ON SMOOTHING METHODS FOR THE P_0 MATRIX LINEAR COMPLEMENTARITY PROBLEM

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Abstract. In this paper, we propose a Big-Γ smoothing method for solving the P_0 matrix linear complementarity problem. We study the trajectory defined by the augmented smoothing equations and global convergence of the method under an assumption that the original P_0 matrix linear complementarity problem has a solution. The method has been tested on the P_0 matrix linear complementarity problem with unbounded solution set. Preliminary numerical results indicate the robustness of the method.

Key words. linear complementarity problem, P_0 matrix, smoothing algorithm.

AMS subject classifications. 65H10, 90C30, 90C33.

1. Introduction.

In this paper we consider the linear complementarity problem (LCP)

$$t^T s = 0, \quad s = Mt + q, \quad \text{and} \quad t, s \geq 0,$$

where M is an $n \times n$ P_0 matrix and q is an n dimensional vector. A matrix $M \in R^{n\times n}$ is called a P_0 matrix if

$$\max_{i:t_i \neq 0} \{t_i(Mt)_i\} \geq 0, \quad \text{for all} \quad t \in R^n, t \neq 0.$$

A linear complementarity problem is called a P_0 matrix LCP if the matrix M is a P_0 matrix. The class of the P_0 matrix LCP includes the monotone LCP and the P matrix LCP. The P_0 matrix LCP has been studied extensively under additional conditions [5, 12].

A differentiable function on R^n is called a P_0 function if its Jacobian is a P_0 matrix at every point in R^n. A nonlinear complementarity problem (NCP) is called a P_0 function NCP if the involved function is a P_0 function. Kojima, Megiddo and Noma [11] proved the existence of a trajectory in the interior of the feasible set of the P_0 function NCP under some additional conditions. Their results influenced the development of interior point methods and non-interior point methods, and led several continuation methods for solving P_0 function NCP.

Recently, Facchinei and Kanzow [7] applied regularization methods for solving a continuously differentiable P_0 function NCP under the following assumption.

Assumption 1.1. *The solution set of the P_0 function NCP is nonempty and bounded.*

This assumption is weaker than that Kojima, Megiddo, Noma and Yoshise used in [11, 12]. Moreover it includes the monotone NCP with an interior point, and the P_0 and R_0 NCP [5]. After Facchinei-Kanzow's encouraging work, several algorithms and theoretical results on regularization methods for the P_0 function NCP have been developed [16, 17, 19] under Assumption 1.1. In particular, Ravindran and Gowda

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[17] generalized the results of Facchinei and Kanzow [7] to a continuous P_0 function variational inequality problem with box constraints. Facchinei and Kanzow [7] gave a counterexample to show that it is not possible to remove the boundedness assumption of the solution set for regularization methods for solving the P_0 matrix LCP, and the P_0 function NCP.

In this paper, we study a "Big-Γ" smoothing method for the P_0 matrix LCP under the following assumption, which removes the boundedness assumption of the solution set from Assumption 1.1.

Assumption 1.2. *The P_0 matrix LCP has a solution.*

Big-Γ interior point methods have been studied for solving the monotone LCP [13]. The methods add one inequality, with a positive number Γ as the right-hand-side bound, to bound the variables of the problem. If this inequality contains an original solution, then the augmented problem has a solution and it is also a solution to the original problem. One can always set Γ sufficiently big such that the inequality does contain a solution, assuming that it exists. However, the techniques used in Big-Γ interior point methods heavily rely on the monotone property, which cannot be carried over from the monotone LCP to the P_0 matrix LCP. One difference, for example, is that the existence of an interior feasible point implies the bounded solution set for the monotone LCP, but it is not held for the P_0 matrix LCP.

Example 1.1.

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix} \quad \text{and} \quad q = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

It is not difficult to verify that M is a P_0-matrix and that this LCP has a strictly feasible point $(t,s) = (1,1,1,1,1,1)^T$. However, the solution set of the LCP contains the unbounded line $(t,s) = (t_1,0,0,0,0,1)^T$ for all $t_1 \geq 0$.

The generalization of Big-Γ methods to the P_0 matrix LCP is nontrivial, see [12]. In order to make the Big-Γ smooth paths and their neighborhood be bounded, we have to slightly destroy the P_0 property. Furthermore, in contrast with the trajectory analysis given by Kojima, Megiddo and Noma [11], the existence of sufficiently short central path is not guaranteed under Assumption 1.2, see Example 2.1.

In section 2, we establish the existence of the Big-Γ smooth trajectory which leads to a solution of the problem. In section 3, we propose an algorithm for tracing the trajectory numerically and show the global convergence property of the algorithm. We tested the algorithm on the P_0 matrix LCP with unbounded solution sets. Numerical results reported in section 4 indicate the robustness of the algorithm.

We use $\|\cdot\|$ to denote $\|\cdot\|_\infty$. We use e for a vector with all entries equal to 1 and I for a diagonal matrix with all diagonal entries equal to 1. We denote the solution set of LCP(M,q) by $S_0(M,q)$.

2. A Big-Γ smoothing model.

Let

$$N = \begin{pmatrix} M & r & 0 \\ 0 & 1 & 0 \\ -e^T & -1 & -1 \end{pmatrix}, \qquad p = \begin{pmatrix} q \\ 0 \\ \Gamma \end{pmatrix},$$

where $r = e - Me - q$ and $\Gamma \geq n+5$ is sufficiently big.

Let $x = (t,\theta,\alpha) \in R^{n+2}$ and $y = (s,\eta,\beta) \in R^{n+2}$. We consider the LCP$(N,p)$

$$x^T y = 0, \quad y = Nx + p, \quad \text{and} \quad x, y \geq 0.$$

By the construction of the model, we have the following lemma whose simple proof is omitted.

LEMMA 2.1.

1. LCP(N,p) has a feasible interior point

$$t = e,\ \theta = 1,\ \alpha = 1$$

$$s = e,\ \eta = 1,\ \beta = \Gamma - (n+2) \geq 1.$$

2. If (t^*, s^*) is a solution of LCP(M,q) with $e^T t^* \leq \Gamma$, then

$$(t^*, 0, \Gamma - e^T t^*, s^*, 0, 0) \quad \text{and} \quad (t^*, 0, 0, s^*, 0, \Gamma - e^T t^*)$$

are solutions of LCP(N,p).

3. If LCP(N,p) has a solution, then in every complementarity solution $(t^*, \theta^*, \alpha^*, s^*, \eta^*, \beta^*)$ of LCP(N,p), (t^*, s^*) is a solution of LCP(M,q), $\eta^* = \theta^* = 0$ and $\alpha^* + \beta^* = \Gamma - e^T t^*$.

4. The feasible set of LCP(N,p) is bounded.

Notice that N is not a P_0 matrix, since $N_{n+2,n+2} = -1$. Although we can easily construct a P_0 matrix which satisfies Results 1–3 of Lemma 2.1, e.g. set $N_{n+2,n+2} = 1$, the resulting LCP may have a unbounded solution set. It seems to the authors that it is hard to construct a Big-Γ model for the P_0 matrix LCP which has both the P_0 property and the boundedness of the solution set. This contrasts with the monotone LCP, for which we always can construct a Big-Γ model having a bounded solution set without loss of the monotone property [13, 21].

Nevertheless, the matrix N is a block lower triangular matrix and its first block is an $(n+1) \times (n+1)$ P_0 matrix, i.e.,

$$N = \begin{pmatrix} \tilde{N} & 0 \\ -e^T & -1 \end{pmatrix}$$

where

$$\tilde{N} := \begin{pmatrix} M & r \\ 0 & 1 \end{pmatrix}.$$

We will often use this fact later.

