Beyond rational monotony: some strong non-Horn rules for nonmonotonic inference relations

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Abstract

Lehmann, Magidor and others have investigated the effects of adding the non-Horn rule of *rational monotony* to the rules for preferential inference in nonmonotonic reasoning. In particular, they have shown that every inference relation satisfying those rules is generated by some ranked preferential model.

We explore the effects of adding a number of other non-Horn rules that are stronger than or incomparable with rational monotony, but which are still weaker than plain monotony. Distinguished among these is a rule of *determinacy preservation*, equivalent to one of *rational transitivity*, for which we establish a representation theorem in terms of *quasi-linear* preferential models. An important tool in the proof of the representation theorem is the following purely semantic result, implicit in work of Freund, but here established by a more direct argument: every ranked preferential model generates the same inference relation as some ranked preferential model that is *collapsed*, in the sense of being both injective and such that each of its states is minimal for some formula.

We also consider certain other non-Horn rules which are incomparable with monotony but are implied by conditional excluded middle, and establish a representation result for a central one among them, which we call *fragmented disjunction*, equivalent to *fragmented conjunction*, in terms of *almost linear* preferential models.

Finally, we consider briefly some curious Horn rules beyond the preferential ones but weaker than monotony, notably those which we call *conjunctive insistence* and *n-monotony*.

Keywords: nonmonotonic reasoning, rational monotony, preferential models.

1 Introduction and Overview

The postulates for preferential inference, as formulated by Kraus, Lehmann and Magidor [7] are intended to gather together some properties for inference relations that may be regarded as in principle desirable, even when the inference relations are not monotonic. They are all Horn conditions, that is of the form: if such and such pairs are in the relation, so too is such another pair. Lehmann and Magidor [8] and [9] have also studied the effects of adding to the preferential postulates a further rule, non-Horn in character, called *rational monotony*. As usually formulated with a negative premise, it is: if $\alpha \mid\!\sim \beta$ and *not* $\alpha \mid\!\sim \neg\gamma$, then $\alpha\wedge\gamma \mid\!\sim \beta$. Equivalently, with positive premises but disjunctive conclusion, it is: if $\alpha \mid\!\sim \beta$ then either $\alpha\wedge\gamma \mid\!\sim \beta$ or $\alpha \mid\!\sim \neg\gamma$. In the two papers mentioned, it is shown that every inference relation satisfying the preferential rules is determined by some model of a certain kind, also called preferential, and that every inference relation satisfying in addition rational monotony is determined by some ranked preferential model.

It is known that rational monotony implies certain other non-Horn conditions of interest, notably *disjunctive rationality*, which in turn implies *negation rationality* - see for example the brief accounts in Makinson [12] and Lehmann and Magidor [9], or the more extensive work in Freund [2] and Freund and Lehmann [3] which provide a semantic characterization of inference relations satisfying these two rules. It is natural to ask whether there are any other rules of interest, stronger than or incomparable with rational monotony, but still weaker than plain monotony.

Makinson [12] drew attention to one such rule, called *determinacy preservation*, showing that it lies between monotony and rational monotony, but without investigating it semantically. Bezzazi and Pino Pérez [1] began a semantic investigation of two other rules, *rational transitivity* and *rational contraposition*. In this paper we study these and related conditions more systematically, establishing interrelations and providing semantic characterizations.

It turns out, as we shall show, that given the preferential rules, rational transitivity and determinacy preservation are equivalent, and are in turn equivalent to the combined force of rational monotony with rational contraposition, as also to the combined force of rational monotony with another rule that we shall consider. Rational transitivity *alias* determinacy preservation thus appears to occupy a rather pivotal position in this region. We show that any inference relation satisfying that rule in addition to preferential ones, is determined by a preferential model that is in a certain sense quasi-linear. The proof makes use of Lehmann and Magidor's representation theorem for rational monotony, but also of an important tool of a purely semantic nature. This is the result, implicit in Freund [2], that every ranked preferential model determines the same inference relation as some ranked preferential model that is *collapsed*, in the sense of being both injective and such that each state is minimal for some formula.

We also consider certain non-Horn rules that are not implied by monotony, and which for this reason are perhaps intuitively less interesting, but which are nevertheless weaker than the well-known rule of *conditional excluded middle* of Stalnaker [16], also called *full determinacy* in Makinson [12]: if not $\alpha \mid\!\sim \beta$ then $\alpha \mid\!\sim \neg\beta$. We isolate two such rules of particular formal interest, which we call *disjunction fragmentation* and *conjunction fragmentation*. We prove that they are equivalent and then we establish a representation theorem for preferential relations satisfying disjunction fragmentation (conjunction fragmentation), using the same semantic tool as for rational transitivity above.

All of the rules so far mentioned as potential additions to those for preferential inference, are non-Horn. Curiously, Horn rules appear to be less plentiful as potential additions. However in a final section we identify some such rules, weaker than monotony but not implied by rational monotony, represent some of them semantically, and raise a number of open questions.

We presume some familiarity with the main lines of at least one of Kraus, Lehmann and Magidor [7], Lehmann and Magidor [9], Makinson [12].

2 Background

In this section we recall some basic definitions and results from Kraus, Lehmann and Magidor [7] and Lehman and Magidor [9], which will be used in the paper.

We consider formulae of classical propositional calculus built over a set of elementary formulae denoted *Var* plus two constants $\top$ and $\bot$ (the formulae **true** and **false** respectively). Let $\mathcal{L}$ be the set of formulae. If *Var* is finite we will say that the language $\mathcal{L}$ is finite. Let $\mathcal{U}$ be the set of valuations (or worlds), *i.e.* functions $v : Var \cup \{\top, \bot\} \longmapsto \{0,1\}$ such that $v(\top) = 1$ and $v(\bot) = 0$. We use lower case letters of the Greek alphabet to denote formulae, and the letters $v, v_1, v_2, \ldots$ to denote worlds. As usual, $\vdash \alpha$ means that α is a tautology and $v \models \alpha$ means that v satisfies α where compound formulae are evaluated using the usual truth-functional rules. We consider certain binary relations between formulae. These relations will be called inference relations and will be written $\mid\!\sim$.

Definition 2.1 A relation $\mid\!\sim$ is said to be preferential iff the following rules hold

$$\text{REF}\quad \frac{}{\alpha\mid\!\sim\alpha} \qquad\qquad \text{LLE}\quad \frac{\alpha\mid\!\sim\beta \quad \vdash \alpha \leftrightarrow \gamma}{\gamma\mid\!\sim\beta}$$

$$\text{RW}\quad \frac{\alpha\mid\!\sim\beta \quad \vdash \beta \rightarrow \gamma}{\alpha\mid\!\sim\gamma} \qquad\qquad \text{AND}\quad \frac{\alpha\mid\!\sim\beta \quad \alpha\mid\!\sim\gamma}{\alpha\mid\!\sim\beta\wedge\gamma}$$

$$\text{OR}\quad \frac{\alpha\mid\!\sim\gamma \quad \beta\mid\!\sim\gamma}{\alpha\vee\beta\mid\!\sim\gamma} \qquad\qquad \text{CM}\quad \frac{\alpha\mid\!\sim\beta \quad \alpha\mid\!\sim\gamma}{\alpha\wedge\gamma\mid\!\sim\beta}$$

These rules are known as the rules of the system P. The abbreviations above are read as follows: REF -reflexivity, LLE -left logical equivalence, RW -right weakening, CM -cautious monotony. AND and OR are self-explanatory.

A relation $\mid\!\sim$ is said to be rational iff it is preferential and the following rule (rational monotony) holds

$$\text{RM}\quad \frac{\alpha\mid\!\sim\beta \quad \alpha\not\mid\!\sim\neg\gamma}{\alpha\wedge\gamma\mid\!\sim\beta}$$

Definition 2.2 A structure $\mathcal{M}$ is defined by a triple $\langle S, \imath, \prec\rangle$ where S is a set (of arbitrary items, called states), $\prec$ is a strict order (*i.e.* transitive and irreflexive) on S and $\imath : S \longmapsto \mathcal{U}$ is a total function (the interpretation function). If the function $\imath$ is injective the structure is said also to be injective.

Let $\mathcal{M} = \langle S, \imath, \prec\rangle$ be a structure. We adopt the following notations: if $T \subseteq S$, then $\min(T)$ is the set of all minimal elements of T with respect to $\prec$, *i.e.* $\min(T) = \{t \in T : \neg\exists t' \ (t' \in T \ \text{and} \ t' \prec t)\}$; $\text{mod}_{\mathcal{M}}(\alpha) = \{s \in S : \imath(s) \models \alpha\}$; $\min_{\mathcal{M}}(\alpha)$ denotes $\min(\text{mod}_{\mathcal{M}}(\alpha))$.

Definition 2.3 A structure $\mathcal{M} = \langle S, \imath, \prec \rangle$ is said to be a preferential model iff for any formula α the following property (smoothness) holds

$$\forall s \in \mathrm{mod}_{\mathcal{M}}(\alpha) \setminus \min_{\mathcal{M}}(\alpha) \quad \exists s' \in \min_{\mathcal{M}}(\alpha) \;\; s' \prec s$$

A structure $\mathcal{M} = \langle S, \imath, \prec \rangle$ is said to be a ranked model iff it is a preferential model and there exists a strict linear order $(\Omega, <)$ and a function $r : S \longmapsto \Omega$ such that for any $s, s' \in S$, $s \prec s'$ iff $r(s) < r(s')$.

Definition 2.4 Let $\mathcal{M} = \langle S, \imath, \prec \rangle$ be a preferential model. The inference relation $\mid\!\sim_{\mathcal{M}}$ is defined by the following

$$\alpha \mid\!\sim_{\mathcal{M}} \beta \Leftrightarrow \min_{\mathcal{M}}(\alpha) \subseteq \mathrm{mod}_{\mathcal{M}}(\beta)$$

The following (representation) theorem is due to Kraus, Lehmann and Magidor [7].

Theorem 2.5 $\mid\!\sim$ is a preferential relation iff there is a preferential model $\mathcal{M} = \langle S, \imath, \prec \rangle$ such that $\mid\!\sim_{\mathcal{M}} = \mid\!\sim$. If the language is finite then S can be chosen finite.

The following (representation) theorem is due to Lehmann and Magidor [9].

Theorem 2.6 $\mid\!\sim$ is a rational relation iff there is a ranked model $\mathcal{M} = \langle S, \imath, \prec \rangle$ such that $\mid\!\sim_{\mathcal{M}} = \mid\!\sim$. If the language is finite then S can be chosen finite.

Proposition 2.7 If $\mid\!\sim$ is a preferential relation then the following rules hold

$$\mathsf{S} \quad \frac{\alpha \wedge \beta \mid\!\sim \gamma}{\alpha \mid\!\sim \beta \rightarrow \gamma} \qquad\qquad \mathsf{CUT} \quad \frac{\alpha \wedge \beta \mid\!\sim \gamma \quad \alpha \mid\!\sim \beta}{\alpha \mid\!\sim \gamma}$$

If $\mid\!\sim$ is a rational relation then the following rules hold

$$\mathsf{DR} \quad \frac{\alpha \vee \beta \mid\!\sim \gamma \quad \alpha \not\mid\!\sim \gamma}{\beta \mid\!\sim \gamma} \qquad\qquad \mathsf{NR} \quad \frac{\alpha \mid\!\sim \beta \quad \alpha \wedge \gamma \not\mid\!\sim \beta}{\alpha \wedge \neg\gamma \mid\!\sim \beta}$$

For the proofs see [7] for S and CUT and see [12] or [9] for DR and NR. The abbreviations above are read as follows: S -Shoham rule (this abbreviation is taken from [7]; note that this rule corresponds to the hard half of the deduction theorem for classical $\vdash$), DR -disjunctive rationality, NR -negation rationality. The term CUT is self-explanatory, but it should be noted that this form of cut, which plays an important role in nonmonotonic logic, is weaker than the forms of cut usually studied in Gentzen-style formulations of classical and intuitionistic logic. The latter imply transitivity of the inference relation; the former does not.

Notation: If n is a natural number, $\overline{n}$ will denote the set $\{0, 1, \ldots, n\}$ linearly ordered with the natural order $<$. If A is a set, the cardinality of A will be denoted by $|A|$. When $\mathcal{M} = \langle S, \imath, \prec \rangle$ is a preferential model, $u \in S$ and α a formula, if there is no ambiguity we shall write $u \models \alpha$, $\mathrm{mod}(\alpha)$ and $\min(\alpha)$ instead of $\imath(u) \models \alpha$, $\mathrm{mod}_{\mathcal{M}}(\alpha)$ and $\min_{\mathcal{M}}(\alpha)$ respectively.

Observation 2.8 It is known that there are preferential models whose inference relation is not generated by any injective one. A simple finite example was given *en passant* by Krauss, Lehmann and Magidor at the end of section 5.2 of [7]. The language is assumed to have just two elementary sentences p, q. The states are s_i $(0 \leq i \leq 3)$ with $s_0 < s_2$ and $s_1 < s_3$, and $s_0 \models p \wedge \neg q$, $s_1 \models \neg p \wedge \neg q$, whilst $s_2, s_3 \models p \wedge q$. Kraus, Lehmann and Magidor leave the verification of the example as an exercise; a verification is sketched by Schlechta in section 1 of [15]. Because of its relation with the theme of this paper, we give the verification in full, using moreover an infinite language so as to make it clear that the example is not an artifact of a limited number of elementary sentences.

Let p_j $(j \in J)$ be all the other elementary sentences and make them behave just like p, *i.e.* put $s_i \models p_j$ iff $s_i \models p$. Let $\mid\!\sim$ the inference relation determined by this preferential model. Then clearly we have the following: (1) $(p \wedge q) \vee \neg q \mid\!\sim \neg q$, whilst (2) $(p \wedge q) \vee (p \wedge \neg q) \not\mid\!\sim \neg q$ (witness s_3) and (3) $(p \wedge q) \vee (\neg p \wedge \neg q) \not\mid\!\sim \neg q$ (witness s_2). Moreover (4) $p \wedge \neg p_j \mid\!\sim \bot$. We claim that any injective preferential model whose inference relation agrees with this one on (1) and (4), disagrees with it on (2) or (3).

