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DECOMPOSING 4-REGULAR GRAPHS INTO TRIANGLE-FREE 2-FACTORS*

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Abstract. There is a polynomial algorithm which finds a decomposition of any given 4-regular graph into two triangle-free 2-factors or shows that such a decomposition does not exist.

Key words. decomposition, triangle-free, 2-factor, polynomial algorithm

AMS subject classifications. 05C70, 05C85, 68R10

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Introduction. A 2-factor of a graph G is a subgraph F of G such that any vertex of G is of degree 2 in F. Hell et al. [5] proved that given a set L of natural numbers, recognizing whether a graph G admits a 2-factor F such that no cycle of F is of length from L is NP-hard unless $L \subseteq \{3, 4\}$. On the other hand, an elegant criterion for deciding if a graph possesses an (unrestricted) 2-factor (i.e., $L = \{\emptyset\}$) was given by Tutte [8], and there is a polynomial algorithm to find such a 2-factor (or to determine that none exist) [3]. Hartvigsen [4] proved that the problem of whether a graph G admits a triangle-free 2-factor ($L = \{3\}$) can also be solved in polynomial time.

In this paper we study a modification of this problem. How difficult is it to recognize whether a 4-regular graph can be decomposed into two triangle-free 2-factors? For a long time it had been thought that the general graph decomposition problem of whether a graph H can be written as an edge-disjoint union of copies of a graph G was difficult. This was confirmed when Dolinski and Tarsi [2] proved that unless G is of the form $tK_2 \cup nP_3$, the decomposition problem is NP-complete. In view of their result it is not surprising that there is interest in restricted decomposition problems. The main result of this paper says that there is a polynomial algorithm for finding a decomposition of any given 4-regular graph into two triangle-free 2-factors (or showing that none exists). In fact, to be able to proceed with the induction, we prove a slightly stronger result.

It would be nice to know the complexity of recognizing $2n$-regular graphs which admit a decomposition into two triangle-free n-factors and the complexity of recognizing $2n$-regular graphs which admit a decomposition into n triangle-free 2-factors. We believe that the following is true.

Conjecture. The two decision problems are NP-complete for all $n \geq 3$.

We point out that using a different approach Koudier and Sabidussi recently published [6] an elegant sufficient condition for a 4-regular graph G to have a decomposition into two triangle-free 2-factors. They showed that G possesses such a decomposition if G has at most two essential cut vertices (a cut vertex is essential if it lies on a triangle). We do not see how to prove the result of Koudier and Sabidussi using methods of this paper. On the other hand, it seems to us that a decision pro-

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cedure, which would determine for all 4-regular graphs whether they have a required decomposition, is not within the methods of [6].

Preliminaries. Except for concepts and notation introduced here we use standard graph-theoretical terminology. A graph G is said to be even (odd) if G has an even (odd) number of edges. We will say that a graph G belongs to class $\mathcal{G}(4,2)$ if all vertices of G are either of degree 4 or of degree 2. Let v be a cut vertex of an even graph $G \in \mathcal{G}(4,2)$. Then $d(v) = 4$ and the graph $G - v$ has two components. We will say that v is an even (odd) cut vertex if the parity of the number of edges of both components is even (odd). It turns out that instead of the language of decompositions it is more convenient to use that of coloring edges. A coloring C of the edges of $G \in \mathcal{G}(4,2)$ with two colors will be called *proper* if

(i) each vertex of G is adjacent to the same number of edges in each color,

(ii) both monochromatic components of C are triangle free.

If G admits a proper coloring C we will also say that G admits a *triangle-free splitting*. Clearly, a triangle-free splitting of a 4-regular graph is a decomposition of G into two triangle-free 2-factors. Further, it makes sense to ask whether a graph $G \in \mathcal{G}(4,2)$ has a triangle-free splitting only if G is even. For an edge e of G we denote by e_t the number of triangles of G containing e. Since $G \in \mathcal{G}(4,2)$, $e_t \leq 3$ for any edge e. Let edges x, y, and z form a triangle T. Then we say that T is of type (x_t, y_t, z_t).

