SIAM J. CONTROL OPTIM.
Vol. 39, No. 4, pp. 1182–1203

PONTRYAGIN'S PRINCIPLE FOR LOCAL SOLUTIONS OF CONTROL PROBLEMS WITH MIXED CONTROL-STATE CONSTRAINTS*

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Abstract. This paper deals with optimal control problems of semilinear parabolic equations with pointwise state constraints and coupled integral state-control constraints. We obtain necessary optimality conditions in the form of a Pontryagin's minimum principle for local solutions in the sense of L^p, $p \le +\infty$.

Key words. optimal control, nonlinear boundary controls, semilinear parabolic equations, state constraints, Pontryagin's minimum principle, unbounded controls

AMS subject classifications. 49K20, 35K20

PII. S0363012998345627

1. Introduction.

Let T be a positive number, Ω be a bounded open subset in $\mathbb{R}^N$ $(N \ge 2)$ with a Lipschitz boundary Γ, and q, σ, and $\bar\sigma$ be numbers satisfying

$$q > N/2 + 1 \quad \text{and} \quad \sigma > \bar\sigma > N + 1.$$

Consider the parabolic system

$$\text{(1.1)} \quad \frac{\partial y}{\partial t} + Ay + f(x,t,y) = 0 \text{ in } Q,\ \frac{\partial y}{\partial n_A} + g(s,t,y,v) = 0 \text{ on } \Sigma,\ y(0) = y_0 \text{ in } \Omega$$

(where $Q := \Omega\times]0,T[$, $\Sigma := \Gamma\times]0,T[$, $T > 0$, v is a boundary control, $y_0 \in C(\overline{\Omega})$, A is a second order elliptic operator) and the following control and state constraints:

$$v \in \widetilde{V}_{ad} := \{v \in L^\sigma(\Sigma) \mid v(s,t) \in V(s,t) \quad \text{for almost every (a.e.) } (s,t) \in \Sigma\},$$

$$\text{(1.2)} \qquad \Phi(y) \in \mathcal{C},$$

$$\text{(1.3)} \qquad \begin{aligned} &\int_\Sigma \Psi_i(s,t,y(s,t),v(s,t))\,dsdt = 0, \quad 1 \le i \le m_0,\\ &\int_\Sigma \Psi_i(s,t,y(s,t),v(s,t))\,dsdt \le 0, \quad m_0+1 \le i \le m. \end{aligned}$$

(V is a measurable set-valued mapping from Σ with closed and nonempty values in $\mathcal{P}(\mathbb{R}^k)$, the set of all subsets of $\mathbb{R}^k$, $\Psi = (\Psi_1, \dots, \Psi_m)$, is a function with values in

*Received by the editors October 5, 1998; accepted for publication (in revised form) June 14, 2000; published electronically December 5, 2000.
http://www.siam.org/journals/sicon/39-4/34562.html

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$\mathbb{R}^m$, Φ is a continuous mapping from $C(\overline{D})$ into $C(\overline{D})$, $\mathcal{C} \subset C(\overline{D})$, $\overline{D}$ is a nonempty compact subset of $\overline{Q}$.) Let us consider the following class of optimal control problems:

$$\text{(P)} \qquad \inf\{J(y,v) \mid y \in W(0,T) \cap C(\overline{Q}), v \in V_{ad}, (y,v) \text{ satisfies } (1.1), (1.2), (1.3)\},$$

where V_{ad} is a subset of $\widetilde{V}_{ad}$ (to be stated precisely later), and the cost functional is defined by

$$J(y,v) = \int_Q F(x,t,y(x,t))\,dx\,dt + \int_\Sigma G(s,t,y(s,t),v(s,t))\,ds\,dt + \int_\Omega L(x,y(x,T))dx.$$

We are mainly interested in optimality conditions for such problems, in the form of Pontryagin's principles. The existence of optimal solutions for (P) is a priori supposed.

In the case where $V_{ad} \equiv \widetilde{V}_{ad}$, and $\widetilde{V}_{ad}$ is a bounded subset in $L^\infty(\Sigma)$ (the case of bounded controls), Pontryagin's principles for (P) have been obtained in [3, 9, 16, 17, 11, 25, 26, 4]. In this case the Pontryagin's principle is of the form

$$H_\Sigma(\bar{y}, \bar{v}, \bar{p}, \bar{\nu}, \bar{\lambda}) = \min_{v \in \widetilde{V}_{ad}} H_\Sigma(\bar{y}, v, \bar{p}, \bar{\nu}, \bar{\lambda}), \tag{1.4}$$

where

$$H_\Sigma(y,v,p,\nu,\lambda) = \int_\Sigma [\nu G(s,t,y,v) - pg(s,t,y,v) + \lambda \Psi(s,t,y,v)]\; dsdt,$$

$(\bar{y}, \bar{v})$ is an optimal solution, $\bar{\lambda}$ is a multiplier associated with the mixed control-state constraints (1.3), $\bar{\nu}$ is a multiplier of the cost functional, $\bar{p}$ is the adjoint state (the multiplier associated with the state constraints (1.2) only intervenes in the adjoint equation satisfied by $\bar{p}$). Notice that (1.4) can also be replaced by a pointwise Pontryagin's principle.

Observe that in [9, 16, 17, 11, 4] there is no mixed control-state constraint. Results with mixed control-state constraint are obtained in [2].

As explained in [8, p. 595] and in [21], the case of unbounded controls, that is, when $V_{ad} \equiv \widetilde{V}_{ad}$ is not bounded in $L^\infty(\Sigma)$, leads to some difficulties. In this case Pontryagin's principles are more recent results [8, 10, 21].

Now consider a control set of the form

$$V_{ad} = \{v \in \widetilde{V}_{ad} \mid v \text{ satisfies } (1.6)\} \tag{1.5}$$

with

$$\begin{aligned} &\int_\Sigma h_i(s,t,v(s,t))\,dsdt = 0, \; 1 \le i \le \ell_0, \\ &\int_\Sigma h_i(s,t,v(s,t))\,dsdt \le 0, \; \ell_0 + 1 \le i \le \ell, \end{aligned} \tag{1.6}$$

where $h = (h_1, \ldots, h_\ell)$ is a function with values in $\mathbb{R}^\ell$. Obviously control constraints (1.6) can be considered as a particular case of mixed control-state constraints (1.3). The corresponding Pontryagin's principle for the problem (P), with the control set V_{ad} defined by (1.5), may be written in the form

$$H_\Sigma(\bar{y}, \bar{v}, \bar{p}, \bar{\nu}, \bar{\lambda}, \hat{\lambda}) = \min_{v \in \widetilde{V}_{ad}} H_\Sigma(\bar{y}, v, \bar{p}, \bar{\nu}, \bar{\lambda}, \hat{\lambda}), \tag{1.7}$$

where $\hat{\lambda}$ is a multiplier for the control constraints (1.6).

The novelty of our paper is the following Pontryagin's principle for the problem (P) (Theorem 2.1):

$$H_\Sigma(\bar{y}, \bar{v}, \bar{p}, \bar{\nu}, \bar{\lambda}) = \min_{v \in V_{ad}} H_\Sigma(\bar{y}, v, \bar{p}, \bar{\nu}, \bar{\lambda}), \tag{1.8}$$

when the control set V_{ad} is defined by (1.5). Let us insist on the fact that the minimum in (1.7) is stated with controls in $\widetilde{V}_{ad}$, while in (1.8) it is stated with controls in V_{ad}. Since V_{ad} takes integral control constraints of isoperimetric type into account, the result is of a different nature. As an application, we are able to prove a Pontryagin's principle (Corollary 2.2) for local solutions of (P) (local in the $L^\sigma(\Sigma)$-sense). To our knowledge, this result is completely new. Control problems for semilinear elliptic equations, with integral control constraints, are considered in [5], but the Pontryagin's principle for local solutions was not obtained there. Also we can deduce from (1.8) the classical pointwise Pontryagin's principle for local solutions in $L^\sigma(\Sigma)$ of the previous control problems; see Corollaries 2.3 and 2.4.

Let us finally mention that we deal with parabolic equations of the form (1.1), where the coefficients of the operator A are not regular, and where the nonlinear terms $f(x,t,\cdot)$ and $g(s,t,\cdot)$ are neither Lipschitz nor monotone with respect to y. When $g(s,t,\cdot,v)$ is Lipschitz and monotone such an equation is studied in [4] for bounded controls. For unbounded controls, when $g(s,t,\cdot,v)$ is neither Lipschitz nor monotone, but when the coefficients of A are time independent and regular, (1.1) is studied in [20, 21] by means of estimates on analytic semigroups. Here we combine these different difficulties. Equation (1.1) and the adjoint state equation are studied in section 3.

Our main results are stated in section 2. Section 4 is devoted to the study of the metric space of the controls and to the existence of diffuse perturbations of controls. These perturbations are the key for the proof of Pontryagin's principle, which is done in section 5.

2. Main results. We set $\overline{\Omega}_0 = \overline{\Omega} \times \{0\}$ and $\overline{\Omega}_T = \overline{\Omega} \times \{T\}$. For every $1 \le \tau \le \infty$, the usual norms of the spaces $L^\tau(\Omega)$, $L^\tau(\Gamma)$, $L^\tau(Q)$, $L^\tau(\Sigma)$ will be denoted by $\|.\|_{\tau,\Omega}$, $\|.\|_{\tau,\Gamma}$, $\|.\|_{\tau,Q}$, $\|.\|_{\tau,\Sigma}$. For every $t > 0$, we define the norm $\|.\|_{Q(t)}$ by $\|y\|^2_{Q(t)} := \|y\|^2_{L^2(0,t;H^1(\Omega))} + \|y\|^2_{L^\infty(0,t;L^2(\Omega))}$. The Hilbert space $W(0,T;H^1(\Omega),(H^1(\Omega))') = \{y \in L^2(0,T;H^1(\Omega)) \mid \frac{dy}{dt} \in L^2(0,T;(H^1(\Omega))')\}$, endowed with its usual norm, will be denoted by $W(0,T)$. We denote by V_{ad} the set of admissible controls

$$V_{ad} := \{v \in \widetilde{V}_{ad} \mid v \text{ satisfies } (1.6)\}.$$

2.1. Assumptions.

(A1) The operator A is defined by

$$Ay(x,t) = -\sum_{i=1}^N D_i\left(\sum_{j=1}^N (a_{ij}(x,t)D_jy(x,t)) + a_i(x,t)y(x,t)\right) + \sum_{i=1}^N (b_i(x,t)D_iy(x,t)),$$

the coefficients a_{ij} belong to $L^\infty(Q)$, a_i and b_i belong to $L^{2q}(Q)$, and

$$\Lambda|\xi|^2 \le \sum_{i,j=1}^N a_{ij}(x,t)\xi_j\xi_i \quad \text{for all } \xi \in \mathbb{R}^N \text{ and for a.e. } (x,t) \in Q \text{ with } \Lambda > 0. \tag{2.1}$$

We make the following assumptions on f, g, F, G, L, Φ, Ψ.

(A2) For every $y \in \mathbb{R}$, $f(\cdot, y)$ is measurable on Q. For almost every $(x,t) \in Q$, $f(x,t,\cdot)$ is of class C^1 on $\mathbb{R}$. The following estimates hold:

$$|f(x,t,0)| \leq M_1(x,t), \qquad C_0 \leq f'_y(x,t,y) \leq M_1(x,t)\eta(|y|),$$

where M_1 belongs to $L^q(Q)$, η is a nondecreasing function from $\mathbb{R}^+$ to $\mathbb{R}^+$ and $C_0 \in \mathbb{R}$. (We have denoted by f'_y the partial derivative of f with respect to y, throughout what follows we adopt the same kind of notation for other functions.)