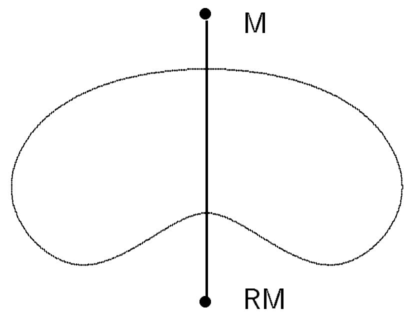
Consider any injective preferential model $\mathcal{M} = \langle S, \imath, < \rangle$ and suppose that (1) and (4) hold. In the case that $p \wedge q \mid\!\sim \bot$ clearly we have $(p \wedge q) \vee (p \wedge \neg q) \mid\!\sim \neg q$ and also $(p \wedge q) \vee (\neg p \wedge \neg q) \mid\!\sim \neg q$, so we may suppose without loss of generality that $p \wedge q \not\mid\!\sim \bot$. Then there is a state $s \in S$ with $s \models p \wedge q$ so $s \models (p \wedge q) \vee \neg q$. By (1) $s \notin \min((p \wedge q) \vee \neg q)$ so there is a $t \in S$ with $t < s$ and $t \in \min((p \wedge q) \vee \neg q)$ so by (1) again $t \models \neg q$. Now either $t \models p$ or $t \models \neg p$. Consider the latter; the argument for the former is similar. Suppose for reductio that (3) holds, *i.e.* there is $u \in S$ with $u \in \min((p \wedge q) \vee (\neg p \wedge \neg q))$ and $u \models q$. Then $u \models p$. Moreover for all $j \in J$, we have $u \models p_j$, for otherwise there is $i \in J$ such that $u \models p \wedge \neg p_i$ so there is u with $u' \leq u$ and $u' \in \min(p \wedge \neg p_i)$ contradicting (4). Similarly $s \models p_j$ for all $j \in J$. Since $s, u \models p \wedge q \wedge p_j$ for all $j \in J$ we have $\imath(s) = \imath(u)$ so by injectivity $s = u$. Thus $s \in \min((p \wedge q) \vee (\neg p \wedge \neg q))$, contradicting $t < s$ and $t \models \neg p \wedge \neg q$.

3 Some strong non-Horn conditions

Rational monotony of course is a restricted form of, and thus implied by, plain monotony (M):

$$\mathsf{M} \quad \frac{\alpha \mid\!\sim \beta}{\alpha \wedge \gamma \mid\!\sim \beta}$$

One of our purposes in this paper is to examine some interesting non-Horn conditions, stronger than rational monotony (or in some cases, independent of it) but still weaker than monotony. In other words, we wish to investigate the enclosed area of the following diagram



Four rules that arise in this connection are *determinacy preservation*, *rational transitivity*, *rational contraposition*, and *weak determinacy*.

Determinacy preservation (DP), briefly considered by Makinson [12], is the rule

$$\mathsf{DP}\quad \frac{\alpha \mid\!\sim \beta \quad \alpha \wedge \gamma \mid\!\not\sim \neg\beta}{\alpha \wedge \gamma \mid\!\sim \beta}$$

This rule evidently is a weak form of monotony. It can also seen as a weak form of Stalnaker's rule [16] of conditional excluded middle (if $\phi \mid\!\not\sim \psi$ then $\phi \mid\!\sim \psi$): a consequence of a formula is either conserved when we add a new hypothesis or we get the negation of this consequence.

Rational transitivity (RT), introduced by Bezzazi and Pino Pérez [1], is the rule

$$\mathsf{RT}\quad \frac{\alpha \mid\!\sim \beta \quad \beta \mid\!\sim \gamma \quad \alpha \mid\!\not\sim \neg\gamma}{\alpha \mid\!\sim \gamma}$$

Obviously this rule is a weak form of transitivity. The intuition behind this rule is the following: when the premises of transitivity hold we get the usual conclusion except when its 'opposite' holds. Note that this rule is also a weak form of conditional excluded middle.

Rational contraposition (RC), also introduced by Bezzazi and Pino Pérez [1], is the rule

$$\mathsf{RC}\quad \frac{\alpha \mid\!\sim \beta \quad \neg\beta \mid\!\not\sim \alpha}{\neg\beta \mid\!\sim \neg\alpha}$$

Obviously this rule is a weak form of contraposition. The intuition behind this rule is the following: when the premise of contraposition holds we get the usual conclusion except when its 'opposite' holds. This rule is again a weak form of conditional excluded middle.

Weak determinacy (WD), formulated by Michael Freund in correspondence with the authors, is the rule

$$\mathsf{WD}\quad \frac{\top \mid\!\sim \neg\alpha \quad \alpha \mid\!\not\sim \beta}{\alpha \mid\!\sim \neg\beta}$$

This rule says that any formula α that is 'exceptional' in the sense of Lehmann and Magidor [9], *i.e.* such that $\top \mid\!\sim \neg\alpha$, is complete in the sense that for every formula, either it or its negation is a consequence of the exceptional formula. Given the preferential rules, this is a special case of both monotony and conditional excluded middle.

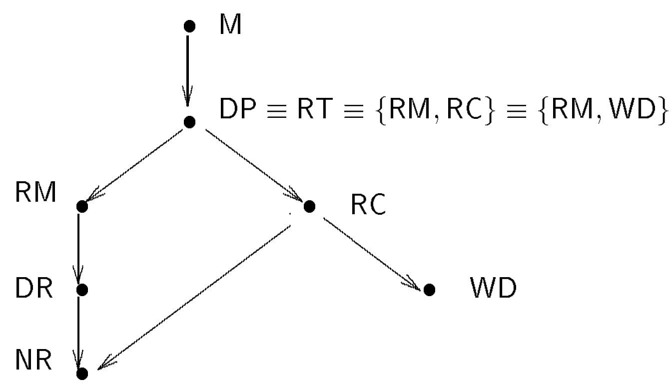
Definition 3.1 A relation $\mid\!\sim$ is said to be determinacy preserving iff it is preferential and the rule DP holds.

A relation $\mid\!\sim$ is said to be rational transitive iff it is preferential and the rule RT holds.

In this section we compare the strength of the rules DP, RT, RC, WD with each other as well as RM on the lower side and M on the upper side. The general picture turns out as follows:

Proposition 3.2 Given the preferential rules P, the rules DP and RT are equivalent, and are implied by monotony. They are also equivalent to the pair $\{\mathsf{RM}, \mathsf{RC}\}$ and also to the pair $\{\mathsf{RM}, \mathsf{WD}\}$. Moreover, given P, RC implies both WD and NR. However given P, none of the following implications hold: RM to WD, RC to DR, WD to NR.

Recalling from [12] that M quite trivially implies DP but not conversely, and that RM implies DR which implies NR but neither conversely, proposition 3.2 gives us the following diagram, where one condition implies another, given P, iff one can follow arrows from the former to the latter.



This proposition suggests a central role for DP *alias* RT. We verify the components of the proposition separately. The positive parts are first proven syntactically, the negative parts are then established semantically.

Observation 3.3 $\mathsf{P} + \mathsf{RT} \Leftrightarrow \mathsf{P} + \mathsf{DP}$

Proof ($\Rightarrow$) Suppose $\alpha \mid\!\sim \gamma$ and $\alpha\wedge\beta \not\mid\!\sim \neg\gamma$. We want to show $\alpha\wedge\beta \mid\!\sim \gamma$. By preferentiality, $\alpha\wedge\beta \mid\!\sim \alpha$. Thus we have $\alpha\wedge\beta \mid\!\sim \alpha$, $\alpha \mid\!\sim \gamma$ and $\alpha\wedge\beta \not\mid\!\sim \neg\gamma$. So by RT $\alpha\wedge\beta \mid\!\sim \gamma$ as desired.

($\Leftarrow$) Suppose $\alpha \mid\!\sim \beta$, $\beta \mid\!\sim \gamma$ and $\alpha \not\mid\!\sim \neg\gamma$. We want to show $\alpha \mid\!\sim \gamma$. Now, $\alpha\wedge\beta \not\mid\!\sim \neg\gamma$ for otherwise since $\alpha \mid\!\sim \beta$ we would have by cut that $\alpha \mid\!\sim \neg\gamma$ contrary to supposition. Hence since $\beta \mid\!\sim \gamma$ we have by DP $\alpha\wedge\beta \mid\!\sim \gamma$, and so since $\alpha \mid\!\sim \beta$ we have by cut that $\alpha \mid\!\sim \gamma$ as desired. ∎

Observation 3.4 $\mathsf{P}+\mathsf{RT}\Rightarrow\mathsf{RM}$. That is, a rational transitive relation is indeed a rational relation.

Proof This is a corollary of observation 3.3 and the fact that $\mathsf{P}+\mathsf{DP}\Rightarrow\mathsf{RM}$ proven in [12]. Here we give a direct proof. Assume $\alpha\mid\!\sim\beta$ and $\alpha\not\mid\!\sim\neg\gamma$. We will show $\alpha\wedge\gamma\mid\!\sim\beta$. First we show $\alpha\wedge\gamma\not\mid\!\sim\neg\beta$. Suppose that it is not true, *i.e.* $\alpha\wedge\gamma\mid\!\sim\neg\beta$. Then, by S, $\alpha\mid\!\sim\gamma\to\neg\beta$. Since $\alpha\mid\!\sim\beta$, by AND and RW we get $\alpha\mid\!\sim\neg\gamma$, a contradiction. Second, we have $\alpha\wedge\gamma\mid\!\sim\alpha$ because of REF and RW. Finally, since $\alpha\wedge\gamma\mid\!\sim\alpha$, $\alpha\mid\!\sim\beta$ and $\alpha\wedge\gamma\not\mid\!\sim\neg\beta$, we conclude using RT. ∎

Observation 3.5 $\mathsf{P}+\mathsf{DP}\Rightarrow\mathsf{RC}$

Proof Suppose $\alpha\mid\!\sim\beta$; we want to show that either $\neg\beta\mid\!\sim\neg\alpha$ or $\neg\beta\mid\!\sim\alpha$.
Case 1: Suppose $\top\not\mid\!\sim\beta$. Now by preferentiality from $\alpha\mid\!\sim\beta$ we have $\top\mid\!\sim\alpha\to\beta$. Hence, applying RM (which we noted follows from DP) we have $\top\wedge\neg\beta\mid\!\sim\alpha\to\beta$ so by preferentiality $\neg\beta\mid\!\sim\neg\alpha$ as desired.
Case 2: Suppose $\top\mid\!\sim\beta$. Then by preferentiality $\top\mid\!\sim\neg\beta\to\neg\alpha$ so by the hypothesis DP either $\top\wedge\neg\beta\mid\!\sim\neg\beta\to\neg\alpha$ or $\top\wedge\neg\beta\mid\!\sim\neg(\neg\beta\to\neg\alpha)$.

Subcase 2.1: Suppose $\top\wedge\neg\beta\mid\!\sim\neg\beta\to\neg\alpha$. Then by preferentiality $\neg\beta\mid\!\sim\neg\alpha$ as desired.

Subcase 2.1: Suppose $\top\wedge\neg\beta\mid\!\sim\neg(\neg\beta\to\neg\alpha)$. Then by preferentiality $\neg\beta\mid\!\sim\alpha$ as desired. ∎

Observation 3.6 $\mathsf{P}+\mathsf{RC}\Rightarrow\mathsf{WD}$

Proof Let $\mid\!\sim$ be an inference relation satisfying the preferential rules, and suppose that it fails WD.

Since $\mid\!\sim$ fails WD, there are α, β with $\top\mid\!\sim\neg\alpha$, $\alpha\not\mid\!\sim\beta$, $\alpha\not\mid\!\sim\neg\beta$. Since $\top\mid\!\sim\neg\alpha$ we have by preferential rules that $\top\mid\!\sim\neg\alpha\vee\neg\beta$, and so combining this with $\top\mid\!\sim\neg\alpha$ again we have by CM that $\neg\alpha\vee\neg\beta\mid\!\sim\neg\alpha$.

On the other hand, since $\alpha\not\mid\!\sim\beta$ we have by preferential rules that $\alpha\not\mid\!\sim\alpha\wedge\beta$ and so $\neg\neg\alpha\not\mid\!\sim\neg(\neg\alpha\vee\neg\beta)$. Also since $\alpha\not\mid\!\sim\neg\beta$ we have by preferentiality that $\alpha\not\mid\!\sim\neg\alpha\vee\neg\beta$ and so $\neg\neg\alpha\not\mid\!\sim\neg\alpha\vee\neg\beta$.

