On Completeness of a Positional Interval Logic with Equality, Overlap and Subinterval Relations

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Abstract

This paper presents an interval-based positional logic and proves its completeness. The proposed logic combines positional operators, defined by time intervals, with modal operators used to express the subinterval relationship. Moreover, the logic can be easily extended either by including more predicates for absolute temporal references or by adding other operators for relative and periodical temporal references. The positional logic can be used to express and reason about time-stamped temporal information, particularly stored in temporal databases. The underlying time structure is motivated by the common-sense calendar-clock style time, where multiple time granularities to any precision, often involved in temporal information, can be very naturally supported. Although each positional/modal operator is normal, the completeness proof of the proposed positional logic is more complicated than the standard completeness proof of normal modal logics.

Keywords: temporal data, time intervals, positional operators, absolute and relative temporal references, completeness

1 Introduction

This paper concerns representing and reasoning about temporal information. A typical system for temporal information processing is a temporal database system, where temporal information is represented into tuples with time stamps, and manipulated by use of a temporal query language [25]. Research into temporal information processing has been very active, and various approaches to temporal databases have been proposed [18, 26, 27], among which 12 different temporal relational algebras and their underlying temporal relational data models are summarized and evaluated in [19]. A recent account of temporal databases can be found in [28]. The work in this paper is concerned with a positional logic which can be used to represent time-stamped information stored in temporal databases.

It is well-known that first-order logic can be used as a logical basis for conventional relational databases, as explained as follows [7, 20, 22]: a relation tuple in a relational database can be represented by an atomic logical assertion in first-order logic, and an atomic logical assertion in first-order logic can also be represented by a relation tuple in a relational database. Thus, on one hand, a relational database can be considered as a finite model of a first-order theory, and evaluating a query amounts to selecting the ground instances of the query that are true in the model in question. On the other hand, a relational database can also be described as a theory expressed in a first-order language, and evaluating a query then amounts to selecting the ground instances of the query that are theorems in the theory. When a time dimension is incorporated into relational databases to get so-called temporal relational databases, which logic can play the same role for temporal relational databases as first-order logic plays for non-temporal relational databases? The answer depends not only on theoretical measures such as expressiveness and deduction capability but also on aesthetic standards. For several reasons first-order logic fails to serve as a satisfactory basis for temporal databases (see Section 2.5 of [13]). The work in [6] made an attempt to define a temporal relational algebra TRA with a close correspondence to a temporal logic US using temporal operators until and since of [5]. The temporal relational algebra TRA has the temporal structure introduced in [29] as its underlying data model, according to which a temporal (historical) database DT is considered as a series of relational databases $\{D_t \mid 0 \leq t \leq n\}$, where the subscript t denotes the time associated with the particular database.

Independent of Gabbay and McBrien's work, we are interested in positional logics as logical basis for temporal databases. Positional logics, as temporal logics, vary with the underlying time structures: discrete or dense, linear or branching, point-based or interval-based. The linear and discrete point-based time structure is the simplest one, and widely used in temporal databases. But one obvious drawback of the linear and discrete point-based time structure is that the smallest time unit is fixed and regarded as a point. In many database applications, multiple time granularities are used [32]. That is to say, the smallest time unit in database applications is not fixed. In some cases such as personnel archive database systems, time with date as granularity might be used, and in others such as scientific experiment recording systems, time with second as granularity might be used. To allow for multiple time granularities, we directly use the common-sense calendar-clock style time as our underlying time structure. It will be seen in Section 2 that the common-sense calendar-clock style time can be one-to-one mapped to time based on pairs of rational numbers.

In general, a time structure S can be represented by a pair (T, R), where T is a set of time elements (points or intervals) and R is a set of relations on T. In the case of points, R usually includes a linear ordering relation $<$ to express the before-after order. In the case of intervals, R may be defined to be the set of the thirteen binary interval relations of [1], or a set of other interval relations, for example, those of [30]. After a time structure S is assumed and axiomatized, a positional logic based on S can be defined as an extension to classic logic by introducing some operators to express temporal references to time. Positional logics date back to [23], where there was a single operator, called temporal realization operator, denoted by $R(t)$, that reads, 'it is the case at time t that ...', where t is a time. Clearly enough, it is very natural and easy to represent temporal databases by use of positional logics.

Example 1.1
Let **Faculty(Name, Rank)** be a temporal relation schema. Consider the following temporal relation table:

Faculty(	Name,	Rank,	Time)
	John	Associate Professor	92
	Mary	Assistant Professor	92

Where the entry under **Time** denotes the time stamp to indicate the valid time of the corresponding tuple.

We want to represent the above temporal relation table into logical formulas. The procedure is simply illustrated as follows: first, for the temporal relation schema **Faculty(Name, Rank)**, we introduce a temporal predicate symbol FACULTY of sort $< names, ranks >$, where *names* is the sort for all the possible personal names, and *ranks* is the sort for all the possible academic ranks; second for each tuple in the table, we introduce an assertion:

$$\mathbf{R}(\%1992\%)\ \text{FACULTY}(\text{`John', `Associate Prof'}) \tag{1.1}$$

$$\mathbf{R}(\%1992\%)\ \text{FACULTY}(\text{`Mary', `Assistant Prof'}) \tag{1.2}$$

where %1992% is intended to represent the year 1992.

In the example above, it is assumed that the smallest time unit is the year. Now consider the following simple query on the relation above: Who are associate professors in September, 1992. Then it can be seen that the temporal realization operator $\mathbf{R}(\%1992\%)$ is too weak to express the *internal structure* of %1992%. This problem can be solved by defining a positional logic with an underlying linear and dense interval-based time structure.

In [14], a temporal formalism is described to specify dynamic behaviour and requirements with temporal references to absolute time, where there are two positional operators, denoted by $[\![\xi\ ]\!]$ and $\big[\xi\big]$ for an interval ξ, which respectively mean that '... is true on every subinterval of ξ' and '... is true on ξ and on all its subintervals'. Using $\big[\xi\big]$ we can represent the relation table above into the following two assertions:

$$\big[\%1992\%\big]\text{FACULTY}(\text{`John', `Associate Prof'}) \tag{1.3}$$

$$\big[\%1992\%\big]\text{FACULTY}(\text{`Mary', `Assistant Prof'}) \tag{1.4}$$

Then, we can answer the query 'Who are Associate Professors in September, 1992' by performing some reasoning procedure based on an axiomatic system for the positional logic with $[\![\xi\ ]\!]$ and $\big[\xi\big]$ as operators. The axioms and inference rules for $[\![\xi\ ]\!]$ and $\big[\xi\big]$ can be found in [14].

In most cases, $[\![\xi\ ]\!]$ and $\big[\xi\big]$ are very expressive. But for non-hereditary information (sometimes also called non-homogeneous information by some authors), $[\![\xi\ ]\!]$ and $\big[\xi\big]$ are weak. For example, when we say that the average temperature of Lisbon is 10°C in January, we do not mean the average temperature of Lisbon is 10°C in the first week of January. For non-hereditary information we introduce another operator, denoted by $\big(\xi\big)$, that means '... is true on interval ξ'. Note that $\big(\xi\big)$ and $\mathbf{R}(t)$ are very similar. The only difference between $\big(\xi\big)$ and R(t) may be that ξ in $\big(\xi\big)$ must be an interval, but t in $\mathbf{R}(t)$ was supposed to be a point. After $\big(\xi\big)$ is introduced, $\big[\xi\big]$ becomes a derived operator: $\big[\xi\big]A$ stands for the formula $[\![\xi\ ]\!]A \wedge \big(\xi\big)A$. Thus, the operators $[\![\xi\ ]\!]$ and $\big(\xi\big)$ may be considered as primitive operators for representation of time-stamped relations (tables). The axioms and inference rules for $[\![\xi\ ]\!]$ and (ξ) can be found in [13].

In addition to time-stamped relations (tables), we also need to consider integrity constraints, which can be denoted by using the usual temporal logic. Thus, we need to unify positional logic and temporal logic in order to represent temporal databases and their integrity constraints. In the case of interval temporal logic, we might need 12 unary temporal operators to express the 12 mutually exclusive relations between two different intervals.

According to [8], Halpern and Shoham's three pairs of temporal operators corresponding to the interval relations *meets, starts* and *finishes* are sufficient to define other three pairs of temporal operators. A comprehensive study of interval temporal logic can be found in [31]. From the 12 unary temporal operators, we can derive many other temporal modal operators to express other disjunctive relations between two intervals, one of which is the operator $\langle\!\langle\rangle\!\rangle$ corresponding to the subinterval relation. $\langle\!\langle\rangle\!\rangle A$ is true on the current observation interval iff A is true on some subinterval of it. The dual operator of $\langle\!\langle\rangle\!\rangle$ is defined as usual: $[\![\ ]\!]A =_{df} \neg \langle\!\langle\rangle\!\rangle\neg A$. After the operator $[\![\ ]\!]$ is introduced, $[\![\xi\ ]\!]$ becomes a derived one since $[\![\xi\ ]\!]A$ and $(\xi)\ [\![\ ]\!]A$ express the same thing.

In this paper we shall consider the complete axiomatization of $[\![\ ]\!]$ and (ξ), and prove the completeness of a propositional temporal logic denoted by BAR, by which we mean Basic formalism for both Absolute and Relative temporal references. The rest of this paper is organized as follows: in Section 2 we briefly discuss the common-sense calendar-clock style time; in Section 3 we present the syntax, semantics and axiomatization of BAR; in Section 4 we prove the completeness of BAR; in Section 5 we make some discussions on related topics for practical applications; and finally in Section 6 we give a short summary of this paper. It should be stressed that completeness is only one of fundamental properties of a logical system. In this paper, however, no effort is made towards other fundamental properties such as decidability, complexity and expressiveness. Completeness is an important and difficult issue on its own, which justifies the interest of this paper.

2 Common-sense time

The nature, understanding and representation of time are, in all the history, very debatable. van Benthem [30] gave a model-theoretic study of temporal ontology and temporal discourse. In temporal databases, the common-sense calendar-clock style time is widely used. For example, '1992/10/2' is a valid calendar-clock style time to represent October 2, 1992. In this paper we choose the calendar-clock style time with arbitrary precision as our intended time structure. According to Ladkin [10, 11] the calendar-clock style time with arbitrary precision can be isomorphically described by rationals-based time structure. Thus, our time structure is in fact a rationals-based time structure. In the following we first show how the calendar-clock style time is related to rational numbers, then we list some properties of the calendar-clock style time, which will be imposed on the interpretation structures of the logic BAR. The techniques used here are adapted from those in [10, 11].

Mathematically, the calendar-clock style time can be represented by a tuple <year, month, day, hour, minute, second, ...>, where the first component denotes the year, the second is the month in that year, the third is the day in that month in that year, etc. For example, <1992, 5, 26, 9, 30, 0> represents 9:30:00 on May 26, 1992. Of course, May 26, 1992 is also a time unit and can be analogously represented by the tuple <1992, 5, 26>.

Times represented by tuples are called basic time units. In addition to the basic time units, one can introduce an operator, say $convexify$, to make arbitrary time intervals. Informally, $convexify(i, j)$ denotes the smallest interval that contains both i and j. For example, $convexify(< 1989 >, < 1992 >)$ denotes the time interval containing all the years from 1989 to 1992. As another example, $convexify(< 1992, 1 >, < 1992, 12 >)$ denotes the interval from January, 1992 to December, 1992, i.e. the year 1992. Thus, $convexify(< 1992, 1 >, < 1992, 12 >) = < 1992 >$. Obviously, all the practical calendar-clock time intervals can be represented by either a basic time unit or by an expression formed by $convexify$ and basic time units. All these intuitive concepts will be made more precise below.