(A3) For every $(y,v) \in \mathbb{R}^2$, $g(\cdot, y, v)$ is measurable on Σ. For almost every $(s,t) \in \Sigma$ and every $v \in \mathbb{R}$, $g(s,t,\cdot,v)$ is of class C^1 on $\mathbb{R}$. For almost every $(s,t) \in \Sigma$, $g(s,t,\cdot)$ and $g'_y(s,t,\cdot)$ are continuous on $\mathbb{R} \times \mathbb{R}$. The following estimates hold:

$$|g(s,t,0,v)| \leq M_2(s,t) + \Lambda_1|v|, \qquad C_0 \leq g'_y(s,t,y,v) \leq (M_2(s,t) + \Lambda_1|v|)\eta(|y|),$$

where M_2 belongs to $L^\sigma(\Sigma)$, $\Lambda_1 > 0$, C_0 and η are as in (A2).

(A4) For every $y \in \mathbb{R}$, $L(\cdot, y)$ is measurable on Ω. For almost every $x \in \Omega$, $L(x,\cdot)$ is of class C^1 on $\mathbb{R}$. The following estimate holds:

$$|L(x,y)| + |L'_y(x,y)| \leq M_3(x)\eta(|y|),$$

where $M_3 \in L^1(\Omega)$, η is as in (A2).

(A5) For every $y \in \mathbb{R}$, $F(\cdot, y)$ is measurable on Q. For almost every $(x,t) \in Q$, $F(x,t,\cdot)$ is of class C^1 on $\mathbb{R}$. The following estimate holds:

$$|F(x,t,y)| + |F'_y(x,t,y)| \leq M_4(x,t)\eta(|y|),$$

where $M_4 \in L^1(Q)$, η is as in (A2).

(A6) For every $(y,v) \in \mathbb{R}^2$, $G(\cdot, y, v)$ is measurable on Σ. For almost every $(s,t) \in \Sigma$ and every $v \in \mathbb{R}$, $G(s,t,\cdot,v)$ is of class C^1 on $\mathbb{R}$. For almost every $(s,t) \in \Sigma$, $G(s,t,\cdot)$ and $G'_y(s,t,\cdot)$ are continuous on $\mathbb{R} \times \mathbb{R}$. The following estimates hold:

$$-M_5(s,t) - \Lambda_1|v|^{\bar{\sigma}} \leq G(s,t,0,v) \leq M_5(s,t) + \Lambda_1|v|^\sigma,$$

$$|G'_y(s,t,y,v)| \leq (M_5(s,t) + \Lambda_1|v|^{\bar{\sigma}})\eta(|y|),$$

where $M_5 \in L^1(\Sigma)$, Λ_1 and η are as in (A3).

(A7) The function $h = (h_1, \ldots, h_\ell)$ is a Carathéodory function from $\Sigma \times \mathbb{R}$ into $\mathbb{R}^\ell$ satisfying

$$|h_i(s,t,v)| \leq M_5(s,t) + \Lambda_1|v|^{\bar{\sigma}} \quad \text{for } i = 1, \ldots, \ell_0,$$

$$-M_5(s,t) - \Lambda_1|v|^{\bar{\sigma}} \leq h_i(s,t,v) \leq M_5(s,t) + \Lambda_1|v|^\sigma \quad \text{for } i = \ell_0 + 1, \ldots, \ell;$$

Λ_1 and M_5 are the same as above.

(A8) The function $\Psi = (\Psi_1, \ldots, \Psi_m)$ is a Carathéodory function from $\Sigma \times \mathbb{R}^2$ into $\mathbb{R}^m$. For almost every $(s,t) \in \Sigma$ and every $v \in \mathbb{R}$, $\Psi(s,t,\cdot,v)$ is of class C^1 on $\mathbb{R}$.

For almost every $(s,t) \in \Sigma$, $\Psi'_y(s,t,\cdot)$ is continuous on $\mathbb{R}\times\mathbb{R}$. The following estimates hold:

$$|\Psi_i(s,t,0,v)| \le M_5(s,t) + \Lambda_1 |v|^{\bar\sigma} \quad \text{for } i = 1,\dots,m_0,$$

$$-M_5(s,t) - \Lambda_1|v|^{\bar\sigma} \le \Psi_i(s,t,0,v) \le M_5(s,t) + \Lambda_1 |v|^{\sigma} \quad \text{for } i = m_0+1,\dots,m,$$

$$|\Psi'_{iy}(s,t,y,v)| \le (M_5(s,t) + \Lambda_1|v|^{\bar\sigma})\eta(|y|) \quad \text{for } i = 1,\dots,m,$$

where Λ_1, M_5, η are as before. We also suppose that the function $\Phi : C(\overline{D}) \to C(\overline{D})$ is of class C^1, and that $\mathcal{C} \subset C(\overline{D})$ is a closed convex subset of finite codimension in $C(\overline{D})$, where $\overline{D}$ is a compact subset of $\overline{Q}$.

2.2. Statement of the main result. We define the boundary Hamiltonian function by

$$H_\Sigma(y,v,p,\nu,\lambda) = \int_\Sigma [\nu G(s,t,y,v) - pg(s,t,y,v) + \lambda\Psi(s,t,y,v)]\, dsdt$$

for every $(y,v,p,\nu,\lambda) \in C(\overline{Q}) \times L^\sigma(\Sigma) \times L^{\sigma'}(\Sigma) \times \mathbb{R}^{1+m}$. (Here $\lambda = (\lambda^1,\dots,\lambda^m)$, $\lambda\Psi(s,t,y,v) = \sum_{i=1}^m \lambda^i \Psi_i(s,t,y,v)$. Throughout the paper we adopt the same kind of notation for scalar products in $\mathbb{R}^m$.)

THEOREM 2.1. *If* (A1)–(A8) *are fulfilled and if* $(\bar y, \bar v)$ *is a solution of* (P), *then there exist* $\bar p \in L^1(0,T;W^{1,1}(\Omega))$, $\bar\nu \in \mathbb{R}$, $\bar\lambda \in \mathbb{R}^m$, $\bar\mu \in \mathcal{M}(\overline{D})$ (*the space of Radon measures on* $\overline{D}$) *such that*

$$(\bar\nu,\bar\lambda,\bar\mu) \ne 0, \quad \bar\nu \ge 0, \; \textit{for } m_0+1 \le i \le m, \; \bar\lambda_i \ge 0, \; \bar\lambda_i \int_\Sigma \Psi_i(s,t,\bar y,\bar v)\, dsdt = 0, \tag{2.2}$$

$$\langle \bar\mu, z - \Phi(\bar y)\rangle_{\overline{D}} \le 0 \quad \textit{for all } z \in \mathcal{C}, \tag{2.3}$$

$$\begin{cases} -\dfrac{\partial \bar p}{\partial t} + A^*\bar p + f'_y(x,t,\bar y)\bar p = \bar\nu F'_y(x,t,\bar y) + [\Phi'(\bar y)^*\bar\mu]|_Q & \textit{in } Q, \\ \dfrac{\partial \bar p}{\partial n_{A^*}} + g'_y(s,t,\bar y,\bar v)\bar p = \bar\nu G'_y(s,t,\bar y,\bar v) + \bar\lambda\Psi'_y(s,t,\bar y,\bar v) + [\Phi'(\bar y)^*\bar\mu]|_\Sigma & \textit{on } \Sigma, \\ \bar p(T) = \bar\nu L'_y(x,\bar y(T)) + [\Phi'(\bar y)^*\bar\mu]|_{\overline{\Omega}_T} & \textit{on } \overline{\Omega}, \end{cases} \tag{2.4}$$

$$\bar p \in L^{\delta'}(0,T;W^{1,d'}(\Omega)) \qquad \textit{for every } (\delta,d) \textit{ satisfying } \frac{N}{2d} + \frac{1}{\delta} < \frac{1}{2}, \tag{2.5}$$

$$H_\Sigma(\bar y,\bar v,\bar p,\bar\nu,\bar\lambda) = \min_{v\in V_{ad}} H_\Sigma(\bar y, v, \bar p, \bar\nu, \bar\lambda), \tag{2.6}$$

where $[\Phi'(\bar y)^*\bar\mu]|_Q$ *is the restriction of* $[\Phi'(\bar y)^*\bar\mu]$ *to* Q, $[\Phi'(\bar y)^*\bar\mu]|_\Sigma$ *is the restriction of* $[\Phi'(\bar y)^*\bar\mu]$ *to* Σ, *and* $[\Phi'(\bar y)^*\bar\mu]|_{\overline{\Omega}_T}$ *is the restriction of* $[\Phi'(\bar y)^*\bar\mu]$ *to* $\overline{\Omega}_T$, $[\Phi'(\bar y)^*\bar\mu]$ *is the Radon measure on* $\overline{D}$ *defined by* $z \longmapsto \langle\bar\mu, \Phi'(\bar y)z\rangle_{\mathcal{M}(\overline{D})\times C(\overline{D})}$ *for* $z \in C(\overline{D})$, $\langle\cdot,\cdot\rangle_{\overline{D}}$

denotes the duality pairing between $\mathcal{M}(\overline{D})$ *and* $C(\overline{D})$, A^* *is the formal adjoint of* A, *that is,*

$$A^*y(x,t) = -\sum_{i=1}^{N} D_i \left(\sum_{i=1}^{N} (a_{ji}(x,t)D_jy(x,t)) + b_i(x,t)y(x,t) \right) + \sum_{i=1}^{N} a_i(x,t)D_iy(x,t).$$

2.3. Pontryagin's principles for local solutions. By definition, a local solution $(\bar{y},\bar{v})$ of (P) in $L^\sigma(\Sigma)$ is a solution of the problem

$$(\mathrm{P}_{\bar{v},\epsilon})\quad \inf\{J(y,v) \mid y \in C(\overline{Q}),\ v \in \widetilde{V}_{ad},\quad (y,v) \text{ satisfies (1.1)–(1.3)},\ \|\bar{v}-v\|_{\sigma,\Sigma} \le \epsilon\}$$

for some $\epsilon > 0$. The following Pontryagin's principle for local solutions of (P) is a direct consequence of Theorem 2.1.