Putting these three facts together we see that RC fails. ∎

Observation 3.7 $\mathsf{P}+\mathsf{RM}+\mathsf{WD}\Rightarrow\mathsf{DP}$

Proof Suppose $\alpha\mid\!\sim\beta$ and $\alpha\wedge\gamma\not\mid\!\sim\neg\beta$; we want to show $\alpha\wedge\gamma\mid\!\sim\beta$. If $\top\mid\!\sim\neg(\alpha\wedge\gamma)$ then the second hypothesis with WD give us what we want. On the other hand, if $\top\not\mid\!\sim\neg(\alpha\wedge\gamma)$ then noting from the first hypothesis that $\top\mid\!\sim\alpha\to\beta$ we get by RM that $\top\wedge(\alpha\wedge\gamma)\mid\!\sim\alpha\to\beta$ so by preferential rules, $\alpha\wedge\gamma\mid\!\sim\beta$ as desired. ∎

Observation 3.8 $\mathsf{P}+\mathsf{RM}+\mathsf{RC}\Rightarrow\mathsf{DP}$

Proof This follows immediately from observations 3.6 and 3.7. For another verification, suppose that $\alpha\mid\!\sim\beta$. We want to show that either $\alpha\wedge\gamma\mid\!\sim\beta$ or $\alpha\wedge\gamma\mid\!\sim\neg\beta$. Now either $\alpha\mid\!\sim\neg\gamma$ or $\alpha\not\mid\!\sim\neg\gamma$.
Case 1: Suppose $\alpha\not\mid\!\sim\neg\gamma$. Then by the hypothesis RM we have $\alpha\wedge\gamma\mid\!\sim\beta$ as desired.
Case 2: Suppose $\alpha\mid\!\sim\neg\gamma$. Then by preferentiality (rule S) $\top\mid\!\sim\alpha\to\neg\gamma$ *i.e.* $\top\mid\!\sim\neg(\alpha\wedge\gamma)$ so by NR which holds by proposition 2.7 either $\beta\mid\!\sim\neg(\alpha\wedge\gamma)$ or $\neg\beta\mid\!\sim\neg(\alpha\wedge\gamma)$, and in each of these two subcases RC tells us that either $\alpha\wedge\gamma\mid\!\sim\beta$ or $\alpha\wedge\gamma\mid\!\sim\neg\beta$, as desired. ∎

Observation 3.9 $\mathsf{P}+\mathsf{RC} \Rightarrow \mathsf{NR}$

Proof We have already that $\mathsf{P}+\mathsf{RC}$ implies WD (observation 3.6). So it will be enough to prove the following two facts:

Fact 1. $\mathsf{P}+\mathsf{RC} \Rightarrow \mathsf{RC}^+$, where RC^+ is the following rule

$$\frac{\alpha\wedge\beta{\mid\!\sim}\gamma \quad \alpha\wedge\neg\gamma{\mid\!\not\sim}\beta}{\alpha\wedge\neg\gamma{\mid\!\sim}\neg\beta}$$

Fact 2. $\mathsf{P}+\mathsf{RC}^+ + \mathsf{WD} \Rightarrow \mathsf{NR}$

Proof of fact 1: Suppose RC holds, and suppose $\alpha\wedge\beta{\mid\!\sim}\gamma$. Then by preferential rules, $\alpha\wedge\beta{\mid\!\sim}\gamma\vee\neg\alpha$, so by RC either $\neg(\gamma\vee\neg\alpha){\mid\!\sim}\neg(\alpha\wedge\beta)$ or $\neg(\gamma\vee\neg\alpha){\mid\!\sim}\alpha\wedge\beta$. In the former case we have by preferential rules $\alpha\wedge\neg\gamma{\mid\!\sim}\neg\alpha\vee\neg\beta$, so by preferential rules again $\alpha\wedge\neg\gamma{\mid\!\sim}\neg\beta$ as desired. In the latter case we have by preferential rules $\alpha\wedge\neg\gamma{\mid\!\sim}\alpha\wedge\beta$ so $\alpha\wedge\neg\gamma{\mid\!\sim}\beta$ as desired.

Proof of fact 2: Suppose RC^+ and WD hold, and suppose $\alpha{\mid\!\sim}\beta$; we want to show that either $\alpha\wedge\gamma{\mid\!\sim}\beta$ or $\alpha\wedge\neg\gamma{\mid\!\sim}\beta$. Since $\alpha{\mid\!\sim}\beta$ we have by preferential rules that $\top{\mid\!\sim}\neg\alpha\vee\beta$. Hence by WD either $\neg(\neg\alpha\vee\beta){\mid\!\sim}\gamma$ or $\neg(\neg\alpha\vee\beta){\mid\!\sim}\neg\gamma$, *i.e.* either $\alpha\wedge\neg\beta{\mid\!\sim}\gamma$ or $\alpha\wedge\neg\beta{\mid\!\sim}\neg\gamma$.
Case 1. Suppose $\alpha\wedge\neg\beta{\mid\!\sim}\gamma$. Then by RC^+, either $\alpha\wedge\neg\gamma{\mid\!\sim}\beta$ or $\alpha\wedge\neg\gamma{\mid\!\sim}\neg\beta$. In the former subcase we are done. In the latter subcase by preferential rules (rule S) $\alpha{\mid\!\sim}\gamma\vee\neg\beta$ which combined with $\alpha{\mid\!\sim}\beta$ gives $\alpha{\mid\!\sim}\gamma$. But this again combined with $\alpha{\mid\!\sim}\beta$ gives, by CM, $\alpha\wedge\gamma{\mid\!\sim}\beta$ and in this subcase we are also done.
Case 2. Suppose $\alpha\wedge\neg\beta{\mid\!\sim}\neg\gamma$. Then by RC^+, either $\alpha\wedge\gamma{\mid\!\sim}\beta$ or $\alpha\wedge\gamma{\mid\!\sim}\neg\beta$. In the former subcase we are done. In the latter subcase by preferential rules we have $\alpha{\mid\!\sim}\beta\rightarrow\neg\gamma$ which combined with $\alpha{\mid\!\sim}\beta$ gives again by preferential rules $\alpha{\mid\!\sim}\neg\gamma$. From this, we conclude as above using CM that $\alpha\wedge\neg\gamma{\mid\!\sim}\beta$ and we are also done. ∎

Given the above positive parts of proposition 3.2, it suffices to show the following negative ones: $\mathsf{P}+\mathsf{RM} \not\Rightarrow \mathsf{WD}$, $\mathsf{P}+\mathsf{RC} \not\Rightarrow \mathsf{DR}$, $\mathsf{P}+\mathsf{WD} \not\Rightarrow \mathsf{NR}$.

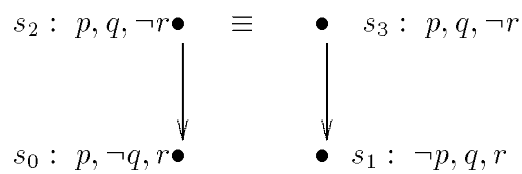
Observation 3.10 Let $\mathcal{M} = \langle S, \imath, \prec\rangle$ be any preferential model with $S = \{s_0, s_1, s_2, s_3\}$, $\prec = \{(s_0, s_2), (s_1, s_3)\}$ and $\imath(s_2) = \imath(s_3)$. Then ${\mid\!\sim}_{\mathcal{M}}$ satisfies RC.

Proof Assume $\alpha{\mid\!\sim}\beta$ and $\neg\beta{\mid\!\not\sim}\neg\alpha$. We want to show $\neg\beta{\mid\!\sim}\alpha$. It will be enough to see that $\mathrm{mod}(\alpha) = S$. Note that $\min(\neg\beta)\cap\mathrm{mod}(\alpha) \neq \emptyset$ because $\neg\beta{\mid\!\not\sim}\neg\alpha$. But neither s_0 nor s_1 can be in $\min(\neg\beta)\cap\mathrm{mod}(\alpha)$ for otherwise as s_0 and s_1 are minimals we would have $\alpha{\mid\!\not\sim}\beta$ contradicting our assumption $\alpha{\mid\!\sim}\beta$. So, either s_2 or s_3 is in $\min(\neg\beta)\cap\mathrm{mod}(\alpha)$, so since $\imath(s_2) = \imath(s_3)$ we have both $s_2, s_3 \in \mathrm{mod}(\neg\beta)\cap\mathrm{mod}(\alpha)$. This and $\alpha{\mid\!\sim}\beta$ imply by smoothness that $s_0, s_1 \in \min(\alpha)$. Thus $\mathrm{mod}(\alpha) = S$, as desired. ∎

Note that the hypothesis $\imath(s_2) = \imath(s_3)$ is necessary in this observation. We can easily find a preferential model $\mathcal{M} = \langle S, \imath, \prec\rangle$ with $S = \{s_0, s_1, s_2, s_3\}$, $\prec = \{(s_0, s_2), (s_1, s_3)\}$ and $\imath(s_2) \neq \imath(s_3)$ which does not satisfy RC.

Observation 3.11 $\mathsf{P}+\mathsf{RC} \not\Rightarrow \mathsf{DR}$

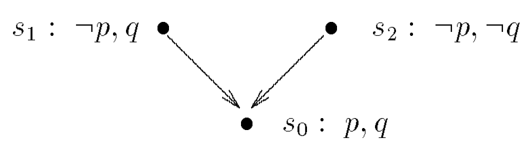
Proof Consider a model as in the previous observation with $\imath(s_2) = \imath(s_3) = \{p, q\}$, $\imath(s_0) = \{p, r\}$, $\imath(s_1) = \{q, r\}$ (we give the valuations as for a Herbrand model, that is identifying the subset of variables with its characteristic function). Graphically



By observation 3.10 RC holds in $\mathcal{M}$ but it is clear that $p \not\mathrel{|}\!\sim r$ (witness s_3), $q \not\mathrel{|}\!\sim r$ (witness s_2) and $p \vee q \mathrel{|}\!\sim r$ so DR fails. ∎

Observation 3.12 $\mathsf{P} + \mathsf{RM} \not\Rightarrow \mathsf{WD}$

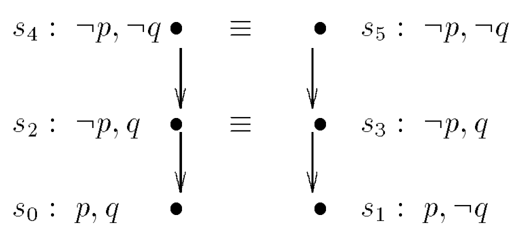
Proof Let $\mathcal{L}$ be the set of all formulae built from the elementary formulae p and q. Consider the following model $\mathcal{M} = \langle S, \imath, \prec \rangle$ where $S = \{s_0, s_1, s_2\}$, $\prec = \{(s_0, s_1), (s_0, s_2)\}$, $\imath(s_0) = \{p, q\}$, $\imath(s_1) = \{q\}$ and $\imath(s_2) = \emptyset$. Graphically



It is clear that $\mathcal{M}$ is ranked, so it satisfies RM. However it fails to satisfy WD since $\top \mathrel{|}\!\sim p$ whilst $\neg p \not\mathrel{|}\!\sim q$ and $\neg p \not\mathrel{|}\!\sim \neg q$. ∎

Observation 3.13 $\mathsf{P} + \mathsf{WD} \not\Rightarrow \mathsf{NR}$

Proof Let $\mathcal{M}$ be the model represented by the following schema:



i.e. $\mathcal{M} = \langle S, \imath, \prec \rangle$ with $S = \{s_0, \ldots, s_5\}$, $\prec$ the transitive closure of the relation $\{(s_0, s_2), (s_2, s_4), (s_1, s_3), (s_3, s_5)\}$, $\imath(s_0) = \{p, q\}$, $\imath(s_1) = \{p\}$, $\imath(s_2) = \imath(s_3) = \{q\}$ and $\imath(s_4) = \imath(s_5) = \emptyset$.

NR fails in this model because $\top \mathrel{|}\!\sim p$, $q \not\mathrel{|}\!\sim p$ (witness s_3), $\neg q \not\mathrel{|}\!\sim p$ (witness s_4). But WD is satisfied in this model. Suppose that $\top \mathrel{|}\!\sim \neg\alpha$, $\alpha \not\mathrel{|}\!\sim \beta$, $\alpha \not\mathrel{|}\!\sim \neg\beta$. Then there must be $u \in \min(\alpha) \cap \mathrm{mod}(\neg\beta)$ and $v \in \min(\alpha) \cap \mathrm{mod}(\beta)$ and $u, v \in \{s_2, s_3, s_4, s_5\}$. But it is clear that each choice of u, v here gives a contradiction. For example if $u = s_2$ and $v = s_5$ then since $\imath(s_2) = \imath(s_3)$ we have $v \notin \min(\alpha)$ giving a contradiction. ∎

Observations 3.11 and 3.13 have been established using non-injective preferential models as examples. In the case of 3.11, at least, there is no injective model that does the job. For by 3.9, any injective model of $\mathsf{P}+\mathsf{RC}$ is an injective model of $\mathsf{P}+\mathsf{NR}$, and it has been shown by Freund and Lehmann [3], that every injective model of $\mathsf{P}+\mathsf{NR}$ is a model of DR.

It is immediate that transitivity (T) of $\mid\!\sim$ implies RT. However, the converse does not hold: given that $\mathsf{P}+\mathsf{RT} \Leftrightarrow \mathsf{P}+\mathsf{DP}$ shown above, and the well-known facts (see *e.g.* [12]) that $\mathsf{P}+\mathsf{T} \Rightarrow \mathsf{M}$ whilst $\mathsf{P}+\mathsf{DP} \not\Rightarrow \mathsf{M}$, we have $\mathsf{P}+\mathsf{RT} \not\Rightarrow \mathsf{T}$. A direct verification can also be made with an appropriate two-state model (see corollary 5.3).

As already remarked, DP, RT, RC and WD are weakened forms not only of monotony but also of Stalnaker's rule of conditional excluded middle which, unlike the principles so far considered, is not implied by monotony but has figured in philosophical discussion of counterfactuals (*e.g.* [6, 10, 13]). We shall study some other rules in the vicinity of conditional excluded middle in section 7.

4 Collapsed models

Our goal in section 5 will be to prove a representation theorem for $\mathsf{P}+\mathsf{RT}$ (equivalently $\mathsf{P}+\mathsf{DP}$). As a preliminary, we shall show in this section that every ranked preferential model is equivalent (in the sense of generating the same inference relation) to one that is both injective and *parsimonious*, in the sense that every one of its states is minimal in at least one formula. This result is indeed implicit in Freund [2] in the more general case of relations satisfying $\mathsf{P}+\mathsf{DR}$, but using different arguments. Our procedure for transforming a ranked model into one with these characteristics is quite straightforward. We proceed in two steps. First, at each level of the model we identify the states of that level that are labelled with the same valuation. Second, we suppress all states that are not minimal in some formula.

Definition 4.1 A model $\mathcal{M} = \langle S, \imath, \prec\rangle$ is said to be horizontally injective iff for all distinct $s, t \in S$, if $s \not\prec t$ and $t \not\prec s$ then $\imath(s) \neq \imath(t)$.

Note that for ranked models, being horizontally injective actually means injectivity by levels.

Lemma 4.2 For any ranked model $\mathcal{M} = \langle S, \imath, \prec\rangle$ there exists a horizontally injective ranked model $\mathcal{M}' = \langle S', \imath', \prec'\rangle$ such that $\mid\!\sim_{\mathcal{M}} = \mid\!\sim_{\mathcal{M}'}$.

