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Truncation invariant copulas for modeling directional dependence: Application to foreign currency exchange data

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Abstract. Directional dependence modeling has been applied to many research areas including economics, finance, biostatistics, and bioinformatics. The concept of directional dependence using copula regression functions has been introduced by Sungur [21]. So we propose a new copula family which incorporates the truncation invariant structure [20] into the generalized Farlie-Gumbel-Morgenstern (FGM) distributions. The directional dependence of the new truncated invariant FGM copulas will be also introduced in this research. We will show that there exists a directional dependence in our truncation invariant FGM copulas *using* Foreign Currency Exchange Data of the Canadian Dollar (CAD/USD), the Japanese Yen (JPY/USD), and the Korean Won (KRW/USD).

Keywords: Copula, regression function, directional dependence, generalized FGM distribution

1. Introduction

Copula is a useful device to express joint distributions of two or more random variables. It explains the dependence structure between variables by eliminating the influence of the marginal distributions of the individual variables. Sklar [18] discussed that any multivariate distribution function, say F, can be represented as a function of marginals, say F_{X_i}, $i = 1, 2, \ldots, m$ by using a m-dimensional copula C, i.e., $F(x_1, x_2, \ldots, x_m) = C(F_{X_1}(x_1), F_{X_2}(x_2), \ldots, F_{X_m}(x_m))$. Since Sklar's theorem [17] has been proposed, numerous copula functions have been introduced. Bairamov et al. [4] has studied distributional properties of concomitants for the generalized FGM distribution, and presented recurrence relations between moments of concomitants. Bairamov and Kotz [3] presented dependence structure and symmetry of Huang-Kotz FGM distributions. Bairamov et al. [5] gave some properties of the local dependence function. Rodrìguez-Lallena and Úbeda-Flores [16] presented the new class of bivariate copulas. Sungur [21] defined two types of directional dependence with the Rodrìguez-Lallena and Úbeda-Flores [16] family of copula, $C(u, v) = uv + f(u)g(v)$, in a regression setting. It considered the general measurement of the directional dependence, because directional dependence can happen from marginal or joint behavior or both. Jung et al. [11], Uhm et al. [22], and Kim and Kim [12] extended the Sungur [21] directional dependence to diverse bivariate distributions and applied their proposed models to show the directional dependence of the financial data and the precipitation data.

But there are no research results about the directional dependence to the multivariate distributions because most copulas fail to satisfy the copula properties when extended from the bivariate case to the multivariate case. To

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overcome the limitation of extending a bivariate copula to a multivariate copula, Aas et al. [1] proposed pair-copula constructions of multiple dependence which allow us to generate families of m-dimensional copulas by using 2-dimensional ones in a probabilistically meaningful way. Aas and Berg [1], and Fischer et al. [8] applied the pair-copula dependence modeling to financial data. Sungur [19,20] also proposed the truncation invariant copula. Oakes [14] offered the truncation invariant copula to the bivariate survival model of Clayton [7] as the only absolute continuous copula that is preserved under bivariate truncation. There are two advantages of introducing truncation invariant copula structure. First, a more detailed dependence structure with two variables when the third variable is given is possible. Second, it is possible to rotate three variables to see the triangle dependence structure. The applications for the proposed model are numerous in biology, engineering and finance. In this paper, we will use the foreign currency exchange data for verifying our proposed model.

In this paper we will apply the truncation invariant copula to the generalized FGM distributions and apply the directional dependence to a financial data using our proposed model. So this paper is organized into 4 sections. Section 2 contains the descriptions of the copula, the directional dependence using the copula, and the truncation invariant copula. Section 3 introduces our new truncation invariant copulas on FGM distribution to determine the directional dependence of a regression line. Section 4 illustrates the application of our proposed copula model to the foreign currency exchange data. Section 5 concludes the paper with future research plans.

2. Definitions and properties of copulas

A copula is a multivariate cumulative distribution function defined on the n-dimensional unit cube $[0,1]^n$ such that every marginal distribution is uniform on the interval $[0,1]$.

Definition 1. *(Copula)*
$C:[0,1]^d \to [0,1]$ *is a d-dimensional copula if*
- $C(u_1,\dots,u_{i-1},0,u_{i+1},\dots,u_d)=0$*, the copula is zero if one of the arguments is zero,*
- $C(1,\dots,1,u,1,\dots,1)=u$*, the copula is equal to* u *if one argument is* u *and all others 1,*
- C *is* d*-increasing, i.e., for each hyperrectangle* $B = X_{i=1}^{d}[x_i,y_i] \subseteq [0,1]^d$ *the C-volume of* B *is non-negative*

$$\int_B dC(u) = \sum_{\mathbf{z}\in X_{i=1}^{d}\{x_i,y_i\}} (-1)^{N(\mathbf{z})} C(\mathbf{z}) \geqslant 0,$$

where the $N(\mathbf{z}) = \sharp\{k : z_k = x_k\}$.

For instance, in the bivariate case, $C:[0,1]\times[0,1]\to[0,1]$ *is a bivariate copula if* $C(0,u)=C(u,0)=0$, $C(1,u)=C(u,1)=u$ *and* $C(y_1,y_2)-C(x_1,y_2)-C(y_1,x_2)+C(x_1,x_2)\geqslant 0$ *for all* $[x_1,y_1]\times[x_2,y_2]\subseteq[0,1]\times[0,1]$.

Sklar [18] showed that any multivariate distribution function, say F, can be represented as a function of its marginals, say G and H, by using a copula C, i.e., $F(x,y)=C(G(x),H(y))$.

2.1. Types of directional dependence

Sungur [21] has proposed two types of directional dependence, one in marginal and the other in joint behavior with examples for the Rodrìguez-Lallena and Úbeda-Flores [16] family of copulas. We also will concentrate on FGM copulas and the Rodrìguez-Lallena and Úbeda-Flores [16] family of copulas because of the simplicity of the model derivations and the property of asymmetry. Other popular copulas such as Normal, Clayton, Rotated Clayton, Plackett, Frank, Gumbel, Rotated Gumbel, Student's t and Symmetrized Joe-Clayton can not be used in this paper because those copulas are symmetric.

Definition 2. *(Rodrìguez-Lallena and Úbeda-Flores Family, 2004)*
The bivariate copula form proposed by the Rodrìguez-Lallena and Úbeda-Flores [16] is as follows:

$$C(u,v) = uv + f(u)g(v), \quad \text{for all } u,v. \tag{1}$$

Example 1. *Let (X,Y) be a continuous random pair whose associated copula C_θ is given by $C_\theta(u,v) = uv + \theta u^a v^b (1-u)^c (1-v)^d$ for every (u,v) in $\mathbb{I}^2$ with $a,b,c,d \geqslant 1$. Rodrìguez-Lallena and Úbeda-Flores [16] proved that C_θ is a copula if and only if $-\frac{1}{\max\{\nu\gamma,\omega\delta\}} \leqslant \theta \leqslant -\frac{1}{\min\{\nu\delta,\omega\gamma\}}$, where $\omega = -\nu = 1$ if $a = c = 1$, $\delta = -\gamma = 1$ if $b = d = 1$ and*

$$\nu = -\left(\frac{a}{a+c}\right)^{a-1}\left(1+\sqrt{\frac{c}{a(a+c-1)}}\right)^{a-1}\left(\frac{c}{a+c}\right)^{c-1}\left(1-\sqrt{\frac{a}{c(a+c-1)}}\right)^{c-1}\sqrt{\frac{ac}{a+c-1}}$$

$$\omega = \left(\frac{a}{a+c}\right)^{a-1}\left(1-\sqrt{\frac{c}{a(a+c-1)}}\right)^{a-1}\left(\frac{c}{a+c}\right)^{c-1}\left(1+\sqrt{\frac{a}{c(a+c-1)}}\right)^{c-1}\sqrt{\frac{ac}{a+c-1}}$$

$$\gamma = -\left(\frac{b}{b+d}\right)^{b-1}\left(1+\sqrt{\frac{d}{b(b+d-1)}}\right)^{b-1}\left(\frac{d}{b+d}\right)^{d-1}\left(1-\sqrt{\frac{b}{d(b+d-1)}}\right)^{d-1}\sqrt{\frac{bd}{b+d-1}}$$

$$\delta = \left(\frac{b}{b+d}\right)^{b-1}\left(1-\sqrt{\frac{d}{b(b+d-1)}}\right)^{b-1}\left(\frac{d}{b+d}\right)^{d-1}\left(1+\sqrt{\frac{b}{d(b+d-1)}}\right)^{d-1}\sqrt{\frac{bd}{b+d-1}},$$

otherwise. Moreover, the range for θ contains the interval $[-1,1]$ for all $a,b,c,d \geqslant 1$. The case $a = b = c = d = 1$ produces the Farlie-Gumbel-Morgenstern family of copulas (and the smallest range for θ, i.e., the interval $[-1,1]$). In general, the larger the parameters a,b,c,d are, the larger the range for θ (for instance: if $a = b = c = d = 2$, then $\theta \in [-27,27]$; if $a = 2, b = 3, c = 4, d = 5$, then $\theta \in [-840.445, 939.403]$; etc.).