In what follows, for simplicity, we use $z := (x, y)$. It is easy to verify that the LCP(N,p) is equivalent to the following system of nonsmooth equations

$$H_0(z) := \begin{pmatrix} Nx + p - y \\ x - \max(x - y, 0) \end{pmatrix} = 0. \tag{2.1}$$

To define a smoothing approximation function of H_0, we employ two density functions

$$\rho_1(\mu) = \frac{2}{(\mu^2 + 4)^{\frac{3}{2}}}$$

and

$$\rho_2(\mu) = \begin{cases} \frac{1}{4} & \text{if } -4 \leq \mu \leq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$\Psi_i(x_i, y_i, \varepsilon) := x_i - \int_{-\infty}^{(x_i - y_i)/\varepsilon} (x_i - y_i - \varepsilon\mu)\rho_1(\mu)d\mu, \qquad i = 1, 2, \ldots, n+1$$

and

$$\Psi_{n+2}(x_{n+2}, y_{n+2}, \varepsilon) := x_{n+2} - \int_{-\infty}^{(x_{n+2} - y_{n+2})/\varepsilon} (x_{n+2} - y_{n+2} - \varepsilon\mu)\rho_2(\mu)d\mu.$$

Calculating the integral, we obtain

$$\Psi_i(x_i, y_i, \varepsilon) = \frac{1}{2}(x_i + y_i - \sqrt{(x_i - y_i)^2 + 4\varepsilon^2}), \quad i = 1, 2, \ldots n+1$$

and

$$\Psi_{n+2}(x_{n+2}, y_{n+2}, \varepsilon) = \begin{cases} x_{n+2} & x_{n+2} - y_{n+2} \le -4\varepsilon \\ y_{n+2} - 2\varepsilon & x_{n+2} - y_{n+2} \ge 0 \\ y_{n+2} - \frac{1}{8\varepsilon}(x_{n+2} - y_{n+2})^2 - 2\varepsilon & \text{otherwise.} \end{cases}$$

Each component of Ψ is continuously differentiable [3]. Moreover, by Lemma 2.4 in [9], for $i = 1, 2, ..., n+1$

$$|\Psi_i(x_i, y_i, \varepsilon_1) - \Psi_i(x_i, y_i, \varepsilon_2)| \le \left(\int_{-\infty}^{\infty} |\mu|\rho_1(\mu)d\mu\right) |\varepsilon_1 - \varepsilon_2| = 2|\varepsilon_1 - \varepsilon_2|,$$

and for $i = n+2$

$$|\Psi_i(x_i, y_i, \varepsilon_1) - \Psi(x_i, y_i, \varepsilon_2)| \le \left(\int_{-\infty}^{\infty} |\mu|\rho_2(\mu)d\mu\right) |\varepsilon_1 - \varepsilon_2| = 2|\varepsilon_1 - \varepsilon_2|.$$

Therefore, letting $\varepsilon > 0$, the smoothing approximation function defined by

$$H(z, \varepsilon) := \begin{pmatrix} Nx + p - y \\ \Psi(x, y, \varepsilon) \end{pmatrix}$$

is continuously differentiable in $R^{2(n+2)}$. Moreover,

$$\|H(z, \varepsilon_1) - H(z, \varepsilon_2)\| \le 2|\varepsilon_1 - \varepsilon_2|, \quad \text{for} \quad z \in R^{2(n+2)} \tag{2.2}$$

and

$$\|H(z, \varepsilon) - H_0(z)\| \le 2\varepsilon, \quad \text{for} \quad z \in R^{2(n+2)}. \tag{2.3}$$

We will show that for every $\varepsilon > 0$ the system

$$H(z, \varepsilon) = 0 \tag{2.4}$$

has at most two solutions, and the two solutions differ possibly in the (x_{n+2}, y_{n+2}) components. Under certain conditions, the solutions of (2.4) forms two paths which never cross each other and converge to two solutions of (2.1).

To study the existence of the trajectory, we first consider the following closed set

$$D(\varepsilon) = \{z \;:\; \|H(z, \varepsilon)\| \le c\varepsilon\}$$

where $c > 2\sqrt{2(n+2)}$ is a constant.

LEMMA 2.2. *Suppose that LCP(N, p) has a solution z^*. Then for every $\varepsilon > 0$, the set $D(\varepsilon)$ is nonempty and compact. Moreover, for every $z \in D(\varepsilon)$,*

$$x_i + c\varepsilon \geq 0, \ \ y_i + c\varepsilon \geq 0 \tag{2.5}$$

and

$$(x_i + c\varepsilon)(y_i + c\varepsilon) \leq 2(c+1)(|x_i| + |y_i|)\varepsilon + (4 + c^2)\varepsilon^2, \tag{2.6}$$

for $i = 1, 2, \ldots, n+2$.

Proof. At the solution z^*, we have

$$\|H(z^*, \varepsilon)\| \leq \|H(z^*, \varepsilon) - H_0(z^*)\| + \|H_0(z^*)\| = 2\varepsilon.$$

Hence $z^* \in D(\varepsilon)$ and so $D(\varepsilon)$ is nonempty.

Suppose $z \in D(\varepsilon)$. Then we have

$$\|H(z, \varepsilon)\| = \| \begin{pmatrix} Nx + p - y \\ \Psi(x, y, \varepsilon) \end{pmatrix} \| \leq c\varepsilon.$$

Set $u = \Psi(x, y, \varepsilon)$. Then $u \in R^{n+2}$ satisfies

$$c\varepsilon \geq u_i \geq -c\varepsilon, \quad i = 1, 2, \ldots, n+2.$$

By construction of Ψ (cf. Lemma 2.1 in [2]), we have

$$\Psi(x - u, y - u, \varepsilon) = 0. \tag{2.7}$$

Since ρ_1 is continuous, symmetric and has an infinite support, by Theorem 2.1 in [4], we have

$$x_i - u_i > 0, \ y_i - u_i > 0, \quad i = 1, 2, ..., n+1.$$

Using $c\varepsilon \geq -u_i$, we obtain (2.5) for $i = 1, 2, \ldots, n+1$. Now we show (2.5) for $i = n+2$. By the definition of Ψ and (2.7), we have the following equalities

$$\begin{aligned} 0 &= \Psi_{n+2}(x_{n+2} - u_{n+2}, y_{n+2} - u_{n+2}, \varepsilon) \\ &= x_{n+2} - u_{n+2} - \int_{-\infty}^{(x_{n+2} - y_{n+2})/\varepsilon} (x_{n+2} - y_{n+2} - \varepsilon\mu)\rho_2(\mu) d\mu. \end{aligned}$$

This implies that $x_{n+2} - u_{n+2} \geq 0$, since the integral part is nonnegative. To show $y_{n+2} - u_{n+2} \geq 0$, we assume on the contrary that $y_{n+2} < u_{n+2}$. Then $x_{n+2} - y_{n+2} > x_{n+2} - u_{n+2} \geq 0$ and

$$\begin{aligned} x_{n+2} - u_{n+2} &= \int_{-\infty}^{(x_{n+2} - y_{n+2})/\varepsilon} (x_{n+2} - y_{n+2} - \varepsilon\mu)\rho_2(\mu) d\mu \\ &= \frac{1}{4} \int_{-4}^{0} (x_{n+2} - y_{n+2} - \varepsilon\mu) d\mu \\ &= x_{n+2} - y_{n+2} + 2\varepsilon, \end{aligned}$$

which implies

$$y_{n+2} - u_{n+2} = 2\varepsilon > 0.$$

This contradicts the assumption that $y_{n+2} - u_{n+2} < 0$. Hence we have

$$x_{n+2} - u_{n+2} \geq 0, \quad \text{and} \quad y_{n+2} - u_{n+2} \geq 0.$$

Using $c\varepsilon \geq -u_{n+2}$, we obtain (2.5) for $i = n+2$.