Now we state several auxiliary results. We start with a parity lemma.

LEMMA 1. *Let $G \in \mathcal{G}(4,2)$ and C be a coloring of the edges of G by two colors such that each vertex of G of degree 4 is incident with two edges of each color. Let N be the number of vertices v of G of degree 2 such that both edges incident with v get the same color in C. Then the parity of N is the same as the parity of the size of G.*

Proof. Since each vertex of G of degree 4 is incident with two edges in each color, the maximum degree of both monochromatic subgraphs in C is 2. Therefore, each component in both monochromatic subgraphs is either a cycle or a path. Further, a vertex v is a terminal vertex of such a monochromatic path if and only if v is of degree 2 in G and the two edges of G incident with v are of distinct colors in C. Since any path has two terminal vertices there is in G an even number of vertices v of degree 2 with edges incident to v being of distinct colors. Clearly, the parity of the size of G equals the parity of the number of vertices of G of degree 2, which yields that the parity of N equals the parity of the size of G. □

As an immediate consequence of Lemma 1 we get the following lemma.

LEMMA 2. *Let a graph $G \in \mathcal{G}(4,2)$ admit a triangle-free splitting and let a vertex v of G be a cut vertex of G. If S is an odd component of $G - v$ then in any proper coloring C of G both edges incident with v and having the other endpoint in S must be colored with the same color.*

Proof. Consider the odd subgraph H of G formed by the edges of S and two edges incident with v having the other end vertex in S. Let C be a proper coloring of G. Then all the vertices of H of degree 2, except v, have the two edges incident with them colored by different colors. The rest of the proof follows from Lemma 1. □

By a *three-triangle* graph, or simply a TT graph, we mean a graph consisting of three triangles with a common edge; see Fig. 1.

Our final lemma will play a crucial role in proving the main result of the paper. First, we introduce one more notion. Let $T = \{v, w, z\}$ be a triangle of G, $d(v) = 4$, and let x, y be the other vertices of G adjacent to v. Then by the *splitting of v with respect to T* we understand the graph $G' = (G - v) \cup \{v'w, v'z, v''x, v''y\}$, where v'

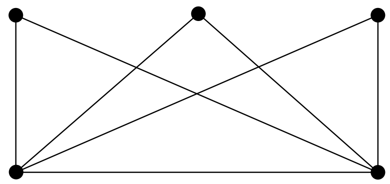


FIG. 1.

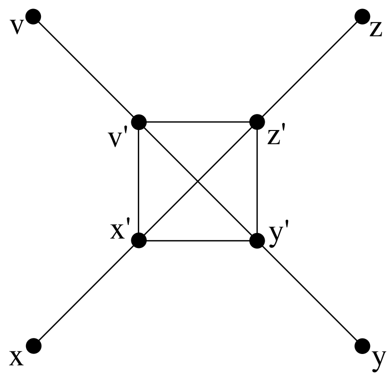


FIG. 2.

and v'' are two new vertices. For the sake of simplicity we view G' as a graph having the same edge set as G.

LEMMA 3. *Let $G \in \mathcal{G}(4,2)$ be an even, connected graph with the following property:* (A) *if T is either a triangle of G of type* $(1,1,1)$ *or an induced TT subgraph of G then one of the vertices of T is a cut vertex of G and the other vertices of T are incident in G only with edges of T. Then G admits a triangle-free splitting.*

Proof. We prove the statement by induction with respect to the number of triangles in G. If there is no triangle in G, then one can get the desired coloring by taking an Eulerian trail of G and alternately coloring its edges. So suppose that there are triangles in G. We will distinguish among five cases. In each of them we construct a graph $G' \in \mathcal{G}(4,2)$ with fewer triangles than G so that each component of G' satisfies the assumptions of the statement. By the induction hypothesis there is a proper coloring C' of G' and we extend (modify) C' to a proper coloring C of G.

