiomatizable, we obtain a decision procedure. The argument below works with minor modifications for $\mathbf{K}^2$ as well. In the case of $\mathbf{K}^2$ one has to take into consideration if a world has horizontal or vertical successors. This can be easily done, but for the sake of brevity we decided to omit these details.

Theorem 2.1 Every $\mathbf{D}^2$-satisfiable formula ξ can be satisfied in a model whose size is bounded by $ex_{d(\xi)+1}(|\xi|)$ (with $d(\xi)$ denoting the modal depth of ξ).

Proof. Let us fix an arbitrary formula ξ with modal depth $d(\xi) = m$. Assume that ξ is satisfied in a model $\mathfrak{M}$ for $\mathbf{D}^2$; i.e., there is a frame $\mathfrak{F} = (W, \mathsf{H}_{\mathfrak{F}}, \mathsf{V}_{\mathfrak{F}})$ and a valuation $\mathfrak{k}$ such that, for some $a \in W$, $\mathfrak{M}, a \Vdash \xi$. As usual in modal logic, we can assume that $\mathfrak{M}$ is generated by the root a.

For every $0 \leq k \leq m$ and $x \in W$, we define a labeling up to modal depth k:

$$\lambda^k(x) = \{\phi \in D_k(\xi) \mid \mathfrak{M}, x \Vdash \phi\}.$$

Recall that $D_k(\xi)$ is the set of all formulas of modal depth less than or equal to k generated from propositional variables in ξ. Note that there are at most $2^{ex_k(|\xi|)}$ different labels $\lambda^k(x)$.

For every $x \in W$, we consider those paths (via the accessibility relations $\mathsf{H}_{\mathfrak{F}}$ and $\mathsf{V}_{\mathfrak{F}}$) leading from a to x which have length at most m. Note that there may be several such paths, since we did not assume anything about the accessibility relations $\mathsf{H}_{\mathfrak{F}}$ and $\mathsf{V}_{\mathfrak{F}}$ (e.g., they might be reflexive).

We define the universe U as the collection of all $\lambda^k(x)$ for $x \in W$ and $0 \leq k \leq m$ for which there is a k-long path from the root a to x. Note that the size of U is bounded by $ex_{d(\xi)+1}(|\xi|)$.

Next we define the accessibility relations for the frame $\mathfrak{G} = (U, \mathsf{H}_{\mathfrak{G}}, \mathsf{V}_{\mathfrak{G}})$: $\mathsf{H}_{\mathfrak{G}}$ and $\mathsf{V}_{\mathfrak{G}}$ are the smallest relations satisfying for all x and y,

- $\lambda^{k+1}(x)\mathsf{H}_{\mathfrak{G}}\lambda^k(y) \iff \diamondminus \bigwedge \lambda^k(y) \in \lambda^{k+1}(x)$;
- $\lambda^0(x)\mathsf{H}_{\mathfrak{G}}\lambda^0(y)$;

and similarly for $\mathsf{V}_{\mathfrak{G}}$. Finally, we define the valuation $\mathfrak{l}$ in the obvious way: for any $\lambda^k(x) \in U$ and propositional variable p occurring in ξ,

$$\lambda^k(x) \in \mathfrak{l}(p) \iff p \in \lambda^k(x).$$

Now we claim that $\mathfrak{G}$ is a frame for $\mathbf{D}^2$. It suffices to show that it satisfies the first-order frame conditions commutativity, confluence and seriality.

First assume that we have

$$\lambda^k(x)\mathsf{V}_{\mathfrak{G}}\lambda^{k'}(y)\mathsf{H}_{\mathfrak{G}}\lambda^{k''}(z).$$

We have to show that there is $\lambda^l(u)$ such that

$$\lambda^k(x)\mathsf{H}_{\mathfrak{G}}\lambda^l(u)\mathsf{V}_{\mathfrak{G}}\lambda^{k''}(z).$$