Sungur [21] showed that the copula regression functions U to V, and V to U have the forms:

$$\begin{aligned} E[V|U=u] &= r_{V|U}(u) = \frac{1}{2} - \rho_C f'(u)\left[12\int_0^1 f(u)du\right]^{-1} \\ E[U|V=v] &= r_{U|V}(v) = \frac{1}{2} - \rho_C g'(v)\left[12\int_0^1 g(v)du\right]^{-1} \end{aligned} \tag{2}$$

where $\rho_C = 12\int_0^1\int_0^1 C(u,v)dudv - 3$ is the Pearson's correlation. When the functions f and g are not the same, the copula function will be asymmetric and the form of the directional regression functions will differ. Thus one can consider two kinds of directional dependence: one is the direction from U to V, and the other is the direction from V to U. We also used two types of directional dependence structure as Sungur [21] considered: one is directional dependence in joint behavior and the other is directional dependence in marginals.

Definition 3. *We can define that any pair (U,V) is directionally dependent in joint behavior if the forms of the regression function for $V|U = u$ and $U|V = v$ differ, i.e., $r_{V|U}(w) \neq r_{U|V}(w)$.*

Definition 4. *The random pair (X,Y) is directionally dependent in marginals if $r_{V|U}(w) = r_{U|V}(w) = r_C(w)$ and $r^C_{Y|X}(z) \neq r^C_{X|Y}(z)$, where $r_{Y|X}(x) = E_C[Y|X=x] = F_Y^{-1}(r_C(F_X(x)))$ and $r_{X|Y}(y) = E_C[X|Y=y] = F_X^{-1}(r_C(F_Y(y)))$.*

From Definitions 3 and 4 in Sungur [21], when we assume that both of the copula regression functions are linear, the random pair (U,V) cannot be directionally dependent, since $r_{U|V}(w) = r_{V|U}(w)$.

2.2. Truncation invariant copulas

We *considered* the truncation invariant structure on FGM copulas [19,20]. Suppose that $\{X_i, i \in (1,2,3)\}$ have the marginal distribution functions (*d.f.*) $F_{X_i}(x_i)$, the joint distribution function $F_{X_1,X_2,X_3}(x_1,x_2,x_3)$, and connecting copula $C_{X_1,X_2,X_3}(u,v,w)$. Let $T_{X_i} = \{x_i; x_i > a_i\}$ be the right-sided marginal truncation region for X_i. Then the joint *d.f.* of (X_1,X_2,X_3) can be written as follows:

$$\begin{aligned} F_{X_1,X_2,X_3}(x_1,x_2,x_3) &= C(F_{X_1}(x_1), F_{X_2}(x_2), F_{X_3}(x_3)) \\ &= P\left(\bigcap_{j=1,j\neq i}^{3}\{X_j \leqslant x_j\} \,|\, \{X_i \leqslant x_i\}\right) F_{X_i}(x_i). \end{aligned}$$

Let $X^{tr}_{j(i)}$ and $X^{tr}_{k(i)}$ represent truncated random variables over the truncated region T_{X_i} with marginal *d.f.*'s $F^{tr}_{X_{j(i)}}$ and $F^{tr}_{X_{k(i)}}$. Then,

$$\begin{aligned}P(X_j \leqslant x_j, X_k \leqslant x_k | X_i \leqslant a_i) =& \frac{P(X_i \leqslant a_i, X_j \leqslant x_j, X_k \leqslant x_k)}{P(X_i \leqslant a_i)}\\ =& C_{a_i}\left(\frac{C_{X_j,X_i}\left(F_{X_j}(x_j), F_{X_i}(a_i)\right)}{F_{X_i}(a_i)}, \frac{C_{X_k,X_i}\left(F_{X_k}(x_k), F_{X_i}(a_i)\right)}{F_{X_i}(a_i)}\right)\end{aligned}$$

where C_{a_i} is the connecting copula of $X^{tr}_{j(i)}$ and $X^{tr}_{k(i)}$ which is a function of the right truncation point a_i.

Theorem 1. *Let $\{X_i, i \in (1,2,3)\}$ be a random vector with a connecting copula C_{X_1,X_2,X_3}. Then, the dependence structure of the random pair (X_j, X_k) over the truncation region $T_{X_i} = \{X_i; X_i > a_i\}$, i.e., $C_{X^{tr}_{j(i)},X^{tr}_{k(i)}}(u_j, u_k)$, is independent of a_i if and only if C_{X_1,X_2,X_3} can be represented as*

$$C_{X_1,X_2,X_3}(u_1,u_2,u_3) = C_{X_j,X_k}\left(\frac{C_{X_j,X_i}(u_j,u_i)}{u_i}, \frac{C_{X_k,X_i}(u_k,u_i)}{u_i}\right)u_i, \quad i \neq j \neq k \in \{1,2,3\}$$

The truncation invariant copula is defined as follows [19].

Definition 5. *If a 3-dimensional copula can be represented as*

$$\begin{aligned}C(u_1,u_2,u_3) =& C_{12}\left(\frac{C_{13}(u_1,u_3)}{u_3}, \frac{C_{23}(u_2,u_3)}{u_3}\right)u_3\\ =& C_{13}\left(\frac{C_{12}(u_1,u_2)}{u_2}, \frac{C_{23}(u_2,u_3)}{u_2}\right)u_2\\ =& C_{23}\left(\frac{C_{12}(u_1,u_2)}{u_1}, \frac{C_{13}(u_1,u_3)}{u_1}\right)u_1,\end{aligned}$$

then it will be called a truncation invariant copula. The class of copulas with this property will be represented by Ψ.

By using the Definitions 2 and 5, we provide a characterization of 3-dimensional multivariate functions.

Example 2. *Define*

$$\begin{aligned}C_{12}(u_1,u_2) &= u_1u_2 + f_1(u_1)f_2(u_2)\\ C_{13}(u_1,u_3) &= u_1u_3 + f_1(u_1)f_3(u_3)\\ C_{23}(u_2,u_3) &= u_2u_3 + f_2(u_2)f_3(u_3).\end{aligned}$$

By Definition 5 and Theorem 2, we end up with:

$$\begin{aligned}C(u_1,u_2,u_3) =& C_{12}\left(\frac{C_{13}(u_1,u_3)}{u_3}, \frac{C_{23}(u_2,u_3)}{u_3}\right)u_3\\ =& C_{12}\left(\frac{u_1u_3 + f_1(u_1)f_3(u_3)}{u_3}, \frac{u_2u_3 + f_2(u_2)f_3(u_3)}{u_3}\right)u_3\\ =& \frac{\Big(u_1u_3 + f_1(u_1)f_3(u_3)\Big)\Big(u_2u_3 + f_2(u_2)f_3(u_3)\Big)}{u_3}\\ &+ f_1\left(\frac{u_1u_3 + f_1(u_1)f_3(u_3)}{u_3}\right)f_2\left(\frac{u_2u_3 + f_2(u_2)f_3(u_3)}{u_3}\right)u_3\\ =& u_1u_2u_3 + u_1f_2(u_2)f_3(u_3) + u_2f_1(u_1)f_3(u_3) + \frac{f_1(u_1)f_2(u_2)f_3^2(u_3)}{u_3}\\ &+ f_1\left(\frac{u_1u_3 + f_1(u_1)f_3(u_3)}{u_3}\right)f_2\left(\frac{u_2u_3 + f_2(u_2)f_3(u_3)}{u_3}\right)u_3\end{aligned}$$

which is a new class of truncation invariant copulas Ψ.

Table 1
Forms of $f(u)$ and $g(v)$ for each type

Type	$f(u)$	$g(v)$
I	$\sqrt{\theta}u(1-u)$	$\sqrt{\theta}v(1-v)$
II	$\sqrt{\theta}u^{\alpha}(1-u)$	$\sqrt{\theta}v^{\beta}(1-v)$
III	$\sqrt{\theta}u(1-u)^{\alpha}$	$\sqrt{\theta}v(1-v)^{\beta}$

3. Truncation invariant FGM copulas

We *considered different types* of the Farlie-Gumbel-Morgenstern (FGM) distribution which have a specific form of Rodrìguez-Lallena and Úbeda-Flores [16] copula family, $C(u, v) = uv + f(u)g(v)$.

In this paper, we introduced three different types of FGM distributions

- Type I: $C(u, v) = uv + \theta uv(1-u)(1-v)$ where $0 \leqslant u, v \leqslant 1$
- Type II: $C(u, v) = uv + \theta u^{\alpha}v^{\beta}(1-u)(1-v)$ where $\alpha \geqslant 1; \beta \geqslant 1; 0 \leqslant u, v \leqslant 1$
- Type III: $C(u, v) = uv + \theta uv(1-u)^{\alpha}(1-v)^{\beta}$ where $\alpha \geqslant 1; \beta \geqslant 1; 0 \leqslant u, v \leqslant 1$,

and applied Definition 5 to the three different types of FGM distributions to propose new truncation invariant copulas. Table 1 shows some special forms of $f(u)$ and $g(v)$ for each type of FGM function considered in this paper.