Proof We define an equivalence relation on S as follows

$$s \equiv s' \Leftrightarrow r(s) = r(s') \text{ and } \imath(s) = \imath(s')$$

Put $S' = S/\equiv$ (the quotient of S by $\equiv$). As usual let $[s]$ denote the equivalent class of s. Define $r' : S' \perp\!\!\to \Omega$, $\prec' \subseteq S' \times S'$ and $\imath' : S' \perp\!\!\to \mathcal{U}$ as follows: $r'([s]) = r(s)$, $[s] \prec' [s']$ iff $s \prec s'$, and $\imath'([s]) = \imath(s)$. It can be easily verified that $r', \prec'$ and $\imath'$ are well defined, *i.e.* their definition does not depend on the choice of the representative of $[s]$. It is also clear that for all $j \in \Omega$, $\imath'$ restricted to $S'_j = \{[s] \in S' : r([s]) = j\}$ is injective. Notice that

$$[s] \prec' [s'] \Leftrightarrow s \prec s' \Leftrightarrow r(s) < r(s') \Leftrightarrow r'([s]) < r'([s'])$$

So, the model $\mathcal{M}'$ defined by $\mathcal{M}' = \langle S', \imath', \prec' \rangle$ is a ranked model. Moreover, we have $\mid\!\sim_{\mathcal{M}} = \mid\!\sim_{\mathcal{M}'}$, since clearly for all $s \in S$ and all $\alpha, \beta \in \mathcal{L}$, $s \in \text{mod}_{\mathcal{M}}(\beta)$ iff $[s] \in \text{mod}_{\mathcal{M}'}(\beta)$, and also $s \in \min_{\mathcal{M}}(\alpha)$ iff $[s] \in \min_{\mathcal{M}'}(\alpha)$. ∎

Definition 4.3 A model $\mathcal{M} = \langle S, \imath, \prec \rangle$ is said to be parsimonious iff for every state $s \in S$ there is a formula α such that $s \in \min_{\mathcal{M}}(\alpha)$.

Proposition 4.4 If $\mathcal{M} = \langle S, \imath, \prec \rangle$ is a preferential model then there exists a preferential model $\mathcal{M}' = \langle S', \imath', \prec' \rangle$ such that $S' \subseteq S$ and the following properties hold

1. $\mathcal{M}'$ is parsimonious.
2. If $\mathcal{M}$ is ranked so is $\mathcal{M}'$.
3. Whenever $s, t \in S'$ with neither $s \prec' t$ nor $t \prec' s$, if $\imath'(s) = \imath'(t)$ then $\imath(s) = \imath(t)$.
4. $\mid\!\sim_{\mathcal{M}} = \mid\!\sim_{\mathcal{M}'}$

Proof It is quite simple. It is enough to suppress the states which are not minimals for any formula. Essentially the same trick has been used by Pavlos Peppas [14] in the context of systems-of-spheres models for belief revision. Define $S' = S \setminus \{s : \neg\exists\alpha \; s \in \min_{\mathcal{M}}(\alpha)\}$. Let $\imath'$ and $\prec'$ be the restrictions of $\imath$ and $\prec$ to S'. Put $\mathcal{M}' = \langle S', \imath', \prec' \rangle$. By definition of $\mathcal{M}'$, it is obvious that

$$\min_{\mathcal{M}}(\alpha) = \min_{\mathcal{M}'}(\alpha) \qquad (*)$$

Hence smoothness of $\mathcal{M}$ implies smoothness of $\mathcal{M}'$. So, $\mathcal{M}'$ is a preferential model which, by its construction, is parsimonious. Clearly if $\mathcal{M}$ is ranked so is $\mathcal{M}'$. Property 3 is trivially verified by definition of $\mathcal{M}'$. Finally, let us verify $\alpha \mid\!\sim_{\mathcal{M}} \beta \Leftrightarrow \alpha \mid\!\sim_{\mathcal{M}'} \beta$.
($\Leftarrow$): Suppose that $\min_{\mathcal{M}'}(\alpha) \subseteq \text{mod}_{\mathcal{M}'}(\beta)$. We want to show $\min_{\mathcal{M}}(\alpha) \subseteq \text{mod}_{\mathcal{M}}(\beta)$. This follows from $(*)$ and the fact that $\text{mod}_{\mathcal{M}'}(\beta) \subseteq \text{mod}_{\mathcal{M}}(\beta)$.
($\Rightarrow$): Suppose that $\min_{\mathcal{M}}(\alpha) \subseteq \text{mod}_{\mathcal{M}}(\beta)$. By definition of S' $\min_{\mathcal{M}}(\alpha) \subseteq \text{mod}_{\mathcal{M}'}(\beta)$ because $\text{mod}_{\mathcal{M}'}(\beta) = \{s \in \text{mod}_{\mathcal{M}}(\beta) : \exists\gamma \; s \in \min_{\mathcal{M}}(\gamma)\}$. So, by $(*)$, $\min_{\mathcal{M}'}(\alpha) \subseteq \text{mod}_{\mathcal{M}'}(\beta)$, *i.e.* $\alpha \mid\!\sim \beta$. ∎

Remark 4.5 When the ranked model $\mathcal{M} = \langle S, \imath, \prec \rangle$ is finite (notice that by theorem 2.6 every rational relation over a finite language is represented by a finite model), *i.e.* S is finite, then S' in the previous lemma can be constructed by an algorithm. In order to see this, first remark that when S is finite, we can suppose the rank function is of the form $r : S \perp\!\!\rightarrow \overline{n}$. Then define $S'_0 = \{s \in S : r(s) = 0\}$ and for $k = 1$ to n, $S'_k = \{s \in S : r(s) = k$ and there exists α with $s \in \min_{\mathcal{M}}(\alpha)\}$. Finally put $S' = \cup_{k=0}^{n} S'_k$.

Theorem 4.6 (Collapsing) If $\mathcal{M} = \langle S, \imath, \prec \rangle$ is a ranked model then there exists a parsimonious, injective ranked model $\mathcal{M}' = \langle S', \imath', \prec' \rangle$ such that $\mid\!\sim_{\mathcal{M}} = \mid\!\sim_{\mathcal{M}'}$.

Proof Let $\mathcal{M}' = \langle S', \imath', \prec' \rangle$ be the model obtained from $\mathcal{M} = \langle S, \imath, \prec \rangle$ by application of lemma 4.2 and then proposition 4.4. Clearly $\mathcal{M}'$ is parsimonious and ranked, and also $\mid\!\sim_{\mathcal{M}} = \mid\!\sim_{\mathcal{M}'}$. It remains to check injectivity. Now by lemma 4.2 and part (3) of proposition 4.4, $\mathcal{M}'$ is horizontally injective. Clearly parsimony implies that $\mathcal{M}'$ is 'vertically injective' in the sense that $s \prec' t$ implies $\imath(s) \neq \imath(t)$. Finally, horizontal and vertical injectivity clearly imply injectivity. ∎

The model $\mathcal{M}'$ obtained from a ranked model $\mathcal{M}$ by successive application of lemma 4.2, and proposition 4.4 will be called the *collapse* of $\mathcal{M}$. Clearly a model is equal to its collapse iff it is both parsimonious and injective. As a corollary of theorem 4.6 we have the following result:

Corollary 4.7 Every rational inference relation is generated by some collapsed ranked preferential model.

Proof Immediate from the Lehmann-Magidor representation theorem (theorem 2.6) and theorem 4.6. ∎

Remark 4.8 (i) In the proof of theorem 4.6 we have applied lemma 4.2 and then proposition 4.4 but the same result is obtained if we reverse the order.
(ii) It is not hard to see that in the finite case (finite language), if a model is injective then each state is minimal for some formula, so the model is parsimonious. But this is not true in general for infinite languages. For instance consider an injective ranked model with two levels: one world in the upper level and the rest of the worlds in the lower level; then there is no formula for which the upper world is minimal.

Remark 4.9 Theorem 4.6 and its corollary 4.7 are implicit in Freund [2] but in reverse order of demonstration and by quite a different strategy. Freund shows that if $\mid\!\sim$ is a rational inference relation (indeed more generally, any preferential inference relation satisfying DR) then we can construct its 'associated standard model', which is a model generating $\mid\!\sim$, that is ranked (or, under the hypothesis of DR, that is 'filtered' in the sense of his definition 5.1) and has additional properties including the following:

1. Every state u is '$\mid\!\sim$-consistent' in the sense (Freund [2], section 2.1) that there is a formula α with $u \models C(\alpha)$, where as usual $C(\alpha) = \{\beta : \alpha \mid\!\sim \beta\}$;

2. The model is 'standard with respect to $\mid\!\sim$' in the sense (Freund [2], definition 3.2) that it is injective and for every formula α and state u, $u \models C(\alpha)$ iff $u \in \min(\alpha)$.

Property 2 explicitly implies injectivity, and the two properties taken together clearly imply parsimony. Conversely, parsimony and injectivity together imply properties 1 and 2, if we assume that the model is ranked: property 1 is immediate from parsimony recalling that $u \in \min(\alpha)$ immediately implies $u \models C(\alpha)$ in every preferential model, whilst to derive property 2 it suffices to show that whenever $u \notin \min(\alpha)$ then $u \not\models C(\alpha)$. Suppose $u \notin \min(\alpha)$. If $u \not\models \alpha$ then we are done, so suppose that $u \models \alpha$. Then there is $v \prec u$ with $v \in \min(\alpha)$. By parsimony, there exists β such that $u \in \min(\beta)$ and thus by rankedness for any $v' \in \min(\alpha)$, $v' \models \neg\beta$. Thus $\neg\beta \in C(\alpha)$ and so since $u \models \beta$ we have $u \not\models C(\alpha)$, establishing property 2.

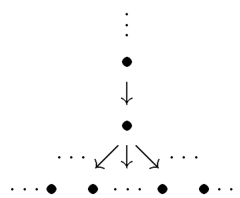
Evidently, each approach has its advantages, depending in part on the purposes for which is used. Since our approach covers only ranked models and thus rationally monotone inference relations, unless it can be generalized it is useless for Freund's purpose, which is to represent preferential inference relations satisfying DR. On the other hand, it provides a simple and natural way of proving representation theorems for conditions such as rational transitivity that are stronger than rational monotony (*e.g.* theorems 5.8, 7.18 and 8.6 below) and a very direct argument for results of independent model-theoretic interest such as lemma 4.2, proposition 4.4, theorem 4.6 and its corollary 4.7.

5 Representation

The goal of this section is to characterize the ranked models that generate rational transitive relations. Our argument exploits corollary 4.7

Definition 5.1 A preferential model (not necessarily injective) $\mathcal{M} = \langle S, \imath, \prec \rangle$ is said to be *quasi-linear* iff it is ranked and it has at most one state at any level above the lowest. In other words quasi-linear means ranked and whenever $r \prec s$, $r \prec t$ then either $s = t$ or $s \prec t$ or $t \prec s$.

Quasi-linear models have the following graphical shape:



Proposition 5.2 If $\mathcal{M} = \langle S, \imath, \prec \rangle$ is quasi-linear then the relation $\mid\!\sim \; = \; \mid\!\sim_{\mathcal{M}}$ is rational transitive.

Proof $\mathcal{M}$ is ranked so $\mid\!\sim$ is a rational relation. We have to prove that $\mid\!\sim$ satisfies RT. So, suppose $\alpha \mid\!\sim \beta$, $\beta \mid\!\sim \gamma$ and $\alpha \not\mid\!\sim \neg\gamma$. We want to show $\alpha \mid\!\sim \gamma$. We consider two cases. First, suppose that $\min(\alpha)$ is contained in the lowest level. As also $\alpha \mid\!\sim \beta$, necessarily $\min(\alpha) \subseteq \min(\beta)$. But $\min(\beta) \subseteq \mathrm{mod}(\gamma)$ because $\beta \mid\!\sim \gamma$. Therefore, $\min(\alpha) \subseteq \mathrm{mod}(\gamma)$, *i.e.* $\alpha \mid\!\sim \gamma$.

Second, suppose $\min(\alpha)$ is not contained in the lowest level. Then $\min(\alpha)$ is a singleton because $\mathcal{M}$ is quasi-linear; suppose $\min(\alpha) = \{s\}$. Then $s \not\models \neg\gamma$ because $\alpha \not\mid\!\sim \neg\gamma$. Thus $s \models \gamma$, so $\alpha \mid\!\sim \gamma$. ∎

Corollary 5.3 There are rational transitive inference relations which are not transitive.

Proof Consider a language $\mathcal{L}$ built on the propositional variables p, q and r. Define $\mathcal{M} = \langle S, \imath, \prec \rangle$ where $S = \{s_0, s_1\}$, $s_0 \prec s_1$ $\imath(s_0) = \{q, r\}$, $\imath(s_1) = \{p, q\}$. By proposition 5.2 the relation $\mid\!\sim \; = \; \mid\!\sim_{\mathcal{M}}$ is a rational transitive relation. But we can easily verify, $p \mid\!\sim q$, $q \mid\!\sim r$ and $p \not\mid\!\sim r$. So $\mid\!\sim$ is not a transitive relation. ∎

Observation 5.4 Suppose that the language is finite and $\mathcal{M} = \langle S, \imath, \prec \rangle$ is an injective ranked model which is not quasi-linear. Then $\mid\!\sim \; = \; \mid\!\sim_{\mathcal{M}}$ does not satisfy RT.