If $k > 1$, then $k'' = k' - 1 = k - 2$. By the definition of $\mathsf{H}_{\mathfrak{G}}$ and $\mathsf{V}_{\mathfrak{G}}$, we have that $x'\mathsf{V}_{\mathfrak{F}}y'\mathsf{H}_{\mathfrak{F}}z'$ in $\mathfrak{F}$ for some $x', y', z' \in W$ such that $\lambda^k(x) = \lambda^k(x')$, $\lambda^{k-1}(y) = \lambda^{k-1}(y')$,

$\lambda^{k-2}(z) = \lambda^{k-2}(z')$. Since $\mathfrak{F}$ satisfies commutativity, there is u such that $x'\mathsf{H}_{\mathfrak{F}}u\mathsf{V}_{\mathfrak{F}}z'$. Thus $\lambda^{k-1}(u)$ meets the requirement. If $k = 1$, then for any $\mathsf{H}_{\mathfrak{F}}$-successor u of x, $\lambda^0(u)$ is suitable. Finally, in case $k = 0$, $\lambda^0(y)$ is a good candidate.

The same argument works for the other version of commutativity. Confluence can be shown by using a similar case distinction. Finally, seriality holds by the definition of $\mathsf{H}_{\mathfrak{G}}$ and $\mathsf{V}_{\mathfrak{G}}$ and the fact that $\mathfrak{F}$ is a serial frame.

Next we claim that $\mathfrak{N} = (U, \mathsf{H}_{\mathfrak{G}}, \mathsf{V}_{\mathfrak{G}}, \mathfrak{l})$ satisfies ξ. We show a truth-lemma stating that for every $\phi \in D_k(\xi)$ and $\lambda^k(x) \in U$,

$$\phi \in \lambda^k(x) \Longleftrightarrow \mathfrak{N}, \lambda^k(x) \Vdash \phi.$$

The only non-trivial case is if ϕ is a diamond formula, say, it has the form $\Phi\psi$.

First assume that $\Phi\psi \in \lambda^{k+1}(x)$. Then $\mathfrak{M}, x \Vdash \Phi\psi$, whence there exists y such that $x\mathsf{V}_{\mathfrak{F}}y$ and $\mathfrak{M}, y \Vdash \psi$. Then, by the induction hypothesis, $\mathfrak{N}, \lambda^k(y) \Vdash \psi$. $x\mathsf{V}_{\mathfrak{F}}y$ implies that $\Phi \bigwedge \lambda^k(y) \in \lambda^{k+1}(x)$, whence $\lambda^{k+1}(x)\mathsf{V}_{\mathfrak{G}}\lambda^k(y)$. Thus we have $\mathfrak{N}, \lambda^{k+1}(x) \Vdash \Phi\psi$.

For the other direction assume that $\mathfrak{N}, \lambda^{k+1}(x) \Vdash \Phi\psi$. Then there is $\lambda^k(y) \in U$ such that $\lambda^{k+1}(x)\mathsf{V}_{\mathfrak{G}}\lambda^k(y)$ and $\mathfrak{N}, \lambda^k(y) \Vdash \psi$. By the induction hypothesis, $\psi \in \lambda^k(y)$. By the definition of $\mathsf{V}_{\mathfrak{G}}$, $\Phi \bigwedge \lambda^k(y) \in \lambda^{k+1}(x)$. Then $\mathfrak{M}, x \Vdash \Phi \bigwedge \lambda^k(y)$. Hence $\mathfrak{M}, x \Vdash \Phi\psi$, i.e., $\Phi\psi \in \lambda^{k+1}(x)$ as desired.

Since our fixed formula ξ occurred in the label $\lambda^m(a)$ of the root, we have that $\mathfrak{N}$ satisfies ξ. ■

The above argument shows that any satisfiable formula is satisfied in a finite model for $\mathbf{D}^2$. Since the size of the model is computable from the formula and it is decidable if a finite model is a model for $\mathbf{D}^2$, decidability of $\mathbf{D}^2$ follows.