The range of θ for each Type is obtained from Example 1:

$$\begin{aligned} &\text{Type I: } -1 \leqslant \theta \leqslant 1 \text{ when } a = b = c = d = 1. \\ &\text{Type II: } -\frac{1}{\max\{\nu\gamma, \omega\delta\}} \leqslant \theta \leqslant -\frac{1}{\min\{\nu\gamma, \omega\delta\}} \text{ when } a = \alpha, b = \beta, c = d = 1. \\ &\text{Type III: } -\frac{1}{\max\{\nu\gamma, \omega\delta\}} \leqslant \theta \leqslant -\frac{1}{\min\{\nu\gamma, \omega\delta\}} \text{ when } a = b = 1, c = \alpha, d = \beta. \end{aligned} \tag{3}$$

Let C_{12}, C_{13} and C_{23} be members of the Farlie-Gumbel-Morgenstern copulas:

$$\{C_{ij}; C_{ij}(u_i, u_j) = u_i u_j \left[1 + \theta(1-u_i)(1-u_j)\right]\}.$$

Also, let θ_{12}, θ_{13}, and θ_{23} be the dependence parameters of C_{12}, C_{13} and C_{23}, respectively. Provided that θ_{12}, θ_{13}, and θ_{23} lead to compatible 2-dimensional copulas, the truncation invariant copula with respect to U_3 is

$$\begin{aligned} C(u_1, u_2, u_3) =& u_1 u_2 u_3 \left[1 + \theta_{13}(1-u_1)(1-u_3)\right] \left[1 + \theta_{23}(1-u_2)(1-u_3)\right] \\ &\times \left\{1 + \theta_{12}(1-u_1)(1-u_2) \left[1 - \theta_{13}u_1(1-u_3)\right] \left[1 - \theta_{23}u_2(1-u_3)\right]\right\}. \end{aligned} \tag{4}$$

If $\theta_1 = \theta_2 = \theta_3 = \theta$ in the Eq. (4), then

$$\begin{aligned} C(u_1, u_2, u_3) =& u_1 u_2 u_3 \left[1 + \theta(1-u_1)(1-u_3)\right] \left[1 + \theta(1-u_2)(1-u_3)\right] \\ &\times \left\{1 + \theta(1-u_1)(1-u_2) \left[1 - \theta u_1(1-u_3)\right] \left[1 - \theta u_2(1-u_3)\right]\right\}. \end{aligned} \tag{5}$$

It is easy to show that such generated copulas are the truncation invariant copulas with respect to all possible truncation regions.

Let $F_X(x)$ and $F_Y(y)$ be CDFs with the density functions $f_X(x)$ and $f_Y(y)$. We considered $F(x, y) = C(F_X(x), F_Y(y))$ shown by Sklar [18]. It yields the joint density function as follows:

$$f(x, y) = f_X(x) f_Y(y) c(F_X(x), F_Y(y))$$

where

$$c(u, v) = \frac{\partial^2 C(u, v)}{\partial u \partial v}.$$

The conditional joint distribution of U_1 and U_2 under the condition $\{U_3 \leqslant a\}$ is

$$C(U_1, U_2|U_3 \leqslant a) = P(U_1 \leqslant u_1, U_2 \leqslant u_2|U_3 \leqslant a) = \frac{P(U_1 \leqslant u_1, U_2 \leqslant u_2, U_3 \leqslant a)}{P(U_3 \leqslant a)} = \frac{C(u_1, u_2, a)}{a} = \frac{C_a(u_1, u_2)}{a}$$

where $C(u_1, u_2, a) = C_a(u_1, u_2)$.

The $C(U_1, U_2|U_3 \leqslant a)$ for each Type with the different parameters, θ_{12}, θ_{13}, and θ_{23}, of C_{12}, C_{13} and C_{23}, can be derived as follows:

– Type I: $C(U_1, U_2|U_3 \leqslant a)$

$$= [u_1 + \theta_{13}u_1(1-u_1)(1-a)]\,[u_2 + \theta_{23}u_2(1-u_2)(1-a)] + \theta_{12}\,[u_1 + \theta_{13}u_1(1-u_1)(1-a)] \times [u_2 + \theta_{23}u_2(1-u_2)(1-a)]\,[1 - u_1 - \theta_{13}u_1(1-u_1)(1-a)]\,[1 - u_2 - \theta_{23}u_2(1-u_2)(1-a)]\,.$$

– Type II: $C(U_1, U_2|U_3 \leqslant a)$

$$= \left[u_1 + \theta_{13}u_1^{\alpha}a^{\beta-1}(1-u_1)(1-a)\right]\left[u_2 + \theta_{23}u_2^{\alpha}a^{\beta-1}(1-u_2)(1-a)\right] + \theta_{12}\left[u_1 + \theta_{13}u_1^{\alpha}a^{\beta-1}(1-u_1)(1-a)\right]^{\alpha}\left[u_2 + \theta_{23}u_2^{\alpha}a^{\beta-1}(1-u_2)(1-a)\right]^{\beta} \times \left[1 - u_1 - \theta_{13}u_1^{\alpha}a^{\beta-1}(1-u_1)(1-a)\right]\left[1 - u_2 - \theta_{23}u_2^{\alpha}a^{\beta-1}(1-u_2)(1-a)\right].$$

– Type III: $C(U_1, U_2|U_3 \leqslant a)$

$$= \left[u_1 + \theta_{13}u_1(1-u_1)^{\alpha}(1-a)^{\beta}\right]\left[u_2 + \theta_{23}u_2(1-u_2)^{\alpha}(1-a)^{\beta}\right] + \theta_{12}\left[u_1 + \theta_{13}u_1(1-u_1)^{\alpha}(1-a)^{\beta}\right]\left[u_2 + \theta_{23}u_2(1-u_2)^{\alpha}(1-a)^{\beta}\right] \times \left[1 - u_1 - \theta_{13}u_1(1-u_1)^{\alpha}(1-a)^{\beta}\right]^{\alpha}\left[1 - u_2 - \theta_{23}u_2(1-u_2)^{\alpha}(1-a)^{\beta}\right]^{\beta}.$$

Cook and Johnson [6] studied the equi-dependence structure for describing data which is not elliptically symmetric. Under the equi-dependence structure ($\theta_{12} = \theta_{13} = \theta_{23} = \theta$), the $C(U_1, U_2|U_3 \leqslant a)$ for each Type is as follows:

– Type I: $C(U_1, U_2|U_3 \leqslant a)$

$$= [u_1 + \theta u_1(1-u_1)(1-a)]\,[u_2 + \theta u_2(1-u_2)(1-a)] + \theta\,[u_1 + \theta u_1(1-u_1)(1-a)] \times [u_2 + \theta u_2(1-u_2)(1-a)]\,[1 - u_1 - \theta u_1(1-u_1)(1-a)]\,[1 - u_2 - \theta u_2(1-u_2)(1-a)]\,.$$

– Type II: $C(U_1, U_2|U_3 \leqslant a)$

$$= \left[u_1 + \theta u_1^{\alpha}a^{\beta-1}(1-u_1)(1-a)\right]\left[u_2 + \theta u_2^{\alpha}a^{\beta-1}(1-u_2)(1-a)\right] + \theta\left[u_1 + \theta u_1^{\alpha}a^{\beta-1}(1-u_1)(1-a)\right]^{\alpha}\left[u_2 + \theta u_2^{\alpha}a^{\beta-1}(1-u_2)(1-a)\right]^{\beta} \times \left[1 - u_1 - \theta u_1^{\alpha}a^{\beta-1}(1-u_1)(1-a)\right]\left[1 - u_2 - \theta u_2^{\alpha}a^{\beta-1}(1-u_2)(1-a)\right].$$

– Type III: $C(U_1, U_2|U_3 \leqslant a)$

$$= \left[u_1 + \theta u_1(1-u_1)^{\alpha}(1-a)^{\beta}\right]\left[u_2 + \theta u_2(1-u_2)^{\alpha}(1-a)^{\beta}\right] + \theta\left[u_1 + \theta u_1(1-u_1)^{\alpha}(1-a)^{\beta}\right]\left[u_2 + \theta u_2(1-u_2)^{\alpha}(1-a)^{\beta}\right] \times \left[1 - u_1 - \theta u_1(1-u_1)^{\alpha}(1-a)^{\beta}\right]^{\alpha}\left[1 - u_2 - \theta u_2(1-u_2)^{\alpha}(1-a)^{\beta}\right]^{\beta}.$$

Then the conditional joint density function of U_1 and U_2 under the condition $\{U_3 \leqslant a\}$ is

$$f_{(U_1,U_2|U_3\leqslant a)}(u_1,u_2) = \frac{1}{a}\frac{\partial^2 C_a(u_1,u_2)}{\partial u_1 \partial u_2}. \tag{6}$$

Under the equi-dependence structure, the $f_{(U_1,U_2|U_3\leqslant a)}(u_1,u_2)$ for each Type is as follows:

– Type I: $f_{(U_1,U_2|U_3\leqslant a)}(u_1,u_2)$

$$\begin{aligned}
= {} & [1+\theta(1-2u_1)(1-a)]\,[1+\theta(1-2u_2)(1-a)] + \theta\,[1+\theta(1-2u_1)(1-a)] \\
& \times [1-u_1-\theta u_1(1-u_1)(1-a)]\,[1+\theta(1-2u_2)(1-a)]\,[1-u_2-\theta u_2(1-u_2)(1-a)] \\
& +\theta\,[1+\theta(1-2u_1)(1-a)]\,[1-u_1-\theta u_1(1-u_1)(1-a)]\,[u_2+\theta u_2(1-u_2)(1-a)] \\
& \times [-1-\theta(1-2u_2)(1-a)] + \theta\,[u_1+\theta u_1(1-u_1)(1-a)]\,[-1-\theta(1-2u_1)(1-a)] \\
& \times [1+\theta(1-2u_2)(1-a)]\,[1-u_2-\theta u_2(1-u_2)(1-a)] + \theta\,[u_1+\theta u_1(1-u_1)(1-a)] \\
& \times [-1-\theta(1-2u_1)(1-a)]\,[u_2+\theta u_2(1-u_2)(1-a)]\,[-1-\theta(1-2u_2)(1-a)]\,.
\end{aligned}$$

– Type II: $f_{(U_1,U_2|U_3\leqslant a)}(u_1,u_2)$

$$\begin{aligned}
= {} & \left[1+\theta a^{\beta-1}(1-a)(\alpha u_1^{\alpha-1}-(\alpha+1)u_1^{\alpha})\right]\left[1+\theta a^{\beta-1}(1-a)(\alpha u_2^{\alpha-1}-(\alpha+1)u_2^{\alpha})\right] \\
& +\theta\alpha\beta\left[u_1+\theta a^{\beta-1}(1-a)(u_1^{\alpha}-u_1^{\alpha+1})\right]^{\alpha-1}\left[1+\theta a^{\beta-1}(1-a)(\alpha u_1^{\alpha-1}-(\alpha+1)u_1^{\alpha})\right] \\
& \times\left[1-u_1-\theta a^{\beta-1}(1-a)(u_1^{\alpha}-u_1^{\alpha+1})\right]\left[u_2+\theta a^{\beta-1}(1-a)(u_2^{\alpha}-u_2^{\alpha+1})\right]^{\beta-1} \\
& \times\left[1+\theta a^{\beta-1}(1-a)(\alpha u_2^{\alpha-1}-(\alpha+1)u_2^{\alpha})\right]\left[1-u_2-\theta a^{\beta-1}(1-a)(u_2^{\alpha}-u_2^{\alpha+1})\right] \\
& +\theta\alpha\left[u_1+\theta a^{\beta-1}(1-a)(u_1^{\alpha}-u_1^{\alpha+1})\right]^{\alpha-1}\left[1+\theta a^{\beta-1}(1-a)(\alpha u_1^{\alpha-1}-(\alpha+1)u_1^{\alpha})\right] \\
& \times\left[1-u_1-\theta a^{\beta-1}(1-a)(u_1^{\alpha}-u_1^{\alpha+1})\right]\left[u_2+\theta a^{\beta-1}(1-a)(u_2^{\alpha}-u_2^{\alpha+1})\right]^{\beta} \\
& \times\left[-1-\theta a^{\beta-1}(1-a)(\alpha u_2^{\alpha-1}-(\alpha+1)u_2^{\alpha})\right]+\theta\beta\left[u_1+\theta a^{\beta-1}(1-a)(u_1^{\alpha}-u_1^{\alpha+1})\right]^{\alpha} \\
& \times\left[-1-\theta a^{\beta-1}(1-a)(\alpha u_1^{\alpha-1}-(\alpha+1)u_1^{\alpha})\right]\left[u_2+\theta a^{\beta-1}(1-a)(u_2^{\alpha}-u_2^{\alpha+1})\right]^{\beta-1} \\
& \times\left[1+\theta a^{\beta-1}(1-a)(\alpha u_2^{\alpha-1}-(\alpha+1)u_2^{\alpha})\right]\left[1-u_2-\theta a^{\beta-1}(1-a)(u_2^{\alpha}-u_2^{\alpha+1})\right] \\
& +\theta\left[u_1+\theta a^{\beta-1}(1-a)(u_1^{\alpha}-u_1^{\alpha+1})\right]^{\alpha}\left[-1-\theta a^{\beta-1}(1-a)(\alpha u_1^{\alpha-1}-(\alpha+1)u_1^{\alpha})\right] \\
& \times\left[u_2+\theta a^{\beta-1}(1-a)(u_2^{\alpha}-u_2^{\alpha+1})\right]^{\beta}\left[-1-\theta a^{\beta-1}(1-a)(\alpha u_2^{\alpha-1}-(\alpha+1)u_2^{\alpha})\right].
\end{aligned}$$

– Type III: $f_{(U_1,U_2|U_3\leqslant a)}(u_1,u_2)$

$$\begin{aligned}
= {} & \left(1+\theta(1-a)^{\beta}\left[(1-u_1)^{\alpha}-\alpha u_1(1-u_1)^{\alpha-1}\right]\right)\left(1+\theta(1-a)^{\beta}\left[(1-u_2)^{\alpha}-\alpha u_2(1-u_2)^{\alpha-1}\right]\right) \\
& +\theta\left(1+\theta(1-a)^{\beta}\left[(1-u_1)^{\alpha}-\alpha u_1(1-u_1)^{\alpha-1}\right]\right)\left[1-u_1-\theta u_1(1-u_1)^{\alpha}(1-a)^{\beta}\right]^{\alpha} \\
& \times\left(1+\theta(1-a)^{\beta}\left[(1-u_2)^{\alpha}-\alpha u_2(1-u_2)^{\alpha-1}\right]\right)\left[1-u_2-\theta u_2(1-u_2)^{\alpha}(1-a)^{\beta}\right]^{\beta} \\
& +\theta\beta\left(1+\theta(1-a)^{\beta}\left[(1-u_1)^{\alpha}-\alpha u_1(1-u_1)^{\alpha-1}\right]\right)\left[1-u_1-\theta u_1(1-u_1)^{\alpha}(1-a)^{\beta}\right]^{\alpha} \\
& \times\left[u_2+\theta u_2(1-u_2)^{\alpha}(1-a)^{\beta}\right]\left[1-u_2-\theta u_2(1-u_2)^{\alpha}(1-a)^{\beta}\right]^{\beta-1} \\
& \times\left(-1-\theta(1-a)^{\beta}\left[(1-u_2)^{\alpha}-\alpha u_2(1-u_2)^{\alpha-1}\right]\right)^{\beta}+\theta\alpha\left[u_1+\theta u_1(1-u_1)^{\alpha}(1-a)^{\beta}\right] \\
& \times\left[1-u_1-\theta u_1(1-u_1)^{\alpha}(1-a)^{\beta}\right]^{\alpha-1}\left(-1-\theta(1-a)^{\beta}\left[(1-u_1)^{\alpha}-\alpha u_1(1-u_1)^{\alpha-1}\right]\right) \\
& \times\left(1+\theta(1-a)^{\beta}\left[(1-u_2)^{\alpha}-\alpha u_2(1-u_2)^{\alpha-1}\right]\right)\left[1-u_2-\theta u_2(1-u_2)^{\alpha}(1-a)^{\beta}\right]^{\beta} \\
& +\theta\alpha\beta\left[u_1+\theta u_1(1-u_1)^{\alpha}(1-a)^{\beta}\right]\left[1-u_1-\theta u_1(1-u_1)^{\alpha}(1-a)^{\beta}\right]^{\alpha-1}
\end{aligned}$$

Table 2
Directional dependence for $U_1|a$ and $U_2|a$

	$r_{U_1\|a}$ and $r_{U_2\|a}$
Type I	$\frac{1}{2}-\frac{\theta(1-a)}{6}$
Type II	$\frac{1}{2}-\frac{\theta(1-a)a^{\beta-1}}{(\alpha+1)(\alpha+2)}$
Type III	$\frac{1}{2}-\frac{\theta(1-a)^{\beta}}{(\alpha+1)(\alpha+2)}$

$$\times\left(-1-\theta(1-a)^{\beta}\left[(1-u_1)^{\alpha}-\alpha u_1(1-u_1)^{\alpha-1}\right]\right)\left[u_2+\theta u_2(1-u_2)^{\alpha}(1-a)^{\beta}\right]$$
$$\times\left[1-u_2-\theta u_2(1-u_2)^{\alpha}(1-a)^{\beta}\right]^{\beta-1}\left(-1-\theta(1-a)^{\beta}\left[(1-u_2)^{\alpha}-\alpha u_2(1-u_2)^{\alpha-1}\right]\right).$$

The marginal density function of U_1 under the condition $\{U_3 \leqslant a\}$ and the marginal density function of U_2 under the condition $\{U_3 \leqslant a\}$ are given by

$$\begin{aligned} f_{(U_1|U_3\leqslant a)}(u_1) =& \frac{1}{a}\int_0^1 \frac{\partial^2 C_a(u_1,u_2)}{\partial u_1 \partial u_2}du_2, \\ f_{(U_2|U_3\leqslant a)}(u_2) =& \frac{1}{a}\int_0^1 \frac{\partial^2 C_a(u_1,u_2)}{\partial u_1 \partial u_2}du_1, \end{aligned} \tag{7}$$

and

$$\int_0^1\int_0^1 f_{(U_1,U_2|U_3\leqslant a)}(u_1,u_2)du_1du_2 = 1.$$

The directional dependence for U_1 with truncation $U_3 \leqslant a$ and the directional dependence for U_2 with truncation $U_3 \leqslant a$ are defined as

$$\begin{aligned} r_{U_1|a}(a) =& E[U_1|U_3 \leqslant a] = \int_0^1 u_1 f_{(U_1|U_3\leqslant a)}(u_1)du_1, \\ r_{U_2|a}(a) =& E[U_2|U_3 \leqslant a] = \int_0^1 u_2 f_{(U_2|U_3\leqslant a)}(u_2)du_2. \end{aligned} \tag{8}$$

Table 1 shows the derivations of both $r_{U_1|a}$ and $r_{U_2|a}$ for the three different FGM distributions by using Eq. (8). We concluded from Table 1 that there exist no directional dependencies defined by Definitions 3 and 4.