Proof As $\mathcal{M}$ is not quasi-linear, necessarily there are three different states s_1, s_2 and s_3 in S such that s_1 is in the lowest level, s_2 and s_3 are in the same level and $s_1 \prec s_i$ for $i = 2, 3$. Let α, β and γ be formulae such that $\mathrm{mod}(\alpha) = \{s_2, s_3\}$, $\mathrm{mod}(\beta) = \{s_1, s_2, s_3\}$ and $\mathrm{mod}(\gamma) = \{s_1, s_2\}$. By finiteness and injectivity such α, β and γ clearly exist. Then, it is clear that $\alpha {\mid\!\sim} \beta$, $\beta {\mid\!\sim} \gamma$ whilst $\alpha {\not\mid\!\sim} \neg\gamma$ and $\alpha {\not\mid\!\sim} \gamma$. Therefore $\mathcal{M}$ does not satisfy RT. ∎

Remark 5.5 When the language is infinite the above observation does not hold. This can be seen by the following example. Let $\mathcal{M}$ be a ranked model whose states are worlds, with two levels: v_0 and v_1 in the upper level and the rest of the valuations in the lower level, *i.e.* the order is $v \prec v_i$ for all valuations $v \neq v_i$, $i = 1, 2$. By definition $\mathcal{M}$ is not quasi-linear. However, ${\mid\!\sim}_{\mathcal{M}}$ satisfies RT (and indeed, satisfies transitivity and monotony) because for any formula α, $\min_{\mathcal{M}}(\alpha)$ lies in the lowest level.

But if instead of injectivity in the observation 5.4 we require that the model $\mathcal{M}$ be collapsed then a similar argument can be used to extend observation 5.4 to the case of infinite languages. More precisely we have the following proposition:

Proposition 5.6 Suppose $\mathcal{M} = \langle S, \imath, \prec \rangle$ is a collapsed ranked model which is not quasi-linear. Then ${\mid\!\sim} = {\mid\!\sim}_{\mathcal{M}}$ does not satisfy RT.

Proof As $\mathcal{M}$ is not quasi-linear, necessarily there are three different states s_1, s_2 and s_3 in S such that s_1 is in the lowest level, s_2 and s_3 are in the same level and $s_1 \prec s_i$ for $i = 2, 3$. We need to find formulae α, β, γ with $\alpha {\mid\!\sim} \beta$, $\beta {\mid\!\sim} \gamma$, $\alpha {\not\mid\!\sim} \neg\gamma$, $\alpha {\not\mid\!\sim} \gamma$. By parsimony there are formulae ϕ_i $(i = 1, 2, 3)$ with $s_i \in \min(\phi_i)$ and by injectivity there are formulae ψ_{ij} $(i, j = 1, 2, 3$ and $i \neq j)$ with $s_i \in \mathrm{mod}(\psi_{ij})$ and $s_j \notin \mathrm{mod}(\psi_{ij})$. Put $\alpha = \phi_2 \vee \phi_3$, $\beta = (\phi_2 \vee \phi_3) \vee (\phi_1 \wedge \psi_{12})$ and $\gamma = (\phi_1 \wedge \psi_{12}) \vee (\phi_3 \wedge \psi_{32})$. Then it is clear that $\alpha \vdash \beta$ so $\alpha {\mid\!\sim} \beta$; and using rankedness $\min(\beta) \subseteq \mathrm{mod}(\phi_1 \wedge \psi_{12})$ so $\beta {\mid\!\sim} \gamma$; whilst again using rankedness $s_2 \in \min(\alpha)$ but $s_2 \notin \mathrm{mod}(\gamma)$ so $\alpha {\not\mid\!\sim} \gamma$ and finally $s_3 \in \min(\alpha)$ but $s_3 \in \mathrm{mod}(\gamma)$ so $\alpha {\not\mid\!\sim} \neg\gamma$. ∎

Propositions 5.2 and 5.6 immediately imply:

Theorem 5.7 Let $\mathcal{M}$ be a collapsed ranked model. Then $\mathcal{M}$ is quasi-linear iff ${\mid\!\sim}_{\mathcal{M}}$ is rational transitive.

This with corollary 4.7 immediately imply the promised representation theorem for rational transitive relations:

Theorem 5.8 ${\mid\!\sim}$ is a rational transitive relation iff there is a quasi-linear model $\mathcal{M}$ such that ${\mid\!\sim} = {\mid\!\sim}_{\mathcal{M}}$.

Putting together theorem 5.8 and proposition 3.2 we clearly have:

Theorem 5.9 The following conditions are equivalent for any preferential inference relation ${\mid\!\sim}$:

1. ${\mid\!\sim}$ is determined by some quasi-linear model.
2. ${\mid\!\sim}$ is determinacy preserving.

3. $\mid\!\sim$ is rational transitive.

4. $\mid\!\sim$ satisfies both RM and RC.

5. $\mid\!\sim$ satisfies both RM and WD.

Remark 5.10 The above results leave open the question of representation theorems for the weaker postulate sets $\mathsf{P}+\mathsf{RC}$ and $\mathsf{P}+\mathsf{WD}$. It may be noted that the techniques used above do not appear to carry over in a straightforward way to those systems. Lemma 4.2 (used for theorem 4.6 and thus corollary 4.7 and thus theorem 5.8) is here proven only for ranked preferential models, and even if the less direct techniques of Freund [2] are used (*cf.* remark 4.9) their scope covers only postulate systems at least as strong as $\mathsf{P}+\mathsf{DR}$.

6 Preferential Orderings and Rational Transitivity

After seeing Bezzazi and Pino Pérez [1] Michael Freund (personal communication) conjectured theorem 6.9 below, which characterizes rational transitive relations in terms of properties of their preferential orders defined as in [2]. The purpose of this section is to prove that characterization. This gives us another way to obtain the representation theorem 5.8. The results of the subsequent sections do not depend upon this one.

Definition 6.1 Let $\mid\!\sim$ be a preferential inference relation. Let α and β be formulae. The preferential order associated with $\mid\!\sim$ is defined by

$$\alpha < \beta \quad \Leftrightarrow \quad \alpha \vee \beta \mid\!\sim \neg\beta$$

The relation $<$ is not a strict order because irreflexivity does not quite hold (for instance $\bot < \bot$). Nevertheless by tradition we conserve the name of preferential order for it.

The following lemma from [2] helps understand better the meaning of this relation:

Lemma 6.2

1. $\alpha < \beta \;\Leftrightarrow\; \alpha \mid\!\sim \neg\beta$ and $\alpha \vee \beta \mid\!\sim \alpha$

2. $\alpha \mid\!\sim \beta \;\Leftrightarrow\; \alpha < \alpha \wedge \neg\beta$

3. $\alpha < \beta$ iff in every preferential model $\mathcal{M} = \langle S, \imath, \prec \rangle$ defining $\mid\!\sim$ the following property holds: for every element $s \in \mathrm{mod}(\beta)$ there exists $t \in \mathrm{mod}(\alpha)$ such that $t \prec s$.

It is easy to show the following corollary of point 3 of this lemma:

Lemma 6.3 Let $\mid\!\sim$ be a rational relation defined by a preferential ranked model $\mathcal{M} = \langle S, \imath, \prec \rangle$ with $r : S \perp\!\!\rightarrow \Omega$ the ranking function (Ω linearly ordered by $\lhd$). For any formula α define its level, $\ell(\alpha)$, as ∞ if $\alpha \mid\!\sim \bot$ and otherwise its level is the unique $a \in \Omega$ such that there exists $s \in \min_{\mathcal{M}}(\alpha)$ with $r(s) = a$. Then, the level is well defined and $\alpha < \beta$ iff $\ell(\alpha) \lhd \ell(\beta)$.

We remark that the relation $<$ of [9] (definition A3, first defined in [8]) is equivalent to that of definition 6.1 in the case of rational inference relations; the idea behind these 'orders' has roots in Lewis [10]. In [2] Freund called *preferential order* any relation $<$ on formulae satisfying the following four properties:

P_0: $\alpha < \bot$
P_1: If $\alpha \vdash \beta$, then

$$\text{(a)}\ \alpha < \gamma \quad \Rightarrow \quad \beta < \gamma$$
$$\text{(b)}\ \delta < \beta \quad \Rightarrow \quad \delta < \alpha$$

P_2: $\alpha < \gamma$ and $\alpha < \delta$ implies $\alpha < \gamma \vee \delta$
P_3: $\alpha \vee \beta < \beta$ implies $\alpha < \beta$

Freund proves that the 'order' associated with a preferential inference relation by definition 6.1 satisfies these properties. Conversely the inference relation $\mid\!\sim$ associated with a relation $<$ satisfying these properties by putting $\alpha \mid\!\sim \beta$ iff $\alpha < \alpha \wedge \neg\beta$ is a preferential inference relation; moreover the order associated with this inference relation by definition 6.1 coincides with $<$. Thus $<$ satisfies properties P_0-P_3 iff it is the preferential order associated with some preferential inference relation in the sense of definition 6.1.

We recall the definition of modular relation (see [9]):

Definition 6.4 A relation $<$ on E is said to be modular iff there exists a linear order $\prec$ on some set Ω and a function $r : E \longrightarrow \Omega$ such that $a < b \Leftrightarrow r(a) \prec r(b)$.

The following characterization of modularity is well-known and easy to verify.

Lemma 6.5 An order $<$ on E is modular iff for any $a, b, c \in E$ if a and b are incomparables and $a < c$ then $b < c$.

The following proposition is due to Freund (personal communication).

Proposition 6.6 $\mid\!\sim$ is a rational relation iff the preferential order between formulae associated by definition 6.1 with $\mid\!\sim$ is modular over the set of $\mid\!\sim$-consistent formulae, *i.e.* those formulae α with $\alpha \not\mid\!\sim \bot$.

Proof The *only if* part follows from the representation theorem 2.6 and lemma 6.3. More precisely, by the representation theorem 2.6 there exists a ranked model $\mathcal{M} = \langle S, l, \prec \rangle$ with $r : S \longrightarrow \Omega$ the ranking function (Ω linearly ordered by $\lhd$) such that $\mid\!\sim \, = \, \mid\!\sim_{\mathcal{M}}$. Define the function ℓ mapping a formula α to its level $\ell(\alpha)$. This mapping and lemma 6.3 prove that $<$ is modular.

Conversely, suppose that the preferential order between formulae $<$ is modular. Assume $\alpha \mid\!\sim \beta$ and $\alpha \not\mid\!\sim \neg\gamma$. We want to show that $\alpha \wedge \gamma \mid\!\sim \beta$. By part 2 of lemma 6.2, this last expression is equivalent to $\alpha \wedge \gamma < \alpha \wedge \gamma \wedge \neg\beta$ and the assumptions are equivalent to $\alpha < \alpha \wedge \neg\beta$ and $\alpha \not< \alpha \wedge \gamma$. Note that either $\alpha \wedge \gamma < \alpha$ or $\alpha \wedge \gamma \not< \alpha$. In the first case we use Freund's property P_1 (b) to obtain $\alpha \wedge \gamma < \alpha \wedge \gamma \wedge \neg\beta$. In the second case, α and $\alpha \wedge \gamma$ are incomparables because $\alpha \not< \alpha \wedge \gamma$ using part 2 of lemma 6.2 again. So by modularity, $\alpha \wedge \gamma < \alpha \wedge \neg\beta$ because $\alpha < \alpha \wedge \neg\beta$. Now, as before using the property P_1 (b), we obtain $\alpha \wedge \gamma < \alpha \wedge \gamma \wedge \neg\beta$. ∎

The following lemma will be useful:

Lemma 6.7 Let $\mid\!\sim$ a preferential relation and $<$ its associated preferential order. For any formulae α and β if $\alpha < \beta$ then $\top < \beta$

Proof Note that $\alpha \vdash \top$. Suppose $\alpha < \beta$. Then by P_1 (a) we have $\top < \beta$. ∎

The *quasi-linear property*, QLP in short, for an order $<$ associated to an inference relation $\mid\!\sim$ is the following property: for any formulae α and β, if $\top < \alpha$ then either $\alpha < \beta$ or $\beta < \alpha$ or α is $\mid\!\sim$-equivalent to β, *i.e.* $\alpha \mid\!\sim \beta$ and $\beta \mid\!\sim \alpha$.

Proposition 6.8 Let $\mathcal{M} = \langle S, \imath, \prec \rangle$ be a ranked collapsed model and put $\mid\!\sim \; = \; \mid\!\sim_{\mathcal{M}}$. If the preferential order associated with $\mid\!\sim$ satisfies the property QLP, then the model $\mathcal{M}$ is quasi-linear.

Proof Suppose that $\mathcal{M}$ is not quasi-linear. We want to show that the preferential order does not satisfy QLP. As $\mathcal{M}$ is not quasi-linear, there are three different states s_1, s_2 and s_3 in S such that s_1 is in the lowest level, s_2 and s_3 are in the same level and $s_1 \prec s_i$ for $i = 2, 3$. By parsimony there are formulae ϕ_2 and ϕ_3 with $s_2 \in \min(\phi_2)$ and $s_3 \in \min(\phi_3)$. By injectivity there exists a formula ζ such that $s_2 \in \mathrm{mod}(\zeta)$ and $s_3 \in \mathrm{mod}(\neg\zeta)$. Put $\alpha = \phi_2 \wedge \zeta$, et $\beta = \phi_3 \wedge \neg\zeta$. Note that $\alpha \not< \beta$ and $\beta \not< \alpha$ because their minimal states lie in the same level. Moreover, $s_2 \in \min_{\mathcal{M}}(\alpha)$ but $s_2 \notin \min_{\mathcal{M}}(\beta)$, so α and β are not $\mid\!\sim$-equivalent. But it is clear that, $\top < \alpha$. Hence the preferential order $<$ does not satisfy QLP. ∎

Note that the argument in this proof "translates" the one of proposition 5.6.

Theorem 6.9 (Conjectured by Freund, personal communication): Let $\mid\!\sim$ be a preferential relation. Then $\mid\!\sim$ satisfies RT iff the preferential order $<$ associated with $\mid\!\sim$ satisfies the property QLP.

Proof The *only if* part is deduced from theorem 5.8 as follows. Suppose that $\mid\!\sim$ is rational transitive. Then by theorem 5.8 there is a quasi-linear model $\mathcal{M}$ such that $\mid\!\sim \; = \; \mid\!\sim_{\mathcal{M}}$. We know that $\mathcal{M}$ is ranked *i.e.* $\mathcal{M} = \langle S, \imath, \prec \rangle$ with $r : S \perp\!\!\rightarrow \Omega$ the ranking function (Ω linearly ordered by $\lhd$) Now, if $\alpha \not< \beta$ and $\beta \not< \alpha$ we have by lemma 6.3 $\ell(\alpha) = \ell(\beta) = a \in \Omega$. But if $\top < \alpha$ then by quasi-linearity and lemma 6.3 there is at most one state $s \in S$ such that $r(s) = a$. So, $\min_{\mathcal{M}}(\alpha) = \min_{\mathcal{M}}(\beta) = \{s\}$ or $\min_{\mathcal{M}}(\alpha) = \min_{\mathcal{M}}(\beta) = \emptyset$. Hence, in either case, $\min_{\mathcal{M}}(\alpha) = \min_{\mathcal{M}}(\beta)$, *i.e.* α is $\mid\!\sim$-equivalent to β.