Now we show (2.6). Suppose that $x_i \leq y_i$. Then we have

$$x_i - u_i - \max(x_i - y_i, 0) = \min(x_i - u_i, y_i - u_i) = x_i - u_i.$$

From (2.3) and (2.7),

$$x_i - u_i = |\Psi_i(x_i - u_i, y_i - u_i, \varepsilon) - (x_i - u_i)| \leq 2\varepsilon.$$

By a simple manipulation, we obtain

$$\begin{aligned}
&(x_i + c\varepsilon)(y_i + c\varepsilon) \\
&= (x_i - u_i)(y_i - u_i) + (x_i + y_i)(u_i + c\varepsilon) - u_i^2 + c^2\varepsilon^2 \\
&= (x_i - u_i)(y_i - x_i) + (x_i - u_i)^2 + |x_i + y_i|(u_i + c\varepsilon) - u_i^2 + c^2\varepsilon^2 \\
&\leq 2(|x_i| + |y_i|)\varepsilon + 4\varepsilon^2 + 2|x_i + y_i|c\varepsilon + c^2\varepsilon^2 \\
&\leq 2(1+c)(|x_i| + |y_i|)\varepsilon + (4 + c^2)\varepsilon^2,
\end{aligned}$$

where the first inequality follows from

$$0 \leq x_i - u_i \leq 2\varepsilon \quad \text{and} \quad -c\varepsilon \leq u_i \leq c\varepsilon.$$

The proof is similar for the case $x_i \geq y_i$.

Now, we show $D(\varepsilon)$ is bounded. Using

$$c\varepsilon \geq H_{n+2}(z, \varepsilon) \geq -c\varepsilon,$$

we have

$$\begin{aligned}
c\varepsilon &\geq (Nx + p - y)_{n+2} \\
&= -e^T x + \Gamma - y_{n+2} \\
&\geq -c\varepsilon.
\end{aligned}$$

This implies

$$\Gamma + c\varepsilon \geq e^T x + y_{n+2} \geq \Gamma - c\varepsilon.$$

Since x_i and y_i are bounded below by (2.5), x and y cannot go to ∞, and so $D(\varepsilon)$ is bounded.

□

THEOREM 2.3. *Suppose that LCP(N, p) has a solution z^*. Then for every $\varepsilon > 0$, there is a $z_\varepsilon \in R^{2(n+2)}$ such that*

$$H'(z_\varepsilon, \varepsilon)^T H(z_\varepsilon, \varepsilon) = 0 \tag{2.8}$$

and

$$\|H(z_\varepsilon, \varepsilon)\| \leq c\varepsilon. \tag{2.9}$$

Proof. By Lemma 2.2, for every $\varepsilon > 0$, $D(\varepsilon)$ is nonempty and bounded. Let

$$\theta(z) = \frac{1}{2}\|H(z,\varepsilon)\|_2^2.$$

Since $H(\cdot,\varepsilon)$ is continuously differentiable and $D(\varepsilon)$ is nonempty and compact, θ has a global minimum point z_ε in $D(\varepsilon)$. Recalling that $\|\cdot\| := \|\cdot\|_\infty$, $z^* \in D(\varepsilon)$ and $H_0(z^*) = 0$, we have

$$\begin{aligned}\|H(z_\varepsilon,\varepsilon)\| &\le \|H(z_\varepsilon,\varepsilon)\|_2 \\ &\le \|H(z^*,\varepsilon)\|_2 \\ &\le \sqrt{2(n+2)}\|H(z^*,\varepsilon)\| \\ &\le \sqrt{2(n+2)}(\|H_0(z^*)\| + 2\varepsilon) \\ &= 2\varepsilon\sqrt{2(n+2)} \\ &< c\varepsilon.\end{aligned}$$

Hence, (2.9) holds. Moreover, this implies $z_\varepsilon \in \text{int } D(\varepsilon)$. By [14, 4.1.3], we have

$$\theta'(z_\varepsilon) = H'(z_\varepsilon,\varepsilon)^T H(z_\varepsilon,\varepsilon) = 0.$$

This completes the proof. □

Theorem 2.3, together with Lemma 2.1, shows that if the LCP(M,q) has a solution, then there is a Γ such that the LCP(N,p) has a solution and for every $\varepsilon > 0$ the system (2.8) has a solution. Clearly, if $H'(z_\varepsilon,\varepsilon)$ is nonsingular, then z_ε is a solution of (2.4). Now, we give a necessary and sufficient condition for the nonsingularity.

LEMMA 2.4. *Let M be a P_0 matrix. Then $H'(z,\varepsilon)$ is nonsingular at $z \in R^{2(n+2)}$ if and only if $x_{n+2} - y_{n+2} \neq -2\varepsilon$.*

Proof. Let

$$d_i(x_i - y_i,\varepsilon) = \int_{-\infty}^{(x_i-y_i)/\varepsilon} \rho_1(\mu)d\mu, \qquad i = 1,2,...,n+1$$

and

$$d_{n+2}(x_{n+2} - y_{n+2},\varepsilon) = \int_{-\infty}^{(x_{n+2}-y_{n+2})/\varepsilon} \rho_2(\mu)d\mu.$$

Let

$$D_{n+1}(x-y,\varepsilon) = \text{diag}(d_1(x_1-y_1,\varepsilon),\ldots,d_{n+1}(x_{n+1}-y_{n+1},\varepsilon))$$

and

$$D(x-y,\varepsilon) = \text{diag}(D_{n+1}(x-y,\varepsilon), d_{n+2}(x_{n+2}-y_{n+2},\varepsilon)).$$

By the definition of $H(z,\varepsilon)$,

$$H'(z,\varepsilon) = \begin{pmatrix} N & -I \\ I - D(x-y,\varepsilon) & D(x-y,\varepsilon)\end{pmatrix}.$$

It is well known that $H'(z,\varepsilon)$ is nonsingular if and only if $I - D(x-y,\varepsilon)(I-N)$ is nonsingular.

Notice that $\tilde{N}$ is a P_0 matrix, and

$$I - D(x-y,\varepsilon)(I-N) = \begin{pmatrix} I_{n+1} - D_{n+1}(x-y,\varepsilon)(I_{n+1}-\tilde{N}) & 0 \\ -d_{n+2}(x_{n+2}-y_{n+2},\varepsilon)e^T & 1-2d_{n+2}(x_{n+2}-y_{n+2},\varepsilon) \end{pmatrix}.$$

Since $\text{supp}\{\rho_1\} = R$ and $\tilde{N}$ is a P_0 matrix, $I_{n+1} - D_{n+1}(x-y,\varepsilon)(I_{n+1}-\tilde{N})$ is nonsingular [9].