3 Formulas of depth less than two

We just saw that our decision procedure grows exponentially with the modal depth of the formula. Here we look at formulas with the smallest modal depth where interesting things (that is, interaction between the modalities) can happen. Instead of modal depth we use a better measure of the interaction between the modalities in a formula, its *switching depth*. The switching depth of a formula ϕ is the minimum of its Φ-depth and its $\ominus$-depth. E.g., the switching depth of $\Phi\Phi p$ is 0, of $\Phi\ominus p$, $\Phi\Phi\ominus p$ and $\Phi\ominus\ominus p$ all 1, but of $\Phi\Phi\ominus p \wedge \Phi\ominus\ominus p$ it is two. We provide a simple formula of switching depth one and modal depth two which —when satisfied— causes that the point of evaluation has exponentially many successors in the length of that formula. We then continue to show that this leads to a non-deterministic exponential time lower bound for satisfiability problem of such formulas. We finish by providing a matching upper bound.

Theorem 3.1 For every n, there exists a $\mathbf{K}^2$-satisfiable formula of switching depth one, modal depth 2 and of length[3] $O(n^2 \log(n))$ which can only be satisfied on (Kripke) models containing at least 2^n elements.

Proof. We start by defining a model in which the formula will hold. The model gives already some insight in the ideas behind the formula. Then we present the formula[4]

[3] We write $f(n) = O(g(n))$ if there is a constant $c > 0$ such that $f(n) \leq c \cdot g(n)$, for all $n \geq 0$.

[4] We assume that the indices of propositional variables p_i are in binary. Thus in a language with n variables $p_1 \dots p_n$, each variable can be written in space $O(\log(n))$.

itself.

Let $\mathcal{T}_n = (T_n, <)$ denote the binary branching tree of depth n. For variables $p_1, \ldots, p_n$ and $d_0, \ldots, d_n$, the standard valuation of these variables is given as follows:

- p_i is true at $x \in T_n$ iff the i-th bit of the binary representation of x is 1 (with p_1 the most significant bit). (If x does not have i bits, p_i is false at x.)
- d_i holds at $x \in T_n$ iff x is i-steps away from the root.

We create the following model $\mathfrak{M}_n = (W, H, V, v)$. W is the disjoint union of the sets

$$\begin{array}{lcl} W_0 & = & \{r\} \\ W_1 & = & T_n \\ W_2 & = & T_n \\ W_3 & = & T_I \uplus T_{II} \uplus \{\mathit{dummy}\}, \text{ where } T_I = T_n \text{ and } T_{II} = T_n \setminus \{\mathit{root}\}. \end{array}$$

The accessibility relations are defined as

$$\begin{array}{lcl} H & = & W_0 \times W_2 \\ & \cup & \{(x, y) \in W_1 \times W_3 \mid x = y \text{ or } y = \mathit{dummy}\} \\ V & = & W_0 \times W_1 \\ & \cup & \{(x, y) \in W_2 \times T_I \mid x = y\} \\ & \cup & \{(x, y) \in W_2 \times T_{II} \mid x < y\} \\ & \cup & W_2 \times \{\mathit{dummy}\}. \end{array}$$

An easy verification shows that

$$(W, H, V) \text{ satisfies commutativity and confluence.} \tag{3.1}$$

For a given n, we use $4n + 3$ propositional variables: $p_1, \ldots, p_n$ for counting and d_i, d_i^I, d_i^{II} for $0 \le i \le n$ to determine the depth in the binary branching tree. In $\mathfrak{M}_n$ they are evaluated as follows:

- The p_i are true at the sets W_1, W_2, T_I and T_{II} just as in the standard valuation on the binary tree of depth n. Everywhere else they are false.
- The d_i are false everywhere except in $W_1 \cup W_2$. Here they get their valuation according to the binary tree.
- d_i^I marks the depth in T_I. d_i^I is false everywhere else.
- d_i^{II} marks the depth minus one in T_{II}. d_i^{II} is false everywhere else.

Thus we have defined a model $\mathfrak{M}_n$.