Now we prove the *if* part. Suppose that $\mid\!\sim$ is a preferential relation which satisfies QLP. We want to show that $\mid\!\sim$ satisfies RT. By theorem 5.8, it will be enough to see that $\mid\!\sim$ is represented by a quasi-linear model. In order to do that, we first show that $<$ is modular. Suppose that $\alpha \not< \beta$, $\beta \not< \alpha$, and $\alpha < \gamma$. We want to show that $\beta < \gamma$. By lemma 6.7, $\top < \gamma$. So, by QLP, $\beta < \gamma$ or $\gamma < \beta$ or γ and β are $\mid\!\sim$-equivalent. But we shall see that the last two cases lead to a contradiction.
Case 1: Suppose that $\gamma < \beta$. Then by transitivity of $<$ we have $\alpha < \beta$, a contradiction.
Case 2: Suppose that γ and β are $\mid\!\sim$-equivalent. Then, in any model $\mathcal{M}$ representing $\mid\!\sim$ we have $\min_{\mathcal{M}}(\gamma) = \min_{\mathcal{M}}(\beta)$. So by the lemma 6.2 using $\alpha < \gamma$ we conclude that $\alpha < \beta$. We find again a contradiction.

Therefore the only possibility is $\beta < \gamma$ as desired. As $<$ is modular, by the proposition 6.6, the relation $\mid\!\sim$ is rational. So there is a ranked model $\mathcal{M}$ representing it, and by theorem 4.6 we can suppose that $\mathcal{M}$ is collapsed. Thus by proposition 6.8, $\mathcal{M}$ is quasi-linear. Therefore $\mid\!\sim$ satisfies RT. ∎

Remark 6.10 We can give a different proof of theorem 6.9 which does not use the representation theorem 5.8. Moreover this proof provides an alternative argument for theorem 5.8.

Proof Here we give only a sketch. The argument uses Freund's notion of 'standard model'. The *only if* part, *i.e.* that RT implies QLP, is proven as follows. Suppose that $\top < \alpha$, *i.e.* $\top\mid\!\sim\neg\alpha$. If $\top\not\mid\!\sim\neg\beta$, *i.e.* $\top \not< \beta$, then β lies at the lowest level. So $\beta < \alpha$. If $\top\mid\!\sim\neg\beta$ we have the following situation: $\alpha\vee\beta\mid\!\sim\top$, $\top\mid\!\sim\neg\alpha$ and $\top\mid\!\sim\neg\beta$. Then, by RT we have $\alpha\vee\beta\mid\!\sim\neg\alpha$, or $\alpha\vee\beta\mid\!\sim\neg\beta$, or (when these two possibilities fail) we have both $\alpha\vee\beta\mid\!\sim\alpha$ and $\alpha\vee\beta\mid\!\sim\beta$. So, $\beta < \alpha$ or $\alpha < \beta$ or α and β are $\mid\!\sim$-equivalents.

The *if* part, *i.e.* that QLP implies RT, is proven as follows. By proposition 5.2 it is enough to show that $\mid\!\sim$ is represented by a quasi-linear model. The relation $\mid\!\sim$ is rational because $<$ is modular as remarked earlier. So, by the theorem 6.3 of [2], $\mid\!\sim$ is generated by its associated standard model (*cf.* remark 4.9) which is ranked. Moreover, this canonical model is quasi-linear: suppose the canonical model is not quasi-linear, *i.e.* there are two different worlds, m and n, both in a non-minimal level. We want to show that the condition QLP does not hold. By standardness (the canonical model is standard), there are formulae α, β (not necessarily different) such that $m \models C(\alpha)$, $n \models C(\beta)$, $m \in \min(\alpha)$ and $n \in \min(\beta)$. But, $m \neq n$ implies that there is a formula γ such that $m \models \gamma$ and $n \models \neg\gamma$. Put $\alpha_1 = \alpha\wedge\gamma$ and $\beta_1 = \beta\wedge\neg\gamma$. It is clear that $m \in \min(\alpha_1)$ and $n \in \min(\beta_1)$, so the minimal elements of α_1 and β_1 are at the same level. Therefore $\alpha_1 \not< \beta_1$ and $\beta_1 \not< \alpha_1$. But it is also clear that $m \notin \min(\beta_1)$ so α_1 and β_1 are not $\mid\!\sim$-equivalent. Thus to see that the property QLP does not hold for α_1 and β_1 it is enough to observe that $\top < \alpha_1$ because the minimal elements of α are in a level above the lowest one. ∎

7 Some Non-Horn rules incomparable with monotony

We consider some non-Horn rules that are stronger than rational monotony, but are not implied by monotony and for this reason are perhaps less interesting than those we have considered so far. We show how they may be characterized by certain subclasses of quasi-linear models.

Definition 7.1 A preferential relation $\mid\!\sim$ is said to be *completely determinated* iff the following rule holds

$$\text{CEM} \quad \frac{\alpha\not\mid\!\sim\beta}{\alpha\mid\!\sim\neg\beta}$$

In other words for any α and β, $\alpha\mid\!\sim\beta$ or $\alpha\mid\!\sim\neg\beta$.

This rule is called conditional excluded middle in Stalnaker [16] and also called full determinacy in Makinson [12].

Remark 7.2
1. $\mathsf{CEM} \Rightarrow \mathsf{DP}$.
2. $\mathsf{P} + \mathsf{M} \not\Rightarrow \mathsf{CEM}$.
3. $\mathsf{P} + \mathsf{CEM} \not\Rightarrow \mathsf{M}$.

Proof 1. This is immediate. Note that, as a consequence, by proposition 3.2 (see diagram), $\mathsf{CEM} + \mathsf{P}$ implies each of RM, DR, NR, RC, WD.
2. This is well known. Take, for instance, $\mid\!\sim$ to be the classical consequence relation. This relation obviously satisfies P and M but does not satisfy CEM.
3. Also well known. To recall: take the preferential structure with just two states, one less than the other. Every model on this structure satisfies P and CEM, whilst an appropriate model on it (*e.g.* the one used in the proof of corollary 5.3) fails to satisfy M. ∎

Definition 7.3 A preferential model (not necessarily injective) $\mathcal{M} = \langle S, \imath, \prec \rangle$ is said to be *linear* iff it is ranked and has at most one state at each level, *i.e.* iff it is of the shape:

$$\begin{array}{c} \vdots \\ \bullet \\ \downarrow \\ \bullet \end{array}$$

The following theorem can be extracted from work of Stalnaker and Lewis on the logic of counterfactual conditionals, but we give a direct verification here.

Theorem 7.4 A preferential inference relation $\mid\!\sim$ is completely determinated iff there exists a linear model $\mathcal{M}$ such that $\mid\!\sim_{\mathcal{M}} = \mid\!\sim$.

Proof The *if* part is evident. We prove the *only if* part. Suppose that $\mid\!\sim$ is completely determinated. Then by remark 7.2 and proposition 3.2, $\mid\!\sim$ satisfies RM. By the Lehmann-Magidor representation theorem 2.6, $\mid\!\sim$ can be represented by a ranked preferential model, which by theorem 4.6 we may suppose collapsed. To show that the model is linear, it suffices to show that there are no two distinct states on the same level. Suppose for reductio that s, t are on the same level and $s \neq t$. By parsimony there are formulae α, β with $s \in \min(\alpha)$ and $t \in \min(\beta)$. By injectivity there is an elementary formula p with $s \in \mathrm{mod}(p)$ and $t \notin \mathrm{mod}(p)$. Then clearly, using rankedness of the model, we have $\{s, t\} \subseteq \min(\alpha \vee \beta)$, so $\alpha \vee \beta \not\mid\!\sim p$ and $\alpha \vee \beta \not\mid\!\sim \neg p$, contrary to complete determination. ∎

Definition 7.5 A preferential model is said to be *almost linear* iff it is ranked and has at most one state at any rank above the lowest and at most two states at the lowest level. In other words, iff it is quasi-linear and has at most two states in the lowest level.

One may wonder whether these models satisfy any interesting new rules. And if so, whether we can characterize those rules by almost linear models. Both answers are positive, as we shall now show. We consider the following two rules

Fragmented Disjunction:

$$\mathsf{FD}\quad \frac{\alpha\mid\!\sim\beta\vee\gamma \quad \alpha\not\mid\!\sim\beta \quad \alpha\not\mid\!\sim\gamma}{\neg\beta\mid\!\sim\gamma}$$

that is, if $\alpha\mid\!\sim\beta\vee\gamma$ then either $\alpha\mid\!\sim\beta$ or $\alpha\mid\!\sim\gamma$ or $\neg\beta\mid\!\sim\gamma$. Its "dual" rule is

Fragmented Conjuction:

$$\mathsf{FC}\quad \frac{\alpha\wedge\beta\mid\!\sim\gamma \quad \alpha\not\mid\!\sim\gamma \quad \beta\not\mid\!\sim\gamma}{\alpha\mid\!\sim\neg\beta}$$

that is, if $\alpha\wedge\beta\mid\!\sim\gamma$ then either $\alpha\mid\!\sim\gamma$ or $\beta\mid\!\sim\gamma$ or $\alpha\mid\!\sim\neg\beta$.

Proposition 7.6 $\mathsf{P}+\mathsf{FD}\Rightarrow\mathsf{FC}$

Proof Suppose FC fails, *i.e.* that $\alpha\wedge\beta\mid\!\sim\gamma$, $\alpha\not\mid\!\sim\gamma$, $\beta\not\mid\!\sim\gamma$ and $\alpha\not\mid\!\sim\neg\beta$. From the first we have (by S and RW) $\alpha\mid\!\sim\neg\beta\vee\gamma$ and from the third (by LLE) we have $\neg\neg\beta\not\mid\!\sim\gamma$. But these together with the second and fourth show the failure of FD. ∎

Proposition 7.7 $\mathsf{P}+\mathsf{FC}\Rightarrow\mathsf{RM}$

Proof Suppose $\alpha\mid\!\sim\beta$, $\alpha\not\mid\!\sim\neg\gamma$. We want to show that $\alpha\wedge\gamma\mid\!\sim\beta$. We consider two cases: $\alpha\mid\!\sim\gamma$ and $\alpha\not\mid\!\sim\gamma$. In the first case we have $\alpha\wedge\gamma\mid\!\sim\beta$ by CM. Now consider the case $\alpha\not\mid\!\sim\gamma$. By preferentiality (REF and RW) we have

$$(\alpha\wedge\beta)\wedge(\alpha\wedge\gamma)\mid\!\sim\alpha\wedge\beta\wedge\gamma \tag{1}$$

We cannot have $\alpha\wedge\beta\mid\!\sim\alpha\wedge\beta\wedge\gamma$, otherwise by preferentiality (CUT and RW) we have $\alpha\mid\!\sim\gamma$ a contradiction. Thus

$$\alpha\wedge\beta\not\mid\!\sim\alpha\wedge\beta\wedge\gamma \tag{2}$$

We cannot have $\alpha\wedge\beta\mid\!\sim\neg(\alpha\wedge\gamma)$, otherwise by preferentiality (RW) we would have $\alpha\wedge\beta\mid\!\sim\alpha\rightarrow\neg\gamma$ and again by preferentiality (CUT,REF,AND and RW) we have $\alpha\mid\!\sim\neg\gamma$ a contradiction. Thus

$$\alpha\wedge\beta\not\mid\!\sim\neg(\alpha\wedge\gamma) \tag{3}$$

By FC, it follows from 1, 2 and 3 that $\alpha\wedge\gamma\mid\!\sim\alpha\wedge\beta\wedge\gamma$ so by RW $\alpha\wedge\gamma\mid\!\sim\beta$. ∎

Proposition 7.8 $\mathsf{P}+\mathsf{FC}\Rightarrow\mathsf{FD}$

Proof Suppose that FD fails, *i.e.* $\alpha\mid\!\sim\beta\vee\gamma$, $\alpha\not\mid\!\sim\beta$, $\alpha\not\mid\!\sim\gamma$ and $\neg\beta\not\mid\!\sim\gamma$. From the first two hypotheses (by RM which holds by proposition 7.7) we have $\alpha\wedge\neg\beta\mid\!\sim\beta\vee\gamma$. By REF and RW we have $\alpha\wedge\neg\beta\mid\!\sim\neg\beta$ so by AND and RW we have $\alpha\wedge\neg\beta\mid\!\sim\gamma$. We have also, from the second, $\alpha\not\mid\!\sim\neg\neg\beta$. But these last two together with the third and fourth hypotheses show the failure of FC. ∎

From propositions 7.6 and 7.8 we have immediately:

Corollary 7.9 Given P, $\mathsf{FD}\Leftrightarrow\mathsf{FC}$

Proposition 7.10 If $\mathcal{M}$ is an almost linear model then $\mid\!\sim_{\mathcal{M}}$ verifies FD and FC.