Hence $H'(z,\varepsilon)$ is nonsingular if and only if $d_{n+2}(x_{n+2}-y_{n+2},\varepsilon) \neq \frac{1}{2}$. By the definition of $d_{n+2}(x_{n+2}-y_{n+2},\varepsilon)$, we have $d_{n+2}(x_{n+2}-y_{n+2},\varepsilon) \neq \frac{1}{2}$ if and only if $x_{n+2}-y_{n+2} \neq -2\varepsilon$. This completes the proof of the lemma. □

Lemma 2.5. *Suppose that M is a P_0 matrix. Then for every $\varepsilon > 0$, (2.4) has at most two solutions, and any two solutions differ possible in the (x_{n+2}, y_{n+2}) components. Moreover, a solution z_ε of (2.4) is unique if and only if $H'(z_\varepsilon,\varepsilon)$ is singular.*

Proof. Let $\tilde{x} = (x_1,\dots,x_{n+1}), \tilde{y} = (y_1,\dots,y_{n+1}), \tilde{z} = (\tilde{x},\tilde{y})$ and

$$\tilde{\Psi}(\tilde{z},\varepsilon) = (\Psi_1(x_1,y_1,\varepsilon),\dots,\Psi_{n+1}(x_{n+1},y_{n+1},\varepsilon))^T.$$

That is, $\tilde{x}$, $\tilde{y}$ and $\tilde{\Psi}$ are the first $n+1$ components of x, y and Ψ, respectively. If z is a solution of (2.4), then $\tilde{z}$ is a solution of

$$\begin{pmatrix} \tilde{N}\tilde{x} + p - \tilde{y} \\ \tilde{\Psi}(\tilde{z},\varepsilon) \end{pmatrix} = 0. \tag{2.10}$$

Since $\tilde{N}$ is a P_0 matrix and $\tilde{\Psi}$ is given by ρ_1, by Theorem 2.3 in [4], $\tilde{z}_\varepsilon$ is the unique solution of (2.10). Hence any two solutions of (2.4) differ possible at the (x_{n+2}, y_{n+2}) components.

Now we show (2.4) has at most two solutions. Since a solution $\tilde{z}$ of (2.10) is unique, we only need to show that the system of the remaining equations in (2.4),

$$\begin{pmatrix} \Gamma - e^T\tilde{x} - x_{n+2} - y_{n+2} \\ \Psi_{n+2}(x_{n+2},y_{n+2},\varepsilon) \end{pmatrix} = 0,$$

has at most two solutions.

Substituting $y_{n+2} = \Gamma - e^T\tilde{x} - x_{n+2}$ into the second equation, we obtain

$$\psi(x_{n+2}) := \Psi_{n+2}(x_{n+2}, \Gamma - e^T\tilde{x} - x_{n+2}, \varepsilon) = 0.$$

The function $\psi : R \to R$ is a polynomial of degree 2 in the interval

$$[\frac{1}{2}(\Gamma - e^T\tilde{x}) - 2\varepsilon,\ \frac{1}{2}(\Gamma - e^T\tilde{x})]$$

and linear outside of this interval. Furthermore, ψ is monotonically decreasing in

$$[\frac{1}{2}(\Gamma - e^T\tilde{x}) - \varepsilon,\ \infty)$$

and monotonically increasing in

$$(-\infty,\ \frac{1}{2}(\Gamma - e^T\tilde{x}) - \varepsilon].$$

Therefore ψ has at most two zero points. Moreover, a zero point x_{n+2} of ψ is unique if and only if

$$x_{n+2} = \frac{1}{2}(\Gamma - e^T\tilde{x}) - \varepsilon = \frac{1}{2}(y_{n+2} + x_{n+2}) - \varepsilon.$$

Hence, the system of (2.4) has at most two solutions, and a solution of (2.4) is unique if and only if $x_{n+2} - y_{n+2} = -2\varepsilon$. By Lemma 2.4, a solution of (2.4) is unique if and only if H' is singular at this solution. □

LEMMA 2.6. *Suppose that M is a P_0 matrix and the solution set of the LCP(M,q) is nonempty and bounded. Then there exist $\Gamma > 0$ and $\varepsilon^0 > 0$ such that LCP(N,p) has a solution and for every $\varepsilon \in (0,\varepsilon^0]$, $H'(z,\varepsilon)$ are nonsingular for all $z \in D(\varepsilon^0)$.*

Proof. Since the solution set $S_0(M,q)$ is bounded, we can choose $\Gamma > 0$ satisfying

$$\Gamma > 4e^Tt, \qquad \text{for all} \quad (t,s) \in S_0(M,q). \tag{2.11}$$

Then from Lemma 2.1, the solution set of the LCP(N,p) is given by

$$S_0(N,q) = \{(t,0,\Gamma - e^Tt, s, 0, 0), (t, 0, 0, s, 0, \Gamma - e^Tt) : \ (t,s) \in S_0(M,q)\}.$$

Hence for a solution $(t^*,s^*) \in S_0(M,q)$, $z^{*,1} = (t^*, 0, \Gamma - e^Tt^*, s^*, 0, 0)$ and $z^{*,2} = (t^*, 0, 0, s^*, 0, \Gamma - e^Tt^*)$ are solutions of LCP(N,p) and

$$\max_{z\in S_0(N,p)} \min(|x - y - x^{*,1} + y^{*,1}|_{n+2}, |x - y - x^{*,2} + y^{*,2}|_{n+2}) = \max_{z\in S_0(N,p)} |e^T(t^* - t)| \le \frac{\Gamma}{4}.$$

By the continuity of $H(z,\varepsilon)$ on ε, for such Γ there exists $\varepsilon^0 \in (0, \frac{\Gamma}{8})$ such that for all $z \in D(\varepsilon^0)$,

$$\max_{z\in D(\varepsilon^0)} \min(|x - y - x^{*,1} + y^{*,1}|_{n+2}, |x - y - x^{*,2} + y^{*,2}|_{n+2}) \le \frac{\Gamma}{2}.$$

Let $z \in D(\varepsilon^0)$. Without loss of generality we may assume $|x - y - x^{*,1} + y^{*,1}|_{n+2} \le |x - y - x^{*,2} + y^{*,2}|_{n+2}$. Then

$$\begin{aligned}
&|x_{n+2} - y_{n+2}| \\
&\ge |x^{*,1}_{n+2} - y^{*,1}_{n+2}| - |x_{n+2} - y_{n+2} - x^{*,1}_{n+2} + y^{*,1}_{n+2}| \\
&\ge \Gamma - e^Tt^* - \max_{z\in D(\varepsilon^0)} \min(|x - y - x^{*,1} + y^{*,1}|_{n+2}, |x - y - x^{*,2} + y^{*,2}|_{n+2}) \\
&\ge \Gamma - \frac{\Gamma}{4} - \frac{\Gamma}{2} = \frac{\Gamma}{4} > 2\varepsilon^0.
\end{aligned}$$

By Lemma 2.4, $H'(z,\varepsilon)$ is nonsingular for $\varepsilon \in (0,\varepsilon^0]$ and $z \in D(\varepsilon^0)$. □

THEOREM 2.7. *Suppose that M is a P_0 matrix and the solution set of the LCP(M,q) is nonempty and bounded. Then there exist $\Gamma > 0$ and $\varepsilon^0 > 0$ such that*

1. *for every $\varepsilon \in (0,\varepsilon^0]$, the system (2.4) has only two solutions $z_\alpha(\varepsilon)$ and $z_\beta(\varepsilon)$, which are continuous in ε and never cross each other;*
2. *$z_\alpha(\varepsilon)$ and $z_\beta(\varepsilon)$ converge to two solutions of LCP(N,q) as $\varepsilon \to 0$.*

Proof. 1. By Theorem 2.3 and Lemma 2.6 there exist $\Gamma > 0$ satisfying (2.11) and $\varepsilon^0 > 0$ such that LCP(N,p) has a solution and for every $\varepsilon \in (0,\varepsilon^0]$, the system (2.4) has a solution in $D(\varepsilon^0)$. Then using Lemma 2.5, the system (2.4) has only two solutions $z_\alpha(\varepsilon), z_\beta(\varepsilon) \in D(\varepsilon^0)$, and $H'(z_\alpha(\varepsilon),\varepsilon)$ and $H'(z_\beta(\varepsilon),\varepsilon)$ are nonsingular.