COROLLARY 2.2. *If* (A1)–(A8) *are fulfilled and if* $(\bar{y},\bar{v})$ *is a solution of* $(\mathrm{P}_{\bar{v},\epsilon})$, *there then exist* $\bar{p} \in L^1(0,T;W^{1,1}(\Omega))$, $\bar{\nu} \in \mathbb{R}$, $\bar{\lambda} \in \mathbb{R}^m$, $\bar{\mu} \in \mathcal{M}(\overline{D})$ *satisfying* (2.2)–(2.5) *along with*

$$H_\Sigma(\bar{y},\bar{v},\bar{p},\bar{\nu},\bar{\lambda}) = \min_{v\in V_{ad}, \|\bar{v}-v\|_{\sigma,\Sigma}\le\epsilon} H_\Sigma(\bar{y},v,\bar{p},\bar{\nu},\bar{\lambda}).$$

As a consequence of this corollary we can get the classical pointwise Pontryagin principle for a local solution in $L^\sigma(\Sigma)$ of the control problem

$$(\tilde{\mathrm{P}})\qquad \inf\{J(y,v) \mid y \in W(0,T) \cap C(\overline{Q}), v \in \widetilde{V}_{ad}, (y,v) \text{ satisfies } (1.1), (1.2), (1.3)\}.$$

COROLLARY 2.3. *If* (A1)–(A8) *are fulfilled and if* $(\bar{y},\bar{v})$ *is a local solution of* $(\tilde{\mathrm{P}})$ *in* $L^\sigma(\Sigma)$, *there then exist* $\bar{p} \in L^1(0,T;W^{1,1}(\Omega))$, $\bar{\nu} \in \mathbb{R}$, $\bar{\lambda} \in \mathbb{R}^m$, $\bar{\mu} \in \mathcal{M}(\overline{D})$ *satisfying* (2.2)–(2.5) *along with*

$$\mathcal{H}_\Sigma(s,t,\bar{y}(s,t),\bar{v}(s,t),\bar{p}(s,t),\bar{\nu},\bar{\lambda}) = \min_{\xi\in V(s,t)} \mathcal{H}_\Sigma(s,t,\bar{y}(s,t),\xi,\bar{p}(s,t),\bar{\nu},\bar{\lambda})$$

for almost all $(s,t) \in \Sigma$, *where*

$$\mathcal{H}_\Sigma(s,t,y,\xi,p,\nu,\lambda) = \nu G(s,t,y,\xi) - pg(s,t,y,\xi) + \lambda\Psi(s,t,y,\xi).$$

Proof. The pointwise Pontryagin's principle stated in the corollary may be derived from the integral Pontryagin's principle of Corollary 2.2 by using the same construction as in [21, proof of Theorem 2.1]. The idea in the proof of [21] is to construct a pointwise perturbation v_n of $\bar{v}$ such that $\lim_{(s,t)\to(s_0,t_0)} v_n(s,t) = \xi$, $\lim_n \mathcal{L}^N(\{(s,t) \in \Sigma \mid v_n(s,t) \ne \bar{v}(s,t)\}) = 0$, where $\xi \in V(s_0,t_0)$, $(s_0,t_0) \in \Sigma$, $\mathcal{L}^N$ denotes the N-dimensional Lebesgue measure. We obtain the pointwise Pontryagin's principle by replacing v by v_n in the integral Pontryagin's principle of Corollary 2.2, by dividing by $\mathcal{L}^N(\{(s,t) \in \Sigma \mid v_n(s,t) \ne \bar{v}(s,t)\}) \ne 0$, and by passing to the limit when n tends to infinity. The only difference with [21] is that v_n must satisfy $\|\bar{v}-v_n\|_{\sigma,\Sigma} \le \epsilon$. Due to the condition $\lim_n \mathcal{L}^N(\{(s,t) \in \Sigma \mid v_n(s,t) \ne \bar{v}(s,t)\}) = 0$, it is clear that this condition will be realized for n big enough. ∎

Let us observe that integral control constraints can be studied in the framework of the problem $(\tilde{\mathrm{P}})$. Indeed, the mixed constraints (1.3) can include the integral control constraints. Then Corollary 2.3 provides a Pontryagin's principle for problems with integral constraints on the control and the state, even with mixed integral constraints, as well as pointwise constraints on the control and state too. The corresponding result is stated in the following corollary.

COROLLARY 2.4. *If* (A1)–(A8) *are fulfilled and if* $(\bar y,\bar v)$ *is a local solution of* $(\tilde{\mathrm{P}})$ *in* $L^\sigma(\Sigma)$, *there then exist* $\bar p\in L^1(0,T;W^{1,1}(\Omega))$, $\bar\nu\in\mathbb{R}$, $\bar\lambda\in\mathbb{R}^m$, $\hat\lambda\in\mathbb{R}^\ell$, *and* $\bar\mu\in\mathcal{M}(\overline{D})$ *such that*

$$(\bar\nu,\bar\lambda,\hat\lambda,\bar\mu)\neq 0,\quad \bar\nu\ge 0,\quad \langle\bar\mu, z-\Phi(\bar y)\rangle_{\overline{D}}\le 0 \qquad \text{for all } z\in\mathcal{C},$$

$$\bar\lambda_i\int_\Sigma \Psi_i(s,t,\bar y,\bar v)\,dsdt=0 \;\; \text{for} \;\; 1\le i\le m,\quad \bar\lambda_i\ge 0 \;\; \text{for} \;\; m_0+1\le i\le m,$$

$$\hat\lambda_i\int_\Sigma h_i(s,t,\bar v)\,dsdt=0 \;\; \text{for} \;\; 1\le i\le \ell,\quad \hat\lambda_i\ge 0 \;\; \text{for} \;\; \ell_0+1\le i\le \ell,$$

$$\begin{cases} -\dfrac{\partial\bar p}{\partial t}+A^*\bar p+\bar f'_y\,\bar p=\bar\nu\bar F'_y+[\Phi'(\bar y)^*\bar\mu]|_Q & \text{in } Q,\\[2mm] \dfrac{\partial\bar p}{\partial n_{A^*}}+\bar g'_y\,\bar p=\bar\nu\bar G'_y+\bar\lambda\bar\Psi'_y+\hat\lambda\bar h+[\Phi'(\bar y)^*\bar\mu]|_\Sigma & \text{on } \Sigma,\\[2mm] \bar p(T)=\bar\nu L'_y(x,\bar y(T))+[\Phi'(\bar y)^*\bar\mu]|_{\overline{\Omega}_T} & \text{on } \overline{\Omega}, \end{cases}$$

where $\bar f'_y$ *stands for* $\bar f'_y(x,t,\bar y)$, $\bar G'_y$ *for* $\bar G'_y(s,t,\bar y,\bar v)$, *and the same convention is used for other functions. Also, the following pointwise Pontryagin's principle holds:*

$$\mathcal{H}_\Sigma(s,t,\bar y(s,t),\bar v(s,t),\bar p(s,t),\bar\nu,\bar\lambda,\hat\lambda)=\min_{\xi\in V(s,t)}\mathcal{H}_\Sigma(s,t,\bar y(s,t),\xi,\bar p(s,t),\bar\nu,\bar\lambda,\hat\lambda)$$

for almost all $(s,t)\in\Sigma$, *where*

$$\mathcal{H}_\Sigma(s,t,y,\xi,p,\nu,\bar\lambda,\hat\lambda)=\nu G(s,t,y,\xi)-pg(s,t,y,\xi)+\bar\lambda\Psi(s,t,y,\xi)+\hat\lambda h(s,t,\xi).$$

3. State and adjoint equations.

3.1. State equation. Existence and regularity results for (1.1) and (2.4) rely on estimates in $C(\overline{Q})$ for solutions of linear equations of the form

$$(3.1)\quad \frac{\partial y}{\partial t}+Ay+ay=\phi-\mathrm{div}\xi \quad \text{in } Q,\quad \frac{\partial y}{\partial n_A}+by=\psi \quad \text{on } \Sigma,\quad y(0)=y_0 \quad \text{in } \Omega.$$

If assumption (A1) is satisfied, if $(a,\phi)\in L^q(Q)\times L^q(Q)$, $(b,\psi)\in L^{\bar\sigma}(\Sigma)\times L^{\bar\sigma}(\Sigma)$, the existence of a unique solution in $C([0,T];L^2(\Omega))\cap L^2([0,T];H^1(\Omega))$ for (3.1) is proved in [12, Chapter 3, Theorem 5.1] when $\xi\equiv 0$. The result can be extended to (3.1) by the same method if the support of ξ is compact in Q and if ξ belongs to $L^\delta(0,T,(L^d(\Omega))^N)$ with $d>1,\delta>1,N/2d+1/\delta<1/2$. Recall that a weak solution in $L^2(0,T;H^1(\Omega))\cap C([0,T];L^2(\Omega))$ of (3.1) is a function $y\in L^2(0,T;H^1(\Omega))\cap C([0,T];L^2(\Omega))$ satisfying

$$\int_Q\left(-y\frac{\partial z}{\partial t}+\sum_{i,j}a_{ij}D_jyD_iz+\sum_i(a_iyD_iz+b_iD_iyz)+ayz\right)dxdt+\int_\Sigma byz\,dsdt$$
$$=\int_Q\left[\phi z+\sum_i\xi_iD_iz\right]dxdt+\int_\Sigma\psi z\,dsdt+\int_\Omega y(0)z(0)\,dx$$

for every $z \in C^1(\overline{Q})$ such that $z(\cdot,T)=0$ on $\overline{\Omega}$. For linear equations with Dirichlet boundary conditions

$$\frac{\partial y}{\partial t}+Ay+ay=\phi-\mathrm{div}\xi \quad \text{in } Q, \qquad y=\psi \quad \text{on } \Sigma, \qquad y(0)=y_0 \quad \text{in } \Omega,$$

estimates of the form

$$\|y\|_{L^\infty(Q)} \le C\left(\|\phi\|_{q,Q}+\|\psi\|_{\infty,\Sigma}+\sum_i \|\xi_i\|_{d,\delta,\Omega}+\|y_0\|_{C(\overline{\Omega})}\right) \tag{3.2}$$

are obtained in [12, Chapter 3, Theorem 7.1] for $d>1, \delta>1, N/2d+1/\delta<1/2$. In this estimate the constant C depends on $T,\Omega,N,\Lambda,q,\bar\sigma,\delta,d,\sum_i\|a_i^2\|_{q,Q},\sum_i\|b_i^2\|_{q,Q}$, but also on $\|a\|_{q,Q}$. The case of Robin boundary conditions is considered in [4] to study nonlinear equations of the form (1.1) when the function $g(s,t,\cdot,v)$, in the boundary condition, is monotone and Lipschitz, and when the boundary control v is bounded [4, Theorem 5.1]. The case when the function $g(s,t,\cdot,v)$ in (1.1) is neither Lipschitz nor monotone (g satisfies (A3)), and when the control v belongs to $L^{\bar\sigma}(\Sigma)$, but when the coefficients of the operator A are regular and independent of the time variable, is studied in [19]. Estimates in $C(\overline{Q})$ are obtained by semigroup techniques and comparison principles [19, Proposition 3.3 and Theorem 3.1]. Here we emphasize the fact that assumptions on the operator A are minimal (bounded leading coefficients, unbounded coefficients of order zero), that we deal with nonhomogeneous boundary conditions, and that source terms in the domain and in the boundary conditions are unbounded.

THEOREM 3.1. *Under assumptions* (A1)–(A3), *if* $v \in L^{\bar\sigma}(\Sigma)$, *then* (1.1) *admits a unique weak solution* y_v *in* $W(0,T)\cap C(\overline{Q})$. *This solution obeys*

$$||y_v||_{C(\overline{Q})} \le C_1(||v||_{\bar\sigma,\Sigma}+1),$$

where $C_1=C_1(T,\Omega,N,C_0,q,\bar\sigma)$. *Moreover, the mapping* $v \longmapsto y_v$ *is continuous from* $L^{\bar\sigma}(\Sigma)$ *into* $C(\overline{Q})$.

Proof. The proof relies on Theorem 3.2 (see [19]). □

THEOREM 3.2. *Suppose that* (A1) *is satisfied,* $(a,\phi)\in L^q(Q)\times L^q(Q)$, $(b,\psi)\in L^{\bar\sigma}(\Sigma)\times L^{\bar\sigma}(\Sigma)$, *and* ξ *belongs to* $(\mathcal{D}(Q))^N$. *If in addition* $a\ge C_0$ *a.e. in* Q *and* $b\ge C_0$ *a.e. in* Σ *(for some* $C_0\in\mathbb{R}$*), then the unique weak solution* y *of* (3.1) *belongs to* $C(\overline{Q})$ *and satisfies the following estimate:*

$$\|y\|_{C(\overline{Q})} \le C_2\left(\|\phi\|_{q,Q}+\|\psi\|_{\bar\sigma,\Sigma}+\sum_i \|\xi_i\|_{L^\delta(0,T;L^d(\Omega))}+\|y_0\|_{C(\overline{\Omega})}\right),$$

where $d>1$, $\delta>1$ *satisfy* $N/2d+1/\delta<1/2$ *and the constant* C_2 *only depends on* T, Ω, N, C_0, Λ, q, $\bar\sigma$, δ, d, $\sum_i\|a_i^2\|_{q,Q}$, $\sum_i\|b_i^2\|_{q,Q}$.