Proof By corollary 7.9 it suffices to consider FD. Let $\mathcal{M} = \langle S, \imath, \prec \rangle$ be an almost linear model and put $\mid\!\sim = \mid\!\sim_{\mathcal{M}}$. Suppose that $\mid\!\sim$ does not verify FD, *i.e.* that $\alpha \mid\!\sim \beta \vee \gamma$, $\alpha \not\mid\!\sim \beta$ $\alpha \not\mid\!\sim \gamma$ and $\neg\beta \not\mid\!\sim \gamma$. From the last three, there are states s_1, s_2 and s_3 such that $s_1 \in \min(\alpha) \cap \mathrm{mod}(\neg\beta)$, $s_2 \in \min(\alpha) \cap \mathrm{mod}(\neg\gamma)$, $s_3 \in \min(\neg\beta) \cap \mathrm{mod}(\neg\gamma)$. Then s_1 and s_2 are on the same level and are distinct (for using $\alpha \mid\!\sim \beta \vee \gamma$ we have $s_1 \in \mathrm{mod}(\gamma)$), and so, by quasi-linearity, they are on the lowest level. Since $s_1 \in \mathrm{mod}(\neg\beta)$, necessarily $\min(\neg\beta)$ is included in the lowest level. So s_3 is also on the lowest level. Since $\alpha \mid\!\sim \beta \vee \gamma$ we have also $s_2 \in \mathrm{mod}(\beta)$ and $s_3 \in \mathrm{mod}(\neg\alpha)$. Hence s_1, s_2 and s_3 are mutually distinct states on the lowest level, contradicting almost linearity. ∎

Remark 7.11 Clearly, P + M does not imply FD; we need only note that classical consequence, which satisfies monotony fails not only CEM as already observed in remark 7.2 but also FD. The same also follows from the fact that are flat models (no states less than any other) that fail FD. For instance consider the model $\mathcal{M} = \langle S, \imath, \prec \rangle$ consisting of just three states s_1, s_2 and s_3 all of the same and hence lowest level. Choose α, β and γ three distinct elementary propositions and put $\imath(s_1) \models \alpha \wedge \neg\beta \wedge \gamma$, $\imath(s_2) \models \alpha \wedge \beta \wedge \neg\gamma$ and $\imath(s_3) \models \neg\alpha \wedge \neg\beta \wedge \neg\gamma$. Then clearly FD fails. Actually we can put this observation in more general form:

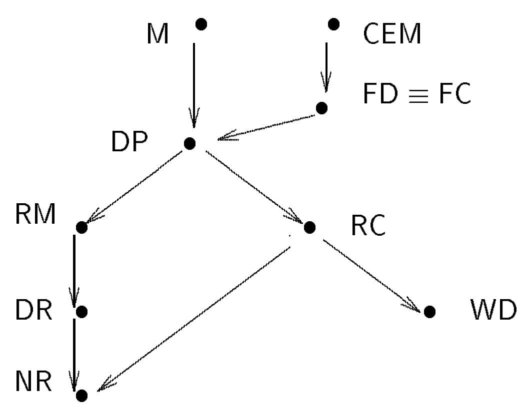
Proposition 7.12 If $\mathcal{M}$ is an injective quasi-linear model which is not almost linear then FD (and FC) fails.

Proof Assume that $\mathcal{M} = \langle S, \imath, \prec \rangle$ is an injective quasi-linear model which is not almost linear. Then there are three different states s_1, s_2 and s_3 on the lowest level. Using injectivity, there are formulae α_1, α_2, α_3 such that $\alpha_i \in \mathrm{mod}(s_j)$ iff $i = j$ (for $i, j = 1, 2, 3$). Now trivially $\alpha_1 \vee \alpha_2 \vdash \alpha_1 \vee \alpha_2$ so $\alpha_1 \vee \alpha_2 \mid\!\sim \alpha_1 \vee \alpha_2$. But $\alpha_1 \vee \alpha_2 \not\mid\!\sim \alpha_1$ (witness s_2) and $\alpha_1 \vee \alpha_2 \not\mid\!\sim \alpha_2$ (witness s_1), and $\neg\alpha_1 \not\mid\!\sim \alpha_2$ (witness s_3), so that FD fails.
∎

We now compare the strength of the rules FD and FC with those implied by monotony that were studied in section 3.

Theorem 7.13 Given P we have CEM ⇒ FC, FD ⇒ DP, but neither converse holds.

Before proving the theorem, we combine the information that it contains with corollary 7.9, remark 7.11, and proposition 3.2, to get the following diagram:



As before, we verify the positive parts of the theorem syntactically and the negative parts semantically. We begin with the positive parts

Proposition 7.14 $\mathsf{P} + \mathsf{CEM} \Rightarrow \mathsf{FC}$

Proof Suppose $\alpha \wedge \beta \mid\!\sim \gamma$ but $\alpha \not\mid\!\sim \gamma$ and $\beta \not\mid\!\sim \gamma$; we want to show $\alpha \mid\!\sim \neg\beta$. Suppose for reductio that $\alpha \not\mid\!\sim \neg\beta$. Then by CEM, $\alpha \mid\!\sim \beta$ and so by the first premise using CUT, $\alpha \mid\!\sim \gamma$ contradicting the second premise. ∎

Note that we also have an easy semantical proof of this proposition using theorem 7.4 and proposition 7.10.

Proposition 7.15 $\mathsf{P} + \mathsf{FD} \Rightarrow \mathsf{DP}$

Proof We have already shown (proposition 7.7 and corollary 7.9) that $\mathsf{P}+\mathsf{FD} \Rightarrow \mathsf{RM}$, and we know from theorem 3.2 that $\mathsf{P} + \mathsf{RM} + \mathsf{RC} \Rightarrow \mathsf{DP}$, so we need only show $\mathsf{P} + \mathsf{FD} \Rightarrow \mathsf{RC}$.

Recall that by preferentiality, whenever $\phi \mid\!\sim \bot$ then $\phi \mid\!\sim \psi$ so by CM $\phi \wedge \psi \mid\!\sim \bot$ so by S $\psi \mid\!\sim \phi \rightarrow \bot$ so $\psi \mid\!\sim \neg\phi$.

Assume P+FD. Suppose that RC does not hold, *i.e.* $\alpha \mid\!\sim \beta$, $\neg\beta \not\mid\!\sim \neg\alpha$ and $\neg\beta \not\mid\!\sim \alpha$. We want to get a contradiction. By supraclassicality (*i.e.* the rule $\alpha \vdash \beta \Rightarrow \alpha \mid\!\sim \beta$, derivable from the preferential postulates REF and RW) we have $\neg\beta \mid\!\sim (\neg\beta \wedge \neg\alpha) \vee (\neg\beta \wedge \alpha)$. We have also $\neg\beta \not\mid\!\sim \neg\beta \wedge \neg\alpha$ and $\neg\beta \not\mid\!\sim \neg\beta \wedge \alpha$. So by FD $\neg(\neg\beta \wedge \neg\alpha) \mid\!\sim \neg\beta \wedge \alpha$. So by LLE $\alpha \vee \beta \mid\!\sim \alpha \wedge \neg\beta$. Thus $\alpha \vee \beta \mid\!\sim \alpha$ and $\alpha \vee \beta \mid\!\sim \neg\beta$. So, by CM $(\alpha \vee \beta) \wedge \alpha \mid\!\sim \neg\beta$. So, by LLE $\alpha \mid\!\sim \neg\beta$. But this together with $\alpha \mid\!\sim \beta$ implies (by AND) $\alpha \mid\!\sim \bot$. By the fact recalled at the beginning of the proof, we get $\neg\beta \mid\!\sim \neg\alpha$, a contradiction as desired. ∎

Proposition 7.16 $\mathsf{P} + \mathsf{FC} \not\Rightarrow \mathsf{CEM}$

Proof Take an almost linear model of only one level, containing two states both of which satisfy α but just one of which satisfies β. Clearly this fails CEM, but by proposition 7.10 it satisfies FC. ∎

Proposition 7.17 $\mathsf{P} + \mathsf{DP} \not\Rightarrow \mathsf{FD}$

Proof Immediate form remark 7.11 and the fact that trivially $\mathsf{M} \Rightarrow \mathsf{DP}$. ∎

Theorem 7.18 Let $\mid\!\sim$ be a preferential relation. Then $\mid\!\sim$ verifies FD (or FC) iff there exists an almost linear model $\mathcal{M} = \langle S, \imath, \prec \rangle$ such that $\mid\!\sim \, = \, \mid\!\sim_{\mathcal{M}}$

Proof The *if* part is proposition 7.10. We prove the *only if* part. By proposition 7.15 and theorem 5.8, there exists a quasi-linear model $\mathcal{M}$ such that $\mid\!\sim \, = \, \mid\!\sim_{\mathcal{M}}$. By collapsing, we can assume that $\mathcal{M}$ is injective. So by proposition 7.12, $\mathcal{M}$ is almost linear. ∎

8 Some Horn rules between Preferential Inference and Monotony

Up to now all the rules studied as potential additions to those of preferential inference, are non-Horn. One may wonder if there are 'interesting' Horn rules beyond those of preferential inference, but still weaker than monotony. One such rule may be called *Conjunctive Insistence*:

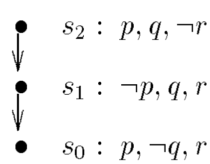
$$\mathsf{CI}\quad \frac{\alpha \mathrel{|\!\sim} \beta \quad \gamma \mathrel{|\!\sim} \beta}{\alpha \wedge \gamma \mathrel{|\!\sim} \beta}$$

Proposition 8.1 Monotony implies CI but the converse does not hold even if we suppose CEM. The preferential rules plus CEM do not imply CI. Moreover CI does not imply NR or WD.

Proof Clearly CI is implied by monotony. To see that monotony is not implied by CI even with CEM recall again the model in the proof of corollary 5.3 and of remark 7.2 (3). We know that this model satisfies P and CEM but fails M. We complete the proof by showing that every model with this structure satisfies CI.

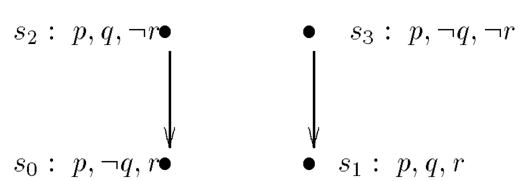
Suppose that $\alpha \wedge \beta \mathrel{|\!\not\sim} \gamma$ in a model with this structure; we need to show that either $\alpha \mathrel{|\!\not\sim} \gamma$ or $\beta \mathrel{|\!\not\sim} \gamma$. By the hypothesis there is a state $s \in \min(\alpha \wedge \beta)$ satisfiying $\neg\gamma$.
Case 1: Suppose s is of level zero. Then $s \in \min(\alpha) \cap \min(\beta)$ and so in fact we have both $\alpha \mathrel{|\!\not\sim} \gamma$ and $\beta \mathrel{|\!\not\sim} \gamma$.
Case 2: Suppose s is of level one. Since $s \in \min(\alpha \wedge \beta)$ the unique state $t \prec s$ of level zero fails to satisfy at least one of α, β. But in this case $s \in \min(\alpha)$ or $s \in \min(\beta)$ giving us in this case $\alpha \mathrel{|\!\not\sim} \gamma$ or $\beta \mathrel{|\!\not\sim} \gamma$.

CI is not implied by preferential rules plus CEM - consider for instance a ranked preferential structure with just three states at three levels, with an appropriate distribution of truth values as in the following figure:



Clearly here $p \mathrel{|\!\sim} r$, $q \mathrel{|\!\sim} r$ but $p \wedge q \mathrel{|\!\not\sim} r$. Note that this model is also linear, so that indeed P plus CEM does not imply CI.

We prove now that CI does not imply NR (which is enough to show that CI does not imply any of DR, RM,RC, DP, FD, CEM). Consider the model defined by the following figure:



Clearly this fails NR, for $p \mathrel{|\!\sim} r$ whilst $p \wedge q \mathrel{|\!\not\sim} r$ (witness s_2) and $p \wedge \neg q \mathrel{|\!\not\sim} r$ (witness s_3). However, it satisfies CI. Suppose for reductio that $\alpha \mathrel{|\!\sim} \gamma$, $\beta \mathrel{|\!\sim} \gamma$ but $\alpha \wedge \beta \mathrel{|\!\not\sim} \gamma$. From the last assumption, there is a $s \in \min(\alpha \wedge \beta)$ such that $s \models \neg\gamma$. Since $\alpha \mid\!\sim \gamma$, $\beta \mid\!\sim \gamma$, $s \models \alpha$, $s \models \beta$, $s \not\models \gamma$ there are $u, u' \prec s$ with $u \in \min(\alpha)$, $u' \in \min(\beta)$. But there is at most one state less than s, so $u = u'$ so $u \models \alpha \wedge \beta$ contradicting the minimality of s in $\mathrm{mod}(\alpha \wedge \beta)$.

The same model shows that $\mathsf{P} + \mathsf{CI} \not\Rightarrow \mathsf{WD}$; we have $\top \mid\!\sim \neg(p \wedge \neg r)$, $p \wedge \neg r \not\mid\!\sim q$ (witness s_3), $p \wedge \neg r \not\mid\!\sim \neg q$ (witness s_2). ∎

We suspect that there are not 'very many' Horn rules which, like CI, are implied by preferential rules with monotony but are not implied by the preferential rules alone. There are some, however, of technical more than conceptual interest. Consider the infinite series of rules of n-monotony ($n \geq 1$), n-M in short, constructed as follows:

$$1\text{-}\mathsf{M} \quad \frac{\alpha_1 \mid\!\sim \phi}{\alpha_1 \wedge \alpha_2 \mid\!\sim \phi}$$

$$2\text{-}\mathsf{M} \quad \frac{\alpha_1 \mid\!\sim \phi \quad \alpha_1 \wedge \alpha_2 \mid\!\sim \neg\phi}{\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \mid\!\sim \neg\phi}$$

and in general

$$n\text{-}\mathsf{M} \quad \frac{\alpha_1 \mid\!\sim \sigma_1(\phi) \quad \ldots \quad \alpha_1 \wedge \ldots \wedge \alpha_n \mid\!\sim \sigma_n(\phi)}{\alpha_1 \wedge \ldots \wedge \alpha_n \wedge \alpha_{n+1} \mid\!\sim \sigma_n(\phi)}$$

where each $\sigma_i(\phi)$ is either ϕ or $\neg\phi$ according as i is odd or even, and noting that the conclusion-rule uses σ_n rather than σ_{n+1}. This rule is evidently reminiscent of the alternating sequence of statements in the party example in section 1.2 of Lewis [10]: *'If Otto had come, it would have been a lively party; but if both Otto and Anna had come, it would have been a dreary party; but if Waldo had come as well, it would have been lively; but ...'* Of course, that is an infinite list of conditional expressions, and not a list of Horn rules about them. The rule number n is a scheme, one of whose instances in effect takes the first n Lewis statements as its n premises, and puts as conclusion a statement that is like Lewis' statement $n+1$ but with opposite consequent.