By the implicit theorem [14, 5.2.4], $z_\alpha(\varepsilon)$ and $z_\beta(\varepsilon)$ are continuous in $\varepsilon \in (0, \varepsilon^0]$.

Now we show that $z_\alpha(\varepsilon)$ and $z_\beta(\varepsilon)$ never cross each other. Assume on the contrary that there is a $\tilde{\varepsilon} \in (0, \varepsilon^0]$ such that $z_\alpha(\tilde{\varepsilon}) = z_\beta(\tilde{\varepsilon})$. Then by Lemma 2.5, $H'(z_\alpha(\tilde{\varepsilon}), \tilde{\varepsilon})$ is singular. This is a contradiction, since $z_\alpha(\tilde{\varepsilon}) \in D(\varepsilon^0)$ and for every $z \in D(\varepsilon^0)$, $H'(z, \varepsilon)$ is nonsingular. Hence $z_\alpha(\varepsilon)$ and $z_\beta(\varepsilon)$ never cross each other, which forms two paths.

2. Since $z_\alpha(\varepsilon), z_\beta(\varepsilon) \subset D(\varepsilon^0)$, and $D(\varepsilon^0)$ is bounded, $z_\alpha(\varepsilon)$ and $z_\beta(\varepsilon)$ has limiting points, respectively, as $\varepsilon \to 0$. By (2.9) in Theorem 2.3, every limiting point is a solution of LCP(N,p). We assume that for some sequence $\epsilon_k \to 0$, $z_\alpha(\varepsilon_k) \to z_\alpha^*$ and $z_\beta(\varepsilon_k) \to z_\beta^*$.

From $H(z_\alpha(\varepsilon), \varepsilon) = H(z_\beta(\varepsilon), \varepsilon) = 0$, we have

$$(x_\alpha(\varepsilon), y_\alpha(\varepsilon)) := z_\alpha(\varepsilon) \geq 0, \quad (x_\beta(\varepsilon), y_\beta(\varepsilon)) := z_\beta(\varepsilon) \geq 0$$

and

$$(x_\alpha(\varepsilon))_i (y_\alpha(\varepsilon))_i = (x_\beta(\varepsilon))_i (y_\beta(\varepsilon))_i = \varepsilon^2, \quad i = 1, 2, \ldots, n+1.$$

Moreover, without loss of generality, we may assume

$$(x_\alpha(\varepsilon))_{n+2} \geq \frac{\Gamma}{4} + 2\varepsilon, \quad (y_\alpha(\varepsilon))_{n+2} \leq 2\varepsilon$$

and

$$(x_\beta(\varepsilon))_{n+2} \leq 2\varepsilon, \quad (y_\beta(\varepsilon))_{n+2} \geq \frac{\Gamma}{4} + 2\varepsilon.$$

(See the proof of Lemma 2.2 and Lemma 2.6.)

Consider the two sets

$$E_1 := \{(x, y, \varepsilon) : \varepsilon > 0, x \geq 0, y = Nx + p \geq 0, x_{n+2} \geq \frac{\Gamma}{4} + 2\varepsilon, y_{n+2} \leq 2\varepsilon,$$

$$x_i y_i = \varepsilon^2, i = 1, 2, \ldots, n+1\}$$

and

$$E_2 := \{(x, y, \varepsilon) : \varepsilon > 0, x \geq 0, y = Nx + p \geq 0, x_{n+2} \leq 2\varepsilon, y_{n+2} \geq \frac{\Gamma}{4} + 2\varepsilon,$$

$$x_i y_i = \varepsilon^2, i = 1, 2, \ldots, n+1\}.$$

The sets E_1 and E_2 are semi-algebraic and $E_1 \cap E_2 = \emptyset$. Furthermore, $z_\alpha(\varepsilon) \subset E_1, z_\alpha^*(\varepsilon) \in E_1, z_\beta(\varepsilon) \subset E_2$ and $z_\beta^*(\varepsilon) \in E_2$. Hence by the similar argument in the proof of Theorem 5.2 in [20] (also see the proof of Theorem 4.4 in [11]), we claim that $z_\alpha(\varepsilon) \to z_\alpha^*(\varepsilon)$ and $z_\beta(\varepsilon) \to z_\beta^*(\varepsilon)$, as $\varepsilon \to 0$. □

Now we consider a special class of the P_0 matrix LCP whose solution set is not empty. Let

$$M = \begin{pmatrix} M_1 & M_{12} \\ 0 & M_2 \end{pmatrix}, \qquad q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \tag{2.12}$$

where $M_1 \in R^{n_1\times n_1}$ is a P_0 matrix, $M_{12} \in R^{n_1\times n_2}$, $M_2 \in R^{n_2\times n_2}$ is a monotone matrix, $q_1 \in R^{n_1}$, $q_2 \in R^{n_2}$, and $n_1 + n_2 = n$.

It is easy to see that M is a P_0 matrix and in every solution $(t_1^*, t_2^*, s_1^*, s_2^*)$ of the LCP(M, q), $(t_2^*, s_2^*) \in S_0(M_2, q_2)$.

We denote the set of all maximal complementarity solutions (the number of positive components in t_2 and s_2 is maximal) of LCP(M_2, q_2) by $\hat{S}_0(M_2, q_2)$. If LCP(M_2, q_2) has a strictly complementarity solution, then $\hat{S}_0(M_2, q_2)$ is the set of all strictly complementarity solutions of LCP(M_2, q_2).

Let us use the standard index set notation

$$T = \{i: \ (t_2^*)_i > 0 = (s_2^*)_i = (M_2 t_2^* + q_2)_i, (t_2^*, s_2^*) \in \hat{S}_0(M_2, q_2)\}$$

$$S = \{i: \ (t_2^*)_i = 0 < (s_2^*)_i = (M_2 t_2^* + q_2)_i, (t_2^*, s_2^*) \in \hat{S}_0(M_2, q_2)\}.$$

To simplify illustration, we assume $T \cup S = \{1, 2, \ldots, n_2\}$, i.e., LCP$(M_2, q_2)$ has a strictly complementarity solution.

Assumption 2.1.

(i) M is a matrix defined by (2.12).

(ii) LCP(M_2, q_2) has a strictly complementarity solution, and $S_0(M_2, q_2)$ is bounded.

(iii) For every $(t_2^*, s_2^*) \in \hat{S}_0(M_2, q_2)$, LCP$(M_1, M_{12}t_2^* + q_1)$ has a solution and

$$\hat{S}_0(M, q) := \{(t_1^*, t_2^*, s_1^*, s_2^*) : (t_1^*, s_1^*) \in S_0(M_1, M_{12}t_2^* + q_1), (t_2^*, s_2^*) \in \hat{S}_0(M_2, q_2)\}$$

is bounded.

(iv) $(r_2)_{i\in T} = (e - M_2 e - q_2)_{i\in T} = 0$.