Remark 3.3. Notice that the constant C_2 does not depend on $\|a\|_{q,Q}$ and $\|b\|_{\bar\sigma,\Sigma}$. As in [12] (see the above estimate (3.2)), the assumption $a\ge C_0$ may be dropped out, and in this case the constant C_2 must be replaced by a constant also depending on $\|a\|_{q,Q}$. But the corresponding estimate cannot be used to treat nonlinear equations of the form (1.1).

Proof. To prove this theorem, we need only to establish the L^∞-estimate; the rest is classical. We prove the L^∞-estimate by using the so-called truncation method as in [12, Chapter 3, proof of Theorem 7.1]. If y is a weak solution of (3.1), then we have

$$\int_\Omega [y(x,t)z(x,t) - y(x,0)z(x,0)]\,dx$$

$$+\int_0^t\int_\Omega \left[-y\frac{\partial z}{\partial t} + \sum_{i,j} a_{ij}D_jyD_iz + \sum_i (a_iyD_iz + b_iD_iyz) + ayz\right] dxd\tau$$

$$+\int_0^t\int_\Gamma byz\,dsd\tau = \int_0^t\int_\Omega \left[\phi z + \sum_i \xi_iD_iz\right] dxd\tau + \int_0^t\int_\Gamma \psi z\,dsd\tau$$

for every $t \in [0,T]$ and every $z \in W_2^{1,1}(Q)$. We establish only the upper bound for y. (The lower bound can be obtained in the same way.) For $k \geq 0$ we set $y^k(x,t) = \max(y(x,t)-k,0)$. By using Steklov averagings, as in [12, p. 183], we prove that

$$\frac{1}{2}\int_\Omega [y^k(x,t)^2 - y^k(x,0)^2]\,dx \tag{3.3}$$

$$+\int_0^t\int_\Omega \left[\sum_{i,j} a_{ij}D_jy^kD_iy^k + \sum_i (a_iyD_iy^k + b_iD_iy^ky^k) + ayy^k\right] dxd\tau$$

$$+\int_0^t\int_\Gamma byy^k\,dsd\tau = \int_0^t\int_\Omega \left[\phi y^k + \sum_i \xi_iD_iy^k\right] dxd\tau + \int_0^t\int_\Gamma \psi y^k\,dsd\tau$$

for every $t \in]0,T]$. Thus, it follows that

$$\frac{1}{2}\int_\Omega y^k(x,t)^2\,dx + \int_0^t\!\!\int_\Omega \left[\sum_{i,j} a_{ij}D_jy^kD_iy^k + (a - C_0 + \Lambda)yy^k\right] dxd\tau \tag{3.4}$$

$$+\int_0^t\!\!\int_\Gamma (b - C_0)yy^k\,dsd\tau = -\int_0^t\!\!\int_\Omega \left[\sum_i (a_iyD_iy^k + b_iD_iy^ky^k) + (C_0 - \Lambda)yy^k\right] dxd\tau$$

$$-\int_0^t\!\!\int_\Gamma C_0yy^k\,dsd\tau + \int_0^t\!\!\int_\Omega \left[\phi y^k + \sum_i \xi_iD_iy^k\right] dxd\tau + \int_0^t\int_\Gamma \psi y^k\,dsd\tau$$

for every $k > \tilde{k} := \|y_0\|_{C(\overline{\Omega})}$. Since $a - C_0 \geq 0$ a.e. in Q, $b - C_0 \geq 0$ a.e. on Σ, and $yy^k \geq (y^k)^2$ a.e. in Q, with (2.1) we obtain

$$\|y^k(t)\|_{2,\Omega}^2 + 2\Lambda\|y^k\|_{L^2(0,t;H^1(\Omega))}^2 \tag{3.5}$$

$$\leq -2\int_0^t\!\!\int_\Omega \left[\sum_i (a_iyD_iy^k + b_iD_iy^ky^k) + (C_0 - \Lambda)yy^k\right] dxd\tau$$

$$-2\int_0^t\!\!\int_\Gamma C_0yy^k\,dsd\tau + 2\int_0^t\!\!\int_\Omega \left[\phi y^k + \sum_i \xi_iD_iy^k\right] dxd\tau + 2\int_0^t\!\!\int_\Gamma \psi y^k\,dsd\tau$$

for every $k > \tilde{k}$. Set $A_k(t) = \{x \in \Omega \mid y(x,t) > k\}$, $B_k(t) = \{s \in \Gamma \mid y(s,t) > k\}$, $Q_k(t) = \{(x,\tau) \in \Omega\times]0,t[\mid y(x,\tau) > k\}$, $\Sigma_k(t) = \{(s,\tau) \in \Gamma\times]0,t[\mid y(s,\tau) > k\}$. We estimate the terms in the right-hand side of (3.5) by means of Young's inequality and we obtain

$$\|y^k(t)\|_{2,\Omega}^2 + \Lambda\|y^k\|_{L^2(0,t;H^1(\Omega))}^2$$

$$\leq \frac{3}{\Lambda}\int_0^t\int_{A_k(\tau)}\left[\sum_i((a_iy)^2 + (b_iy^k)^2) + (C_0-\Lambda)^2y^2\right] dxd\tau + \frac{3K^2}{\Lambda}\int_0^t\int_{B_k(\tau)} C_0^2y^2\,dsd\tau$$

$$+2\int_0^t\int_{A_k(\tau)}\left[|\phi||y^k| + \sum_i|\xi_i||D_iy^k|\right] dxd\tau + 2\int_0^t\int_{B_k(\tau)}|\psi||y^k|\,dsd\tau,$$

where $K > 0$ satisfies $\|\varphi\|_{2,\Gamma} \leq K\|\varphi\|_{H^1(\Omega)}$ for all $\varphi \in H^1(\Omega)$. Since $y = y^k + k$ in $A_k(\tau)$ and $B_k(\tau)$ for a.e. τ, it follows that

$$\|y^k(t)\|_{2,\Omega}^2 + \Lambda\|y^k\|_{L^2(0,t;H^1(\Omega))}^2$$

$$\leq \frac{6}{\Lambda}\int_0^t\int_{A_k(\tau)}\left[\sum_i(a_i^2+b_i^2) + (C_0-\Lambda)^2\right]((y^k)^2+k^2)\,dxd\tau$$

$$+\frac{6K^2}{\Lambda}\int_0^t\int_{B_k(\tau)} C_0^2((y^k)^2+k^2)\,dsd\tau + 2\int_0^t\int_{A_k(\tau)}\left[|\phi||y^k| + \sum_i|\xi_i||D_iy^k|\right] dxd\tau$$

$$+2\int_0^t\int_{B_k(\tau)}|\psi||y^k|\,dsd\tau$$

for every $t \in [0,T]$ and every $k > \tilde{k}$. With Hölder's inequality we have

(3.6)
$$\|y^k(t)\|_{2,\Omega}^2 + \Lambda\|y^k\|_{L^2(0,t;H^1(\Omega))}^2$$

$$\leq \left(K_1(|Q_k(t)|^{\frac{1}{q'}} + |Q_k(t)|) + K_2|\Sigma_k(t)|\right)k^2$$

$$+K_1(|Q_k(t)|^{\frac{2}{N+2}} + |Q_k(t)|^{\frac{1}{q'}-\frac{N}{N+2}})\|y^k\|_{\frac{2(N+2)}{N},\Omega\times]0,t[}^2$$

$$+2\|\phi\|_{q,Q}|Q_k(t)|^{\frac{1}{q'}-\frac{N}{2(N+2)}}\|y^k\|_{\frac{2(N+2)}{N},\Omega\times]0,t[}$$

$$+K_2|\Sigma_k(t)|^{\frac{1}{N+1}}\|y^k\|_{\frac{2(N+1)}{N},\Gamma\times]0,t[}^2 + 2\|\psi\|_{\bar{\sigma},\Sigma}|\Sigma_k(t)|^{\frac{1}{\bar{\sigma}'}-\frac{N}{2(N+1)}}\|y^k\|_{\frac{2(N+1)}{N},\Gamma\times]0,t[}$$

$$+\frac{\Lambda}{2}\|y^k\|_{L^2(0,t;H^1(\Omega))}^2 + \frac{2K_3}{\Lambda}\left(\int_0^t|A_k(\tau)|^{\frac{\delta(d-2)}{d(\delta-2)}}\,d\tau\right)^{\frac{\delta-2}{\delta}},$$

where

$$K_1 = \frac{6}{\Lambda}\left[\sum_i(\|a_i^2\|_{q,Q} + \|b_i^2\|_{q,Q}) + (C_0-\Lambda)^2\right],$$

$$K_2 = \frac{6K^2}{\Lambda} C_0^2 \quad \text{and} \quad K_3 = \sum_i \|\xi_i\|^2_{L^\delta(0,T;L^d(\Omega))},$$

$|Q_k(t)|$ denotes the $(N+1)$-dimensional Lebesgue measure of $Q_k(t)$, $|\Sigma_k(t)|$ denotes the N-dimensional Lebesgue measure of $\Sigma_k(t)$, and $|A_k(\tau)|$ denotes the N-dimensional Lebesgue measure of $A_k(\tau)$. Notice that $\frac{N(d-2)}{2d} + \frac{\delta-2}{\delta} > \frac{N}{2}$. Then there exists $\tilde r > \frac{2\delta}{\delta-2} > 2$ such that $\frac{N(d-2)}{2d} + \frac{\delta-2}{\delta} > \frac{N}{2}\frac{\tilde r(\delta-2)}{2\delta} > \frac{N}{2}$. For such an $\tilde r$ we have $\frac{1}{\tilde r}(\frac{N}{2}\frac{\delta}{\delta-2}\frac{d-2}{d} + 1) > \frac{N}{4}$. We define $r > 2$ by $\frac{1}{r} = \frac{1}{\tilde r}\frac{\delta}{\delta-2}\frac{d-2}{d} < \frac{d-2}{2d} < \frac{1}{2}$ and we obtain $\frac{N}{2r} + \frac{1}{\tilde r} > \frac{N}{4}$. Thus the imbedding from $L^2(0,t;H^1(\Omega)) \cap C([0,t];L^2(\Omega))$ into $L^{\tilde r}(0,t;L^r(\Omega))$ is continuous; see [12, p. 75]. Observe that

$$|Q_k(t)|^{\frac{2}{N+2}} + |Q_k(t)|^{\frac{1}{q'}-\frac{N}{N+2}} \le (t|\Omega|)^{\frac{2}{N+2}} + (t|\Omega|)^{\frac{1}{q'}-\frac{N}{N+2}}, \qquad |\Sigma_k(t)|^{\frac{1}{N+1}} \le (t|\Gamma|)^{\frac{1}{N+1}}.$$