To define the basic time units $\Im$:

Let $f_1, f_2, \ldots$ be an infinite collection of integer functions, and f_k be of arity k for each $k \geq 1$. For simplicity, we identify the arguments of f_k with a single sequence of length k, so we can write $f_k(s)$, where s is a sequence of length k. We require that for any $k \geq 1$, for any sequence s of length k, $f_k(s) \geq 2$. In practice, some time units start to count from 1 and others from 0. For example, months and days are counted from 1, while hours and minutes are counted from 0. For later development, we define the following function to indicate types of time units: $type : \{f_1, f_2, ...\} \rightarrow \{1, 0\}$. Then, the set $\Im$ of basic time units is defined through the following procedure:

- $\Im_1 = \{< n >: n \in Z\}$, where Z is the set of integers. That is, the sequences of length one have integer elements.
- $\Im_{k+1} = \{s\bullet < n >: s \in \Im_k \text{ and } n \in N_k\}$, where $\bullet$ is concatenation of sequences, and N_k is defined as follows: $N_k = \{p : type(f_k) \leq p \leq f_k(s) - 1 + type(f_k)\}$.
- $\Im$ is defined as the set union of all $\Im_k$ for $k \geq 1$.

In the definition above, if $type(f_k) = 0$, then $N_k = \{0, 1, ..., f_k(s) - 1\}$; if $type(f_k) = 1$, then $N_k = \{1, ..., f_k(s)\}$. Intuitively, $\Im$ is a collection of sequences of the form $< a_1, a_2, ..., a_n >$. Suitably defining $f_1, f_2, ...$, one can have his intended calendar-clock style time. For example, one can make use of the following $f_1, f_2, f_3, ...$ to get the standard calendar-clock style time:

Let f_1 be the constant function 12 (there are 12 months in a year), and $type(f_1) = 1$; Define $f_2(< m, 1 >) = 31, f_2(< m, 2 >) = 29$ if m is divisible by 4 and not divisible by 100, or m is divisible by 400, $f_2(< m, 2 >) = 28$ in other cases, $f_2(< m, 3 >) = 31, f_2(< m, 4 >) = 30$, ...and $type(f_2) = 1$; Let f_3 be the constant function 24 (there are 24 hours in a day), and $type(f_3) = 0$; Etc.

Let Q be the set of the rational numbers. Define $\text{INT(Q)} = \{(x, y) : x, y \in \text{Q and } x < y)\}$. Note that the elements of INT(Q) are simply pairs of rational numbers. Now we want to map $\Im$ into INT(Q). Define a mapping $G_1 : \Im \rightarrow \text{INT(Q)}$ by induction on the length of the sequence $< a_1, a_2, ..., a_n >$ as follows:

- $G_1(< n >) = (n, n+1)$. That is, a year is mapped to a pair of consecutive integers.
- Suppose $G_1(< a_1, a_2, ..., a_n >) = (a, b)$. Let $p = f_n(< a_1, a_2, ..., a_n >)$, and $d = (b - a) \div p$, where $-$ and $\div$ are two standard minus and division arithmetic

operations on the rational numbers. We divide the interval (a, b) into p equal parts of length d, and map the sequences of form $< a_1, a_2, ..., a_n, a_{n+1} >$ into the equal parts, consecutively. To be more specific, if $type(f_n) = 0$, then we define

$$\begin{aligned} G_1(< a_1, a_2, \ldots, a_n, 0 >) &= (a, a+d) \\ G_1(< a_1, a_2, \ldots, a_n, 1 >) &= (a+d, a+2d) \\ &\ldots \\ G_1(< a_1, a_2, \ldots, a_n, p-1 >) &= (a+(p-1)d, b) \end{aligned}$$

Otherwise $type(f_n) = 1$, and we define

$$\begin{aligned} G_1(< a_1, a_2, \ldots, a_n, 1 >) &= (a, a+d) \\ G_1(< a_1, a_2, \ldots, a_n, 2 >) &= (a+d, a+2d) \\ &\ldots \\ G_1(< a_1, a_2, \ldots, a_n, p >) &= (a+(p-1)d, b) \end{aligned}$$

Consider, for example, the pair $(1992, 1994)$ of rational numbers. There is no sequence $s \in \Im$ such that $G_1(s) = (1992, 1994)$. In order to get the one-to-one mapping between calendar-clock style time and pairs of rational numbers, we need to extend $\Im$. Let $C = \{conv(i, j) : i, j \in \Im\}$, where *conv* can be considered as a symbol. C is intended to represent the intervals obtained by applying the operator $convexify$. We should point out that $conv(i, i)$ and i actually represent the same interval. In fact many elements of C can represent the same interval. So, we want to transform C into another set D so that any two different elements of D represent different intervals. First, by using G_1 we define a mapping G_2 which maps C into INT(Q), then we define an equivalence relation $=_C$ on C, and finally we consider the equivalence class $C \backslash =_C$ of C under the equivalence relation $=_C$.

Define a mapping G_2 from C to INT(Q) as follows: For any i, j, let $G_1(i) = (a_1, b_1)$ and $G_1(j) = (a_2, b_2)$, and define $G_2(conv(i, j)) = (a, b)$, where $a = min(a_1, a_2)$ and $b = max(b_1, b_2)$. Note that $G_2(conv(i, i)) = G_1(i)$ for any $i \in \Im$. Define an equivalence relation $=_C$ on C as follows: $(t_1 =_C t_2)$ iff $(G_2(t_1) = G_2(t_2))$, where $=$ is the standard pair-wise equality on pairs of rational numbers. Let D be $C \backslash =_C$, the equivalence classes $C \backslash =_C$ of C under $=_C$. Now we define G_3 over D as follows: For any $I \in D, G_3(I) = (a, b)$ iff there is an element $s \in C$ such that $G_2(s) = (a, b)$ and s is a member of the equivalence class I. Then we can show that G_3 is a one-to-one mapping between D and INT(Q). Thus we have the following result:

PROPOSITION 2.1
There is a one-to-one mapping between D and INT(Q).

It is tedious but straightforward to prove this result (See [10, 11] for a discussion in the background of an interval calculus of [1]).

In this paper we are particularly interested in information with absolute temporal references. For our purpose we now define two relations on D as follows. Let I_1 and I_2 be any two elements of D and suppose that $G_3(I_1) = (a_1, b_1)$ and $G_3(I_2) = (a_2, b_2)$. Then we define:

$$\begin{aligned} (I_1 \subset I_2) &=_{df} (a_2 \leq a_1 < b_1 < b_2) \vee (a_2 < a_1 < b_1 \leq b_2) \\ (I_1 \# I_2) &=_{df} (\exists I_3 \in D)((I_3 \subset I_1) \wedge (I_3 \subset I_2)) \end{aligned}$$

The binary relations $\subset$ and $\#$ are called subinterval and overlap relations on D, respectively. It is then easy to prove that $\subset$ is irreflexive, transitive, left-endless and dense:

(i) $(\forall I \in D)\neg(I \subset I)$
(ii) $(\forall I_1, I_2, I_3 \in D)(((I_1 \subset I_2) \wedge (I_2 \subset I_3)) \rightarrow (I_1 \subset I_3))$
(iii) $(\forall I_1 \in D)(\exists I_2 \in D)(I_2 \subset I_1)$
(iv) $(\forall I_1, I_2 \in D)((I_1 \subset I_2) \rightarrow (\exists I_3)((I_1 \subset I_3) \wedge (I_3 \subset I_2)))$

The properties above will be used to characterize our underlying interpretation structure of the logic BAR.

3 A propositional temporal logic BAR

In this section, we define syntax, semantics and axiomatization of BAR. For our purposes we need to consider static information and dynamic information. A temporal database may include a collection of tables, some of which are time-stamped and the others may not. For example, in a personnel archives system, relations on the relation schema **Birth**(Name, Date, Place) may be used for some static information. In this paper we will only consider a propositional positional logic and represent the atomic assertions simply as propositional symbols. However, we distinguish different classes of propositional symbols in order to capture main properties of temporal information that they are supposed to abstractly represent. The truth values of static propositional symbols for static information, do not vary with time, whereas the truth values of dynamic propositional symbols for dynamic information, may vary with time. The dynamic propositional symbols can be further divided into two classes: hereditary and non-hereditary. Informally, a formula is said to be hereditary when a formula is true on an interval implies it is true on all its subintervals. Note that the time interval constants, which are of interest to us, are generally a subset of D. Now we start with the alphabet of BAR.

An alphabet of the propositional positional logic BAR is a tuple (C, R, S, H, D), where C is a non-empty set of interval constant symbols, R the set $\{=, \#, \subset\}$ of predicate symbols, S a set of static propositional symbols, H a set of hereditary propositional symbols and D a set of non-hereditary propositional symbols. All C, R, S, H and D are assumed to be disjoint and countable.

In what follows, we assume that an arbitrary alphabet $\Sigma(\text{BAR}) = (C, R, S, H, D)$ is given and fixed. Moreover, for brevity we assume that C, S, H and D are subsets of $\{\xi_1, \xi_2, \ldots\}$, $\{X_1, X_2, \ldots\}$, $\{H_1, H_2, \ldots\}$ and $\{D_1, D_2, \ldots\}$, respectively.

The constant symbols of C are intended to represent the time intervals and used to define the positional operators of the form $\big(\xi\big)$. The predicate symbols $=, \subset$ and $\#$ are used to represent the interval relationships in which we are interested. As we discuss in Section 5, other predicates could be added to R in order to express other interval relationships.

Definition 3.1 (Well-formed formulas)
The well-formed formulas, or simply formulas, of BAR are inductively defined as follows:

1. A is a well-formed formula if $A \in (S \cup H \cup D)$;
2. $(\xi_i = \xi_j)$, $(\xi_i \# \xi_j)$ and $(\xi_i \subset \xi_j)$ are well-formed formulas if $\xi_i, \xi_j \in C$;

3. $(\neg A)$ and $(\llbracket\ \rrbracket A)$ are well-formed formulas if A is;
4. $(A \rightarrow B)$ is a well-formed formula if A and B are;
5. $((\xi)A)$ is a well-formed formula for any $\xi \in C$ if A is.

As usual, we will omit some unnecessary parentheses in formulas when there is no confusion. The other Boolean connectives $\wedge, \vee$ and $\leftrightarrow$ are defined as usual. From (ξ) and $\llbracket\ \rrbracket$ we can define more operators as follows:

$$\begin{aligned}
\langle\!\langle\rangle\!\rangle A &=_{df} \neg \llbracket\ \rrbracket \neg A \\
\llbracket\xi\ \rrbracket A &=_{df} (\xi) \llbracket\ \rrbracket A \\
[\xi] A &=_{df} \llbracket\xi\ \rrbracket A \wedge (\xi) A \\
\langle\!\langle\xi\ \rangle\!\rangle A &=_{df} \neg \llbracket\xi\ \rrbracket \neg A \\
< \xi > A &=_{df} \neg [\xi] \neg A
\end{aligned}$$

Their informal meanings are as follows:

$\llbracket\ \rrbracket A$: A is always true during the current observation interval, i.e., A is true on every subinterval of the current observation interval.

$\langle\!\langle\rangle\!\rangle A$: A is sometimes true during the current observation interval. That is, there is a subinterval I of the current observation interval such that A is true on I.

$(\xi)A$: A is true on the interval denoted by ξ.

$\llbracket\xi\ \rrbracket A$: A is always true during the interval denoted by ξ, i.e., A is true on all the subintervals of the interval denoted by ξ.

$\langle\!\langle\xi\ \rangle\!\rangle A$: A is sometimes true during the interval denoted by ξ. That is, there exists a subinterval of the interval denoted by ξ where A is true.

$[\xi]A$: A is true on the interval denoted by ξ and on all its subintervals.

$< \xi > A$: A is true on the interval denoted by ξ or on some of its subintervals.

In this paper we call (ξ), $\llbracket\xi\ \rrbracket$, $\langle\!\langle\xi\ \rangle\!\rangle$, $[\xi]$ and $< \xi >$ positional operators. Sometimes we also call them modal operators defined by intervals, or interval-defined modal operators. Positional operators can be used to represent information with absolute temporal references. The concept of using explicit time to define modal operators has origin in work on positional logic and dates back to [23], where there is a single operator $\mathbf{R}(t)$, called temporal realization operator. In [12], the realization operator $\mathbf{R}(t)$ is incorporated into PROLOG for temporal reasoning. Our interval-defined modal operators (ξ), $\llbracket\xi\ \rrbracket$, $\langle\!\langle\xi\ \rangle\!\rangle$, $[\xi]$ and $< \xi >$ can be regarded as an extension and development of the temporal realization operator $\mathbf{R}(t)$. We should point out that semantically the formula $[\xi]A$ means the same thing as HOLDS(A, ξ) of [2] does. The difference between $[\xi]A$ and HOLDS(A, ξ) lies in that A in HOLDS(A, ξ) is treated at the term level while A in $[\xi]A$ is still at the formula level. It seems to us more reasonable to regard Boolean-valued propositions at the formula level.