Example 1.1 satisfies Assumption 2.1, here

$$M_1 = (0), \quad M_{12} = (1, 0), \quad q_1 = (0)$$

$$M_2 = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \quad q_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$S_0(M_2, q_2) = \{(\tau, 0, 0, 1-\tau): \ 0 \le \tau \le 1\}$$

$$\hat{S}_0(M_2, q_2) = \{(\tau, 0, 0, 1-\tau): \ 0 < \tau < 1\}$$

$$S_0(M, q) = \{(t_1, 0, 0, 0, 0, 1): \ t_1 \ge 0\} \cup \{(0, t_2, 0, t_2, 0, 1-t_2): \ 0 < t_2 \le 1\}$$

$$\hat{S}_0(M, q) = \{(0, t_2, 0, t_2, 0, 1-t_2): \ 0 < t_2 < 1\}$$

and $T = \{1\}$.

Although the solution set $S_0(M, q)$ is unbounded, its subset $\hat{S}_0(M, q)$ is bounded.

Theorem 2.8. *Under Assumption 2.1, the conclusion of Theorem 2.7 holds.*

Proof. We first consider the problem LCP(M_2, q_2). By Theorem 2.7, there exist Γ_2 and ε_2^0 such that two paths $z_{\alpha_2}(\varepsilon)$ and $z_{\beta_2}(\varepsilon)$ for $\varepsilon \in (0, \varepsilon_2^0]$ exist and converge to two solutions of the corresponding problem LCP(N_2, p_2),

$$z_{\alpha_2} = (\bar{t}_2, 0, \Gamma_2 - e^T\bar{t}_2, \bar{s}_2, 0, 0) \quad \text{and} \quad z_{\beta_2} = (\bar{t}_2, 0, 0, \bar{s}_2, 0, \Gamma_2 - e^T\bar{t}_2).$$

By Lemma 2.1, $(\bar{t}_2, \bar{s}_2)$ is a solution of LCP(M_2, q_2).

Now we show $(\bar{t}_2, \bar{s}_2)$ is a strictly complementarity solution. Choose any element $(t_2^*, s_2^*) \in \hat{S}_0(M_2, q_2)$.

By Lemma 2.5, for every $\varepsilon \in (0, \varepsilon_2]$, the two solutions of (2.4) are identical in the (t_2, s_2) components. Moreover, by Lemma 2.1 in [10] and the construction of the Big-Γ model, we have

$$\theta_2 = \beta_2 = \varepsilon$$

and

$$(t_2)_i(s_2 + \varepsilon r_2)_i = (t_2)_i(M_2 t_2 + q_2 + \varepsilon r_2)_i = \varepsilon^2.$$

Hence, by (iv) of Assumption 2.1,

$$\begin{aligned}
&\sum_{i \in T} \varepsilon^2 + \varepsilon^2 \sum_{i \in S} \frac{(s_2)_i}{(s_2 + \varepsilon r_2)_i} \\
&= \sum_{i \in T} (t_2)_i (s_2 + \varepsilon r_2)_i + \sum_{i \in S} \frac{(t_2)_i (s_2)_i (s_2 + \varepsilon r_2)_i}{(s_2 + \varepsilon r_2)_i} \\
&= t_2^T s_2 \geq s_2^T t_2^* + t_2^T s_2^* \\
&= \sum_{i \in T} (s_2)_i (t_2^*)_i + \sum_{i \in S} (t_2)_i (s_2^*)_i \\
&= \sum_{i \in T} \frac{(s_2 + \varepsilon r_2)_i (t_2)_i (t_2^*)_i}{(t_2)_i} + \sum_{i \in S} \frac{(s_2 + \varepsilon r_2)_i (t_2)_i (s_2^*)_i}{(s_2 + \varepsilon r_2)_i} \\
&= \varepsilon^2 \sum_{i \in T} \frac{(t_2^*)_i}{(t_2)_i} + \varepsilon^2 \sum_{i \in S} \frac{(s_2^*)_i}{(s_2 + \varepsilon r_2)_i},
\end{aligned}$$

where the inequality uses the monotone property of M_2. Therefore, we have

$$\sum_{i \in T} \frac{(t_2^*)_i}{(t_2)_i} + \sum_{i \in S} \frac{(s_2^* - s_2)_i}{(s_2 + \varepsilon r_2)_i} \leq \sum_{i \in T} 1.$$

This implies that $(t_2)_{i \in T}$ and $(s_2)_{i \in S}$ cannot go to zero. Hence $(\bar{t}_2, \bar{s}_2) \in \hat{S}_0(M_2, q_2)$.

Now, we claim that there exist Γ satisfying

$$\Gamma > 4e^T t, \quad \text{for all} \quad t \in \hat{S}_0(M, q) \tag{2.13}$$

and $\varepsilon^0 > 0$ such that the conclusion of Theorem 2.7 holds. Indeed, since M is block triangular, we claim that the system

$$\hat{H}(t_1, x_{n+2}, s_1, y_{n+2}, \varepsilon) := \begin{pmatrix} M_1 t_1 + M_{12} t_2 + \varepsilon r_1 + q_1 - s_2 \\ -e^T(t_1 + t_2) - \varepsilon - x_{n+2} - y_{n+2} + \Gamma \\ \Psi_1((t_1 - s_1)_1, \varepsilon) \\ \cdots \\ \Psi_{n_1}((t_1 - s_1)_{n_1}, \varepsilon) \\ \Psi_{n+2}(x_{n+2} - y_{n+2}, \varepsilon) \end{pmatrix} = 0$$

has two solutions for all $\varepsilon \in (0, \varepsilon^0]$.

Assume on the contrary that this claim is not true. By (iii) of Assumption 2.1, the solution set $S_0(M_1, M_{12}\bar{t}_2 + q_1)$ is nonempty and bounded. There is $\varepsilon_0 > 0$, such that $S_0(M_1, M_{12}t_2+q_1)$ is nonempty for all t_2 satisfying $\|t_2-\bar{t}_2\| \le \varepsilon_0$ [8, 17]. Hence by Theorem 2.3 for all $\varepsilon \in (0, \varepsilon_0)$, there is a $z_\varepsilon^1 := (t_1, x_{n+2}, s_1, y_{n+2})_\varepsilon$, such that

$$\hat{H}'(z_\varepsilon^1, \varepsilon)^T \hat{H}(z_\varepsilon^1, \varepsilon) = 0$$

$$\|\hat{H}(z_\varepsilon^1, \varepsilon)\| \le c\varepsilon \le c\varepsilon_0.$$

However, by our assumption, there exists a sequence $\{\varepsilon_k\}$ with $\varepsilon_{k-1} \le \varepsilon_k \le \varepsilon_0$ and $\varepsilon_k \to 0$ such that $\hat{H}'(z_{\varepsilon_k}^1, \varepsilon_k)$ is singular. Then by Lemma 2.4, at the point $z_{\varepsilon_k}^1$

$$-2\varepsilon_k = x_{n+2} - y_{n+2} \le 2x_{n+2} + e^T(t_1+t_2) + \varepsilon_k - \Gamma + c\varepsilon_k.$$

Since $D(\varepsilon_0)$ is bounded, by passing to a subsequence, we may assume that $z_{\varepsilon_k}^1 \to (\bar{t}_1, \bar{t}_2, \bar{x}_{n+2}, \bar{s}_1, \bar{s}_2, \bar{y}_{n+2})$. By (2.5) and (2.6), $(\bar{t}_1, \bar{t}_2, \bar{s}_1, \bar{s}_2) \in \hat{S}_0(M,q)$ and $\bar{x}_{n+2} = \bar{y}_{n+2} = 0$. Hence, we have

$$e^T(\bar{t}_1 + \bar{t}_2) \ge \Gamma.$$

This contradicts (2.13). This completes the proof. □

Let us end this section by the following example, which shows

1. the central path can be very short when the solution set is bounded;
2. the central path may not exist for all $\varepsilon > 0$ if the solution set is unbounded.