Let us choose $\bar t > 0$ small enough to have

$$K_1((\bar t|\Omega|)^{\frac{2}{N+2}} + (\bar t|\Omega|)^{\frac{1}{q'}-\frac{N}{N+2}})\|y\|^2_{\frac{2(N+2)}{N},\Omega\times]0,\bar t[} + K_2(\bar t|\Gamma|)^{\frac{1}{N+1}}\|y\|^2_{\frac{2(N+1)}{N},\Gamma\times]0,\bar t[} \le \frac{1}{2}\min\left(1,\frac{\Lambda}{2}\right)\|y\|^2_{Q(\bar t)} \tag{3.7}$$

for every $y \in L^2(0,\bar t;H^1(\Omega)) \cap C([0,\bar t];L^2(\Omega))$. Then from (3.6) and imbedding theorems, it follows that

$$\begin{aligned} &\nu(\|y^k\|_{\frac{2(N+2)}{N},\Omega\times]0,\bar t[} + \|y^k\|_{\frac{2(N+1)}{N},\Gamma\times]0,\bar t[} + \|y^k\|_{L^{\tilde r}(0,\bar t;L^r(\Omega))}) \le \\ &\le \|y\|_{Q(\bar t)} \le K_4\left(|Q_k(\bar t)|^{\frac{1}{2q'}} + |Q_k(\bar t)|^{\frac12} + |\Sigma_k(\bar t)|^{\frac12}\right)k \\ &+K_4\left(|Q_k(\bar t)|^{\frac{1}{q'}-\frac{N}{2(N+2)}} + |\Sigma_k(\bar t)|^{\frac{1}{\bar\sigma'}-\frac{N}{2(N+1)}}\right) + K_4\left(\int_0^{\bar t}|A_k(\tau)|^{\frac{\delta(d-2)}{d(\delta-2)}}\,d\tau\right)^{\frac{\delta-2}{2\delta}}, \end{aligned} \tag{3.8}$$

for $k > \tilde k$, where $\nu > 0$ depends on Λ, and where K_4 depends on K_1, K_2, K_3, $\|\phi\|_{q,Q}$, $\|\psi\|_{\bar\sigma,\Sigma}$, and Λ. Now, we set $\theta(k) = |Q_k(\bar t)|^{\frac{N}{2(N+2)}} + |\Sigma_k(\bar t)|^{\frac{N}{2(N+1)}} + (\int_0^{\bar t}|A_k(\tau)|^{\frac{\tilde r}{r}}d\tau)^{\frac{1}{\tilde r}}$. Observe that, for every $\ell \ge k \ge 0$, we have $y^k \ge \ell - k$ a.e. in $Q_\ell(\bar t)$, a.e. on $\Sigma_\ell(\bar t)$, and a.e. in $A_\ell(\tau)$ for a.e. $\tau \in]0,\bar t[$; therefore

$$(\ell-k)\theta(\ell) \le \|y^k\|_{\frac{2(N+2)}{N},\Omega\times]0,\bar t[} + \|y^k\|_{\frac{2(N+1)}{N},\Gamma\times]0,\bar t[} + \|y^k\|_{L^{\tilde r}(0,\bar t;L^r(\Omega))}. \tag{3.9}$$

Taking $k = 0$ in the above inequality, with the definition of the function θ we first obtain $\ell\theta(\ell) \le K_0$ for all $\ell \ge 0$, where $K_0 = \|y\|_{\frac{2(N+2)}{N},\Omega\times]0,\bar t[} + \|y\|_{\frac{2(N+1)}{N},\Gamma\times]0,\bar t[} + \|y\|_{L^{\tilde r}(0,\bar t;L^r(\Omega))}$. In particular, for $\ell = K_0$, this implies $\theta(K_0) \le 1$. On the other hand, (3.8) and (3.9) give

$$\begin{aligned} (\ell-k)\theta(\ell) \le{}& \frac{K_4}{\nu}\left(|Q_k(\bar t)|^{\frac{1}{2q'}} + |Q_k(\bar t)|^{\frac12} + |\Sigma_k(\bar t)|^{\frac12}\right. \\ &\left.+|Q_k(\bar t)|^{\frac{1}{q'}-\frac{N}{2(N+2)}} + |\Sigma_k(\bar t)|^{\frac{1}{\bar\sigma'}-\frac{N}{2(N+1)}}\right)k + \frac{K_4}{\nu}\left(\int_0^{\bar t}|A_k(\tau)|^{\frac{\delta(d-2)}{d(\delta-2)}}d\tau\right)^{\frac{\delta-2}{2\delta}} \end{aligned} \tag{3.10}$$

for all $\ell \geq k > \max(K_0, 1, \tilde{k})$. Set

$$\alpha_1 = \frac{N+2}{Nq'}, \quad \alpha_2 = \frac{N+1}{N\bar{\sigma}'}, \quad \alpha_3 = \tilde{r}\frac{\delta-2}{2\delta}, \quad \alpha = \min(\alpha_1, \alpha_2, \alpha_3),$$

and observe that $\alpha > 1$. Since $\theta(K_0) \leq 1$, and since θ is a nonincreasing function, we also have $|Q_k(\bar{t})| \leq 1$, $|\Sigma_k(\bar{t})| \leq 1$, and $\int_0^{\bar{t}} |A_k(\tau)|^{\frac{\tilde{r}}{r}} d\tau \leq 1$ for all $k \geq K_0$. Thus it follows that

$$|Q_k(\bar{t})|^{\frac{1}{2q'}} + |\Sigma_k(\bar{t})|^{\frac{1}{2}} + |Q_k(\bar{t})|^{\frac{1}{2}}$$

$$+|Q_k(\bar{t})|^{\frac{1}{q'} - \frac{N}{2(N+2)}} + |\Sigma_k(\bar{t})|^{\frac{1}{\bar{\sigma}'} - \frac{N}{2(N+1)}} + \left(\int_0^{\bar{t}} |A_k(\tau)|^{\frac{\tilde{r}}{r}} \, d\tau \right)^{\frac{\delta-2}{2\delta}} \leq 3\theta(k)^\alpha.$$

From (3.10), we deduce

$$(\ell - k)\theta(\ell) \leq K_5 \theta(k)^\alpha k \tag{3.11}$$

for every $\ell \geq k > \max(K_0, 1, \tilde{k})$. With the same arguments as in [12, Chapter 3, p. 186], still using (3.8), we finally obtain

$$\|y\|_{\infty,Q} \leq K_6, \tag{3.12}$$

where K_6 depends not only on $T, \Omega, N, C_0, \Lambda, q, \bar{\sigma}, \delta, d, \sum_i \|a_i^2\|_{q,Q}, \sum_i \|b_i^2\|_{q,Q}$, but also on K_0, $\|y_0\|_{C(\overline{\Omega})}$, $\|\phi\|_{q,Q}$, $\|\psi\|_{\bar{\sigma},\Sigma}$, and $\sum_i \|\xi_i\|^2_{L^\delta(0,T;L^d(\Omega))}$. The constant K_6 depends on $K_0 = \|y\|_{\frac{2(N+2)}{N},\Omega\times]0,\bar{t}[} + \|y\|_{\frac{2(N+1)}{N},\Gamma\times]0,\bar{t}[} + \|y\|_{L^{\tilde{r}}(0,\bar{t};L^r(\Omega))} \leq C\|y\|_{Q(\bar{t})}$. By using the same trick as in (3.4), we can obtain an estimate of $\|y\|_{Q(\bar{t})}$ depending on T, Ω, N, C_0, Λ, q, $\bar{\sigma}$, δ, d, $\sum_i \|a_i^2\|_{q,Q}$, $\sum_i \|b_i^2\|_{q,Q}$, $\|y_0\|_{C(\overline{\Omega})}$, $\sum_i \|\xi_i\|^2_{L^\delta(0,T;L^d(\Omega))}$, $\|\phi\|_{q,Q}$, and $\|\psi\|_{\bar{\sigma},\Sigma}$, but independent of $\|a\|_{q,Q}$ and $\|b\|_{\bar{\sigma},\Sigma}$. Since (3.1) is linear, the estimate given in Theorem 3.2 can be easily deduced from (3.12). □

3.2. Adjoint equation. Let (a, b) be in $L^q(Q) \times L^{\bar{\sigma}}(\Sigma)$ with $a \geq C_0$ and $b \geq C_0$. We consider the terminal boundary value problem

$$-\frac{\partial p}{\partial t} + A^* p + ap = \mu_Q \text{ in } Q, \quad \frac{\partial p}{\partial n_{A^*}} + bp = \mu_\Sigma \text{ on } \Sigma, \quad p(T) = \mu_{\overline{\Omega}_T} \text{ on } \overline{\Omega}, \tag{3.13}$$

where $\mu = \mu_Q + \mu_\Sigma + \mu_{\overline{\Omega}_T}$ is a bounded Radon measure on $\overline{Q} \setminus \overline{\Omega}_0$, μ_Q is the restriction of μ to Q, μ_Σ is the restriction of μ to Σ, and $\mu_{\overline{\Omega}_T}$ is the restriction of μ to $\overline{\Omega}_T$. A function $p \in L^1(0, T; W^{1,1}(\Omega))$ is a weak solution of (3.13) if

$$ap \in L^1(Q), \quad bp \in L^1(\Sigma), \quad a_i D_i p \in L^1(Q), \quad \text{and} \quad b_i p \in L^1(Q) \quad \text{for } i = 1, \ldots, N,$$

$$\int_Q \left(p\frac{\partial y}{\partial t} + \sum_{i,j} a_{ji} D_j p D_i y + \sum_i (a_i D_i p y + b_i p D_i y) + apy \right) dxdt + \int_\Sigma bpy \, dsdt$$
$$= \int_{\overline{Q} \setminus \overline{\Omega}_0} y d\mu(x,t) \quad \text{for every } y \in C^1(\overline{Q}) \text{ satisfying } y(x,0) = 0 \text{ on } \overline{\Omega}.$$

As for elliptic equations [23], it is well known that (3.13) may admit more than one solution. However, uniqueness is guaranteed if we look for solutions of (3.13) satisfying some Green formula. (Such uniqueness results are proved in [1] for elliptic equations and in [4] for parabolic equations.)

THEOREM 3.4. *Let* μ *be in* $\mathcal{M}_b(\overline{Q}\setminus\overline{\Omega}_0)$ *and let* (a,b) *be in* $L^q(Q)\times L^{\bar\sigma}(\Sigma)$ *satisfying* $a\ge C_0$ *a.e. in* Q, $b\ge C_0$ *a.e.* Σ, *for some* $C_0\in\mathbb{R}$. *Equation* (3.13) *admits a unique solution* p *in* $L^1(0,T;W^{1,1}(\Omega))$ *satisfying*

$$\int_Q p\left\{\frac{\partial y}{\partial t}+Ay+ay\right\}dxdt+\int_\Sigma p\left\{\frac{\partial y}{\partial n_A}+by\right\}dsdt=\langle y,\mu\rangle_{C_b(\overline{Q}\setminus\overline{\Omega}_0)\times\mathcal{M}_b(\overline{Q}\setminus\overline{\Omega}_0)}$$

for every $y\in\{y\in W(0,T)\cap C(\overline{Q})\mid \frac{\partial y}{\partial t}+Ay\in L^q(Q),\ \frac{\partial y}{\partial n_A}\in L^{\bar\sigma}(\Sigma),\ y(x,0)=0$ *on* $\overline{\Omega}\}$. *Moreover* p *belongs to* $L^{\delta'}(0,T;W^{1,d'}(\Omega))$ *for every* $\delta>2$, $d>2$ *satisfying* $\frac{N}{2d}+\frac{1}{\delta}<\frac12$ *and we have*

$$\|p\|_{L^{\delta'}(0,T;W^{1,d'}(\Omega))}\le C_4(\delta,d)\|\mu\|_{\mathcal{M}_b(\overline{Q}\setminus\overline{\Omega}_0)},$$

where $C_4(\delta,d)=C_4(T,\Omega,N,C_0,q,\bar\sigma,\delta,d,\|a_i\|_{L^{2q}(Q)},\|b_i\|_{L^{2q}(Q)})$, *but* C_4 *is independent of* a *and* b.