Definition 3.2 (Interpretation structure)
An interpretation structure `IS` for BAR is a tuple $(INT, @, M, V)$, where

- INT is a non-empty set and @ a binary relation on INT satisfying the following four constraints: (i) irreflexivity; (ii) transitivity; (iii) left-endless; and (iv) density.

- $M : C \to INT$ is a mapping.
- $V : (S \cup H \cup D) \times INT \to \{true, false\}$ is a mapping satisfying
  - for every $h \in H$ and every $I \in INT$, if $V(h, I) = true$, then $(\forall J\ @\ I)V(h, J) = true$.
  - for every $s \in S$, if there is an $I \in INT$ such that $V(s, I) = true$, then for every $J \in INT, V(s, J) = true$.

An interpretation structure is also simply called interpretation, and its component $(INT, @)$ is called a frame for BAR. Obviously, $(D, \subset)$ is a frame for BAR. It should be stressed that there are many other frames for BAR. For instance, any of the structures below is a frame for BAR, where $\subset$ denotes the usual strict (or proper) containment relation between intervals of numbers:

$$
\begin{array}{ll}
(\{]x,y] : x < y\}, \subset), & \text{where }]x,y] = \{z \in \mathtt{Q} : x < z \le y\} \\
(\{[x,y[: x < y\}, \subset), & \text{where } [x,y[= \{z \in \mathtt{Q} : x \le z < y\} \\
(\{]x,y[: x < y\}, \subset), & \text{where }]x,y[= \{z \in \mathtt{Q} : x < z < y\} \\
(\{[x,y] : x < y\}, \subset), & \text{where } [x,y] = \{z \in \mathtt{Q} : x \le z \le y\} \\
(\{]x,y] : x < y\}, \subset), & \text{where }]x,y] = \{z \in \Re : x < z \le y\} \\
\ldots &
\end{array}
$$

In literature some people prefer the first two frames or the fifth one to the others. Herein we do not want to go into deeper discussions, but we stress that the proposed logic is compatible with any of the previous choices. On the contrary, structures such as $(\{[x, y] : x, y \in \mathtt{Q} \text{ and } x \le y\}, \subset)$ and $(\{]x, y] : x, y \in \mathrm{Z} \text{ and } x < y\}, \subset)$, where $]x, y] = \{z \in \mathrm{Z} : x < z \le y\}$, are not frames for BAR,

since the former is not left-endless and the latter is neither left-endless nor dense. In Section 5 we will make some more discussions.

Let $\mathtt{IS} = (INT, @, M, V)$ be an interpretation for BAR. If $I \in INT$, we also say that I is an interval of $\mathtt{IS}$.

Definition 3.3 (Semantics)
Let $\mathtt{IS} = (INT, @, M, V)$ be an interpretation for BAR. The truth of a formula A on an interval I of $\mathtt{IS}$, denoted by $\mathtt{IS} \models_I A$, where I is also called the current observation interval, is inductively defined as follows:

- If $A \in (S \cup H \cup D)$, then $\mathtt{IS} \models_I A$ iff $V(A, I) = true$.
- $\mathtt{IS} \models_I (\xi_i = \xi_j)$ iff $M(\xi_i) = M(\xi_j)$.
- $\mathtt{IS} \models_I (\xi_i \subset \xi_j)$ iff $M(\xi_i)\ @\ M(\xi_j)$.
- $\mathtt{IS} \models_I (\xi_i \# \xi_j)$ iff there is a $J \in INT$ such that $J\ @\ M(\xi_i)$ and $J\ @\ M(\xi_j)$.
- $\mathtt{IS} \models_I \neg B$ iff $\mathtt{IS} \not\models_I B$. That is, it is not the case that $\mathtt{IS} \models_I B$.
- $\mathtt{IS} \models_I B \to C$ iff $\mathtt{IS} \not\models_I B$ or $\mathtt{IS} \models_I C$.
- $\mathtt{IS} \models_I [\![\ ]\!]B$ iff for every $J\ @\ I, \mathtt{IS} \models_J B$.
- $\mathtt{IS} \models_I (\xi)B$ iff $\mathtt{IS} \models_{M(\xi)} B$.

Definition 3.4 (Satisfiability and validity)
Let A be a formula.

- A is true in an interpretation $\mathtt{IS} = (INT, @, M, V)$, denoted by $\mathtt{IS} \models A$, iff for every$I \in INT, \mathtt{IS} \models_I A$.

- A is (logically) valid, denoted by $\models A$, iff $\texttt{IS} \models A$ for any interpretation $\texttt{IS}$.
- A is satisfiable iff there is an interpretation $\texttt{IS} = (INT, @, M, V)$ and an $I \in INT$ such that $IS \models_I A$. In this case, $\texttt{IS}$ is also called a model of A.

Before going on we give a definition to be used later.

DEFINITION 3.5
A formula A is said to be static iff (i) A is a static propositional symbol, i.e. $A \in S$; or (ii) A is of the form $(\xi_i = \xi_j)$, $(\xi_i \subset \xi_j)$ or $(\xi_i \# \xi_j)$, for any $\xi_i, \xi_j \in C$; or (iii) A is of the form $\big(\xi_i\big)B$ for any $\xi_i \in C$ and any formula B.

Obviously, if A is a static formula then either A or $\neg A$ is true in $\texttt{IS}$ for any interpretation $\texttt{IS}$. This is also the case for the Boolean combinations of static formulas. Thus, we could extend definition 3.5 in order to include these combinations. However, the simpler definition above suffices for our future purposes.

By definition 3.4 we have the notion of $\models$. Now we define the notion of $\vdash$ by giving the axioms (schemas) and inference rules as follows:

Axioms

(1) All the propositional tautologies.

(2) Axioms about R. Let $\xi_i, \xi_j, \xi_k \in C$ be any constant symbols. Then we have the axioms:

(Ax2-1) $\neg(\xi_i \subset \xi_i)$
(Ax2-2) $(\xi_i \subset \xi_j) \wedge (\xi_j \subset \xi_k) \to (\xi_i \subset \xi_k)$
(Ax2-3) $(\xi_i \subset \xi_j) \to (\xi_i \# \xi_j)$
(Ax2-4) $(\xi_k \subset \xi_i) \wedge (\xi_k \subset \xi_j) \to (\xi_i \# \xi_j)$
(Ax2-5) $(\xi_i \# \xi_j) \wedge (\xi_j \subset \xi_k) \to (\xi_i \# \xi_k)$
(Ax2-6) $(\xi_i = \xi_j) \to (\xi_i \# \xi_j)$
(Ax2-7) $(\xi_i \# \xi_j) \to (\xi_j \# \xi_i)$
(Ax2-8) $(\xi_i = \xi_i)$
(Ax2-9) $(\xi_i = \xi_j) \to (A \to A[\xi_i \backslash \xi_j])$

where $A[\xi_i \backslash \xi_j]$ is obtained by replacing some, but not necessarily all, occurrences of ξ_i with ξ_j.

(3) Axioms about $[\![\ ]\!]$ in isolation.

(Ax3-1) $[\![\ ]\!](A \to B) \to ([\![\ ]\!]A \to [\![\ ]\!]B)$
(Ax3-2) $[\![\ ]\!]A \to \langle\!\langle\rangle\!\rangle A$
(Ax3-3) $[\![\ ]\!]A \leftrightarrow [\![\ ]\!][\![\ ]\!]A$

(4) Axioms about $\big(\xi\big)$ in isolation.

(Ax4-1) $\big(\xi\big)\neg A \leftrightarrow \neg\big(\xi\big)A$
(Ax4-2) $\big(\xi\big)(A \to B) \to (\big(\xi\big)A \to \big(\xi\big)B)$
(Ax4-3) $\big(\xi\big)A \leftrightarrow \big(\xi\big)\big(\xi\big)A$

(5) Axioms about combinations of modal operators.

(Ax5-1) $\big(\xi_j\big)A \leftrightarrow \big(\xi_i\big)\big(\xi_j\big)A$
(Ax5-2) $\big(\xi\big)A \leftrightarrow [\![\ ]\!]\big(\xi\big)A$
(Ax5-3) $(\xi_i \subset \xi_j) \to (\big(\xi_j\big)[\![\ ]\!]A \to \big(\xi_i\big)A)$
(Ax5-4) $(\xi_i \# \xi_j) \to ((\big(\xi_j\big)[\![\ ]\!]A \wedge \big(\xi_j\big)[\![\ ]\!]B) \to \big(\xi_i\big)\langle\!\langle\rangle\!\rangle(A \wedge B))$

(6) Let $A(\xi_i, \xi_j)$ be any of $(\xi_i = \xi_j)$, $(\xi_i \# \xi_j)$ and $(\xi_i \subset \xi_j)$. Then we have the axioms:

(Ax6-1)	$A(\xi_i, \xi_j) \leftrightarrow [\![\ ]\!] A(\xi_i, \xi_j)$
(Ax6-2)	$\neg A(\xi_i, \xi_j) \leftrightarrow [\![\ ]\!] \neg A(\xi_i, \xi_j)$
(Ax6-3)	$A(\xi_i, \xi_j) \leftrightarrow \big(\xi_k\big) A(\xi_i, \xi_j)$
(Ax6-4)	$\neg A(\xi_i, \xi_j) \leftrightarrow \big(\xi_k\big) \neg A(\xi_i, \xi_j)$

(7) Other axioms. Let X be any element of S, and Y any element of H.

(Ax7-1)	$X \leftrightarrow [\![\ ]\!] X$
(Ax7-2)	$\neg X \leftrightarrow [\![\ ]\!] \neg X$
(Ax7-3)	$X \leftrightarrow \big(\xi\big) X$
(Ax7-4)	$\neg X \leftrightarrow \big(\xi\big) \neg X$
(Ax7-5)	$Y \rightarrow [\![\ ]\!] Y$

Inference rules

(MP)	From A and $A \rightarrow B$, infer B,	written as: $A, A \rightarrow B \vdash B$
$([\![\ ]\!]+)$	From A, infer $[\![\ ]\!] A$,	written as: $A \vdash [\![\ ]\!] A$
$(\big(\xi\big)+)$	From A, infer $\big(\xi\big) A$,	written as: $A \vdash \big(\xi\big) A$

We should point out that the axiom (Ax6-4) can be derived from (Ax6-3) and (Ax4-1), and (Ax7-4) can be derived from (Ax7-3) and (Ax4-1). This is a matter of independence of axioms. For convenience we have listed (Ax6-4) and (Ax7-4) as axioms. It is easy to see that both $\big(\xi\big)$ and $[\![\ ]\!]$ are normal in the sense that they have the axiom K and the necessity rule of [3].