However, the big-M smooth paths exist for all $\varepsilon \in (0,1]$.

EXAMPLE 2.1. *Let*

$$M = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix} \qquad q = \begin{pmatrix} \delta \\ -\frac{1}{2} \end{pmatrix}, \qquad \text{where} \quad \delta \ge \frac{1}{2}.$$

It is easy to verify that M is a P_0 matrix and the LCP(M,q) has a unique solution

$$(t^*, s^*) = (0, \frac{1}{2}, \delta - \frac{1}{2}, 0), \quad \text{if} \quad \delta > \frac{1}{2}.$$

However for any $\delta^0 > 0$ the complementarity level set

$$\{(t,s) \ : \ t^T s \le \delta^0 + \delta(\delta - \frac{1}{2}),\ s = Mt + q, (t,s) > 0\}$$

contains the unbounded line

$$(k, \delta - \frac{\delta^0}{k}, \frac{\delta^0}{k}, \delta - \frac{\delta^0}{k} - \frac{1}{2})$$

for all $k > \delta^0/(\delta - \frac{1}{2})$.

The central path

$$\begin{aligned} &\{(t,s) \ : \ t_i s_i = \varepsilon,\ i = 1,2,\ s = Mt + q, (t,s) > 0\} \\ &= \{(\frac{\varepsilon}{\delta - t_2}, t_2),\ t_2 = \frac{1 + \sqrt{1 + 16\varepsilon}}{4} < \delta\} \end{aligned}$$

does not exist for $\varepsilon \ge \delta(\delta - \frac{1}{2})$.

If $\delta = \frac{1}{2}$, this problem has a unbounded solution set

$$\{(t_1, \frac{1}{2}, 0, 0) : \ t_1 \ge 0\}.$$

In this case the central path does not exist for all $\varepsilon > 0$.

Now we show that if we choose $\Gamma \geq n + 5 = 7$, then for all $\varepsilon \leq 1$, (2.4) has a solution, i.e. the Big-Γ smooth paths exist for all $\varepsilon \leq 1$.

By Lemma 2.1 in [10] and Lemma 2.2, at the Big-Γ smooth paths, $(x, y) \geq 0$ and

$$\begin{aligned} x_1 y_1 &= x_1(\delta - x_2 + (2-\delta)x_3) = \varepsilon^2 \\ x_2 y_2 &= x_2(-\frac{1}{2} + x_2 + \frac{1}{2}x_3) = \varepsilon^2 \\ x_3 y_3 &= x_3^2 = \varepsilon^2 \\ y_4 &= \Gamma - x_1 - x_2 - x_3 - x_4 \\ x_4 &= \int_{-\infty}^{(x_4 - y_4)/\varepsilon} (x_4 - y_4 - \varepsilon\mu)\rho_2(\mu) d\mu. \end{aligned}$$

We can calculate the point on the Big-Γ smooth paths

$$x_3 = y_3 = \varepsilon \leq 1$$

$$x_2 = \frac{1 - \varepsilon + \sqrt{(1-\varepsilon)^2 + 16\varepsilon^2}}{4} \leq 1$$

$$x_1 = \frac{\varepsilon^2}{\delta - x_2 + (2-\delta)\varepsilon} \leq 1$$

$$y_1 = \delta - x_2 + (2-\delta)x_3, \; y_2 = x_2 + \frac{1}{2}x_3 - \frac{1}{2}$$

$$x_4 = \Gamma - x_1 - x_2 - x_3 - 2\varepsilon \geq 2\varepsilon, \; y_4 = 2\varepsilon$$

or

$$x_4 = 0, \; y_4 = \Gamma - x_1 - x_2 - x_3 \geq 4\varepsilon,$$

where we use $\rho(\mu) = 0$, for $\mu \notin [-4, 0]$ to calculate x_4 and y_4.

There are two Big-Γ smooth paths which are bounded, continuous in ε, never cross each other and converge to two solutions

$$z^{*,1} = (0, \frac{1}{2}, 0, \Gamma - \frac{1}{2}, \delta - \frac{1}{2}, 0, 0, 0)$$

and

$$z^{*,2} = (0, \frac{1}{2}, 0, 0, \delta - \frac{1}{2}, 0, 0, \Gamma - \frac{1}{2}),$$

as $\varepsilon \to 0$. Both of them contain the solution of the original LCP$(M, q) : (x_1^{*,1}, x_2^{*,1}, y_1^{*,1}, y_2^{*,1}) = (x_1^{*,2}, x_2^{*,2}, y_1^{*,2}, y_2^{*,2}) = (0, \frac{1}{2}, \delta - \frac{1}{2}, 0)$.

3. Algorithm and its convergence. In this section we propose an algorithm and prove its global convergence.

ALGORITHM 3.1. *Given $\sigma \in (0,1)$, and $\alpha_i \in (0,1)$ for $i = 1,2$.*

Step 0 *(Initial Step)*
Choose x^0, y^0, ε^0 such that $\|H(z^0,\varepsilon^0)\| \le c\varepsilon^0$ and $H'(z^0,\varepsilon^0)$ is nonsingular.

Step 1 *(Newton Step)*
If $H(z^k,\varepsilon^k) = 0$, set $z^{k+1} = z^k$ and go to Step 3. Otherwise, Let Δz^k solve the equation

$$H(z^k,\varepsilon^k) + H'(z^k,\varepsilon^k)\Delta z^k = 0. \tag{3.1}$$

Step 2 *(Line Search)*
Let λ_k be the maximum of the values $1, \alpha_1, \alpha_1^2, \ldots$ such that

$$\|H(z^k + \lambda_k \Delta z^k, \varepsilon^k)\| \le (1 - \sigma\lambda_k)\|H(z^k,\varepsilon^k)\|. \tag{3.2}$$

Set $z^{k+1} = z^k + \lambda_k \Delta z^k$.

Step 3 *(ε Reduction)*
Let γ_k be the maximum of the values $\alpha_2^2, \alpha_2^3, \ldots$ such that

$$\|H(z^{k+1}, (1-\gamma_k)\varepsilon^k)\| \le (1-\gamma_k)c\varepsilon_k. \tag{3.3}$$

If $x_{n+2}^{k+1} - y_{n+2}^{k+1} \ne -2(1-\gamma_k)\varepsilon^k$, set $\varepsilon^{k+1} = (1-\gamma_k)\varepsilon^k$. Otherwise, set $\varepsilon^{k+1} = (1-\alpha_2\gamma_k)\varepsilon^k$.

Algorithm 3.1 is similar to the smoothing method introduced by Burke and Xu [1]. The main difference is that the definition of ε^{k+1} in Step 3 ensures the nonsingularity of $H'(z^{k+1},\varepsilon^{k+1})$ for the P_0 matrix LCP(M,q).

It is easy to verify that if $y^0 = Nx^0 + p$, then $y^k = Nx^k + p$ for all $k \ge 0$.

The following lemma shows that we can easily find a starting point (z^0,ε^0) satisfying these conditions in the initial step of Algorithm 3.1.