Case 1. There are vertices x', y', z', w' in G so that the subgraph induced by them is K_4. The other neighbors of these vertices are x, y, z, w, respectively; see Fig. 2. To get the graph G' we remove a cycle K of length 4 on the vertices x', y', z', and v'. Clearly, for any choice of such a cycle K one cannot create a new induced TT subgraph, and all possible new triangles of type (1,1,1) (this can happen when some of the vertices x, y, z, v are identical) satisfy (A). A choice of K could lead to a disconnected graph G' with possibly odd components to which we cannot apply the induction hypothesis. This could happen only when $H = G–\{x', y', z', v'\}$ is a disconnected graph. However, then H has exactly two components. We choose K in such a way that the remaining two edges of K_4 make G' a connected graph. Clearly, G' has a smaller number of triangles than G and by the induction hypothesis there is a proper coloring C' of G'. There are now two possibilities for coloring the edges of the cycle K alternatively by two colors and one of them, in some cases either of them,

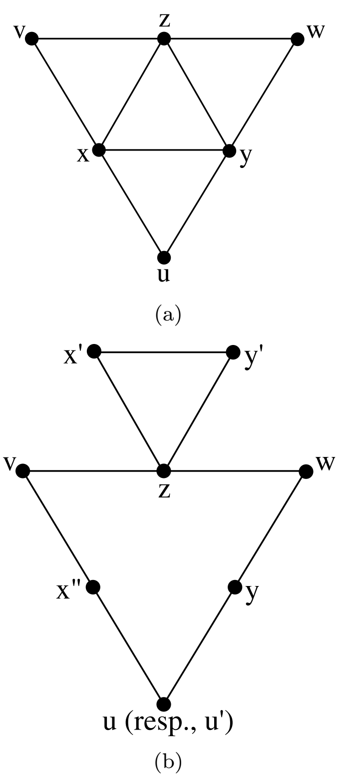


FIG. 3.

gives the extension of C' to a proper coloring of G. Indeed, one must be careful only in the case when some of vertices x, y, z, v coincide and some additional conditions are met by C'. First suppose that, say $x \equiv y$, the edge $x'y'$ is in K, and the edges xx', yy' get the same color in C'. In order not to get a monochromatic triangle start coloring K from the edge $x'y'$ and assign to $x'y'$ the color not assigned to xx'. We note that if also $z \equiv v$ then the edges zz', vv' must have the same color as xx' and yy'. Alternating the coloring of the edges of K guarantees that $z'v'$ gets the same color as $x'y'$ and no monochromatic triangle is created. By the same token one can deal with the cases when three or all four vertices of x, y, z, v coincide.

Case 2. There is in G a triangle $T = \{x, y, z\}$ of type (2,2,2) and G has no K_4. Suppose first that one of the vertices u, v, w (see Fig. 3(a)), say u, is not an odd cut vertex. Then to construct G' we first, if u is of degree 4, split at u with respect to the triangle $\{u, x, y\}$ and then split at x and y with respect to the triangle T; see Fig. 3(b). Clearly, G' satisfies condition (A) and by the induction hypothesis there is a proper coloring of G'. The same coloring (we view G' as having the same edge set as G) provides a proper coloring of G. Indeed, the vertices x, y, u are incident with an equal number of edges in both colors as the vertices x', x'', y', y'',u', u'' have that property. Further, by Lemma 2, the edges zx', zy' are of the same color. This means the edges zv, zw are of the other color and hence the triangles $\{v, x, z\}$ and $\{w, y, z\}$ are not monochromatic. The triangle $\{u, x, y\}$ cannot be monochromatic because the edges ux, uy are of different colors. To finish the proof of this case we suppose that all vertices u, v, w are odd cut vertices. To get the graph G' we first split at w with

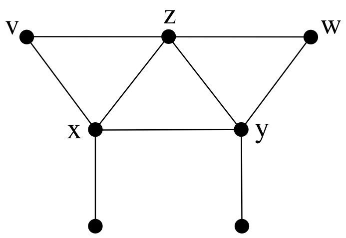


FIG. 4.