By use of the axioms and inference rules above we can define the notion of theorem as usual. We write $\vdash A$ to denote that A is a theorem of BAR. It is easy to show that the axioms and inference rules above are sound, and hence all theorems are valid. Some useful theorems and derived inference rules follow:

(P1)	$\vdash (\xi_i \subset \xi_j) \rightarrow ([\![\xi_j]\!] A \rightarrow [\![\xi_i]\!] A)$
$(\langle\!\langle\rangle\!\rangle \wedge)$	$\vdash \langle\!\langle\rangle\!\rangle (A \wedge B) \rightarrow (\langle\!\langle\rangle\!\rangle A \wedge \langle\!\langle\rangle\!\rangle B)$
$([\![\xi]\!] \rightarrow)$	$\vdash [\![\xi]\!] (A \rightarrow B) \rightarrow ([\![\xi]\!] A \rightarrow [\![\xi]\!] B)$
$(\big(\xi\big) \wedge)$	$\vdash \big(\xi\big)(A \wedge B) \leftrightarrow (\big(\xi\big) A \wedge \big(\xi\big) B)$
$([\![\xi]\!] \wedge)$	$\vdash [\![\xi]\!] (A \wedge B) \leftrightarrow ([\![\xi]\!] A \wedge [\![\xi]\!] B)$
$([\![\xi]\!] \langle\!\langle \xi \rangle\!\rangle)$	$\vdash [\![\xi]\!] A \rightarrow \langle\!\langle \xi \rangle\!\rangle A$
$([\![\xi]\!]+)$	$A \vdash [\![\xi]\!] A$
$(\langle\!\langle\rangle\!\rangle +)$	$A \vdash \langle\!\langle\rangle\!\rangle A$
$(\rightarrow [\![\xi]\!]+)$	$A \rightarrow B \vdash [\![\xi]\!] A \rightarrow [\![\xi]\!] B$

The proof for them is easy. By $([\![\xi]\!] \rightarrow)$ and $([\![\xi]\!]+)$ we have that the derived modal operator $[\![\xi]\!]$ is also normal, and by $([\![\xi]\!] \langle\!\langle \xi \rangle\!\rangle)$ it is even a KD-system for $[\![\xi]\!]$ in Chellas classification [3] (as a matter of fact it is stronger than KD-systems, but we will not discuss it herein). Now we give a metatheorem that will be used later. The proof for it is omitted.

LEMMA 3.6
Let A be any static formula. Then:

(i) $\vdash A \leftrightarrow [\![\;]\!]A$
(ii) $\vdash \neg A \leftrightarrow [\![\;]\!]\neg A$
(iii) $\vdash A \leftrightarrow (\xi)A$, for $\xi \in C$
(iv) $\vdash \neg A \leftrightarrow (\xi)\neg A$, for $\xi \in C$

4 Completeness

Although each of the modal operators (ξ), for $\xi \in C$, and $[\![\;]\!]$ is normal, the proof of the completeness of the logic BAR is more complicated than the standard completeness proof for normal modal logics. In this section we present a proof of the completeness of BAR. As usual, we will prove that any consistent formula is satisfiable, from which it follows that if A is valid then $\vdash A$.

The notions of consistency and maximal consistency are assumed to be known and defined as in e.g. [9]. Moreover, we will also use some standard results of [3] for normal modal operators without giving proofs. Two preliminary results on consistency are stated in lemma 4.1. The proof of lemma 4.1 is omitted.

LEMMA 4.1
Let ? be a consistent set of formulas.

- $\{A : (\xi)A \in ?\}$ is consistent for any $\xi \in C$.
- If $\xi_i \# \xi_j \in ?$, then $\{A : (\xi_i)\, [\![\;]\!]A \in ?\} \cup \{B : (\xi_j)\, [\![\;]\!]B \in ?\}$ is consistent.

Throughout the rest of this paper we use $A, B, A_1, B_1, \ldots$ to refer to formulas and $?, ?_1, \ldots$ to refer to sets of formulas, and we consider the following standard abbreviations: $A_1 \wedge \ldots \wedge A_n =_{df}$ `TRUE` if $n = 0$, and $A_1 \vee \ldots \vee A_n =_{df}$ `FALSE` if $n = 0$, where `TRUE` and `FALSE` are defined as usual: e.g. `TRUE` may be defined as $X_1 \vee \neg X_1$.

In the construction of the desired model for a consistent formula, we will make use of the following assumptions, notations and conventions:

- We assume that the formulas of BAR have been enumerated in an arbitrary but fixed order.
- The Lindenbaum extension of ? is denoted by $?^*$ and is defined as follows (see e.g. [9]): $?^*$ is the union of all $?_n$ for $n \geq 0$, where (1) $?_0 = ?$, and (2) $?_{n+1} = ?_n \cup \{A_{n+1}\}$ if $?_n \cup \{A_{n+1}\}$ is consistent, and $?_{n+1} = ?_n \cup \{\neg A_{n+1}\}$ otherwise, where A_{n+1} is the formula in the $(n+1)$–place in the previously assumed enumeration. It is well known that if ? is consistent then $?^*$ is maximal consistent. We will often write A^* to stand for $\{A\}^*$ for convenience, where A is a formula.
- The set of all the subformulas of A is defined as usual and denoted by $\texttt{Sub}(A)$. In particular, a formula is a subformula of itself.
- $\texttt{not}? =_{df} \{\neg A : A \in ?\}$ and $\texttt{Subn}(A) =_{df} \texttt{Sub}(A) \cup \texttt{notSub}(A)$.
- $\texttt{CONST}(A) =_{df} \{\xi : \xi \in C$ and ξ appears in $A\}$.
- Given a finite set ? of formulas, we define the set of all the descriptors formed by ?, denoted by $\texttt{DESCRIPTORS}(?)$, as follows:

$$\texttt{DESCRIPTORS}(?) \;=_{df}\; \{A_1 \wedge \ldots \wedge A_{\#\Gamma}$$
$$: \text{ for } 1 \leq i \leq \#?, A_i \text{ is either } a_i \text{ or } \neg a_i\}$$

where $a_1, \ldots, a_{\#\Gamma}$ is the enumeration of ? according to the assumed enumeration order of all the formulas and #? denotes the cardinality of ?. In particular, $\mathtt{DESCRIPTORS}(\emptyset) =_{df} \{\mathtt{TRUE}\}$.

- For a descriptor $A = A_1 \wedge \ldots \wedge A_{\#\Gamma} \in \mathtt{DESCRIPTORS}(?)$ and a formula B, we write $B \propto A$ iff there exists $1 \leq i \leq \#?$ such that B is A_i, i.e. B is a conjunct of A.
- For a finite set of formulas ?, we define a function $\mathtt{CH}_\Gamma$, called ?-characteristic function, from the set of all the maximal consistent sets into $\mathtt{DESCRIPTORS}(?)$ as follows: $\mathtt{CH}_\Gamma(?_1) = B$ iff $B \in \mathtt{DESCRIPTORS}(?) \cap ?_1$. Since $?_1$ is maximal consistent, $\mathtt{DESCRIPTORS}(?) \cap ?_1$ contains one and only one formula. Thus, $\mathtt{CH}_\Gamma$ is well-defined.

Schema of the proof

Let Ω be a consistent formula. We will construct an interpretation $\mathtt{IS}_\Omega = (INT_\Omega, @_\Omega, M_\Omega, V_\Omega)$ for BAR such that for any $A \in \mathtt{Sub}(\Omega)$ and any $I \in INT_\Omega, \mathtt{IS}_\Omega \models_I A$ iff $A \in \textsc{Form}(I)$, where $\textsc{Form}(I)$ will be defined later. Since there is an $I \in INT_\Omega$ such that $\Omega \in \textsc{Form}(I)$, we will have that Ω is satisfiable, as we wish. In order to construct the desired interpretation structure $\mathtt{IS}_\Omega$ we proceed as follows:

1. For any consistent set ? of formulas we define an infinite tree of labels and a mapping $\textsc{Form}_{\textsc{Tree}}$ that associates a maximal consistent set of formulas to each label in the tree. In particular, $\textsc{Form}_{\textsc{Tree}}$ associates $?^*$ to the root.
   Since the tree is uniquely determined by the root and ?, we use $\textsc{Tree}(\xi, ?)$ to denote such a tree having $\lfloor \xi \rfloor$ as the label of the root, where ξ is any symbol.
   We also define a relation $\prec_{\textsc{Tree}}$ over $\textsc{Tree}(\xi, ?)$ as follows: $r \prec_{\textsc{Tree}} s$ iff r is a (direct) child of s. Intuitively, a label r of $\textsc{Tree}(\xi, \{\Omega\})$ is meant to denote an interval and $\textsc{Form}_{\textsc{Tree}}(r)$ contains the formulas that are true on r.
2. Using the previous tree construction, we can construct a model for Ω step by step. The main ideas are as follows.
   (a) Let $IS_{\textsc{Tree}} = (\textsc{Tree}(\xi_0, \{\Omega\}), \prec_{\textsc{Tree}}, V_{\textsc{Tree}})$, where $V_{\textsc{Tree}}$ is defined as follows: $V_{\textsc{Tree}}(A, r) = \mathit{true}$ iff $A \in \textsc{Form}_{\textsc{Tree}}(r)$, for each propositional symbol $A \in \mathtt{Sub}(\Omega)$.
   The label $\lfloor \xi_0 \rfloor$ is meant to denote the current observation interval. It should be noted that $IS_{\textsc{Tree}}$ is not an interpretation for BAR: neither does it include the component M, nor does $\prec_{\textsc{Tree}}$ satisfy the transitivity and density properties. If we ignored all references to interval constants (i.e. we did not consider positional operators and formulas about interval relationships), then $IS_{\textsc{Tree}}$ would suffice in the sense $IS_{\textsc{Tree}} \models_r A$ iff $A \in \textsc{Form}_{\textsc{Tree}}(r)$ for any $A \in \mathtt{Sub}(\Omega)$ and any $r \in \textsc{Tree}(\xi_0, \{\Omega\})$, where the notation $\models$ is abused. In particular, $IS_{\textsc{Tree}} \models_{\lfloor \xi_0 \rfloor} A$, since $\Omega \in \textsc{Form}_{\textsc{Tree}}(\lfloor \xi_0 \rfloor)$.
   In general, however, we do have references to interval constants. Hence, $IS_{\textsc{Tree}}$ does not suffice for our purpose.
   (b) Let $\mathtt{IS}_{\textsc{Forest}} = (\textsc{Forest}(\xi_0, \Omega), \prec_{\textsc{Forest}}, V_{\textsc{Forest}})$ be the union of the $\mathtt{IS}_{\textsc{Tree}}$ associated to the trees $\textsc{Tree}(\xi_0, \{\Omega\})$ and $\textsc{Tree}(\xi, \{A : (\xi) A \in \Omega^*\})$ for $\xi \in \mathtt{CONST}(\Omega)$, and $\textsc{Form}_{\textsc{Forest}}$ the union of the relevant $\textsc{Form}_{\textsc{Tree}}$.
   If we were required to consider only positional operators without taking interval relationships into account, then $\mathtt{IS}_{\textsc{Forest}}$ would suffice for our purpose: if

we define $M_{\text{FOREST}}(\xi) = \lfloor \xi \rfloor$, then we have that $\mathtt{IS}_{\text{FOREST}} \models_r A$ iff $A \in \text{FORM}_{\text{FOREST}}(r)$ for any $A \in \mathtt{Sub}(\Omega)$ and any r in the forest.
In general, however, we need to consider formulas about interval relationships. Hence, $\mathtt{IS}_{\text{FOREST}}$is not enough for our purpose. In the following (c) and (d) we consider interval relationships.

(c) For the formulas of the form $(\xi_i = \xi_j)$ and $(\xi_i \subset \xi_j)$ we only need to re-define $M_{\text{FOREST}}(\xi)$, take out some trees associated to equal interval constants from the forest, and extend the binary relation $\prec_{\text{FOREST}}$.

(d) For the formulas of the form $(\xi_i \# \xi_j)$, if $(\xi_i \# \xi_j) \in \Omega^*$, for different ξ_i and ξ_j, we introduce new trees with their roots labelled by $\lfloor \xi_{i,j} \rfloor$ and impose that $\lfloor \xi_{i,j} \rfloor \prec_{\text{FOREST}} \lfloor \xi_i \rfloor$ and $\lfloor \xi_{i,j} \rfloor \prec_{\text{FOREST}} \lfloor \xi_j \rfloor$.

(e) Finally, we supplement the previous forest with more labels and extend $\prec_{\text{FOREST}}$ in order to get the desired density, then take the transitive closure for the transitivity.

In what follows we will make the above ideas more precise. In order to make the presentation concise and to the point, we will omit some simple or standard proofs in normal modal logics. In some longer proofs we will also omit some technical details if they are not essential to the understanding of the main idea of the proofs.