The directional dependences from U_1 to U_2 with truncation $U_3 \leqslant a$ and from U_2 to U_1 with truncation $U_3 \leqslant a$ are

$$\begin{aligned} r_{U_1|U_2,a}(u_2) =& E[U_1|U_2=u_2, U_3 \leqslant a] \\ =& 1-\int_0^1 \frac{\partial C(u_1,u_2,a)}{\partial u_2}du_1 = 1-\frac{\partial}{\partial u_2}\int_0^1 C(u_1,u_2,a)du_1, \\ r_{U_2|U_1,a}(u_1) =& E[U_2|U_1=u_1, U_3 \leqslant a] \\ =& 1-\int_0^1 \frac{\partial C(u_1,u_2,a)}{\partial u_1}du_2 = 1-\frac{\partial}{\partial u_1}\int_0^1 C(u_1,u_2,a)du_2. \end{aligned} \tag{9}$$

Table 3 shows the derivations of the directional dependencies for $U_1|U_2, a$ and $U_2|U_1, a$ of Type I, II, and III by using Eq. (9). From Table 3, we see $r_{U_2|U_1,a}(u_1) \neq r_{U_1|U_2,a}(u_2)$ for Type II, and III which means that there exists the directional dependency for $U_1|U_2, a$ and $U_2|U_1, a$. Since Type I is derived from a symmetric FGM distribution, the fact that there is no directional dependence of Type I is verified from Tables 2 and 3. But when we deal with Type II and Type III FGM copulas which are two variables under truncation applied to the third variable, the fact there exist directional dependencies between two variables is also verified from Tables 2 and 3. These facts can be used

Table 3
Summary of directional dependence by Type I, II, III

Type I
$r_{U_2\|U_1,a}(u_1) = \left(\frac{1}{2} - \frac{\theta(1-a)}{6}\right) - \theta(1-a)\left(\frac{1}{2} + \frac{\theta(1-a)}{6}\right)(1-2u_1) - \theta\left(\frac{1}{6} - \frac{\theta^2(1-a)^2}{30}\right)[1+\theta(1-a)(1-2u_1)]\left[1-2\left\{u_1+\theta(1-a)u_1(1-u_1)\right\}\right]$
$r_{U_1\|U_2,a}(u_2) = \left(\frac{1}{2} - \frac{\theta(1-a)}{6}\right) - \theta(1-a)\left(\frac{1}{2} + \frac{\theta(1-a)}{6}\right)(1-2u_2) - \theta\left(\frac{1}{6} - \frac{\theta^2(1-a)^2}{30}\right)[1+\theta(1-a)(1-2u_2)]\left[1-2\left\{u_2+\theta(1-a)u_2(1-u_2)\right\}\right]$
Type II
$r_{U_2\|U_1,a}(u_1) = 1 - \left[1+\theta(1-a)a^{\beta-1}(\alpha-(\alpha+1)u_1)u_1^{\alpha-1}\right]\left\{\frac{1}{2} + \frac{\theta(1-a)a^{\beta-1}}{(\alpha+1)(\alpha+2)}\right\} - \theta\left[1+\theta(1-a)a^{\beta-1}(\alpha-(\alpha+1)u_1)u_1^{\alpha-1}\right]\left[u_1+\theta(1-a)a^{\beta-1}(1-u_1)u_1^{\alpha}\right]^{\alpha-1} \times \left[\alpha-(\alpha+1)\left\{u_1+\theta(1-a)a^{\beta-1}(1-u_1)u_1^{\alpha}\right\}\right] \times \frac{1}{(\beta+1)(\beta+2)\left[1-\theta(1-a)a^{\beta-1}\right]}$
$r_{U_1\|U_2,a}(u_2) = 1 - \left[1+\theta(1-a)a^{\beta-1}(\alpha-(\alpha+1)u_2)u_2^{\alpha-1}\right]\left\{\frac{1}{2} + \frac{\theta(1-a)a^{\beta-1}}{(\alpha+1)(\alpha+2)}\right\} - \theta\left[1+\theta(1-a)a^{\beta-1}(\alpha-(\alpha+1)u_2)u_2^{\alpha-1}\right]\left[u_2+\theta(1-a)a^{\beta-1}(1-u_2)u_2^{\alpha}\right]^{\alpha-1} \times \left[\beta-(\beta+1)\left\{u_2+\theta(1-a)a^{\beta-1}(1-u_2)u_2^{\alpha}\right\}\right] \times \frac{1}{(\alpha+1)(\alpha+2)\left[1-\theta(1-a)a^{\beta-1}\right]}$
Type III
$r_{U_2\|U_1,a}(u_1) = 1 - \left[1+\theta(1-a)^{\beta}(1-u_1-\alpha u_1)(1-u_1)^{\alpha-1}\right]\left\{\frac{1}{2} + \frac{\theta(1-a)^{\beta}}{(\alpha+1)(\alpha+2)}\right\} - \theta\left[1+\theta(1-a)^{\beta}(1-u_1-\alpha u_1)(1-u_1)^{\alpha-1}\right]\left[1-\left\{u_1+\theta(1-a)^{\beta}u_1(1-u_1)^{\alpha}\right\}\right]^{\alpha-1} \times \left[(1-u_1-\alpha u_1)-(\alpha+1)\theta(1-a)^{\beta}u_1(1-u_1)^{\alpha}\right] \times \left(\frac{1}{(\beta+1)(\beta+2)\left(1+\theta(1-a)^{\beta}\right)}\right)$
$r_{U_1\|U_2,a}(u_2) = 1 - \left[1+\theta(1-a)^{\beta}(1-u_2-\alpha u_2)(1-u_2)^{\alpha-1}\right]\left\{\frac{1}{2} + \frac{\theta(1-a)^{\beta}}{(\alpha+1)(\alpha+2)}\right\} - \theta\left[1+\theta(1-a)^{\beta}(1-u_2-\alpha u_2)(1-u_2)^{\alpha-1}\right]\left[1-\left\{u_2+\theta(1-a)^{\beta}u_2(1-u_2)^{\alpha}\right\}\right]^{\beta-1} \times \left[(1-u_2-\beta u_2)-(\beta+1)\theta(1-a)^{\beta}u_2(1-u_2)^{\alpha}\right] \times \left(\frac{1}{(\alpha+1)(\alpha+2)\left[1+\theta(1-a)^{\beta}\right]}\right)$

to investigate a directional dependence of a financial data. To check for the existence of he directional dependence, the following general measures of directional dependence can be considered:

$$\begin{aligned} \rho^{(k)}_{X\to Y} &= \frac{E\Big[r_{Y|X}(X)-E[Y]\Big]^k}{\mu_k(Y)} \quad \text{if } \mu_k(Y)=E[Y-E[Y]]^k \neq 0; \\ \rho^{(k)}_{Y\to X} &= \frac{E\Big[r_{X|Y}(Y)-E[X]\Big]^k}{\mu_k(X)} \quad \text{if } \mu_k(X) \neq 0, \end{aligned} \tag{10}$$

where $\rho^{(k)}_{X\to Y}$ is the proportion of the k-th central moment of Y explained by the regression of Y on X. For example, $\rho^{(2)}_{X\to Y}$ can be interpreted as the proportion of variation explained by the regression of Y on X with respect to total variation of Y.

To calculate Eqs (9) and (10) with for the real data set, the parameter θ in each truncation invariant FGM copula should be estimated. So we define $U_i := F_X(X_i)$ and $V_i := F_Y(Y_i)$ for the continuous empirical marginal distribution functions F_X and F_Y and assume that U_i and V_i have uniform distribution $U(0,1)$. Hence we can reduce our empirical likelihood function to

$$L(\theta; U, V) = \prod_{i=1}^{n} c(U_i, V_i). \tag{11}$$

For a computational convenience of a MLE of θ, the logarithmic form of Eq. (11) is as follows:

$$\hat{\theta} = \arg\max_{\theta\in R} \sum_{i=1}^{n} logL(\theta; U_i, V_i) \tag{12}$$

where R is the set of all possible θ's.