Clearly, 1-M is plain monotony. Moreover we have the following:

Observation 8.2 1. For all n, $\mathsf{P} + n$-M implies $(n+1)$-M.

2. For all n, $\mathsf{P} + (n+1)$-M does not imply n-M, even if CEM is also assumed.

3. $\mathsf{P} + \mathsf{CI}$ implies 2-M.

4. $\mathsf{P} +$ 2-M does not imply CI, even if FD is also assumed.

Proof Here we give only an outline.

1. Simply treat $\alpha_1 \wedge \alpha_2$ as a single formula in the premises of the rule of $(n+1)$-M, relabel letters, and apply the rule of n-M to the last n premises.

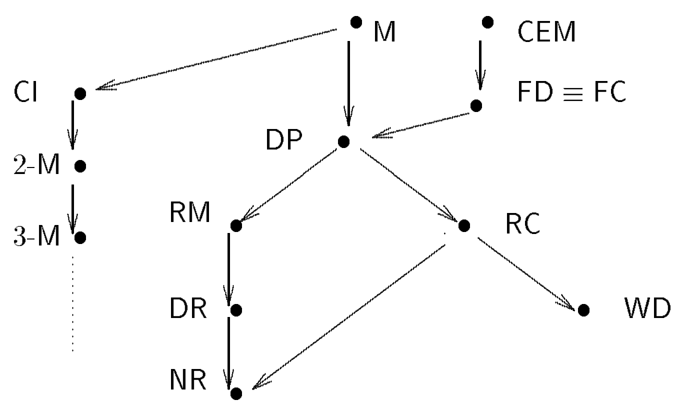
2. It is easy to see that (a) any ranked model with at most n levels satisfies n-M, (b) there is a linear preferential model with $n+1$ levels that fails n-M. These two facts give the desired result, using theorem 7.4.

3. Let $\mid\!\sim$ be a preferential relation that fails 2-M; we want to show that it fails CI. Since it fails 2-M, there are formulae $\alpha_1, \alpha_2, \alpha_3, \phi$ with $\alpha_1 \mid\!\sim \phi$, $\alpha_1 \wedge \alpha_2 \mid\!\sim \neg\phi$, $\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \not\mid\!\sim \neg\phi$. We need to find formulae β, γ, δ with $\beta \mid\!\sim \delta$, $\gamma \mid\!\sim \delta$, $\beta \wedge \gamma \not\mid\!\sim \delta$. Put $\beta = \alpha_1 \wedge (\neg\alpha_2 \vee \alpha_3)$, $\gamma = \alpha_1 \wedge \alpha_2$, $\delta = \neg\alpha_2 \vee \neg\phi$.
To show that $\beta \mid\!\sim \delta$, *i.e.* that $\alpha_1 \wedge (\neg\alpha_2 \vee \alpha_3) \mid\!\sim \neg\alpha_2 \vee \neg\phi$ note that since $\alpha_1 \wedge \alpha_2 \mid\!\sim \neg\phi$ we have $\alpha_1 \mid\!\sim \neg\alpha_2 \vee \neg\phi$, so since $\alpha_1 \mid\!\sim \phi$ we have $\alpha_1 \mid\!\sim \neg\alpha_2$ by preferential rules, so that by further preferential rules, $\alpha_1 \mid\!\sim \neg\alpha_2 \vee \alpha_3$ and also $\alpha_1 \mid\!\sim \neg\alpha_2 \vee \neg\phi$, so finally by the preferential rule CM, $\alpha_1 \wedge (\neg\alpha_2 \vee \alpha_3) \mid\!\sim \neg\alpha_2 \vee \neg\phi$ as desired. To show that $\gamma \mid\!\sim \delta$, *i.e.* that $\alpha_1 \wedge \alpha_2 \mid\!\sim \neg\alpha_2 \vee \neg\phi$ simply apply the preferential rule RW to the assumption $\alpha_1 \wedge \alpha_2 \mid\!\sim \neg\phi$. Finally, to show that $\beta \wedge \gamma \not\mid\!\sim \delta$, *i.e.* that $\alpha_1 \wedge (\neg\alpha_2 \vee \alpha_3) \wedge (\alpha_1 \wedge \alpha_2) \not\mid\!\sim \neg\alpha_2 \vee \neg\phi$, suppose the contrary and apply LLE to get $\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \mid\!\sim \neg\alpha_2 \vee \neg\phi$ so that $\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \mid\!\sim \neg\phi$ contrary to hypothesis.

4. To prove this part, it suffices by proposition 7.10 and 2(a) above, to find an almost linear model with two levels that fails CI. Clearly the following model will do: the language is built over the elementary formulae p, q, r; the lowest level has two states s_1, s_2 with $\imath(s_1) = \{p, r\}$, $\imath(s_2) = \{q, r\}$, whilst the next level has one state only s_3 with $\imath(s_3) = \{p, q\}$, so $p \mid\!\sim r$, $q \mid\!\sim r$, $p \wedge q \not\mid\!\sim r$. ∎

We note that since by proposition 8.1, $\mathsf{P} + \mathsf{CI} \not\Rightarrow \mathsf{NR}|\mathsf{WD}$, points 1 and 3 above tells us that $\mathsf{P} + n\text{-}\mathsf{M} \not\Rightarrow \mathsf{NR}|\mathsf{WD}$, whenever $n > 1$.

We may thus extend our diagram as follows:



Contrasting with point 4 of observation 8.2 we have:

Observation 8.3 $\mathsf{P} + 2\text{-}\mathsf{M} + \mathsf{CEM} \Rightarrow \mathsf{CI}$

Proof It is possible to verify this semantically, but the following is a direct syntactic proof. Suppose $\mathsf{P} + 2\text{-}\mathsf{M} + \mathsf{CEM}$. Let α, β, ϕ be formulae. Suppose $\alpha \mid\!\sim \phi$, $\beta \mid\!\sim \phi$; we want to show $\alpha \wedge \beta \mid\!\sim \phi$. We divide the argument into two cases.

Case 1. Suppose $\alpha \vee \beta \mid\!\sim \alpha \leftrightarrow \phi$ and $\alpha \vee \beta \mid\!\sim \beta \leftrightarrow \phi$. Since $\alpha \mid\!\sim \phi$, $\beta \mid\!\sim \phi$ we have by OR $\alpha \vee \beta \mid\!\sim \phi$, so by preferentiality, $\alpha \vee \beta \mid\!\sim \alpha \wedge \beta$. Now using CM we have $(\alpha \vee \beta) \wedge (\alpha \wedge \beta) \mid\!\sim \phi$, *i.e.* $\alpha \wedge \beta \mid\!\sim \phi$ as desired.

Case 2. Suppose $\alpha \vee \beta \not\mid\!\sim \alpha \leftrightarrow \phi$ or $\alpha \vee \beta \not\mid\!\sim \beta \leftrightarrow \phi$. We consider the former; the latter is similar. By CEM, $\alpha \vee \beta \mid\!\sim \neg(\alpha \leftrightarrow \phi)$. But also since $\alpha \mid\!\sim \phi$ we clearly have $\alpha \mid\!\sim \alpha \leftrightarrow \phi$, *i.e.* $(\alpha \vee \beta) \wedge \alpha \mid\!\sim \alpha \leftrightarrow \phi$. Hence by 2-M we have $((\alpha \vee \beta) \wedge \alpha) \wedge \beta \mid\!\sim \alpha \leftrightarrow \phi$, *i.e.* $\alpha \wedge \beta \mid\!\sim \alpha \leftrightarrow \phi$, so $\alpha \wedge \beta \mid\!\sim \phi$ as desired. ∎

8.1 Semantics for n-Monotony

Let $\mathcal{M} = \langle S, \imath, \prec \rangle$ be a preferential model. Define the *height* of $\mathcal{M}$ to be the maximal length of any chain of states $s_1, \ldots, s_n$ with $s_1 \prec s_2 \prec \cdots \prec s_n$. If there is no maximal length, put the height of the model to be ∞. The following observations generalize points 2(a) and 2(b) in the proof of observation 8.2.

Observation 8.4 Every preferential model of height $\leq n$ satisfies n-M.

Proof Consider a preferential model that fails n-M. We show that it has height $\geq n+1$. Since n-M fails in the model, there are formulae $\alpha_1, \ldots \alpha_n, \alpha_{n+1}, \phi$ with

$$\begin{array}{ll} \alpha_1 \mid\!\sim \phi & (1) \\ \alpha_1 \wedge \alpha_2 \mid\!\sim \neg\phi & (2) \\ \vdots & \\ \alpha_1 \wedge \ldots \wedge \alpha_n \mid\!\sim \sigma_n(\phi) & (n) \\ \alpha_1 \wedge \ldots \wedge \alpha_n \wedge \alpha_{n+1} \not\mid\!\sim \sigma_n(\phi) & (n+1) \end{array}$$

where each $\sigma_i(\phi)$ is ϕ or $\neg\phi$ according as i is odd or even. From $(n+1)$ there is a state s_{n+1} with $s_{n+1} \in \min(\alpha_1 \wedge \ldots \wedge \alpha_{n+1})$ but $\imath(s_{n+1}) \not\models \sigma_n(\phi)$. But by (n), $s_{n+1} \notin \min(\alpha_1 \wedge \ldots \wedge \alpha_n)$, so there is $s_n \prec s_{n+1}$ with $s_n \in \min(\alpha_1 \wedge \ldots \wedge \alpha_n)$ and $\imath(s_n) \models \sigma_n(\phi)$. Continuing down like this we get a sequence $s_{n+1} \succ s_n \succ \cdots \succ s_1$ of length $n+1$ of states of the model, so that its height is $\geq n+1$. ∎

Observation 8.5 Every parsimonious ranked model of height $> n$ fails n-M.

Proof Consider any parsimonious ranked model. Let $\mid\!\sim$ be the inference relation generated by the model. Since the model is of height $> n$ there is a sequence of states $s_1, \ldots, s_n, s_{n+1}$ with $s_1 \prec s_2 \prec \cdots \prec s_{n+1}$. By parsimony there exists a sequence of formulae $\gamma_1, \ldots, \gamma_{n+1}$ with $s_i \in \min(\gamma_i)$ for $i = 1, \ldots, n+1$. We define the formulae α_i for $i = 1, \ldots, n+1$ and ϕ as follows:

$$\alpha_i = \bigvee_{k=i}^{n+1} \gamma_k \qquad \phi = \bigvee_{k \geq 0}^{2k \leq n} (\gamma_{2k+1} \wedge \bigwedge_{j=2k+2}^{n+1} \neg\gamma_j)$$

Then we have:

1. For every $i = 1, \ldots, n$ and for every state u at the same level as s_i, $u \models \neg\gamma_k$ for any $k = i+1, \ldots, n+1$.
2. $\min(\alpha_i) = \min(\gamma_i)$ for any $i = 1, \ldots, n+1$.
3. $\vdash \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_i \leftrightarrow \alpha_i$ for every $i = 1, \ldots, n+1$.

The points 1 and 2 are easy consequences of rankedness and point 3 is evident by definition of α_i. From points 1 and 2 is easy to see that $\alpha_i \mid\!\sim \sigma_i(\phi)$ for any $i =$

$1, \ldots, n+1$. Thus we have

$$\begin{array}{l} \alpha_1 \mid\!\sim \phi \\ \alpha_1 \wedge \alpha_2 \mid\!\sim \neg\phi \\ \vdots \\ \alpha_1 \wedge \ldots \wedge \alpha_n \mid\!\sim \sigma_n(\phi) \\ \alpha_1 \wedge \ldots \wedge \alpha_n \wedge \alpha_{n+1} \mid\!\sim \sigma_{n+1}(\phi) \end{array}$$

From this it is evident that to show that n-M fails it is enough to see that $\alpha_1 \wedge \ldots \wedge \alpha_n \wedge \alpha_{n+1} \not\mid\!\sim \sigma_n(\phi)$. From points 2 and 3, $\alpha_1 \wedge \ldots \wedge \alpha_n \wedge \alpha_{n+1}$ is $\mid\!\sim$-consistent and since $\alpha_1 \wedge \ldots \wedge \alpha_n \wedge \alpha_{n+1} \mid\!\sim \sigma_{n+1}(\phi)$, necessarily $\alpha_1 \wedge \ldots \wedge \alpha_n \wedge \alpha_{n+1} \not\mid\!\sim \sigma_n(\phi)$, as desired. ∎

Theorem 8.6 Let $\mid\!\sim$ be any preferential inference relation. Then the following conditions are equivalent:

1. $\mid\!\sim$ satisfies n-M and RM (resp. RT, FD).
2. $\mid\!\sim$ is generated by some ranked (resp. quasi-linear, almost linear) preferential model of height $\leq n$.

Proof For the implication 2 ⇒ 1, apply observation 8.4 together with theorem 2.6 (resp. 5.2, 7.10). For the implication 1 ⇒ 2, apply observation 8.5 together with corollary 4.7 (plus 5.6, 7.12 respectively). ∎

Conjecture and open questions

We conclude with a conjecture and some open questions.

Conjecture: There are no Horn-rules *with a single premise* which, given the preferential rules, are implied by monotony, but do not imply monotony, and are not implied by the preferential rules alone (recall that CI has two premises, and n-monotony has n premises).

Open questions:

1. Determine whether the non implication $\mathsf{P} + \mathsf{WD} \not\Rightarrow \mathsf{NR}$ can be witnessed by *injective* preferential models (*cf.* observation 3.13)

2. Determine whether the construction used to prove lemma 4.2 can be adapted for a class of preferential models broader than the ranked ones -*e.g.* to the class of all models that are filtered in the sense of Freund [2] (*cf.* the discussion in remark 4.9).

3. Find appropriate classes of preferential models to provide representation theorems for RC, WD, CI (*cf.* the discussion in remark 5.10).

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