In the following we write Θ to stand for $\mathtt{DESCRIPTORS}(\mathtt{Sub}(\Omega))$ and $\mathtt{CH}$ for $\mathtt{CH}_{\mathtt{Sub}(\Omega)}$. We always assume that $a_1, \ldots, a_{\#\Theta}$ is the enumeration of Θ according to the assumed enumeration order of all the formulas, and assume that $\mathtt{CONST}(\Omega) = \{\xi_{j_1}, \ldots, \xi_{j_n}\}$, where $n \geq 0$, and $1 \leq j_1 < \ldots < j_n$ if $n > 0$. Moreover, we assume ξ_0 and $\xi_{k,j}, 1 \leq k \leq j \leq n$, are disjoint from the alphabet of BAR.

Construction of $\text{TREE}(\xi, ?), \text{FORM}_{\text{TREE}}$ **and** $\prec_{\text{TREE}}$**:**

Let ξ be any element and $?$ a consistent set of formulas.

- Define $\text{FORM}_{\text{TREE}}$ and L_m for $m \geq 0$ as follows:
  - $L_0 = \{\lfloor \xi \rfloor\}$ and $\text{FORM}_{\text{TREE}}(\lfloor \xi \rfloor) = ?^*$.
  - For $i \geq 0$:

$$L_{i+1} = \{\lfloor r, a, 1 \rfloor : r \in L_i, a \in \Theta,\ \langle\!\langle\rangle\!\rangle a \in \text{FORM}_{\text{TREE}}(r)\}$$

  and

$$\text{FORM}_{\text{TREE}}(\lfloor r, a, 1 \rfloor) = (\{A : [\![\ ]\!] A \in \text{FORM}_{\text{TREE}}(r)\} \cup \{a\})^*$$

- $\text{TREE}(\xi, ?)$ is the union of all L_m for $m \geq 0$.
- $\prec_{\text{TREE}}$ is a binary relation on $\text{TREE}(\xi, ?)$: $r \prec_{\text{TREE}} s$ iff r is $\lfloor s, a, 1 \rfloor$ for some $a \in \Theta$.

When no confusion may arise, we will often simply write FORM instead of $\text{FORM}_{\text{TREE}}$. We should point out that $\text{TREE}(\xi, ?)$ also depends on Ω. Note that $\Theta = \mathtt{DESCRIPTORS}(\mathtt{Sub}(\Omega))$.

LEMMA 4.2
$\text{FORM}(r)$ is maximal consistent for any $r \in \text{TREE}(\xi, ?)$.

Note that $\text{TREE}(\xi, ?)$ can be really represented as a tree whose root is $\lfloor \xi \rfloor$ and whose edge relation is as follows: for any $r, s \in \text{TREE}(\xi, ?)$, r is a child of s iff r is of the form $\lfloor s, a, 1 \rfloor$. Obviously, any internal node has a finite number of children, since $\#L_{i+1}(\xi, ?) \leq \#L_i(\xi, ?) \times \#\Theta$. Moreover, as we will show (see lemma 4.4), $\text{TREE}(\xi, ?)$ has no leaves, and hence is infinite.

LEMMA 4.3
Assume that ? is maximal consistent and $?_1$ is finite. For any $A \in (?_1 \cup \texttt{not}?_1)$, if $\langle\!\langle\rangle\!\rangle A \in ?$, then there is a formula $a \in \texttt{DESCRIPTORS}(?_1)$ such that $A \propto a$ and $\langle\!\langle\rangle\!\rangle a \in ?$.

PROOF. We will construct a formula $a \in \texttt{DESCRIPTORS}(?_1)$ such that $\langle\!\langle\rangle\!\rangle(a \wedge A) \in ?$, from which it follows that $\langle\!\langle\rangle\!\rangle a \in ?$ and $A \propto a$. Let $e_1, \ldots, e_k$ be an enumeration of $?_1$. Then $a = b_1 \wedge \ldots \wedge b_k$, where $b_i, 1 \leq i \leq k$, is inductively defined in such a way that b_i is either e_i or $\neg e_i$, and $\langle\!\langle\rangle\!\rangle(b_1 \wedge \ldots \wedge b_i \wedge A) \in ?$:

- b_1 is determined as follows. Either $\langle\!\langle\rangle\!\rangle(e_1 \wedge A) \in ?$ or $\langle\!\langle\rangle\!\rangle(e_1 \wedge A) \notin ?$. If $\langle\!\langle\rangle\!\rangle(e_1 \wedge A) \in ?$, just let $b_1 = e_1$. If $\langle\!\langle\rangle\!\rangle(e_1 \wedge A) \notin ?$, then $\neg\langle\!\langle\rangle\!\rangle(e_1 \wedge A) \in ?$, thus $[\![\,]\!]\neg(e_1 \wedge A) \in ?$. Since $\langle\!\langle\rangle\!\rangle A \in ?$, we can prove $\langle\!\langle\rangle\!\rangle(\neg e_1 \wedge A) \in ?$. Now just let $b_1 = \neg e_1$.
- Assume that $b_i, 1 \leq i < k$, has been determined so that $\langle\!\langle\rangle\!\rangle(b_1 \wedge \ldots \wedge b_i \wedge A) \in ?$. We want to determine b_{i+1}. Analogously, if $\langle\!\langle\rangle\!\rangle(b_1 \wedge \ldots \wedge b_i \wedge e_{i+1} \wedge A) \in ?$, then let $b_{i+1} = e_{i+1}$, otherwise let $b_{i+1} = \neg e_{i+1}$.

∎

The next lemma states a series of 'good' properties that we would like to maintain, with some adjustments, in the richer structures of labels to be constructed.

LEMMA 4.4
Let ? be a consistent set of formulas.

1. For any formula A and any $r, s \in \textsc{Tree}(\xi, ?)$, if $s \prec_{\textsc{Tree}} r$ and $[\![\,]\!]A \in \textsc{Form}(r)$, then $(A \wedge [\![\,]\!]A) \in \textsc{Form}(s)$.
2. For any $A \in \texttt{Subn}(\Omega)$ and any $r \in \textsc{Tree}(\xi, ?)$, if $\langle\!\langle\rangle\!\rangle A \in \textsc{Form}(r)$, then there exists $s \in \textsc{Tree}(\xi, ?)$ such that $s \prec_{\textsc{Tree}} r$ and $A \in \textsc{Form}(s)$.
3. For any static formula A and any $r \in \textsc{Tree}(\xi, ?)$, $A \in \textsc{Form}(r)$ iff $A \in ?^*$.
4. For any hereditary proposition $Y \in H$ and any $r, s \in \textsc{Tree}(\xi, ?)$, if $s \prec_{\textsc{Tree}} r$ and $Y \in \textsc{Form}(r)$, then $Y \in \textsc{Form}(s)$.
5. $\prec_{\textsc{Tree}}$ is irreflexive.
6. $\prec_{\textsc{Tree}}$ is left-endless.

PROOF. We will only prove (3) and (6). The proof of the rest is omitted.

(3). By induction on the complexity of r:
Basis: $r = \lfloor \xi \rfloor$.
It follows immediately from $\textsc{Form}(\lfloor \xi \rfloor) = ?^*$.
Induction: Suppose $A \in \textsc{Form}(r)$ iff $A \in ?^*$. We want to prove that $A \in \textsc{Form}(\lfloor r, a, 1 \rfloor)$ iff $A \in ?^*$. Assume that $A \in ?^*$. By the induction hypothesis $A \in \textsc{Form}(r)$, and by lemma 3.6 $[\![\,]\!]A \in \textsc{Form}(r)$. Hence, $A \in \textsc{Form}(\lfloor r, a, 1 \rfloor)$. On the other hand, if $A \notin ?^*$, then by the induction hypothesis $A \notin \textsc{Form}(r)$. Analogously we can obtain $\neg A \in \textsc{Form}(\lfloor r, a, 1 \rfloor)$, which implies $A \notin \textsc{Form}(\lfloor r, a, 1 \rfloor)$.

(6). By (2) it suffices to prove that there is $A \in \texttt{Subn}(\Omega)$ such that $\langle\!\langle\rangle\!\rangle A \in \textsc{Form}(r)$. Let x_i be a subformula of Ω. Either $\langle\!\langle\rangle\!\rangle x_i \in \textsc{Form}(r)$ or $\neg\langle\!\langle\rangle\!\rangle x_i \in \textsc{Form}(r)$. If $\neg\langle\!\langle\rangle\!\rangle x_i \in \textsc{Form}(r)$, then $[\![\,]\!]\neg x_i \in \textsc{Form}(r)$, thus $\langle\!\langle\rangle\!\rangle\neg x_i \in \textsc{Form}(r)$ by axiom (Ax3-2).

∎

Recall that we have assumed that $\mathtt{CONST}(\Omega) = \{\xi_{j_1}, \ldots, \xi_{j_n}\}$, where $n \geq 0$, and $1 \leq j_1 < \ldots < j_n$ if $n > 0$. Moreover, we assume ξ_0 and $\xi_{k,j}, 1 \leq k \leq j \leq n$, are disjoint from the alphabet of BAR. Now we construct a collection of trees of labels.

Construction of $\textsc{Forest}(\xi_0, \Omega), \textsc{Form}_{\textsc{Forest}}$ **and** $\prec_{\textsc{Forest}}$

- $\textsc{Forest}(\xi_0, \Omega)$ is defined as the union of all $\textsc{Base}^i(\Omega)$ for $0 \leq i \leq n$, where $\textsc{Base}^i(\Omega)$ is defined as follows:
  - $\textsc{Base}^0(\Omega) = \textsc{Tree}(\xi_0, \{\Omega\})$
  - $\textsc{Base}^1(\Omega) = \textsc{Tree}(\xi_{j_1}, \{A : (\xi_{j_1})A \in \Omega^*\})$
  - For $1 \leq i < n$, $\textsc{Base}^{i+1}(\Omega)$ is defined as follows:
    * If there is $1 \leq k \leq i$ such that $(\xi_{j_k} = \xi_{j_{i+1}}) \in \Omega^*$, then let $\textsc{Base}^{i+1}(\Omega)$ be empty.
    * Otherwise, let $\textsc{Base}^{i+1}(\Omega) = \textsc{Tree}(\xi_{j_{i+1}}, \{A : (\xi_{j_{i+1}})A \in \Omega^*\}) \cup \Delta$, where Δ is the union of $\Delta_{k,i+1}$ for $1 \leq k \leq i$ in which $\Delta_{k,i+1}$ is defined as follows:
      If $\neg(\xi_{j_k} \# \xi_{j_{i+1}}) \in \Omega^*$ or $\textsc{Base}^k(\Omega)$ is empty, then let $\Delta_{k,i+1}$ be empty, otherwise we define $\Delta_{k,i+1} = \textsc{Tree}(\xi_{k,i+1}, (\{A : (\xi_{j_k}) [\![\,]\!] A \in \Omega^*\} \cup \{B : (\xi_{j_{i+1}}) [\![\,]\!] B \in \Omega^*\}))$.
      Note that $\textsc{Forest}(\xi_0, \Omega)$ is well-defined, since $\{A : (\xi_{j_i})A \in \Omega^*\}$, $1 \leq i \leq n$, and $\{A : (\xi_{j_k}) [\![\,]\!] A \in \Omega^*\} \cup \{B: (\xi_{j_{i+1}}) [\![\,]\!] B \in \Omega^*\}$, $1 \leq k \leq i$, are consistent by lemma 4.1.
- For any r of $\textsc{Forest}(\xi_0, \Omega)$, $\textsc{Form}_{\textsc{Forest}}(r) = \textsc{Form}_{\textsc{Tree}}(r)$ if $r \in \textsc{Tree}(\xi, ?)$ for some ξ and $?$.
- Let $r, s \in \textsc{Forest}(\xi_0, \Omega)$. If $r, s \in \textsc{Tree}(\xi, ?)$ for some ξ and $?$, then $r \prec_{\textsc{Forest}} s$ iff $r \prec_{\textsc{Tree}} s$, otherwise $r \prec_{\textsc{Forest}} s$ iff one of the following two conditions is satisfied:
  - There is $1 \leq k < i \leq n$ such that $r = \lfloor \xi_{k,i} \rfloor$ and ($s = \lfloor \xi_{j_k} \rfloor$ or $s = \lfloor \xi_{j_i} \rfloor$).
  - There is $1 \leq k, i \leq n$ such that $r = \lfloor \xi_{j_k} \rfloor$ and $s = \lfloor \xi_{j_i} \rfloor$ and $(\xi_{j_k} \subset \xi_{j_i}) \in \Omega^*$.