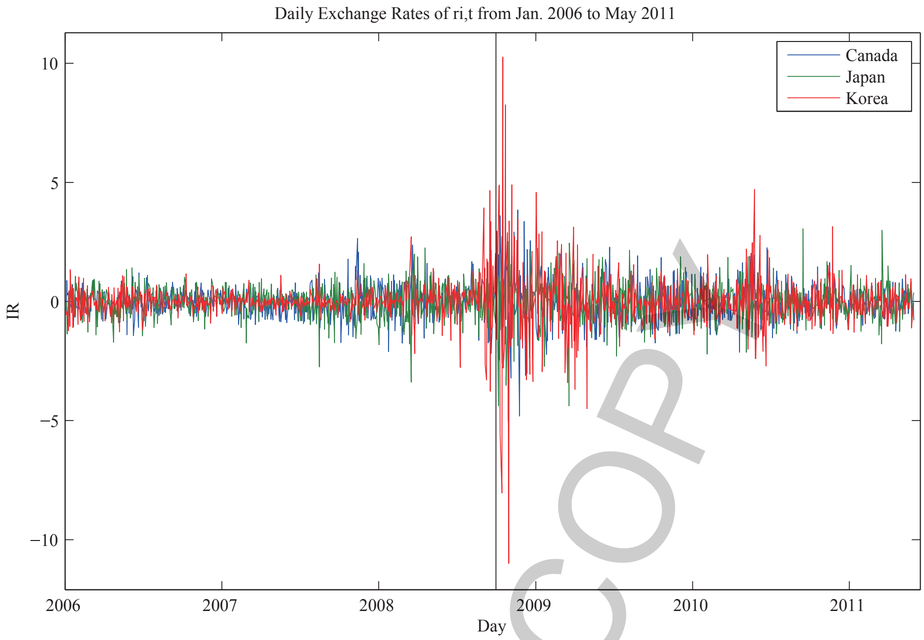

Fig. 1. Daily exchange rates of $r_{i,t}$ from Jan 2006 to May 2011.

For copula parameter estimation, Joe [10] proposed inference functions for margins and Genest et al. [9] proposed a semiparametric estimation procedure of dependence parameters in multivariate families of distributions. With these methods of parameter estimation, it is difficult to estimate θ's in our proposed copulas because $\frac{\partial L(\theta;U,V)}{\partial\theta}$ (or $\frac{\partial logL(\theta;U,V)}{\partial\theta}$) is not a function of θ. As an alternative method, we considered the optimization techniques based on Nelder-Mead, quasi-Newton, conjugate-gradient, and Broyden-Fletcher-Goldfarb-Shanno (BFGS) algorithms which are suitable for finding a value of θ in Eq. (11). We selected Nelder-Mead optimization to find an estimate of θ because Nelder-Mead optimization is a commonly used nonlinear optimization technique, which is a well-defined numerical method for twice differentiable and unimodal problems. The procedure for finding an estimate of θ is introduced in the following section.

4. Data analysis

This section demonstrates how our proposed directional copula method can be applied to real data. The data used in this paper is the monthly exchange market data from January 2006 to May 2011, from the *Pacific Exchange Rate Service* in Sauder School of Business, Univ. of British Columbia. Three traded currencies quoted against the US Dollar (USD) are used. They are the Canadian Dollar (CAD/USD), the Japanese Yen (JPY/USD), and the Korean Won (KRW/USD). These currencies were grouped into three pairs as the base. That is, three currency pairs (CAD,JPY), (CAD,KRW), and (JPY,KRW) against USD was analyzed to check for the existence of directional dependencies in at least one of the six pairs by using the proposed truncation invariant FGM copulas.

Figure 1 shows the exchange ratio plot for CAD, JPY, and KRW against USD from Jan. 2006 to May 2011. We partitioned the whole data into two groups as the pre-financial-crisis group (Jan. 2006 to Sept. 2008) and the post-financial-crisis group (Oct. 2008 to May 2011). The black color vertical line is the partition of the two groups in the Fig. 1.

Table 4 shows the estimates of the different parameters θ_{12}, θ_{13}, and θ_{23} in Type I truncation invariant FGM copula based on the values of the truncation ($a = 0.1, 0.3, 0.5, 0.7, 0.9$) for two different time groups with the three possible pairs (CAD,JPY), (CAD,KRW), and (JPY,KRW) against USD.

Table 5 shows the estimates of the common parameter θ based on the values of the truncation ($a = 0.1, 0.3, 0.5, 0.7, 0.9$) for two different time groups and the three possible pairs (CAD,JPY), (CAD,KRW), and

Table 4
θ Estimation by nelder and mead optimization

		Type I with Different θ					
	Trunc	Jan. 1, 2006 to Sep. 30, 2008			Oct. 1, 2008 to May 31, 2011		
	(a)	θ_{12}	θ_{13}	θ_{23}	θ_{12}	θ_{13}	θ_{23}
CAD/JPY	0.1	3.31854149	0.186364609	−0.183155839	3.259078	0.32852417	0.03462140
	0.3	3.37868942	0.233254790	−0.241834309	3.330302	0.44456352	0.06442161
	0.5	3.53404289	0.315614219	−0.354618596	3.728157	0.80832637	0.29690586
	0.7	4.25390830	0.547661732	−0.639270353	16.247513	1.27540829	1.52952479
	0.9	16.18338681	1.254840771	−1.308370484	14.625260	−0.58287114	−1.46865839
CAD/KRW	0.1	0.06858840	0.003204654	0.009179639	22.613838	−1.05380677	−0.98971809
	0.3	0.06871136	0.004154343	0.011912484	41.381769	−1.31972015	−1.13597612
	0.5	0.06861631	0.005844697	0.016639921	54.380067	−1.38296714	−1.21829699
	0.7	0.06868526	0.010040986	0.027964761	11.236623	0.07857219	−0.03333117
	0.9	2.72606679	2.221995150	−2.043425704	21.447700	1.07081163	0.46622846
JPY/KRW	0.1	−2.62948266	0.372920644	−0.340385258	2.707794	−0.34945843	−0.16921195
	0.3	−2.82718599	0.576454573	−0.487107673	2.844137	−0.47238704	−0.24733599
	0.5	−3.53369377	0.983753629	−0.758596325	3.249731	−0.70173914	−0.42133475
	0.7	−5.84730036	1.260586946	−1.175976936	4.915797	−1.12334280	−0.63190625
	0.9	−22.87217550	1.588564075	1.370445518	12.931441	−1.51883217	−0.39668568

Table 5
θ Estimation by nelder and mead optimization

	Type I With Common θ					
Trunc	Jan. 1, 2006 to Sep. 30, 2008			Oct. 1, 2008 to May 31, 2011		
(a)	CAD/JPY	CAD/KRW	JPY/KRW	CAD/JPY	CAD/KRW	JPY/KRW
0.1	0.3710937	0.01333008	−0.2413086	0.4141602	0.4707031	0.3842773
0.3	0.5104492	0.02050781	−0.3378906	0.5619141	0.6085938	0.5264648
0.5	0.6705078	0.03310547	−0.4907227	0.7351563	0.7570313	0.7021484
0.7	0.8188477	0.05351562	−0.7343750	0.8964844	0.8916016	0.8791016
0.9	0.9044922	0.06943359	−1.0042969	0.9873047	0.9787109	0.9882813
	Type II With Common θ					
Trunc	Jan. 1, 2006 to Sep. 30, 2008			Oct. 1, 2008 to May 31, 2011		
(a)	CAD/JPY	CAD/KRW	JPY/KRW	CAD/JPY	CAD/KRW	JPY/KRW
0.1	4.137500	−1.6505859	−2.2052734	3.192285	6.660840	4.273047
0.3	2.797363	−0.9406250	−1.1484375	2.661133	3.482617	2.695898
0.5	2.462598	−0.7760742	−0.9574219	2.196875	2.789844	2.252734
0.7	2.797363	−0.9406250	−1.1484375	2.661133	3.482617	2.695898
0.9	4.137500	−1.6505859	−2.2052734	3.192285	6.660840	4.273047
	Type III With Common θ					
Trunc	Jan. 1, 2006 to Sep. 30, 2008			Oct. 1, 2008 to May 31, 2011		
(a)	CAD/JPY	CAD/KRW	JPY/KRW	CAD/JPY	CAD/KRW	JPY/KRW
0.1	0.2339844	0.1462402	−0.1583008	0.2559937	0.2687500	0.2121338
0.3	0.5353516	0.3707520	−0.3913086	0.6048828	0.6193359	0.4888672
0.5	1.3826172	1.0191406	−0.9520508	1.7183594	1.7041016	1.2863281
0.7	2.8953125	1.5207031	−1.7488281	6.2746094	6.1601563	2.3039063
0.9	3.1019531	1.5020508	−2.1050781	4.2523438	8.4526367	2.5052734

(JPY,KRW) against USD for each Type truncation invariant FGM copula especially when $\alpha = 1$ and $\beta = 2$ in both Type II truncation invariant FGM copula and Type III truncation invariant FGM copula. We find a maximum likelihood estimate of θ within the range of θ obtained from Example 1 with the given values of α and β. The procedures for finding an estimate of θ are as follows:

Step 0 Select one truncation invariant FGM copula which is Type I, II, or III.
Step 1 Find the range of θ from (?) under $\alpha = 1, \beta = 2$ only for Type II and III.
Step 2 Find a optimized value of θ by Nelder-Mead method in the range of of θ for Type I, II, or III.
Step 3 Calculate the variation of the directional dependence with θ obtained from Step 2.
Step 4 Calculate the proportion of the variation for the directional dependence.