Proof. Due to Theorem 3.2, the proof of Theorem 3.3 follows the lines of the proofs of Theorem 6.3 in [4] and of Theorem 4.2 in [18]. Since we improve the results given in [4, 18], we sketch the main points of the proof. Let $(h_n)_n$ be a sequence in $C_c(Q)$ (the space of continuous functions with compact support in Q), $(k_n)_n$ be a sequence in $C_c(\Sigma)$, and $(\ell_n)_n$ be a sequence in $C(\overline{\Omega})$ such that

$$\|h_n\|_{L^1(Q)}=\|\mu_Q\|_{\mathcal{M}_b(Q)},\quad \|k_n\|_{L^1(\Sigma)}=\|\mu_\Sigma\|_{\mathcal{M}_b(\Sigma)},\ \|\ell_n\|_{L^1(\Omega)}=\|\mu_{\overline{\Omega}_T}\|_{\mathcal{M}(\overline{\Omega}_T)},$$

$$\lim_n\int_Q h_n\phi dxdt=\langle\phi,\mu_Q\rangle_{C_b(Q)\times\mathcal{M}_b(Q)},$$

$$\lim_n\int_\Sigma k_n\phi dsdt=\langle\phi,\mu_\Sigma\rangle_{C_b(\Sigma)\times\mathcal{M}_b(\Sigma)},$$

$$\lim_n\int_\Omega \ell_n\phi dx=\langle\phi,\mu_{\bar\Omega_T}\rangle_{C(\overline{\Omega}_T)\times\mathcal{M}(\overline{\Omega}_T)}$$

for every $\phi\in C(\overline{Q})$. Let $(p_n)_n$ be the sequence in $W(0,T)$ defined by

$$-\frac{\partial p_n}{\partial t}+Ap_n+ap_n=h_n\quad\text{in } Q,\qquad \frac{\partial p_n}{\partial n_A}+bp_n=k_n\quad\text{on }\Sigma,\qquad p_n(T)=\ell_n\quad\text{in }\Omega.$$

Due to Theorem 3.2, and by using the same arguments as in [4, 18], we can prove that there exists a constant $C_5(\delta,d)=C_5(T,\Omega,N,C_0,q,\bar\sigma,\delta,d,\|a_i\|_{L^{2q}(Q)},\|b_i\|_{L^{2q}(Q)})$ such that

$$\|p_n\|_{L^{\delta'}(0,T;W^{1,d'}(\Omega))}\le C_5(\delta,d)\|\mu\|_{\mathcal{M}_b(\bar Q\setminus\bar\Omega_0)}$$

for every (δ,d) satisfying $\frac{N}{2d}+\frac{1}{\delta}<\frac12$. Since $q>\frac{N}{2}+1$ and $\bar\sigma>N+1$, there exist (δ_1,d_1), (δ_2,d_2), (δ_3,d_3) satisfying $\frac{N}{2d_i}+\frac{1}{\delta_i}<\frac12$ for $i=1,2,3$, such that $\delta_1'\ge q'$, $d_1'^*=\frac{Nd_1'}{N-d_1'}\ge q'$, $\delta_2'\ge\bar\sigma'$, $\frac{(N-1)d_2d_2'}{(N-1)d_2-d_2'}\ge\bar\sigma'$, $\delta_3'\ge(2q)'$, and $d_3'\ge(2q)'$. Therefore

$$\|p_n\|_{L^{q'}(Q)}\le C\|p_n\|_{L^{\delta'_1}(0,T;W^{1,d'_1}(\Omega))}\le CC_5(\delta_1,d_1)\|\mu\|_{\mathcal{M}_b(\bar Q\setminus\bar\Omega_0)},$$

$$\|p_n\|_{L^{\bar\sigma'}(\Sigma)} \le C\|p_n\|_{L^{\delta_2'}(0,T;W^{1,d_2'}(\Omega))} \le CC_5(\delta_2, d_2)\|\mu\|_{\mathcal{M}_b(\bar Q\setminus\bar\Omega_0)},$$

$$\|p_n\|_{L^{(2q)'}(0,T;W^{1,(2q)'}(\Omega))} \le C\|p_n\|_{L^{\delta_3'}(0,T;W^{1,d_3'}(\Omega))} \le CC_5(\delta_3, d_3)\|\mu\|_{\mathcal{M}_b(\bar Q\setminus\bar\Omega_0)}.$$

Then, there exist a subsequence, still indexed by n, and p such that $(p_n)_n$ converges to p for the weak-star topology of $L^{\delta'}(0,T;W^{1,d'}(\Omega))$ for every (δ, d) satisfying $\frac{N}{2d} + \frac{1}{\delta} < \frac{1}{2}$. By passing to the limit in the variational formulation satisfied by $(p_n)_n$, we prove that p is a solution of (3.13). The uniqueness can be proved as in [1, 4]. □

4. Technical results.

4.1. Metric space of controls. To apply the Ekeland variational principle, we have to define a metric space of controls in such a way that the mapping $v \longmapsto y_v$ be continuous from this metric space to $C(\overline{Q})$. Due to Theorem 3.1, this continuity condition will be realized if convergence in the metric space of controls implies convergence in $L^{\bar\sigma}(\Sigma)$. In the case where boundary controls are bounded, convergence in (V_{ad}, d) (where d is the so-called Ekeland's distance) implies convergence in $L^{\bar\sigma}(\Sigma)$. This condition is no longer true for unbounded controls; see [10, p. 227]. To overcome this difficulty, we proceed as in [5] and we define a new metric space in the following way. Let $\tilde v$ be in V_{ad}. (In section 5, $\tilde v$ will be an optimal boundary control that we want to characterize.) For $0 < M < \infty$, we define the set

$$V_{ad}(\tilde v, M) = \{v \in V_{ad} \mid \|v - \tilde v\|_{\sigma,\Sigma} \le M\}.$$

We endow the set $V_{ad}(\tilde v, M)$ with the Ekeland metric

$$d(v_1, v_2) = \mathcal{L}^N(\{(s,t) \in \Sigma \mid v_1(s,t) \ne v_2(s,t)\}).$$

PROPOSITION 4.1. *Let $\tilde v$ be in V_{ad}. Let $M > 0$ and $\{(v_n)_n, v\} \subset V(\tilde v, M)$. If $(v_n)_n$ tends to v in $(V(\tilde v, M), d)$, then $(v_n)_n$ tends to v in $L^{\bar\sigma}(\Sigma)$.*

Proof. Since $1 \le \bar\sigma < \sigma$, the proof is immediate if we notice that we have

$$\int_\Sigma |v - v_n|^{\bar\sigma}\, ds \le \|v - v_n\|^{\bar\sigma}_{\sigma,\Sigma}(d(v_n, v))^{\frac{\sigma-\bar\sigma}{\sigma}} \le (2M)^{\bar\sigma}(d(v_n, v))^{\frac{\sigma-\bar\sigma}{\sigma}}. \quad □$$

PROPOSITION 4.2. *For every $M > 0$, we have that*
- (i) $(V_{ad}(\tilde v, M), d)$ *is a complete metric space;*
- (ii) *the mapping which associates y_v with v is continuous from $(V_{ad}(\tilde v, M), d)$ into $C(\overline{Q})$;*
- (iii) *the mappings $v \to J(y_v, v)$ and $v \to \int_\Sigma \Psi_i(s, t, y_v, v)\, dsdt$ are continuous (respectively, lower semicontinuous) on $(V_{ad}(\tilde v, M), d)$ for $1 \le i \le m_0$ (respectively, $m_0 + 1 \le i \le m$).*

Proof. Claims (i) and (ii) are proved in [5], for control problems of elliptic equations; this proof can be repeated here with the obvious modifications. Contrary to [4], [21], the mapping $v \to J(y_v, v)$ is not necessarily continuous on the space of "truncated controls" endowed with the Ekeland metric. We can prove only a lower semicontinuity result. This result is stated in [5, Proposition 3.1] under the additional assumption that $G(s, t, y, \cdot)$ is convex. In fact we can prove the same result without this convexity assumption. Let $(v_n)_n$ be a sequence converging to v in $(V_{ad}(\tilde v, M), d)$. From Proposition 4.1 and Theorem 3.1 we know that $(v_n)_n$ converges to v in $L^{\bar\sigma}(\Sigma)$

and $(y_{v_n})_n$ converges to y_v uniformly on $\overline{Q}$. With assumption (A6), with Fatou's lemma, and with Lebesgue's dominated convergence theorem we have

$$\liminf_n \int_\Sigma G(s,t,0,v_n)\,dsdt \geq \int_\Sigma G(s,t,0,v)\,dsdt,$$

$$\lim_n \int_\Sigma \int_0^1 G'_y(s,t,\theta y_{v_n},v_n)y_{v_n} d\theta\,dsdt = \int_\Sigma \int_0^1 G'_y(s,t,\theta y_v,v)y_v d\theta\,dsdt.$$

Therefore we obtain

$$\liminf_n \int_\Sigma G(s,t,y_{v_n},v_n)\,dsdt = \liminf_n \int_\Sigma G(s,t,0,v_n)\,dsdt$$
$$+\lim_n \int_\Sigma \int_0^1 G'_y(s,t,\theta y_{v_n},v_n)y_{v_n} d\theta\,dsdt$$
$$\geq \int_\Sigma G(s,t,0,v)\,dsdt + \int_\Sigma \int_0^1 G'_y(s,t,\theta y_v,v)y_v d\theta\,dsdt = \int_\Sigma G(s,t,y_v,v)\,dsdt.$$

Following the same ideas, we can prove the continuity (for $1 \leq i \leq m_0$) or the lower semicontinuity (for $m_0+1 \leq i \leq m$) of $v \to \int_\Sigma \Psi_i(s,t,y_v,v)\,dsdt$. ◻

4.2. Existence of diffuse perturbations. Let $\tilde{v}$ be an admissible control, and let v_1 and v_2 be in $V_{ad}(\tilde{v},M)$. A diffuse perturbation of v_1 by v_2 is a family of functions $(v_\rho)_{\rho>0}$ defined by

$$v_\rho(s,t) = \begin{cases} v_1(s,t) & \text{on } \Sigma \setminus E_\rho, \\ v_2(s,t) & \text{on } E_\rho, \end{cases}$$

where E_ρ is a measurable subset of Σ satisfying some conditions. Such perturbations are used to derive Pontryagin's principles from the Ekeland variational principle. In the case of bounded controls (when $V_{ad}(\tilde{v},M) \equiv V_{ad}$) the use of this kind of perturbations goes back to Yao [24, 25] and Li [13] (see also [17, 11, 26, 14]). Some variants have been developed in [4] for bounded controls, and in [21] for unbounded controls. In [5] we have investigated the case of unbounded controls with integral control constraints. Here we prove that the diffuse perturbations defined in [21] may be extended to derive a Pontryagin's principle for problems with integral coupled control-state constraints. Before proving the existence of such diffuse perturbations let us state an auxiliary lemma analogous to Lemma 3.2 of [5].

LEMMA 4.3. *Let ρ be such that $0<\rho<1$. For every $v_1, v_2, v_3 \in V_{ad}$, there exists a sequence of measurable sets $(E^n_\rho)_n$ in Σ such that*

$$\mathcal{L}^N(E^n_\rho) = \rho\mathcal{L}^N(\Sigma), \tag{4.1}$$

$$\int_{E^n_\rho} |v_i - v_3|^\sigma\,dsdt = \rho\int_\Sigma |v_i - v_3|^\sigma\,dsdt \quad \textit{for } i=1,2, \tag{4.2}$$

$$\int_{E^n_\rho} h(s,t,v_i)\,dsdt = \rho\int_\Sigma h(s,t,v_i)\,dsdt \quad \textit{for } i=1,2, \tag{4.3}$$

$$\frac{1}{\rho}\chi_{E^n_\rho} \rightharpoonup 1 \quad \textit{weakly-star in } L^\infty(\Sigma) \textit{ when } n \textit{ tends to infinity}, \tag{4.4}$$

where $\chi_{E^n_\rho}$ is the characteristic function of E^n_ρ.