Now we define a satisfiable formula ϕ_n which is such that $\mathfrak{M}_n$ is the smallest model in which it can be satisfied. ϕ_n is the conjunction of the following formulas.

$$\begin{array}{lll} \mathit{start} & \neg d_0 \wedge \diamondbar d_0 & \\ \mathit{copy}^{HI} & \boxbar(d_i \to \diamondminus d_i^I) & (0 \le i \le n) \\ \mathit{copy}^{VI} & \boxminus(\diamondbar d_i^I \to d_i) & (0 \le i \le n) \\ \mathit{branch} & \boxminus(d_i \to \diamondbar(d_i^{II} \wedge p_{i+1}) \wedge \diamondbar(d_i^{II} \wedge \neg p_{i+1})) & (0 \le i < n) \\ \mathit{copy}^{HII} & \boxbar(\diamondminus d_i^{II} \to d_{i+1}) & (0 \le i < n) \end{array}$$

The order of these formulas and their names correspond to what happens when one tries to satisfy this conjunction: a kind of ping-pong behavior from the vertical successors to the horizontal successors of the root and vice versa. The next set of formulas will make sure that when creating a model, we cannot re-use worlds and in fact we obtain copies of the binary branching tree of depth n as horizontal successors and vertical successors of the root of the model. We use the abbreviation $\pm p$ as follows: a formula ϕ in which $\pm p$ occurs abbreviates the conjunction $\phi(p) \wedge \phi(\neg p)$, in which we substitute p and $\neg p$, respectively for $\pm p$.

mem−n	$\boxminus(d_i \wedge \pm p_j \rightarrow \boxbar(d_i^{II} \rightarrow \pm p_j))$	$(0 \leq j \leq i \leq n)$
mem−w	$\boxbar(\diamondminus(d_i^{II} \wedge \pm p_j) \rightarrow \pm p_j)$	$(0 \leq j \leq i \leq n)$
mem−e	$\boxbar(d_i \wedge \pm p_j \rightarrow \boxminus(d_i^{I} \rightarrow \pm p_j))$	$(0 \leq j \leq i \leq n)$
mem−s	$\boxminus(\diamondbar(d_i^{I} \wedge \pm p_j) \rightarrow \pm p_j)$	$(0 \leq j \leq i \leq n)$.

The last conjunct *dis* states that all the d_i, d_i^I, d_i^{II} are evaluated disjointly.

Let ϕ_n be the conjunction of all these formulas. The length of ϕ_n is $O(n^2 \log(n))$. The formula ϕ_n is satisfied in the model $\mathfrak{M}_n$ at the root, as the reader can easily verify. Moreover the formula does what we want: if $\mathfrak{M}, w \Vdash \phi_n$, then w has an isomorphic copy of T_n as a subset of its vertical successors. We prove this by induction. For $n = 0$, this holds by the formula *start*. Suppose it holds for n, and $\mathfrak{M}, w \Vdash \phi_{n+1}$. Now apply $copy^{HI}$ and $mem - e$, commutativity, $copy^{VI}$ and $mem - s$, *branch* and $mem - n$, and finally commutativity, $mem - w$ and $copy^{HII}$ to obtain the desired result. Note that the confluence axiom is not even needed to get an exponential number of successors. ∎

The formula ϕ_n quickly leads to the following result, in which the confluence axiom is heavily used.

Theorem 3.2 The $\mathbf{K}^2$-satisfiability problem for formulas of switching depth at most one is hard for nondeterministic exponential time.

We give a proof sketch only. [9] contains a reduction to the square tiling problem (with the boundary given in binary notation) which shows NEXPTIME-hardness for arbitrary deep $\mathbf{K}^2$-satisfiable formulas. Using the formula ϕ_n above, we can easily redo that tiling proof in modal formulas of switching depth at most one. Comparing the colors of tiles and ensuring that they match will be done in the region where before we had the dummy point. In more detail, instead of one dummy point we have a set $T_n \times T_n$. We then set the relations as follows: for $x \in W_1$, $y \in W_2$, $(a, b) \in T_n \times T_n$, $xH(a, b)$ iff x and a have the same value in the binary tree, and $yV(a, b)$ iff y and b have the same value in the binary tree. We now make sure that the tiling is expressed both in W_1 and W_2 in the same way. Using the connections in $T_n \times T_n \subseteq W_3$ we make sure that colors match. We leave the details to the reader.