Table 6
Variation of directional dependence at Type I

	Trunc	Jan. 1, 2006 to Sep. 30, 2008			Oct. 1, 2008 to May 31, 2011		
	(a)	CAD/JPY	CAD/KRW	JPY/KRW	CAD/JPY	CAD/KRW	JPY/KRW
$Var(r_{v\|u})$	0.1	0.001248632	1.647561e−06	0.0005348320	0.001546817	0.001981709	0.001336830
	0.3	0.002354840	3.899505e−06	0.0010467909	0.002838306	0.003311484	0.002500966
	0.5	0.004075685	1.016165e−05	0.0022060174	0.004877493	0.005163571	0.004459910
	0.7	0.006142456	2.655385e−05	0.0049520864	0.007345256	0.007266606	0.007067005
	0.9	0.007573322	4.470168e−05	0.0093329776	0.009021309	0.008865244	0.009039040
$Var(r_{u\|v})$	0.1	0.001248553	1.647571e−06	0.0005348476	0.001546791	0.001981584	0.001336772
	0.3	0.002354686	3.899529e−06	0.0010468199	0.002838257	0.003311275	0.002500855
	0.5	0.004075427	1.016172e−05	0.0022060773	0.004877411	0.005163254	0.004459716
	0.7	0.006142113	2.655401e−05	0.0049522296	0.007345148	0.007266206	0.007066720
	0.9	0.007572973	4.470194e−05	0.0093333359	0.009021220	0.008864895	0.009038772

Table 7
Variation of directional dependence at Type II

	Trunc	Jan. 1, 2006 to Sep. 30, 2008			Oct. 1, 2008 to May 31, 2011		
	(a)	CAD/JPY	CAD/KRW	JPY/KRW	CAD/JPY	CAD/KRW	JPY/KRW
$Var(r_{v\|u})$	0.1	0.01462250	0.001663722	0.002866616	0.008271450	0.04317522	0.01571009
	0.3	0.04118091	0.002841463	0.004104621	0.036677624	0.06905499	0.03779658
	0.5	0.04594396	0.002748179	0.004048351	0.035234599	0.06166803	0.03734107
	0.7	0.04118091	0.002841463	0.004104621	0.036677624	0.06905499	0.03779658
	0.9	0.01462250	0.001663722	0.002866616	0.008271450	0.04317522	0.01571009
$Var(r_{u\|v})$	0.1	0.01462188	0.001663731	0.002866752	0.008271398	0.04317405	0.01570977
	0.3	0.04117918	0.002841478	0.004104816	0.036677395	0.06905312	0.03779579
	0.5	0.04594202	0.002748193	0.004048544	0.035234379	0.06166636	0.03734029
	0.7	0.04117918	0.002841478	0.004104816	0.036677395	0.06905312	0.03779579
	0.9	0.01462188	0.001663731	0.002866752	0.008271398	0.04317405	0.01570977

Table 8
Variation of directional dependence at Type III

	Trunc	Jan. 1, 2006 to Sep. 30, 2008			Oct. 1, 2008 to May 31, 2011		
	(a)	CAD/JPY	CAD/KRW	JPY/KRW	CAD/JPY	CAD/KRW	JPY/KRW
$Var(r_{v\|u})$	0.1	3.001820e−03	1.171969e−03	1.370682e−03	0.0035937746	0.0039611620	2.467148e−03
	0.3	5.872353e−03	2.797221e−03	3.017705e−03	0.0075188954	0.0078872281	4.887632e−03
	0.5	1.115233e−02	5.887106e−03	4.359687e−03	0.0176859196	0.0173743649	9.580441e−03
	0.7	7.388237e−03	1.804803e−03	1.728131e−03	0.0454954616	0.0434754818	4.443990e−03
	0.9	9.431097e−05	2.038595e−05	3.289447e−05	0.0001875498	0.0008994835	5.970556e−05
$Var(r_{u\|v})$	0.1	3.001693e−03	1.171975e−03	1.370747e−03	0.0035937522	0.0039610548	2.467097e−03
	0.3	5.872105e−03	2.797236e−03	3.017849e−03	0.0075188485	0.0078870148	4.887530e−03
	0.5	1.115186e−02	5.887137e−03	4.359894e−03	0.0176858093	0.0173738950	9.580241e−03
	0.7	7.387925e−03	1.804812e−03	1.728214e−03	0.0454951780	0.0434743060	4.443898e−03
	0.9	9.430699e−05	2.038606e−05	3.289603e−05	0.0001875486	0.0008994591	5.970432e-05

Step 5 Calculate Akaike's Information Criterion (AIC).

With the maximum likelihood estimates of θ, the following Tables 6 and 9 are made for Type I. Table 6 shows the variation of each directional dependence based on the values of the truncation ($a = 0.1, 0.3, 0.5, 0.7, 0.9$) for two different time groups and the three possible pairs (CAD,JPY), (CAD,KRW), and (JPY,KRW) against USD. In Table 6, we found that the post-financial-crisis group had a larger variation of directional dependence than the pre-financial-crisis group. Table 9 shows the proportion of the variation of the directional dependence for Type I. We found that there existed no directional dependence for three possible pairs (CAD,JPY), (CAD,KRW), and (JPY,KRW) against USD by using Type I model. Tables 7 and 10 are made for Type II. Table 7 shows the variation of the directional dependence based on the values of the truncation ($a = 0.1, 0.3, 0.5, 0.7, 0.9$) for two different time groups and the three possible pairs (CAD,JPY), (CAD,KRW), and (JPY,KRW) against USD. In terms of (CAD,JPY) pair against USD in Table 7, the pre-financial-crisis group had a higher variation than the post-financial-crisis group. This result

Table 9
Proportion of variation for directional dependence at Type I

	Trunc	Jan. 1, 2006 to Sep. 30, 2008			Oct. 1, 2008 to May 31, 2011		
	(a)	CAD/JPY	CAD/KRW	JPY/KRW	CAD/JPY	CAD/KRW	JPY/KRW
$\rho^{(2)}_{v\to u}$	0.1	0.01496310	1.974276e−05	0.006408904	0.01853472	0.02374631	0.01601889
	0.3	0.02821944	4.672785e−05	0.012543718	0.03400997	0.03968066	0.02996842
	0.5	0.04884134	1.217673e−04	0.026434756	0.05844452	0.06187373	0.05344195
	0.7	0.07360868	3.181953e−04	0.059340961	0.08801446	0.08707385	0.08468208
	0.9	0.09075560	5.356612e−04	0.111837278	0.10809775	0.10622992	0.10831247
$\rho^{(2)}_{u\to v}$	0.1	0.01496152	1.974299e−05	0.006409395	0.01853430	0.02374417	0.01601786
	0.3	0.02821640	4.672839e−05	0.012544662	0.03400917	0.03967708	0.02996648
	0.5	0.04883619	1.217687e−04	0.026436729	0.05844317	0.06186826	0.05343850
	0.7	0.07360146	3.181990e−04	0.059345495	0.08801263	0.08706670	0.08467691
	0.9	0.09074759	5.356671e−04	0.111846883	0.10809601	0.10622286	0.10830700

Table 10
Proportion of variation for directional dependence at Type II

	Trunc	Jan. 1, 2006 to Sep. 30, 2008			Oct. 1, 2008 to May 31, 2011		
	(a)	CAD/JPY	CAD/KRW	JPY/KRW	CAD/JPY	CAD/KRW	JPY/KRW
$\rho^{(2)}_{v\to u}$	0.1	0.1752300	0.01993642	0.03435072	0.09911257	0.5173574	0.1882500
	0.3	0.4934952	0.03404931	0.04918577	0.43948929	0.8274680	0.4529066
	0.5	0.5505736	0.03293148	0.04851148	0.42219826	0.7389520	0.4474484
	0.7	0.4934952	0.03404931	0.04918577	0.43948929	0.8274680	0.4529066
	0.9	0.1752300	0.01993642	0.03435072	0.09911257	0.5173574	0.1882500
$\rho^{(2)}_{u\to v}$	0.1	0.1752153	0.01993663	0.03435399	0.09911134	0.5173295	0.1882421
	0.3	0.4934536	0.03404968	0.04919044	0.43948381	0.8274232	0.4528877
	0.5	0.5505271	0.03293183	0.04851609	0.42219300	0.7389120	0.4474297
	0.7	0.4934536	0.03404968	0.04919044	0.43948381	0.8274232	0.4528877
	0.9	0.1752153	0.01993663	0.03435399	0.09911134	0.5173295	0.1882421