When no confusion arises, we will write $\textsc{Forest}(\Omega)$ for $\textsc{Forest}(\xi_0, \Omega)$, and $\textsc{Form}$ for $\textsc{Form}_{\textsc{Forest}}$. The lemma below indicates that the properties stated in lemma 4.4, with some obvious adjustments, are preserved in the new structure $\textsc{Forest}(\Omega)$. The proof of lemma 4.5 is omitted.

Lemma 4.5

(1)-(2), (4)-(6): The items (1)-(2), (4)-(6) of lemma 4.4 still hold, when $\textsc{Tree}(\xi, ?)$ and $\prec_{\textsc{Tree}}$ are replaced by $\textsc{Forest}(\Omega)$ and $\prec_{\textsc{Forest}}$, respectively.

(3): $A \in \textsc{Form}(r)$ iff $A \in \Omega^*$ for any static formula A and any $r \in \textsc{Forest}(\Omega)$.

In order to get the desired density we supplement $\textsc{Forest}(\Omega)$ with some new labels.

Construction of INT_Ω **and** $\textsc{Form}_{\mathrm{INT}_\Omega}$

First we define $\mathrm{INT}^m(\Omega)$ for $m \geq 0$ and $\textsc{Form}_{\mathrm{INT}_\Omega}$ as follows:

- $\mathrm{INT}^0(\Omega) = \textsc{Forest}(\Omega)$. For any $r \in \mathrm{INT}^0(\Omega)$, $\textsc{Form}_{\mathrm{INT}_\Omega}(r) = \textsc{Form}_{\textsc{Forest}}(r)$.
- $\mathrm{INT}^{m+1}(\Omega) = \{\lfloor r, \langle\!\langle\rangle\!\rangle a, \frac{1}{2^{m+1}}\rfloor : r \in L_m \text{ and } a \in \Theta \text{ and } \langle\!\langle\rangle\!\rangle a \in \textsc{Form}_{\mathrm{INT}_\Omega}(r)\}$, where L_m is the union of all $\mathrm{INT}^i(\Omega)$ for $0 \leq i \leq m$. For any $s = \lfloor r, \langle\!\langle\rangle\!\rangle a, \frac{1}{2^{m+1}}\rfloor \in \mathrm{INT}^{m+1}(\Omega)$, $\textsc{Form}_{\mathrm{INT}_\Omega}(s) = (\{A : [\![\,]\!]A \in \textsc{Form}_{\mathrm{INT}_\Omega}(r)\} \cup \{\langle\!\langle\rangle\!\rangle a\})^*$.

Then let INT_Ω be the union of all $\mathrm{INT}^m(\Omega)$ for $m \geq 0$.

The notation $\lfloor r, x, \frac{1}{2^{m+1}}\rfloor$ above suggests that the 'edge' between r and $\lfloor r, x, \frac{1}{2^m}\rfloor$ is split into two. For convenience we will simply write $\textsc{Next}_x^{m+1}(r)$ to stand for $\lfloor r, x, \frac{1}{2^{m+1}}\rfloor$. Moreover, we will also simply write INT^i for $\mathrm{INT}^i(\Omega)$, $\textsc{Form}$ for $\textsc{Form}_{\mathrm{INT}_\Omega}$, and L_m for the union of all INT^i for $0 \leq i \leq m$. Using $\vdash \langle\!\langle\rangle\!\rangle a \rightarrow \langle\!\langle\rangle\!\rangle \langle\!\langle\rangle\!\rangle a$, we can easily show that $\textsc{Form}(r)$ is maximal consistent.

Construction of $\prec_{\mathrm{INT}_\Omega}$

- We define a series of binary relations $\prec_m$ on $L_m, m \geq 0$, as follows:
  - $\prec_0$ is defined as $\prec_{\textsc{Forest}}$.
  - Let $m \geq 0$. For any $r, s \in L_m$:
    - $\textsc{Next}_x^{m+1}(r) \prec_{m+1} r$.
    - $s \prec_{m+1} \textsc{Next}_x^{m+1}(r)$ iff $s \prec_m r$ and $\langle\!\langle\rangle\!\rangle\mathtt{CH}(\textsc{Form}(s)) \in \textsc{Form}(\textsc{Next}_x^{m+1}(r))$
- For any labels $r, s \in \mathrm{INT}_\Omega, r \prec_{\mathrm{INT}_\Omega} s$ iff there is $m \geq 0$ such that $r, s \in L_m$ and $r \prec_m s$.

We will simply write $\prec$ for $\prec_{\mathrm{INT}_\Omega}$. Now we want to see whether the properties of $(\textsc{Forest}(\Omega), \prec_{\textsc{Forest}}, \textsc{Form})$ in lemma 4.5 are enjoyed by the new structure $(\mathrm{INT}_\Omega, \prec, \textsc{Form})$. It turns out that we are only able to prove some weaker results in some cases, namely (1) and (4), which are, however, sufficient for our purposes. Now we prove a preliminary result.

Lemma 4.6
For any $r, s \in \mathrm{INT}_\Omega$, let $s \prec r$.

1. $\langle\!\langle\rangle\!\rangle\mathtt{CH}(\textsc{Form}(s)) \in \textsc{Form}(r)$.
2. For any $A \in \mathtt{Subn}(\Omega)$, if $[\![\,]\!]A \in \textsc{Form}(r)$, then $A \in \textsc{Form}(s)$.

Proof. 1. Let $a = \mathtt{CH}(\textsc{Form}(s))$ and $s \prec r$. We need to consider the following three cases:

- $s \prec_0 r$:
  Suppose $\langle\!\langle\rangle\!\rangle a \notin \textsc{Form}(r)$. Then $[\![\,]\!]\neg a \in \textsc{Form}(r)$. By lemma 4.5-(1), $\neg a \in \textsc{Form}(s)$. However, $a \in \textsc{Form}(s)$ by the definition of CH.
- $s \prec_{m+1} r, r \in L_m$ and $s = \textsc{Next}_x^{m+1}(r)$ for some $m \geq 0$ and x:
  If $\langle\!\langle\rangle\!\rangle a \notin \textsc{Form}(r)$, then $[\![\,]\!]\neg a \in \textsc{Form}(r)$ and $\neg a \in \textsc{Form}(s)$ by the definition of Form.
- $s \prec_{m+1} r, s \in L_m$ and $r = \textsc{Next}_x^{m+1}(r_1)$ for some $m \geq 0, r_1 \in L_m$ and x:
  Then, by the definition of $\prec_{m+1}$, we have $\langle\!\langle\rangle\!\rangle a \in \textsc{Form}(r)$.

2. Let $A \in \texttt{Subn}(\Omega)$ and $s \prec r$. Suppose $A \notin \textsc{Form}(s)$. Then $\neg A \in \textsc{Form}(s)$ and there is $B \in \texttt{Subn}(\Omega)$ such that $\vdash \neg A \leftrightarrow B$. Note that $A \in \texttt{notSub}(\Omega)$ does not imply $\neg A \in \texttt{Subn}(\Omega)$. Let $a = \texttt{CH}(\textsc{Form}(s))$. Then $B \propto a$. On the other hand, by (1) we have $\langle\!\langle\rangle\!\rangle a \in \textsc{Form}(r)$, which makes $\langle\!\langle\rangle\!\rangle B \in \textsc{Form}(r)$, and hence $[\![\,]\!]A \notin \textsc{Form}(r)$.

∎

Lemma 4.7

(1): For any $[\![\,]\!]A \in \texttt{Sub}(\Omega)$ and any $r, s \in \text{INT}_\Omega$, if $s \prec r$ and $[\![\,]\!]A \in \textsc{Form}(r)$, then $(A \wedge [\![\,]\!]A) \in \textsc{Form}(s)$.

(2)-(3), (5)-(6): The items (2)-(3) and (5)-(6) of lemma 4.5 still hold, when $\textsc{Forest}(\Omega)$ and $\prec_{\textsc{Forest}}$ are replaced by INT_Ω and $\prec$, respectively.

(4): For any $Y \in H$ and any $r, s \in \text{INT}_\Omega$, if $s \prec r, Y \in \textsc{Form}(r)$ and $Y \in \texttt{Sub}(\Omega)$, then $Y \in \textsc{Form}(s)$.

(7): $\prec$ is dense, i.e., if $r_1 \prec r_2$, then there is $r \in \text{INT}_\Omega$ such that $r_1 \prec r$ and $r \prec r_2$.

Proof. We will only prove (2) and (7). The proof of the rest is omitted.

(2): By induction on $m \geq 0$ we prove that if $a \in \Theta$ and $\langle\!\langle\rangle\!\rangle a \in \textsc{Form}(r)$ and $r \in \text{INT}^m$ then there exists $s \in L_m$ such that $s \prec_m r$ and $a \in \textsc{Form}(s)$.

Basis: $m = 0$. Since $r \in \text{INT}^0(\Omega)$ implies $r \in \textsc{Tree}(\xi, ?)$ for some ξ and $?$, the result follows from the definitions of $\textsc{Tree}(\xi, ?)$ and $\prec_0$.

Induction: Let $m \geq 0$ and suppose $r \in \text{INT}^{m+1}$. We have $r = \textsc{Next}_x^{m+1}(t)$ for some x and $t \in L_m$. If $\langle\!\langle\rangle\!\rangle a \in \textsc{Form}(r)$ then $\langle\!\langle\rangle\!\rangle\, \langle\!\langle\rangle\!\rangle a \in \textsc{Form}(t)$, which implies $\langle\!\langle\rangle\!\rangle a \in \textsc{Form}(t)$ by (Ax3-3), and there is $s \in L_m$ such that $s \prec_m t$ and $a \in \textsc{Form}(s)$ by induction hypothesis. Observe that this in turn implies $s \prec_{m+1} r$, since $a \in \textsc{Form}(s)$ and $a \in \Theta$ implies $a = \texttt{CH}(\textsc{Form}(s))$.

Now, (2) follows from the above result and lemma 4.3.

(7): If $r_1 \prec r_2$, there is $m \geq 0$ such that $r_1, r_2 \in L_m$ and $r_1 \prec_m r_2$. Let $a = \texttt{CH}(\textsc{Form}(r_1))$. By lemma 4.6-(1), $\langle\!\langle\rangle\!\rangle a \in \textsc{Form}(r_2)$. Thus, $r = \textsc{Next}^{m+1}_{\langle\!\langle\rangle\!\rangle a}(r_2) \in \text{INT}_\Omega$ and $r \prec_{m+1} r_2$. On the other hand, $\langle\!\langle\rangle\!\rangle a \in \textsc{Form}(r)$ and $r_1 \prec_m r_2$ implies $r_1 \prec_{m+1} r$. So, $r_1 \prec r$ and $r \prec r_2$.

∎

In order to have the transitivity, we just take the transitive closure of $\prec$. As the following lemma 4.8 indicates, the new structure also enjoys the properties stated in lemma 4.7.

Construction of $@_\Omega$

$@_\Omega$ is defined to be the transitive closure of $\prec$.

Lemma 4.8

The items (1)–(7) of lemma 4.7 still hold, when $\prec$ is replaced by $@_\Omega$. In addition, $@_\Omega$ is transitive.

The proof of lemma 4.8 is omitted for brevity. So far we have constructed INT_Ω and $@_\Omega$. In order to construct the desired interpretation structure $\texttt{IS}_\Omega = (\text{INT}_\Omega, @_\Omega, M_\Omega, V_\Omega)$, we need to define M_Ω and V_Ω.

Definition of M_Ω

We define a mapping M_Ω from C to INT_Ω as follows: for any $\xi \in C$, if $\xi \in \mathtt{CONST}(\Omega)$, then let $M_\Omega(\xi) = \lfloor \xi_{\mathrm{min}} \rfloor$, where min = minimum$\{k : (\xi = \xi_k) \in \Omega^*$ and $\xi_k \in \mathtt{CONST}(\Omega)\}$, otherwise let $M_\Omega(\xi)$ be any arbitrary element of INT_Ω.