We now show the corresponding upper bound for formulas of switching depth at most one. A variation on this proof can be used to show that satisfiable formulas ξ of modal depth less than or equal to two can be satisfied in models whose size is exponential in $|\xi|$.

Theorem 3.3 (i) Every $\mathbf{K}^2$-satisfiable formula ξ of switching depth at most one can be satisfied on a model of size bounded by $2^{O(|\xi|^2)}$.

(ii) Thus the $\mathbf{K}^2$-satisfiability problem for formulas of switching depth at most one is complete for NEXPTIME.

PROOF. Let ξ be of switching depth at most one and satisfied in a model $\mathfrak{M}$ at state r and $\mathfrak{M}$ a model over a frame of the form $(U_0 \times U_1, H, V)$. Let ξ's $\diamondbar$-depth be one and its $\diamondminus$-depth be k. The other case when the $\diamondminus$-depth is one is treated similarly.

The case in which one of the depths is zero is left to the reader. A *path* in $\mathfrak{M}$ is a finite sequence of V and H transitions. A sequence of m H-transitions is denoted by H^m. For any path $P = Z; H^m$, where $Z \in \{V, \epsilon\}$, define $P_{\mathfrak{M}} = \{x \in M \mid rPx\}$. We will define a filtration $\mathfrak{M}^*$ of $\mathfrak{M}$ by filtrating each set $P_{\mathfrak{M}}$. Let $\equiv$ be the equivalence relation on states in $\mathfrak{M}$ defined by $x \equiv y$ if x and y satisfy the same subformulas of ξ in $\mathfrak{M}$.

We recursively define $P^*_{\mathfrak{M}}$ as follows:

- $\epsilon^*_{\mathfrak{M}} = (H^0_{\mathfrak{M}})^* = \{r\}$;
- $(V; H^m_{\mathfrak{M}})^* = \bigcup_{x \in (H^m_{\mathfrak{M}})^*} (\{y \in V; H^m_{\mathfrak{M}} \mid xVy\}/\equiv)$;
- $(H^{m+1}_{\mathfrak{M}})^*$ is the smallest set such that
  - $\bigcup_{x \in (H^m_{\mathfrak{M}})^*} (\{y \in H^{m+1}_{\mathfrak{M}} \mid xHy\}/\equiv) \subseteq (H^{m+1}_{\mathfrak{M}})^*$ and
  - for all $z \in (H^m_{\mathfrak{M}})^*$, for all $x \in (V; H^m_{\mathfrak{M}})^*$, if zVx and xHy, then there exists a $w \in (H^{m+1}_{\mathfrak{M}})^*$ such that zHw and wVy' for some $y' \equiv y$.

Now define the model $\mathfrak{M}^*$ as follows. Its domain consists of all elements from the sets $P^*_{\mathfrak{M}}$ for P a path $Z; H^m$ with m less than or equal to the $\ominus$-depth of ξ. It is convenient to create the following abbreviation: $x\tilde{H}y$ abbreviates "for every subformula $\ominus\phi$ of ξ, $\mathfrak{M}, x \not\Vdash \ominus\phi \Rightarrow \mathfrak{M}, y \not\Vdash \phi$". Note that xHy and $y \equiv y'$ and $x \equiv x'$ implies that $x\tilde{H}y', x'\tilde{H}y'$ and $x'\tilde{H}y$. The accessibility relations in $\mathfrak{M}^*$ are defined as follows.

- $V^* = V_{\restriction \mathfrak{M}^*}$ and
- $H^* = H_{\restriction \mathfrak{M}^*} \cup \{(x, y) \mid (\exists z, w \in M^*) : zVx, zHwVy \text{ and } x\tilde{H}y\}$

The valuation of the propositional variables is set just as in $\mathfrak{M}$. So we can view $\mathfrak{M}^*$ as a substructure of $\mathfrak{M}^*$ with some additional H relations added. The next claim says that $\mathfrak{M}^*$ satisfies our purposes.