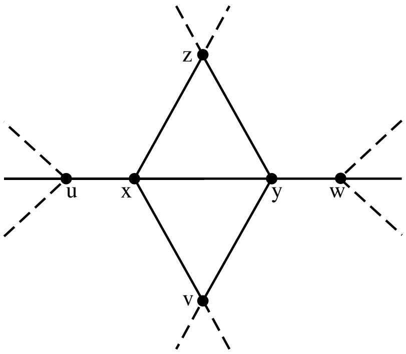
respect to triangle $\{w, y, z\}$, obtaining a graph with two components. Then as before we split at x and y with respect to the triangle T. Finally, we add to vertices w', w'' a loop and then subdivide both loops by four new vertices. Clearly, both components are even and satisfy the assumptions of the statement. We can choose proper colorings of C', C'' of the two components of G' in such a way that the edges of G incident to w' have different colors from the edges incident to w'' (by Lemma 2 the edges incident to w' (to w'') have the same color). Now it is a routine matter to check that a coloring of G given by the restriction of C' and C'' to the edges of G is a proper coloring, since the edges xz, yz, and uy are of the same color and the edges vz, wz, and xy are of the other color. This implies that no triangle on the vertex set from $\{x, y, z, u, v, w\}$ is monochromatic.

Case 3. There is in G a triangle $T = \{x, y, z\}$ of type (1,2,2); see Fig. 4. This case is simpler then the previous one, and we will use an argument very similar to that of the first part of Case 3. To construct G' we split at the vertices x and y with respect to T. By the induction hypothesis there is a proper coloring C' of G'. As before, C' also provides a proper coloring of G (again we view G' as having the same edge set as G).

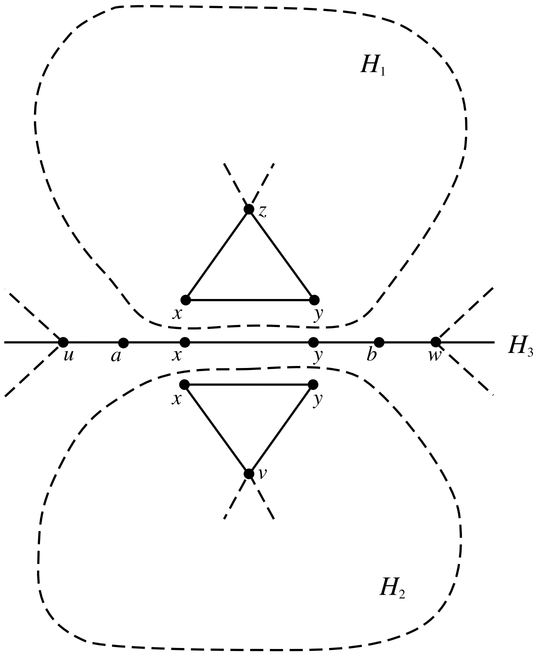
Case 4. There is in G a triangle T=$\{x, y, z\}$ of type (1,1,2) as in Fig. 5(a). The edges depicted by broken lines may or may not be in G.

We assume that edges uv, vw, uz, zw are not in G, for otherwise there would be a triangle of type (1,2,2). Suppose first that both vertices v and z are odd cut vertices. In this case the broken edges incident with v and z are in G. Then we construct three even connected graphs H_1, H_2, H_3 as in Fig. 5(b), a, b being new vertices. Any of them satisfies the assumptions of the statement and has fewer triangles than G. By induction we may take such proper colorings of these graphs where the edge xy has in all three of them the same color. The union of the three colorings provides a proper coloring of G, where the edges ux, yw get the color of the edge xy. Thus we may now assume that z is not an odd cut vertex of G. To get G' first, if z is of degree 4, we split at z with respect to T and then modify the obtained graph (possibly having two even components) as in Fig. 5(c). A proper coloring of G can be obtained from any proper coloring of G' by giving the edge vy the color of the edge va and giving the edge zy the color of zb.