Table 11
Proportion of variation for directional dependence at Type III

	Trunc	Jan. 1, 2006 to Sep. 30, 2008			Oct. 1, 2008 to May 31, 2011		
	(a)	CAD/JPY	CAD/KRW	JPY/KRW	CAD/JPY	CAD/KRW	JPY/KRW
$\rho^{(2)}_{v\to u}$	0.1	0.035972578	0.0140437298	0.0164249076	0.043062371	0.04746558	0.0295631951
	0.3	0.070371880	0.0335191664	0.0361612308	0.090095096	0.09451061	0.0585672229
	0.5	0.133644951	0.0705453237	0.0522422306	0.211921371	0.20819250	0.1147999347
	0.7	0.088537610	0.0216269916	0.0207082377	0.545148957	0.52095541	0.0532511805
	0.9	0.001130184	0.0002442853	0.0003941751	0.002247314	0.01077828	0.0007154362
$\rho^{(2)}_{u\to v}$	0.1	0.035969544	0.0140438790	0.0164264678	0.043061834	0.04746301	0.0295619647
	0.3	0.070365944	0.0335195225	0.0361646659	0.090093973	0.09450549	0.0585647853
	0.5	0.133633677	0.0705460732	0.0522471932	0.211918729	0.20818124	0.1147951567
	0.7	0.088530141	0.0216272214	0.0207102048	0.545142161	0.52092723	0.0532489642
	0.9	0.001130089	0.0002442879	0.0003942126	0.002247286	0.01077769	0.0007154064

is a little different from Table 6. Table 10 shows the proportion of the variation of the directional dependence based on the values of the truncation ($a = 0.1, 0.3, 0.5, 0.7, 0.9$) for two different groups and the three possible pairs (CAD,JPY), (CAD,KRW), and (JPY,KRW) against USD. Looking at (CAD,JPY) pair against USD in Table 10, the Japanese Yen had more influence on the Canadian dollar in both the pre-financial-crisis and the post-financial-crisis. In cases of (CAD,KRW) and (JPY,KRW) in Table 10, the Canadian dollar and the Japanese Yen had more influence on the Korean Won in the pre-financial-crisis group but the Korean Won had more influence on the Canadian dollar and the Japanese Yen in the post-financial-crisis group. Table 8 and Table 11 are made for Type III. Table 8 shows the variation of the directional dependence based on the values of the truncation ($a = 0.1, 0.3, 0.5, 0.7, 0.9$) for two different groups and the three possible pairs (CAD,JPY), (CAD,KRW), and (JPY,KRW) against USD. The results in Table 8 are the same as the ones in Table 6. Table 11 shows the proportion of variation of the directional dependence based on the values of the truncation ($a = 0.1, 0.3, 0.5, 0.7, 0.9$) for two different groups and the three possible pairs

Table 12
Akaike's information criterion (AIC)

Type	Trunc	Jan. 1, 2006 to Sep. 30, 2008			Oct. 1, 2008 to May 31, 2011		
	(a)	CAD/JPY	CAD/KRW	JPY/KRW	CAD/JPY	CAD/KRW	JPY/KRW
I	0.1	−51.31922	1.931598	−29.15149	−67.05200	−96.93801	−58.97322
	0.3	−73.88208	1.895552	−42.16850	−94.04119	−129.63061	−83.24268
	0.5	−102.25826	1.831273	−62.72962	−127.53892	−166.55852	−114.67904
	0.7	−129.49236	1.728270	−94.08629	−159.79578	−200.15485	−147.16244
	0.9	−144.12085	1.648237	−128.43586	−178.16411	−221.03251	−167.84318
II	0.1	−271.0288	−61.69668	−165.20292	−261.5085	−686.3156	−366.8978
	0.3	−195.2294	−35.39278	−88.95704	−191.1735	−365.9465	−228.1332
	0.5	−157.9789	−28.97516	−74.49838	−159.4347	−297.9168	−189.6386
	0.7	−195.2294	−35.39278	−88.95704	−191.1735	−365.9465	−228.1332
	0.9	−271.0288	−61.69668	−165.20292	−261.5085	−686.3156	−366.8978
III	0.1	−19.57616	−4.371337	−5.010598	−21.41270	−25.39721	−14.72210
	0.3	−46.76723	−13.896630	−15.992775	−52.08842	−59.64442	−35.65240
	0.5	−119.93749	−40.044332	−44.525314	−141.13396	−155.77402	−90.97352
	0.7	−270.31254	−65.928365	−83.877995	−411.04666	−460.65150	−184.19854
	0.9	−345.79527	−63.664413	−90.132604	−392.88053	−742.36037	−222.06603

Table 13
Tail dependencies for copulas

	Jan 2006 to Sep 2008								
	CAD vs JPY			CAD vs KRW			JPY vs KRW		
Copula type	τ^L	τ^U	AIC	τ^L	τ^U	AIC	τ^L	τ^U	AIC
Normal	0	0	−3.1567	0	0	−17.6271	0	0	−0.0408
Clayton	0	0	0.0085	0.0097	0	−10.5633	0	0	−1.0889
Rotated clayton (in upper tail)	0	0	0.0065	0	0.0294	−19.2261	0	0	0.0030
Plackett	0	0	−1.5910	0	0	−16.2655	0	0	−0.1790
Frank	0	0	0.0039	0	0	−15.6003	0	0	−0.1598
Gumbel	0	0.1221	18.7799	0	0.1289	−20.8133	0	0.1221	10.7094
Rotated gumbel	0.1221	0	20.8212	0.1221	0	−14.3362	0.1221	0	6.8271
Student's t	0.0308	0.0308	−17.9415	0.0062	0.0062	−20.5415	0.0690	0.0690	−27.2141
Symmetrised Joe-Clayton	0	0	3.4347	0.0078	0.0648	−21.4226	0	0	−0.1159
	Oct 2008 to May 2011								
	CAD vs JPY			CAD vs KRW			JPY vs KRW		
Copula type	τ^L	τ^U	AIC	τ^L	τ^U	AIC	τ^L	τ^U	AIC
Normal	0	0	−38.8800	0	0	−128.6688	0	0	−21.2690
Clayton	0	0	0.0209	0.3055	0	−106.6845	0	0	0.0171
Rotated clayton (in upper tail)	0	0	0.0255	0	0.3253	−117.8446	0	0	0.0190
Plackett	0	0	−32.9930	0	0	−128.6814	0	0	−16.9699
Frank	0	0	0.0075	0	0	−119.8372	0	0	0.0063
Gumbel	0	0.1221	49.6337	0	0.3469	−141.4106	0	0.1221	40.4421
Rotated gumbel	0.1221	0	41.6721	0.3414	0	−133.5163	0.1221	0	36.0730
Student's t	0.0319	0.0319	−63.7206	0.1999	0.1999	−160.7055	0.0291	0.0291	−40.3390
Symmetrised Joe-Clayton	0	0	11.3796	0.2398	0.2803	−152.4032	0	0	8.7947

(CAD,JPY), (CAD,KRW), and (JPY,KRW) against USD. The results in Table 11 are the same as the ones in Table 10. From Tables 6–8, more variation in the directional dependence happened in the three possible pairs (CAD,JPY), (CAD,KRW), and (JPY,KRW) against USD after the financial crisis. From Tables 10 and 11, we concluded that the Canadian dollar and the Japanese Yen had more influence on the Korean Won in the pre-financial-crisis group but Korean Won had more influence on the Canadian dollar and the Japanese Yen in the post-financial-crisis group.

Table 12 shows Akaike's Information Criterion for three different truncation invariant copulas based on the values of the truncation ($a = 0.1, 0.3, 0.5, 0.7, 0.9$) for two different groups and the three possible pairs (CAD,JPY), (CAD,KRW), and (JPY,KRW) against USD. Table 12 showed that Type II model was better model than Type I and Type III models in terms of the values of the truncation ($a = 0.1, 0.3, 0.5$) because the values of Akaike's Information Criterion for Type II model is smaller than the value of the other two models. And Type III model was better model than other two models in terms of the values of the truncation ($a = 0.7, 0.9$).

To show an asymmetric dependence for financial data, Longin and Solnik [13] and Patton [15] used the difference of the lower tail dependence and the upper tail dependence by showing which tail dependence is greater or lower than the other tail dependence. Table 13 shows the tail dependencies for copulas used by Patton [15] for both the pre-financial-crisis group and the post-financial-crisis group. Looking at Student's t Copula and Symmetrized Joe-Clayton Copula in Table 13, the tail dependencies, lower tail dependence (τ^L) and Upper tail dependence (τ^U), for the Canadian dollar and the Korean Won against the US dollar during October 2008 to May 2011 are a significant increase compared to the ones during January 2006 to Septemb 2008. Three other pairs, (CAD,JPY), (CAD,KRW), and (JPY,KRW) against USD are not different at two different time groups. In terms of Akaike's Information Criterion for nine different copula models (Normal, Clayton, Rotated Clayton, Plackett, Frank, Gumbel, Rotated Gumbel, Student's t and Symmetrized Joe-Clayton) for two different groups, Student's t Copula had the smallest value of Akaike's Information Criterion. It means the Student's t Copula is the best copula of the nine copulas for both the pre-financial-crisis group and the post-financial-crisis group. We found two results that were meaningful based on the data analyzed. First, using our truncation invariant FGM copulas, we can check the directional dependence for two different foreign currency exchange rate data in a given period. Second, the asymmetric dependence for two different foreign currency exchange rate data can be calculated by the tail dependencies of the most commonly used copulas.

5. Conclusion

In this paper, we proposed three different truncation invariant FGM copulas. Using the proposed copulas, we showed the directional dependence with foreign currency exchange data for both the pre-financial-crisis group and the post-financial-crisis group. Furthermore, the asymmetric dependence with foreign currency exchange data was investigated by the tail dependencies with the general copula models. In a future study, we will construct the multivariate pair-copula models and then investigate the directional dependence of the multivariate pair-copula models.

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