Claim (i) ξ is satisfied in $\mathfrak{M}^*$.
(ii) $\mathfrak{M}^*$ is a model satisfying the confluence and commutativity properties.
(iii) The size of the domain of $\mathfrak{M}^*$ bounded by $2^{O(|\xi|^2)}$.

For (i) we show a truth lemma relativised by vertical and horizontal depth: for all subformulas ϕ of ξ of $\ominus$ depth less than m,

- for all $x \in (H^l_{\mathfrak{M}})^*$, for l smaller than the modal depth of ξ minus m: $\mathfrak{M}, x \Vdash \phi \iff \mathfrak{M}^*, x \Vdash \phi$
- if ϕ of $\oplus$ depth zero, then for all $x \in (V; H^l_{\mathfrak{M}})^*$, for l smaller than the modal depth of ξ minus m: $\mathfrak{M}, x \Vdash \phi \iff \mathfrak{M}^*, x \Vdash \phi$.

The proof of the truth lemma is by a double induction on the complexity of the formula. All cases in the first part go trivially through because a witness for every V and H successor is added in the filtration and the relations from states in $(H^l_{\mathfrak{M}})^*$ are all coming from the model. For the second part, the only non trivial case is when $x \in (V; H^l_{\mathfrak{M}})^*$ and $\mathfrak{M}, x \Vdash \ominus\psi$.

Then there exists a y such that xHy and $\mathfrak{M}, y \Vdash \psi$. Let $z \in (H^l_{\mathfrak{M}})^*$ such that zVx. By definition of M^*, there exists $w, y' \in M^*$ such that $y' \equiv y$ and $zHwVy'$. Thus also $x\tilde{H}y'$, and xH^*y'. By the inductive hypothesis $\mathfrak{M}^*, y' \Vdash \psi$, whence by the truth definition, $\mathfrak{M}^*, x \Vdash \ominus\psi$. This finishes the proof of part (i).

We continue with part (ii). We start with confluence. Let zV^*x and zH^*w. Then zVx and zHw, thus by confluence in $\mathfrak{M}$, there is a y such xHy and wVy. But then we have for some $y' \equiv y$ and wVy', that $y' \in M^*$, whence also wV^*y'. Since xHy and $y \equiv y'$, $x\tilde{H}y'$ holds. But then xH^*y', as desired.

For commutativity we have to check two cases. The case zV^*xH^*y holds directly by construction. For the other case suppose zH^*wV^*y. Then $zHwVy$. So there exists a x in M such that $zVxHy$. Then in M^* there is a $x' \equiv x$ such that zVx'. Since xHy and $x \equiv x'$, $x'\tilde{H}y$ holds. But then $x'H^*y$, as desired.

Finally we compute the size of M^*. Let m be the number of $\equiv$-equivalence classes in $\mathfrak{M}$ and k the $\ominus$-depth of ξ. Let $l = 1 + m + 2m + m \cdot 2m$. l is an upper bound for the size of the union of the sets $\epsilon^*_{\mathfrak{M}}, V^*_{\mathfrak{M}}, \epsilon; H^*_{\mathfrak{M}}, V; H^*_{\mathfrak{M}}$. In the recursive definition of the $(H^{m+1}_{\mathfrak{M}})^*$, at most l elements for every element in $(H^m_{\mathfrak{M}})^*$ are chosen. Since we only go up to depth $k \leq |\xi|$, we obtain that the size of M^* is less than $k \cdot l^k$. Thus as m is bounded by $2^{|\xi|}$, $|M^*|$ is bounded by $2^{O(|\xi|^2)}$.

Thus the claim has been proved, whence part (i) of the Theorem.

The lower bound of part (ii) of the Theorem follows from Theorem 3.2. The upper bound follows from part (i) in the standard way, since $\mathbf{K}^2$ Kripke models are finitely first order axiomatizable. ∎

Conjecture

We conjecture that the satisfiability problem for $\mathbf{K}^2$ has a non–elementary lower bound. In particular, we conjecture that the satisfiability problem for formulas of switching depth n is hard for non deterministic n exponential time. The corresponding upper bound can be obtained by a filtration similar to the one in the proof of Theorem 2.1.

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