Note that if $\xi \in \mathtt{CONST}(\Omega)$, then the set $\{k : (\xi = \xi_k) \in \Omega^*$ and $\xi_k \in \mathtt{CONST}(\Omega)\}$ is not empty since $(\xi = \xi)$ is an axiom. Moreover, by the construction of INT_Ω we have $\lfloor \xi_{\mathrm{min}} \rfloor \in \mathrm{INT}_\Omega$. Thus, M_Ω is a well-defined mapping from C to INT_Ω. When $\xi \notin C$, there are many choices of $M_\Omega(\xi)$. For our later purpose we can pick up any one.

Definition of V_Ω

Let r be any element of INT_Ω.

- For any $A \in (S \cup D)$, we define $V_\Omega(A, r) = \mathit{true}$ iff $A \in \textsc{Form}(r)$.
- For any $A \in (H \cap \mathtt{Sub}(\Omega))$, we define $V_\Omega(A, r) = \mathit{true}$ iff $A \in \textsc{Form}(r)$.
- For any $A \in H$ but $A \notin \mathtt{Sub}(\Omega)$, we define $V_\Omega(A, r) = \mathit{false}$.

Let $\mathtt{IS}_\Omega = (\mathrm{INT}_\Omega, @_\Omega, M_\Omega, V_\Omega)$, where $\mathrm{INT}_\Omega, @_\Omega, M_\Omega$, and V_Ω are defined as above.

Corollary 4.9
$\mathtt{IS}_\Omega = (\mathrm{INT}_\Omega, @_\Omega, M_\Omega, V_\Omega)$ is an interpretation structure for BAR.

This corollary can be proved by using lemma 4.8. The detailed proof is omitted. corollary 4.9 is one of our main desired results. In order to prove our completeness theorem, we just need to show that Ω is satisfiable in $\mathtt{IS}_\Omega = (\mathrm{INT}_\Omega, @_\Omega, M_\Omega, V_\Omega)$.

Proposition 4.10
For any $A \in \mathtt{Sub}(\Omega)$ and any $r \in \mathrm{INT}_\Omega$, $\mathtt{IS}_\Omega \models_r A$ iff $A \in \textsc{Form}(r)$.

Proof. Let A be any element of $\mathtt{Sub}(\Omega)$. We prove it by induction on the complexity of A.

Basis: Let r be any element of INT_Ω.

Case 1: $A \in (S \cup D \cup H)$.
By lemma 4.8.

Case 2: A is $(\xi_i = \xi_j)$, where $\xi_i, \xi_j \in C$.
Note that $\mathtt{IS}_\Omega \models_r (\xi_i = \xi_j)$ iff $M_\Omega(\xi_i) = M_\Omega(\xi_j)$. Since $A \in \mathtt{Sub}(\Omega)$, we have $\xi_i, \xi_j \in \mathtt{CONST}(\Omega)$, and hence $M_\Omega(\xi_i) = \lfloor \xi_{\mathrm{min}(i)} \rfloor$ and $M_\Omega(\xi_j) = \lfloor \xi_{\mathrm{min}(j)} \rfloor$, where

$$\begin{aligned} \mathrm{min}(i) &= \mathrm{minimum}\{k : (\xi_k = \xi_i) \in \Omega^* \text{ and } \xi_k \in \mathtt{CONST}(\Omega)\} \\ \mathrm{min}(j) &= \mathrm{minimum}\{k : (\xi_k = \xi_j) \in \Omega^* \text{ and } \xi_k \in \mathtt{CONST}(\Omega)\} \end{aligned}$$

By lemma 4.8–(3), it suffices to prove that $\mathrm{min}(i) = \mathrm{min}(j)$ iff $(\xi_i = \xi_j) \in \Omega^*$, which follows from axioms (Ax2-8) and (Ax2-9).

Case 3: A is $(\xi_i \subset \xi_j)$.

As in Case 2, $M_\Omega(\xi_i) = \lfloor \xi_{\min(i)} \rfloor$ and $M_\Omega(\xi_j) = \lfloor \xi_{\min(j)} \rfloor$. It suffices to show that $\xi_i \subset \xi_j \in \Omega^*$ iff

$\lfloor \xi_{\min(i)} \rfloor \ @_\Omega \ \lfloor \xi_{\min(j)} \rfloor$. Since $\xi_i = \xi_{\min(i)} \in \Omega^*$ and $\xi_j = \xi_{\min(j)} \in \Omega^*$, by (Ax2–9) we have $(\xi_i \subset \xi_j) \in \Omega^*$ iff $(\xi_{\min(i)} \subset \xi_{\min(j)}) \in \Omega^*$.

Thus we want to prove $(\xi_{\min(i)} \subset \xi_{\min(j)}) \in \Omega^*$ iff $\lfloor \xi_{\min(i)} \rfloor \ @_\Omega \ \lfloor \xi_{\min(j)} \rfloor$. We need to consider the following two basic cases:

Case 3–1: $\min(i) = \min(j)$.

By (Ax2–1), $(\xi_{\min(i)} \subset \xi_{\min(i)}) \notin \Omega^*$. By lemma 4.8 –(5), $\lfloor \xi_{\min(i)} \rfloor \ @_\Omega \ \lfloor \xi_{\min(i)} \rfloor$ does not hold.

Case 3–2: $\min(i) \neq \min(j)$.

- If $\xi_{\min(i)} \subset \xi_{\min(j)} \in \Omega^*$, then $\lfloor \xi_{\min(i)} \rfloor \prec_0 \lfloor \xi_{\min(j)} \rfloor$, thus $\lfloor \xi_{\min(i)} \rfloor \ @_\Omega \ \lfloor \xi_{\min(j)} \rfloor$.
- Suppose $\lfloor \xi_{\min(i)} \rfloor \ @_\Omega \ \lfloor \xi_{\min(j)} \rfloor$. We want to prove $(\xi_{\min(i)} \subset \xi_{\min(j)}) \in \Omega^*$. Since $@_\Omega$ is the transitive closure of $\prec$, there are $r_1, \ldots, r_m \in \mathrm{INT}_\Omega$, $m \geq 0$, such that $\lfloor \xi_{\min(i)} \rfloor \prec r_1 \prec \ldots \prec r_m \prec \lfloor \xi_{\min(j)} \rfloor$. The result can be proved by induction on $m \geq 0$.

Case 4: A is $(\xi_i \# \xi_j)$.

It suffices to prove that $(\xi_i \# \xi_j) \in \Omega^*$ iff there is $s \in \mathrm{INT}_\Omega$ such that $s \ @_\Omega \ \lfloor \xi_{\min(i)} \rfloor$ and $s \ @_\Omega \ \lfloor \xi_{\min(j)} \rfloor$. It is easy to show that $(\xi_i \# \xi_j) \in \Omega^*$ iff $(\xi_{\min(i)} \# \xi_{\min(j)}) \in \Omega^*$. The rest of the proof is lengthy. We omit it.

Induction:

Case i: A is $\neg B$ or $(B \to C)$.

As usual.

Case ii: A is $\big(\xi_i\big)B$.

Note that $\xi_i \in \mathtt{CONST}(\Omega)$ and $B \in \mathtt{Sub}(\Omega)$. By the induction hypothesis we have that $\mathtt{IS}_\Omega \models_s B$ iff $B \in \mathrm{F{\scriptsize ORM}}(s)$ for any $s \in \mathrm{INT}_\Omega$. Let $r \in \mathrm{INT}_\Omega$. We have $\mathtt{IS}_\Omega \models_r \big(\xi_i\big)B$

iff $\mathtt{IS}_\Omega \models_{M_\Omega(\xi_i)} B$ iff $B \in \mathrm{F{\scriptsize ORM}}(M_\Omega(\xi_i))$.

On the other hand, by lemma 4.8–(3), $\big(\xi_i\big)B \in \mathrm{F{\scriptsize ORM}}(r)$ iff $\big(\xi_i\big)B \in \Omega^*$. Define $\min(i)$ as in Case 2. Then $M_\Omega(\xi_i) = \lfloor \xi_{\min(i)} \rfloor$, and $\big(\xi_i\big)B \in \Omega^*$ iff $\big(\xi_{\min(i)}\big)B \in \Omega^*$. By using axiom (Ax4-1) we can prove $B \in \mathrm{F{\scriptsize ORM}}(\lfloor \xi_{\min(i)} \rfloor)$ iff $\big(\xi_{\min(i)}\big)B \in \Omega^*$.

Case iii: A is $[\![\ ]\!]B$.

Since $[\![\ ]\!]B \in \mathtt{Sub}(\Omega)$, we have $\neg B \in \mathtt{notSub}(\Omega)$. We can prove it by using (1) and (2) of lemma 4.8.

∎

Corollary 4.11 (Completeness Theorem)
Let Ω be any formula. If Ω is consistent, then Ω has a model.

5 Discussion

In previous sections we have presented a logic and proved its completeness. In this section we make some discussions on related topics for practical applications.

5.1 Intervals and points

This paper uses intervals as basic time elements. Another possible choice of basic time elements is time points.

Intuitively, an interval can be defined as a set of time points $\{p : I^- \leq p \leq I^+\}$, or $\{p : I^- < p \leq I^+\}$, or $\{p : I^- \leq p < I^+\}$, or $\{p : I^- < p < I^+\}$, where I^- and I^+ denote the bounds of the interval. In order to avoid the fire-on-and-off paradox [1], some researchers seem to prefer $\{p : I^- < p \leq I^+\}$ or $\{p : I^- \leq p < I^+\}$.

In temporal relational algebras, some special operators are introduced to relate intervals to points. For example, in [17] there are two basic operators UNFOLD and FOLD: the operator UNFOLD extracts all the points $t_m, t_{m+1}, \ldots, t_n$ from a time interval $\{p : t_m \leq p < t_{n+1}\}$, and FOLD does the inverse. Operations having similar functions or playing similar roles can be found in other proposals [28]. The reader is referred to van Benthem [30] for a comprehensive model-theoretical study of time points, intervals and their relationships.

To choose time intervals or points as basic time elements is concerned with expressiveness and complexity of information representation and retrieval, and practical requirements. In fact, it might never be conclusive, or convincing, to say time intervals are better than time points or to say the converse, since we can always relate them to each other by mathematics. From the point of view of practice, different people may have different choices. In what follows we shall briefly analyse why we have chosen time intervals as basic time elements from the point of view of practice.

First, we observe that multiple time units are widely used in practice. If time is viewed as point-based, we need to assume what the smallest time unit is. The smallest time unit is then taken as time point. After the smallest time unit is assumed, we may later need to deal with temporal information related to smaller time units. Thus, we have to give up the old assumption of the smallest time unit and assume a new one, then relate old time stamps to the new smallest time unit (point). This could be very cumbersome. It can be seen that time points are not so flexible as time intervals, since we do not have to assume smallest time units in our interval-based time structure.

Second, we observe that it is hard to define validity of non-hereditary information in terms of time points if time points are not big enough. For example, if a time point is not a month, it is not clear how to represent something like that the average temperature of Lisbon is 10°C in January, but it is very easy and natural to represent it in terms of time intervals.

It should be stressed that our positional logic allows for static, dynamic and hereditary propositions. If all propositions are hereditary, the completeness proof above still works.

5.2 Interval relations

From Section 3 it can be seen that the logic BAR and its semantics can talk about interval containment ($\subset$), interval overlap (#) and interval equality (=). Thus, BAR can be used to represent information where only interval relations $\subset, \#$ and $=$ are involved.

As is known, there are thirteen exclusive binary interval relations[1]. A natural question is whether BAR can be used to talk about other interval relations such as before and after. Assume we are given

$$[\![\%1992\%\,]\!]\text{FACULTY (`John', `Associate Prof')}$$

where %1992% designates the year 1992.

In addition to the query 'Is John an Associate Professor in September 1992?' , which is concerned with interval containment, we may want to answer a query like 'Is John an Associate Professor before August 1993?'. The answer to it should be 'yes', since John is an Associate Professor in 1992 and $(\%1992\%\ before\ \%1993, 8\%)$ should be valid.