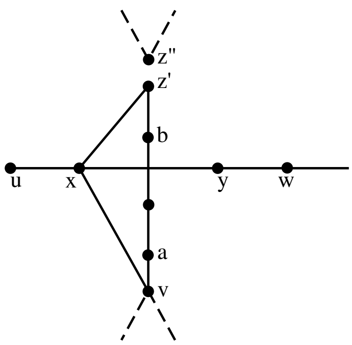
Case 5. There is in G a subgraph T which is either a triangle of type (1,1,1) or an induced TT subgraph of G, satisfying (A), and a vertex x of T is a cut vertex of G. To obtain G' we subdivide the two edges of T incident with x by two new vertices y, z. Since G' has fewer triangles than G and G' satisfies the assumptions of the statement there is a proper coloring C' of G'. To obtain a proper coloring C of G we color the edges of G which do not belong to T by the same color as in C', the edges of T incident with x get the color used in C' for edges xy, zy (by Lemma 2 the



(a)



(b)



(c)

Fig. 5.

two edges have in C' the same color), and the other edges (edge) of T are colored in an obvious way to get a proper coloring. □

The main result. The following statement constitutes the main result of this paper.

THEOREM 1. *The decision problem "Given $G \in \mathcal{G}(4,2)$, does G have a triangle-free splitting?" can be solved in polynomial time.*

Proof. To prove the statement we show that the decision problem can be reduced to a special case of the general f-factor problem; for a detailed discussion of the matter see [7]. For the sake of completeness we recall that the general f-factor problem asks whether there exists a subgraph of a graph $G = (V, E)$, say $F = (V, E')$, $E' \subset E$, such that $d_F(v) \in B_v$, where B_v is a subset of the set $\{1, \ldots, d_G(v)\}$ for all $v \in V$. This problem is NP-complete. However, Cornuéjols [1] showed that if in each B_v all the gaps (if any) have length 1 then the problem can be solved in polynomial time (a set B_v is said to have a gap of length p if there is an integer $k \in B_v$ so that $k+1+p \in B_v$ but no number between these two is in B_v).

If there is no triangle of type (1,1,1) and no induced TT subgraph in G we are done using Lemma 3. So suppose that $T = \{T_1, \ldots, T_n\}, n > 0$, is the set of all triangles of G of type (1,1,1) and of all induced TT subgraphs of G. Denote by $G(T)$ the graph obtained from G by removing all the edges of the subgraphs from T; if $T_i \in T$ is an induced TT subgraph then we also remove from G the two vertices of T_i which are of degree 4 in T_i. Let $O_1, \ldots, O_s$ and $E_1, \ldots, E_r$ be odd and even components of $G(T)$, respectively. A component comprising a single vertex is considered an even component. We construct a bipartite graph B with bipartition $(T', E \cup O)$, where the vertices of $T' = \{t_1, \ldots, t_n\}$ represent the subgraphs from T and the vertices of $E \cup O$=$\{o_1, \ldots, o_s\} \cup \{e_1, \ldots, e_r\}$ represent the components of $G(T)$. Further, $t_i o_j (t_i e_j)$ is an edge of B if an edge of the subgraph T_i is incident with a vertex of O_j (a vertex of E_j). Clearly, $d(t_i) \leq 3$ for $i = 1, \ldots, n$. Now we prove the following.

(*) The graph G has a triangle-free splitting if and only if there is a subgraph F of B such that $d_F(t_i) = 1$ for $i = 1, \ldots, n$, $d_F(o_j)$ is odd for $j = 1, \ldots, s$, and $d_F(e_j)$ is even for $j = 1, \ldots, r$.

First we prove the necessity of the condition. Let C be a proper coloring of G. Each subgraph $T_i \in T$ has three vertices of degree 2 in T_i. Exactly one of them has both edges incident with the vertex colored with the same color. We call this vertex the monochromatic vertex of T_i. We define a subgraph F of B by letting an edge $t_i o_j$ ($t_i e_j$) belong to F if the monochromatic vertex of T_i is in the component O_j (E_j). Since each subgraph T_i has exactly one monochromatic vertex, $d_F(t_i) = 1$. Further, if $v \in O_j$ ($v \in E_j$) is a monochromatic vertex of T_i then v must be of degree 4 in G and the two edges incident to v which are not in T_i must be of the same color. Thus the coloring C restricted to a component K of $G(T)$ provides a coloring of K such that each vertex v of degree 2 in K has both edges incident with it of the same color if and only if v is a monochromatic vertex of a subgraph from T. By Lemma 1 the parity of the number of such vertices coincides with the parity of the size of the component. Hence the parity of the degree of the vertex k in F, where k is the vertex representing the component K, is the same as the parity of the size of K. This finishes the proof of this part of the statement.