Proof. We follow the ideas of [21, Lemma 4.1]. Let us take a sequence $(\varphi_n)_n$ dense in $L^1(\Sigma)$. For $n \geq 1$ we define $f^n \in (L^1(\Sigma))^{n+2\ell+3}$ by

$$f^n = (1, \varphi_1, \dots, \varphi_n, |v_1 - v_3|^\sigma, |v_2 - v_3|^\sigma, h(\cdot,\cdot,v_1), h(\cdot,\cdot,v_2)).$$

Thanks to Lyapunov's convexity theorem, for every $n \geq 1$ and every $\rho \in (0,1)$, there exists a measurable subset $E^n_\rho \subset \Sigma$ satisfying

$$\int_{E^n_\rho} f^n \, dsdt = \rho \int_\Sigma f^n \, dsdt.$$

As in [21], it is easy to prove that (4.1)–(4.4) hold for the sequence $(E^n_\rho)_n$. ∎

THEOREM 4.4. *Let ρ be such that $0 < \rho < 1$. For every $v_1, v_2, v_3 \in V_{ad}$, there exists a measurable subset $E_\rho \subset \Sigma$ such that*

$$\mathcal{L}^N(E_\rho) = \rho \mathcal{L}^N(\Sigma), \tag{4.5}$$

$$\begin{aligned} &\int_{\Sigma\setminus E_\rho} |v_1 - v_3|^\sigma \, dsdt + \int_{E_\rho} |v_2 - v_3|^\sigma \, dsdt \\ &= (1-\rho)\int_\Sigma |v_1 - v_3|^\sigma \, dsdt + \rho\int_\Sigma |v_2 - v_3|^\sigma \, dsdt, \end{aligned} \tag{4.6}$$

$$\begin{aligned} &\int_{\Sigma\setminus E_\rho} h(s,t,v_1) \, dsdt + \int_{E_\rho} h(s,t,v_2) \, dsdt \\ &= (1-\rho)\int_\Sigma h(s,t,v_1) \, dsdt + \rho\int_\Sigma h(s,t,v_2) \, dsdt, \end{aligned} \tag{4.7}$$

$$y_\rho = y_1 + \rho z + r_\rho \quad \textit{with} \quad \lim_{\rho\to 0} \frac{1}{\rho} ||r_\rho||_{C(\bar{Q})} = 0, \tag{4.8}$$

$$J(y_\rho, v_\rho) = J(y_1, v_1) + \rho[J'_y(y_1, v_1)z + J(y_1, v_2) - J(y_1, v_1)] + o(\rho), \tag{4.9}$$

$$\int_\Sigma \Psi(s,t,y_\rho,v_\rho) \, dsdt \tag{4.10}$$

$$= \int_\Sigma \Big(\Psi(s,t,y_1,v_1) + \rho[\Psi'_y(s,t,y_1,v_1)z + \Psi(s,t,y_1,v_2) - \Psi(s,t,y_1,v_1)] \Big) \, dsdt + o(\rho),$$

where v_ρ is the control defined by

$$v_\rho(s,t) = \begin{cases} v_1(s,t) & \textit{on } \Sigma \setminus E_\rho, \\ v_2(s,t) & \textit{on } E_\rho, \end{cases} \tag{4.11}$$

y_ρ, y_1 are the solutions of (1.1) *corresponding, respectively, to v_ρ and to v_1, z is the weak solution of*

$$\begin{cases} \dfrac{\partial z}{\partial t} + Az + f'_y(x,t,y_1)z = 0 & \textit{in } Q, \\ \dfrac{\partial z}{\partial n_A} + g'_y(s,t,y_1,v_1)z = g(s,t,y_1,v_1) - g(s,t,y_1,v_2) & \textit{on } \Sigma, \\ z(0) = 0 & \textit{in } \Omega. \end{cases} \tag{4.12}$$

Proof. Using Lemma 4.3, the proof is similar to the one of Theorem 4.1 in [21] and the one of Theorem 3.4 in [4]. The relation (4.10), which does not appear in our previous papers, is deduced with the help of (4.4) and (4.8). ∎

5. Proof of Pontryagin's principle.

5.1. Penalized problem. Following [15, 16], since $C(\overline{D})$ is separable, there exists a norm $|\cdot|_{C(\overline{D})}$, which is equivalent to the usual norm $\|\cdot\|_{C(\overline{D})}$ such that $(C(\overline{D}), |\cdot|_{C(\overline{D})})$ is strictly convex, and $\mathcal{M}(\overline{D})$, endowed with the dual norm of $|\cdot|_{C(\overline{D})}$ (denoted by $|\cdot|_{\mathcal{M}(\overline{D})}$), is also strictly convex; see [7, Corollary 2, p. 148 or Corollary 2, p. 167]. We define the distance function to $\mathcal{C}$ (for the new norm $|\cdot|_{C(\overline{D})}$) by

$$d_{\mathcal{C}}(\varphi) = \inf_{z\in\mathcal{C}} |\varphi - z|_{C(\overline{D})}.$$

Since $\mathcal{C}$ is convex, then $d_{\mathcal{C}}$ is convex and Lipschitz of rank 1, and we have

$$\limsup_{\substack{\rho\searrow 0,\\ \varphi'\to\varphi}} \frac{d_{\mathcal{C}}(\varphi'+\rho z) - d_{\mathcal{C}}(\varphi')}{\rho} = \max\{\langle \xi, z\rangle_{\mathcal{M}(\overline{D})\times C(\overline{D})} \mid \xi \in \partial d_{\mathcal{C}}(\varphi)\} \tag{5.1}$$

for every $\varphi, z \in C(\overline{D})$, where $\partial d_{\mathcal{C}}$ is the subdifferential in the sense of convex analysis (see [6]). Therefore, for a given $\varphi \in C(\overline{D})$ we have

(5.2)

$$\langle \xi, z-\varphi\rangle_{\mathcal{M}(\overline{D})\times C(\overline{D})} + d_{\mathcal{C}}(\varphi) \le d_{\mathcal{C}}(z) \quad \text{for all } \xi\in\partial d_{\mathcal{C}}(\varphi) \text{ and for all } z\in C(\overline{D}),$$
$$|\xi|_{\mathcal{M}(\overline{D})} \le 1 \quad \text{for every } \xi \in \partial d_{\mathcal{C}}(\varphi).$$

Moreover it is proved in [16, Lemma 3.4] that, since $\mathcal{C}$ is a closed convex subset of $C(\overline{D})$, for every $\varphi \notin \mathcal{C}$, and every $\xi \in \partial d_{\mathcal{C}}(\varphi)$, then $|\xi|_{\mathcal{M}(\overline{D})} = 1$. Since $\partial d_{\mathcal{C}}(\varphi)$ is convex in $\mathcal{M}(\overline{D})$ and $(\mathcal{M}(\overline{D}), |\cdot|_{\mathcal{M}(\overline{D})})$ is strictly convex, if $\varphi\notin\mathcal{C}$, then $\partial d_{\mathcal{C}}(\varphi)$ is a singleton and $d_{\mathcal{C}}$ is Gâteaux-differentiable at φ.

Let $(\bar y, \bar v)$ be an optimal solution of (P). Consider the penalized functional

$$J_k(y,v) = \left\{ \left[\left(J(y,v) - J(\bar y,\bar v) + \frac{1}{k^2}\right)^+\right]^2 + (d_{\mathcal{C}}(\Phi(y)))^2 \right.$$
$$\left. + \sum_{i=1}^{m_0}\left[\int_\Sigma \Psi_i(s,t,y,v)\,dsdt\right]^2 + \sum_{i=m_0+1}^{m}\left[\left(\int_\Sigma \Psi_i(s,t,y,v)\,dsdt\right)^+\right]^2 \right\}^{\frac12}.$$

We easily verify that $(\bar y,\bar v)$ is a $\frac{1}{k^2}$-solution of the penalized problem

$$(\mathrm{P}_k^M) \qquad \inf\{J_k(y,v) \mid y\in W(0,T)\cap C(\overline{Q}), v\in V_{ad}(\bar v, M),\ (y,v) \text{ satisfies } (1.1)\}$$

for every $M > 0$ and every $k > 0$. For every $k > 0$, we set $M_k = k^{(\frac{1}{2\bar\sigma} - \frac{1}{2\sigma})}$ and we denote by (P^k) the penalized problem $(\mathrm{P}_k^{M_k})$.

5.2. Proof of Theorem 2.1. Step 1. For every $k \ge 1$, the metric space $(V_{ad}(\bar v, M_k), d)$ is complete; see Proposition 4.2. Let us prove that the functional $v \longmapsto J_k(y_v, v)$ is lower semicontinuous on this metric space. Since the mappings

$v \to J(y_v, v)$ and $v \to \int_\Sigma \Psi_i(s,t,y_v,v)\, dsdt$ $(m_0+1 \le i \le m)$ are lower semicontinuous on $(V_{ad}(\tilde{v}, M_k), d)$, it is clear that $v \to \left(J(y_v, v) - J(\bar{y}, \bar{v}) + \frac{1}{k^2}\right)^+$ and $v \to \left(\int_\Sigma \Psi_i(s,t,y_v,v)\, dsdt\right)^+$ $(m_0+1 \le i \le m)$ are also lower semicontinuous on $(V_{ad}(\tilde{v}, M_k), d)$ because $r \to r^+$ is a nondecreasing continuous mapping from $\mathbb{R}$ into $\mathbb{R}^+$. On the other hand, the mappings $v \to \int_\Sigma \Psi_i(s,t,y_v,v)\, dsdt$ $(1 \le i \le m_0)$ are continuous on $(V_{ad}(\tilde{v}, M_k), d)$. Since the mappings $r \to r^2$ and $r \to r^{\frac{1}{2}}$ are nondecreasing and continuous from $\mathbb{R}^+$ into $\mathbb{R}^+$, then $v \to J_k(y_v, v)$ is lower semicontinuous. Due to Ekeland's variational principle, for every $k \ge 1$, there exists $v_k \in V_{ad}(\bar{v}, M_k)$ such that

$$(5.3)\quad d(v_k, \bar{v}) \le \frac{1}{k} \text{ and } J_k(y_k, v_k) \le J_k(y_v, v) + \frac{1}{k} d(v_k, v) \text{ for every } v \in V_{ad}(\bar{v}, M_k).$$

(y_k and y_v are the solutions of (1.1) corresponding, respectively, to v_k and v.) Let v_0 be in V_{ad}. Let k_0 be large enough so that v_0 belong to $V_{ad}(\bar{v}, M_k)$ for every $k \ge k_0$. Observe that, for the above choice of M_k, $(v_k)_k$ tends to $\bar{v}$ in $L^{\bar{\sigma}}(\Sigma)$. Let us check this. Denote by Σ_k the set of points $(s,t) \in \Sigma$ where $v_k(s,t) \ne \bar{v}(s,t)$. From (5.3) we know that $\mathcal{L}^N(\Sigma_k) \le 1/k$. Then

$$(5.4)\quad \begin{aligned} \int_\Sigma |\bar{v} - v_k|^{\bar{\sigma}} dsdt = \int_{\Sigma_k} |\bar{v} - v_k|^{\bar{\sigma}} dsdt &\le \|\bar{v} - v_k\|^{\bar{\sigma}}_{\sigma,\Sigma} \mathcal{L}^N(\Sigma_k)^{1-\frac{\bar{\sigma}}{\sigma}} \\ &\le M_k^{\bar{\sigma}} k^{\frac{\bar{\sigma}}{\sigma}-1} = k^{\frac{1}{2}(\frac{\bar{\sigma}}{\sigma}-1)} \longrightarrow 0 \quad \text{when } k \to +\infty. \end{aligned}$$