There is a simple way to deal with other interval relations than containment, overlap and equality. Let ξ and ζ be any two intervals and $\oplus$ any of binary interval relations. One can simply extend the clause (2) of definition 3.1 by admitting $(\xi \oplus \zeta)$ as a wff. Notice that the truth value of $(\xi \oplus \zeta)$ does not depend on the observation interval. In any interpretation I, either $(\xi \oplus \zeta)$ or $\neg(\xi \oplus \zeta)$ is valid in IS. This means that $(\xi \oplus \zeta)$ is actually static. If expressions of the form$(\xi \oplus \zeta)$ are added as explicit formulas, some axioms are needed to characterize the properties of $\oplus$ so that formulas like $(\%1992\%\ \ before\ \%1993, 8\%)$ could be taken as axioms or derived as theorems. Whether the extended logic is complete depends on what new requirements are imposed on the interpretation structure of definition 3.2. The main concern of this paper is positional operators like $(\!\xi\!)$ and $[\![\xi\,]\!]$. Their axiomatization need not have something to do with all the interval relations. For axiomatic properties of $(\!\xi\!)$ and $[\![\xi\,]\!]$, it suffices only to consider interval containment, overlap and equality, as shown as in previous sections. In fact, the proof schema of the completeness of BAR can be adapted to show the completeness of the enhanced BAR (at least in some cases). As an illustrative example, let's add $(\xi\ before \zeta)$ to BAR, and denote the extended BAR as BAR$^+$. As said before, whether the extended logic is complete depends on what new requirements are imposed on the interpretation structure of definition 3.2. For simplicity we assume BEFORE satisfies the following properties: (i) irreflexivity; (ii) transitivity; and (iii) monotonicity:

$$(\forall I_1, I_2, I_3 \in \text{INT})(((I_1\ \ @\ I_2) \wedge (I_2 \text{B}\text{EFORE}\ I_3)) \rightarrow (I_1 \text{B}\text{EFORE}\ I_3))$$
$$(\forall I_1, I_2, I_3 \in \text{INT})(((I_1\ \ @\ I_2) \wedge (I_3 \text{B}\text{EFORE}\ I_2)) \rightarrow (I_3 \text{B}\text{EFORE}\ I_1))$$

The interpretation structure of BAR$^+$ now becomes $(\text{INT}, @, \text{BEFORE}, M,\ V)$ with BEFORE as interpretation of $before$. Let $\xi_i, \xi_j, \xi_k \in C$ be any constant symbols, add the following formulas to the axioms of BAR:

(B1)	$\neg(\xi_i\ before\ \xi_i)$
(B2)	$(\xi_i\ before\ \xi_j) \wedge (\xi_j\ before\ \xi_k) \rightarrow (\xi_i\ before\ \xi_k)$
(B3)	$(\xi_i \subset \xi_j) \wedge (\xi_j\ before\ \xi_k) \rightarrow (\xi_i\ before\ \xi_k)$
(B4)	$(\xi_i \subset \xi_j) \wedge (\xi_k\ before\ \xi_j) \rightarrow (\xi_k\ before\ \xi_i)$
(B5)	$(\xi_i\ before\ \xi_j) \leftrightarrow [\![\ ]\!](\xi_i\ before\ \xi_j)$
(B6)	$\neg(\xi_i\ before\ \xi_j) \leftrightarrow [\![\ ]\!]\neg(\xi_i\ before\ \xi_j)$
(B7)	$(\xi_i\ before\ \xi_j) \leftrightarrow \big(\xi_k\big)(\xi_i\ before\ \xi_j)$
(B8)	$\neg(\xi_i\ before\ \xi_j) \leftrightarrow \big(\xi_k\big)\neg(\xi_i\ before\ \xi_j)$

Extend definition 3.5 such that formulas of the form $(\xi_i\ before\ \xi_j)$ are static formulas.

Let Ω be a consistent formula. We can follow the same procedure to construct and define INT_Ω, $@_\Omega$, M_Ω, V_Ω. Note that formulas of the form $(\xi_i\ before\ \xi_j)$ are static formulas. It is easy to show that lemmas 4.1- 4.8 still hold. In order to complete the construction of the intended interpretation, we define $\mathrm{B{\scriptstyle EFORE}}_\Omega$ to be the smallest binary relation on INT_Ω satisfying:

(i) $\lfloor\xi_{j_k}\rfloor\ \mathrm{B{\scriptstyle EFORE}}_\Omega\ \lfloor\xi_{j_i}\rfloor$ iff $(\xi_{j_k}\ before\ \xi_{j_i}) \in \{\Omega\}^*$.

(ii) if $r\ @_\Omega\ \lfloor\xi_{j_i}\rfloor$ and $\lfloor\xi_{j_i}\rfloor\ \mathrm{B{\scriptstyle EFORE}}_\Omega\ \lfloor\xi_{j_k}\rfloor$, then $r\ \mathrm{B{\scriptstyle EFORE}}_\Omega\ \lfloor\xi_{j_k}\rfloor$.

(iii) if $r\ @_\Omega\ \lfloor\xi_{j_k}\rfloor$ and $\lfloor\xi_{j_i}\rfloor\ \mathrm{B{\scriptstyle EFORE}}_\Omega\ \lfloor\xi_{j_k}\rfloor$, then $\lfloor\xi_{j_i}\rfloor\ \mathrm{B{\scriptstyle EFORE}}_\Omega\ r$.

(iv) if $r\ \mathrm{B{\scriptstyle EFORE}}_\Omega\ s$ and $s\ \mathrm{B{\scriptstyle EFORE}}_\Omega\ t$, then $r\ \mathrm{B{\scriptstyle EFORE}}_\Omega\ t$.

Since $\{\Omega\}^*$ is maximal consistent, we can show that $\mathrm{B{\scriptstyle EFORE}}_\Omega$ satisfies irreflexivity, transitivity and monotonicity (there is no technical interest or difficulty here), which makes $(\mathrm{INT}_\Omega, @_\Omega, \mathrm{B{\scriptstyle EFORE}}_\Omega, M_\Omega, V_\Omega)$ an interpretation structure of BAR^+. In the proof of the counterpart of proposition 4.10, it suffices to additionally prove: $\mathtt{IS}_\Omega \models_r (\xi_i\ before\ \xi_j)$ iff $(\xi_i\ before\ \xi_j) \in \mathrm{F{\scriptstyle ORM}}(r)$, for any $(\xi_i\ before\ \xi_j) \in \mathtt{Sub}(\Omega)$ and any $r \in \mathrm{INT}_\Omega$. This can be proved by using the properties of the maximal consistent sets and (3) of lemma 4.8. Note that $(\xi_i\ before\ \xi_j)$ is a static formula.

The argument above should have made it clear that other interval relations can be easily dealt with by explicitly adding some formulas into BAR. The proof schema of the completeness of BAR can be adapted for the extended logic. Since the main concern of this paper is positional operators like $\big(\xi\big)$ and $[\![\xi\ ]\!]$, we only took into account the interval relations which have a close and essential relationship with the axiomatization of $\big(\xi\big)$ and $[\![\xi\ ]\!]$, namely interval containment, overlap and equality.

The positional operators are mainly used for representation of knowledge with absolute temporal references, which are usually dates such as September 1992. In the next section we discuss extensions to BAR for other temporal references.

5.3 Extensions

By a careful examination of BAR it can be found that BAR (extended with other interval relations) is not expressive for some other purposes.

As studied in [13], there are three kinds of temporal references involved in temporal information: absolute, relative and periodical temporal references. An absolute temporal reference is indicated by, e.g. 'in 1992' in 'John is an Associate Professor in 1992'; a relative temporal reference is indicated by, e.g. 'before' in 'John had been promoted to Associate Professor before Mary became Assistant Professor'; a periodical temporal reference is indicated by, e.g. 'at seven o'clock every morning' in 'John gets up at 7 o'clock every morning'.

The logic BAR is not expressive if all the three kinds of temporal references are needed. But BAR can be extended by adding temporal operators for relative temporal references and periodical references. In [13] a very complex and expressive temporal formalism $\mathcal{A}$ for all the three kinds of temporal references, and actions and their effects, is proposed and studied, where the logic BAR seems to play the central and basic role. That is also why BAR is called Basic formalism for both Absolute and Relative temporal references. The completeness of $\mathcal{A}$ is, however, unknown to us, although its core part BAR has been shown to be complete. In fact, no effort has been made towards the completeness of $\mathcal{A}$. Once the operators for all the three kinds of temporal references are admitted in a unifying formalism such as $\mathcal{A}$, the use of their combinations will greatly enhance the expressiveness. For example, let $\mathbf{P}A$ to denote A is true on an interval in the past as usual in interval temporal logic, combining (ξ) and $\mathbf{P}$ we can write $(\xi)\mathbf{P}A$ to express'A is true before ξ'. For example, 'John was an Assistant Professor before 1992' can be expressed as $(\%1992\%)\mathbf{P}$FACULTY('John', 'Assistant Prof'), where FACULTY('John', 'Assistant Prof') is an atomic hereditary assertion.

In practice, to choose or to design a formalism depends on many factors such as completeness, expressiveness and decidability, and some aesthetic standards. This makes it very hard and complicated to compare alternative solutions. This paper is not intended to be involved in disputes. More discussions and technical details can be found in [13].

5.4 Implementation

Detailed discussion on the implementation of BAR or $\mathcal{A}$ is out of this paper, but in the sequel we indicate two implementation schemes of the implementation of positional operators for the reader who is interested in implementations and applications.

The first scheme is to use reified positional logic. That is, one can express positional logic in a first-order logic. In [21, 24], some main ideas on reified temporal logic are discussed. Actually, we can follow the same idea to reify our positional logic by first-order logic. After positional logic is expressed by first-order logic, we can then make use of any known mechanisms for first-order logic to do reasoning in positional logic.

The second scheme is to use automated theorem-proving algorithms in positional logic. In general, automated theorem-proving in positional logic is more difficult than that in first-order logic. The resolution principle and Herbrand theorem serve as a theoretic basis for general automated theorem-proving in first-order logic. In positional logic, we can analogously develop positional resolution rules. For example, from formulas $(\zeta \subset \xi)$, $[\![\xi\ ]\!](L \vee L_1 \vee \ldots \vee L_m)$ and $[\![\zeta\ ]\!](\neg L \vee L_1' \vee \ldots \vee L_m')$,

we can have a resolvent $[\![\zeta\ ]\!](L_1 \vee \ldots \vee L_m \vee L_1' \vee \ldots \vee L_m')$. The key point in positional resolution is that: Two clauses can be resolved if they have the same positional modal context. Positional resolution principle is similar to modal resolution principle of [4] to some extent. Following the ideas of first-order temporal tableau and transitional tableau rules of [16], we can also develop positional tableau rules for positional operators when we always keep in mind that a set of formulas appear in the same tableau if they are in the same positional modal context. The dependency relation among tableaux can be defined similarly to that in [16]. We will not go into deeper discussions.

6 Concluding remarks

This paper has presented an attempt to combine positional operators and linebreak modal/temporal operators to construct a logic BAR for temporal knowledge representation in a single unifying interval-based framework. The underlying dense time structure of the positional logic was motivated by the common-sense calendar-clock style time, where multiple time granularities to any precision, often involved in temporal information, can be very naturally supported. The syntax, semantics, and axiomatization of BAR have been presented, and its completeness has also been proved. From the axiomatization of BAR it can be seen that the operators in BAR are normal in the sense that they have the axiom K and the necessity rule of [3], but the proof of its completeness is more complicated than the standard proof of the completeness of normal modal logics. In practical applications, BAR may be extended either by admitting more interval relations for absolute temporal references or by adding other kinds of temporal operators for relative and periodical temporal references. BAR can also be extended by adding actions and their effects [13].

The focus of this paper is on positional operators and on a proof of the completeness of BAR, a basic formalism for absolute and relative temporal references. Other topics such as decidability are out of the scope of this paper and constitute a direction for future research. The positional logic in this paper has been applied to contractual knowledge representation in our recent work, which is involved in many absolute temporal references expressed by time elements such as dates. The positional assertions play a very important role in contractual knowledge representation. Positional operators can also be used to design a logic-based language for general purposes of representing and querying historical knowledge [15].

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