Suppose now that F is a subgraph of B as in (*). We show how to construct a proper coloring of G. If $t_i o_j$ ($t_i e_j$) is an edge of F then we choose a vertex of T_i which is in O_j (E_j) to be a monochromatic vertex of the subgraph T_i. Now we take G and split at each vertex of T_i which is of degree 2 in T_i and is not its monochromatic vertex.

Denote the obtained graph as G^*. Components of G^* can be matched in a natural way with the components of $G(T)$. In fact, if D is a component of $G(T)$ then the match of D in G^* is a component D^*, where D^* comprises all the edges of D and the edges of those subgraphs from T which have their monochromatic vertex in D. The size of D^* is even since the parity of the number of subgraphs from T which are "attached" to D to form D^* equals the parity of the size of D. Clearly, each component of G^* satisfies the assumptions of Lemma 3, and therefore each component of G^* has a proper coloring. A coloring C of the edges of G which is the union of proper colorings of components of G^* is a proper coloring. Indeed, if a vertex v was split during the procedure of constructing G^* then both new vertices v_1 and v_2 are of degree 2 and the edges incident with $v_i, i = 1, 2$, are of different colors. Thus, in G, v is incident with two edges of each color. We note that a vertex v which is the monochromatic vertex of T_i is really incident with the edges of T_i of the same color since v is an odd cut vertex in the component of G^* containing the edges of T_i; cf. Lemma 2.

It is obvious that the reductions from the graph G to the graph $G(T)$ and from $G(T)$ to the graph B are polynomial. From the mentioned result of Cornuéjols it also follows that the decision problem of whether B possesses a required subgraph F described in the condition (*) can be solved in polynomial time. Thus our decomposition problem is polynomial. □

Remark. Clearly, the proof of Lemma 3 provides a polynomial algorithm for finding a proper coloring of the components of the graph G^*. Thus, following the proof of Theorem 1, together with the polynomial algorithm for the special case of the general f-factor problem, one can easily obtain a polynomial algorithm for the decomposition problem.

Finally, we show how to construct even graphs from $\mathcal{G}(4, 2)$ which do not have a triangle-free splitting. We will make use of the following theorem. Here, the bipartite graph B and the set T of subgraphs of G are the same as in the proof of Theorem 1.

THEOREM 2. *Let $G \in \mathcal{G}(4, 2)$ admit a triangle-free splitting. Then the number of odd components of B is at most the cardinality of T.*

Proof. By the condition (*), B has a subgraph F such that each vertex of B representing a subgraph of T is of degree 1 in F and each vertex of B representing an odd connectivity component of $B(T)$ is of degree at least 1. Thus the number of odd components of $B(T)$ is at most the cardinality of T. □

Theorem 2 provides a hint toward constructing some even graphs $G \in \mathcal{G}(4, 2)$ which do not have a triangle-free splitting. Let H be a graph having a triangle T of type (1,1,1) such that all vertices of T are odd cut vertices. By Lemma 2, in any proper coloring of H all edges of T must have the same color, which is a contradiction. Thus, H does not admit a triangle-free splitting. Suppose now that u, v are vertices of H of degree 2. Consider an even graph H' consisting of a triangle $T' = \{x, y, z\}$, where x, y are of degree 2 and z is an odd cut vertex. Construct a new graph H'' by identifying vertices x and u, obtaining a new vertex of degree 4, and possibly also identifying y and v. H'' does not admit a proper coloring since this coloring restricted to the edges of H would have to be a proper coloring of H. Thus, in this way, we can construct an infinite class of graphs not admitting a triangle-free splitting.

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