Step 2. Theorem 3.1 gives the existence of measurable sets $E_\rho^k \subset \Sigma$, such that $\mathcal{L}^N(E_\rho^k) = \rho \mathcal{L}^N(\Sigma)$,

$$(5.5)\quad \begin{aligned} &\int_{\Sigma \setminus E_\rho^k} |v_k - \bar{v}|^\sigma\, dsdt + \int_{E_\rho^k} |v_0 - \bar{v}|^\sigma\, dsdt \\ &= (1-\rho)\int_\Sigma |v_k - \bar{v}|^\sigma\, dsdt + \rho \int_\Sigma |v_0 - \bar{v}|^\sigma\, dsdt, \end{aligned}$$

$$(5.6)\quad \begin{aligned} &\int_{\Sigma \setminus E_\rho^k} h(s,t,v_k)\, dsdt + \int_{E_\rho^k} h(s,t,v_0)\, dsdt \\ &= (1-\rho)\int_\Sigma h(s,t,v_k)\, dsdt + \rho \int_\Sigma h(s,t,v_0)\, dsdt, \end{aligned}$$

$$(5.7)\quad \begin{aligned} &\int_\Sigma (\Psi(s,t,y_\rho^k, v_\rho^k) - \Psi(s,t,y_k,v_k))\, dsdt \\ &= \rho \int_\Sigma (\Psi'_y(s,t,y_k,v_k) z_k + \Psi(s,t,y_k,v_0) - \Psi(s,t,y_k,v_k))\, dsdt + o(\rho), \end{aligned}$$

$$(5.8)\quad y_\rho^k = y_k + \rho z_k + r_\rho^k, \quad \lim_{\rho \to 0} \frac{1}{\rho} \|r_\rho^k\|_{C(\overline{Q})} = 0,$$

$$(5.9)\quad J(y_\rho^k, v_\rho^k) = J(y_k, v_k) + \rho \Delta J_k + o(\rho),$$

where v_ρ^k is defined by

$$(5.10)\quad v_\rho^k(s,t) = \begin{cases} v_k(s,t) & \text{on } \Sigma \setminus E_\rho^k, \\ v_0(s,t) & \text{on } E_\rho^k, \end{cases}$$

y_ρ^k is the state corresponding to v_ρ^k, z_k is the weak solution of

$$\begin{cases} \dfrac{\partial z_k}{\partial t} + Az_k + f'_y(x,t,y_k)z_k = 0 & \text{in } Q, \\ \dfrac{\partial z_k}{\partial n_A} + g'_y(s,t,y_k,v_k)z_k = g(s,t,y_k,v_k) - g(s,t,y_k,v_0) & \text{on } \Sigma, \\ z_k(0) = 0 & \text{in } \Omega, \end{cases}$$

and

$$\Delta J_k = \int_Q F'_y(x,t,y_k(x,t))z_k(x,t)\,dxdt + \int_\Sigma G'_y(s,t,y_k(s,t),v_k(s,t))z_k(s,t)\,dsdt$$

$$+ \int_\Sigma [G(s,t,y_k(s,t),v_0(s,t)) - G(s,t,y_k(s,t),v_k(s,t))]\,dsdt + \int_\Omega L'_y(x,y_k(T))z_k(T)\,dx.$$

On the other hand, for every $k > k_0$ and every $0 < \rho < 1$, due to (5.5) and (5.6), v_ρ^k belongs to $V_{ad}(\bar{v}, M_k)$. If we set $v = v_\rho^k$ in (5.3), it follows that

$$\lim_{\rho\to 0} \frac{J_k(y_k,v_k) - J_k(y_\rho^k, v_\rho^k)}{\rho} \le \frac{1}{k}\mathcal{L}^N(\Sigma). \tag{5.11}$$

Taking (5.1), (5.7), (5.9), and the definition of J_k into account, we obtain

$$- \nu_k \Delta J_k \lambda_k \int_\Sigma \left[\Psi(\cdot,y_k,v_0) - \Psi(\cdot,y_k,v_k) + \Psi'_y(\cdot,y_k,v_k)z_k\right]\,dsdt \tag{5.12}$$

$$-\langle \mu_k, \Phi'(y_k)z_k\rangle_{\overline{D}} \le \frac{1}{k}\mathcal{L}^N(\Sigma),$$

where

$$\lambda_k^i = \frac{\int_\Sigma \Psi_i(s,t,y_k,v_k)\,dsdt}{J_k(y_k,v_k)} \quad \text{for } 1 \le i \le m_0,$$

$$\lambda_k^i = \frac{\left(\int_\Sigma \Psi_i(s,t,y_k,v_k)\,dsdt\right)^+}{J_k(y_k,v_k)} \quad \text{for } m_0 + 1 \le i \le m,$$

$$\nu_k = \frac{(J(y_k,v_k) - J(\bar{y},\bar{v}) + \frac{1}{k^2})^+}{J_k(y_k,v_k)}, \quad \mu_k = \begin{cases} \dfrac{d_{\mathcal{C}}(\Phi(y_k))\nabla d_{\mathcal{C}}(\Phi(y_k))}{J_k(y_k,v_k)} & \text{if } \Phi(y_k) \notin \mathcal{C}, \\ 0 & \text{otherwise.} \end{cases}$$

For every $k > 0$, we consider the weak solution p_k of

$$\begin{cases} -\dfrac{\partial p_k}{\partial t} + A^* p_k + f'_y(x,t,y_k)p_k = \nu_k F'_y(x,t,y_k) + [\Phi'(y_k)^*\mu_k]|_Q, \\ \dfrac{\partial p_k}{\partial n_{A^*}} + g'_y(\cdot,y_k,v_k)p_k = \nu_k G'_y(\cdot,y_k,v_k) + \lambda_k \Psi'_y(\cdot,y_k,v_k) + [\Phi'(y_k)^*\mu_k]|_\Sigma, \\ p_k(T) = \nu_k L'_y(x,y_k(T)) + [\Phi'(y_k)^*\mu_k]|_{\overline{\Omega}_T}, \end{cases} \tag{5.13}$$

where $[\Phi'(y_k)^*\mu_k]|_Q$, $[\Phi'(y_k)^*\mu_k]|_\Sigma$, and $[\Phi'(y_k)^*\mu_k]|_{\bar{\Omega}_T}$ have the same meaning as in Theorem 2.1. By using the Green formula of Theorem 3.4, we obtain

$$\nu_k \int_Q F'_y(x,t,y_k) z_k \, dxdt + \nu_k \int_\Sigma G'_y(s,t,y_k,v_k) z_k \, dsdt + \nu_k \int_\Omega L'_y(x, y_k(T)) z_k(T) \, dx$$

$$+\lambda_k \int_\Sigma \Psi'_y(s,t,y_k,v_k) z_k \, dsdt + \langle \mu_k, \Phi'(y_k) z_k \rangle_{\overline{D}}$$

$$= \int_Q p_k \left(\frac{\partial z_k}{\partial t} + A z_k + f'_y(x,t,y_k) z_k \right) dxdt + \int_\Sigma p_k \left(\frac{\partial z_k}{\partial n_A} + g'_y(s,t,y_k,v_k) z_k \right) dsdt$$

$$= \int_\Sigma p_k [g(s,t,y_k,v_k) - g(s,t,y_k,v_0)] \, dsdt.$$

With this equality, (5.12), and the definition of ΔJ_k, we have

$$\int_\Sigma [\nu_k G(s,t,y_k,v_k) + \lambda_k \Psi(s,t,y_k,v_k) - p_k g(s,t,y_k,v_k)] \, dsdt$$
$$\leq \int_\Sigma [\nu_k G(s,t,y_k,v_0) + \lambda_k \Psi(s,t,y_k,v_0) - p_k g(s,t,y_k,v_0)] \, dsdt + \frac{1}{k} \mathcal{L}^N(\Sigma) \tag{5.14}$$

for every $k \geq k_0$.

Step 3. Notice that $\nu_k^2 + \sum_i (\lambda_k^i)^2 + |\mu_k|^2_{\mathcal{M}(\overline{D})} = 1$. Then there exist an element $(\bar{\nu}, \bar{\lambda}, \bar{\mu})$ in $\mathbb{R}^{1+m} \times \mathcal{M}(\overline{D})$ with $\bar{\nu} \geq 0$ and $\bar{\lambda}_i \geq 0$ for $m_0 + 1 \leq i \leq m$, and a subsequence, still denoted by $(\nu_k, \lambda_k, \mu_k)_k$, such that

$$(\nu_k, \lambda_k) \longrightarrow (\bar{\nu}, \bar{\lambda}) \text{ in } \mathbb{R}^{1+m}, \ \mu_k \rightharpoonup \bar{\mu} \text{ weak-star in } \mathcal{M}(\overline{D}).$$

From Theorem 3.4, we obtain the estimate

$$\|p_k\|_{L^{\delta'}(0,T;W^{1,d'}(\Omega))} \leq C_4(\delta, d) \Big\{ \|F'_y(\cdot, y_k)\|_{1,Q} + \|G'_y(\cdot, y_k, v_k)\|_{1,\Sigma} +$$

$$\|L'_y(\cdot, y_k(T))\|_{1,\Omega} + |\lambda_k| \|\Psi'_y(., y_k, v_k)\|_{1,\Sigma} + |\mu_k|_{\mathcal{M}(\overline{D})} \|\Phi'_y(y_k)\|_{\mathcal{L}(C(\overline{D}); C(\overline{D}))} \Big\}$$

for every (δ, d) satisfying $\frac{N}{2d} + \frac{1}{\delta} < \frac{1}{2}$, where $\mathcal{L}(C(\overline{D}); C(\overline{D}))$ denotes the space of linear continuous mappings from $C(\overline{D})$ to $C(\overline{D})$.

Since the sequences $(\nu_k)_k, (\lambda_k)_k, (\mu_k)_k, (y_k)_k$, and $(v_k)_k$ are bounded, respectively, in $\mathbb{R}$, $\mathbb{R}^m$, $\mathcal{M}(\overline{D})$, $C(\overline{Q})$, and in $L^{\bar{\sigma}}(\Sigma)$, the sequence $(p_k)_k$ is bounded in $L^{\delta'}(0,T;W^{1,d'}(\Omega))$. Then there exist $\bar{p} \in L^{\delta'}(0,T;W^{1,d'}(\Omega))$ and a subsequence, still denoted by $(p_k)_k$, such that $(p_k)_k$ weakly converges to $\bar{p}$ in $L^{\delta'}(0,T;W^{1,d'}(\Omega))$ for every (δ, d) satisfying $\frac{N}{2d} + \frac{1}{\delta} < \frac{1}{2}$. By using the same arguments as in [21], we can prove that $\bar{p}$ is the weak solution of (2.4).

Step 4. Recall that $(v_k)_k$ tends to $\bar{v}$ in $L^{\bar{\sigma}}(\Sigma)$ (see (5.4)).

By passing to the limit when k tends to infinity in (5.14), with Fatou's lemma (applied to the sequence of functions $(\nu_k G(\cdot, 0, v_k(\cdot)), \lambda_k \Psi(\cdot, 0, v_k(\cdot)))_k$ and the convergence results stated in Step 2, we obtain

$$H_\Sigma(\bar{y}, \bar{v}, \bar{p}, \bar{\nu}, \bar{\lambda}) \leq H_\Sigma(\bar{y}, v_0, \bar{p}, \bar{\nu}, \bar{\lambda}), \tag{5.15}$$

for every $v_0 \in V_{ad}$. On the other hand, from definitions of μ_k and λ_k, and from (5.2), we deduce

$$\lambda_k^i \int_\Sigma \Psi_i(s,t,y_k,v_k)\,dsdt = 0,\ m_0+1 \le i \le m,$$

$$\langle \mu_k, z - \Phi(y_k)\rangle_{\mathcal{M}(\overline{D})\times C(\overline{D})} \le 0 \quad \text{for all } z \in \mathcal{C}.$$

We obtain (2.2) and (2.3) by passing to the limit in these expressions. Since $\mathcal{C}$ is of finite codimension, by using the same arguments as in [22], we prove that $(\bar{\nu}, \bar{\lambda}, \bar{\mu})$ is nonzero. □

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