LEMMA 3.1. *Suppose that M is a P_0 matrix. Let*

$$x^0 = (e,1,0), \quad y^0 = (e,1,\Gamma-(n+1)), \quad \frac{1}{c+1} \le \varepsilon^0 \le 1. \tag{3.4}$$

Then $y^0 = Nx^0 + p \ge 0$, $\|H(z^0,\varepsilon^0)\| \le c\varepsilon^0$, and $H'(z^0,\varepsilon^0)$ is nonsingular.

Proof. Obviously $y^0 = Nx^0 + p$. Since $\Gamma \ge n+5$, $y^0 \ge 0$. Thus $H_i(z^0,\varepsilon^0) = 0, i = 1,2,\ldots,n+2$. By a simple calculation, we have

$$\Psi_i(x_i^0, y_i^0, \varepsilon^0) = \frac{1}{2}(x_i + y_i - \sqrt{(x_i - y_i)^2 + 4\varepsilon^2}) = 1 - \varepsilon^0, \quad i = 1,2,\ldots,n+1,$$

and

$$\Psi_{n+2}(x_{n+2}^0, y_{n+2}^0, \varepsilon^0) = x_{n+2} = 0.$$

Hence (z^0,ε^0) satisfies $\|H(z^0,\varepsilon^0)\| \le c\varepsilon^0$.

Moreover, from $\Gamma \ge n+5$ and $\varepsilon^0 \le 1$, we have

$$x_{n+2}^0 - y_{n+2}^0 = n + 1 - \Gamma \le -4 < -2\varepsilon^0.$$

Therefore $H'(z^0,\varepsilon^0)$ is nonsingular by Lemma 2.4. □

THEOREM 3.2. *If M is a P_0 matrix, then Algorithm 3.1 is well defined, and the sequence $\{z^k\}$ satisfies*

$$\|H(z^k, \varepsilon^k)\| \leq c\varepsilon^k. \tag{3.5}$$

Proof. We prove this theorem by induction.

For $k = 0$, by Lemma 3.1, $z^0 = (e, 1, 0, e, 1, \Gamma-(n+1))$ satisfies (3.5) and $H'(z^0, \varepsilon^0)$ is nonsingular.

We suppose that z^k satisfies (3.5) and $H'(z^k, \varepsilon^k)$ is nonsingular. Then Step 1 is well defined. If $H(z^k, \varepsilon^k) \neq 0$, then $\Delta z^k \neq 0$. Hence Δz^k is a strictly decent direction of $\|H(\cdot, \varepsilon^k)\|$ at z^k, and so the line search procedure is finite by construction in Step 2.

Step 3 is well defined since if $H(z^k, \varepsilon^k) = 0$ then $z^{k+1} = z^k$ and

$$\|H(z^{k+1}, \varepsilon^k)\| = 0 < c\varepsilon^k.$$

Otherwise, by the construction of Step 2,

$$\|H(z^{k+1}, \varepsilon^k)\| < \|H(z^k, \varepsilon^k)\| \leq c\varepsilon^k,$$

which implies that there is a finite number $\gamma_k > 0$ such that (3.3) holds.

By the construction of Step 3, $H'(z^{k+1}, \varepsilon^{k+1})$ is nonsingular.

Now we show that (3.5) holds at $(z^{k+1}, \varepsilon^{k+1})$.

If $x^{k+1}_{n+2} - y^{k+1}_{n+2} \neq -2(1-\gamma_k)\varepsilon^k$, then by construction of Step 3, (3.5) holds. Hence we only need to consider the case

$$x^{k+1}_{n+2} - y^{k+1}_{n+2} = -2(1-\gamma_k)\varepsilon^k,$$

i.e.,

$$\varepsilon^{k+1} = (1 - \alpha_2\gamma_k)\varepsilon^k.$$

Notice that

$$\varepsilon^k > (1 - \alpha_2\gamma_k)\varepsilon^k > (1 - \gamma_k)\varepsilon^k$$

and Step 3 provides that

$$\begin{aligned} c\varepsilon^{k+1} &\geq c(1 - \alpha_2\gamma_k)\varepsilon^k \\ &\geq c(1 - \gamma_k)\varepsilon_k \\ &\geq H(z^{k+1}, (1 - \gamma_k)\varepsilon_k) \\ &\geq -c(1 - \gamma_k)\varepsilon^k. \end{aligned} \tag{3.6}$$

By Result 3 of Proposition 2.1 in [2], for $i = 1, 2, \ldots, n+1$, $\Psi_i(x_i^{k+1}, y_i^{k+1}, \cdot)$ is strictly decreasing with respect to ε, which gives

$$\begin{aligned} c\varepsilon^{k+1} &\geq \Psi_i(x_i^{k+1}, y_i^{k+1}, (1 - \gamma_k)\varepsilon^k) \\ &> \Psi_i(x_i^{k+1}, y_i^{k+1}, \varepsilon^{k+1}) \\ &= \Psi_i(x_i^{k+1}, y_i^{k+1}, (1 - \alpha_2\gamma_k)\varepsilon^k) \\ &\geq \Psi_i(x_i^{k+1}, y_i^{k+1}, (1 - \gamma_k)\varepsilon^k) + (\alpha_2 - 1)\gamma_k\varepsilon_k \end{aligned}$$

$$
\begin{aligned}
&> -c(1-\gamma_k)\varepsilon^k - (1-\alpha_2)\gamma_k\varepsilon^k \\
&= -c\varepsilon_k + c\gamma_k\varepsilon^k - \gamma_k\varepsilon^k + \alpha_2\gamma_k\varepsilon^k \\
&\geq -c\varepsilon^k + \alpha_2(c-1)\gamma_k\varepsilon^k + \alpha_2\gamma_k\varepsilon^k \\
&> -c\varepsilon^k + c\alpha_2\gamma_k\varepsilon^k \\
&= -c\varepsilon^{k+1},
\end{aligned}
$$

where the third inequality follows from Result 4 of Proposition 2.1 in [2] (also see [9]), and the fourth inequality follows from (3.6). Hence (3.5) holds for $i = 1, 2, \dots, n+1$.

Let $w = x_{n+2}^{k+1} - y_{n+2}^{k+1}$, and

$$\phi(\varepsilon) = \int_{-\infty}^{\frac{w}{\varepsilon}} (w - \varepsilon\mu)\rho_2(\mu)d\mu.$$

Then $\omega = -2(1-\gamma_k)\varepsilon^k < 0$ and

$$\phi'(\varepsilon) = -\int_{-\infty}^{\frac{w}{\varepsilon}} \mu\rho_2(\mu)d\mu \geq 0.$$

Hence ϕ is monotonically increasing function. This implies that Ψ_{n+2} is monotonically decreasing and Lipschitz continuous with respect to the parameter ε. Hence we obtain

$$
\begin{aligned}
c\varepsilon^{k+1} &\geq \Psi_{n+2}(x_{n+2}^{k+1}, y_{n+2}^{k+1}, (1-\gamma_k)\varepsilon^k) \\
&\geq \Psi_{n+2}(x_{n+2}^{k+1}, y_{n+2}^{k+1}, (1-\alpha_2\gamma_k)\varepsilon^k) \\
&\geq \Psi_{n+2}(x_{n+2}^{k+1}, y_{n+2}^{k+1}, (1-\gamma_k)\varepsilon^k) - 2(1-\alpha_2)\gamma_k\varepsilon^k \\
&\geq -c(1-\gamma_k)\varepsilon^k - 2(1-\alpha_2)\gamma_k\varepsilon_k \\
&= -c\varepsilon^k + (c - 2(1-\alpha_2))\gamma_k\varepsilon^k \\
&= -c\varepsilon^k + c\alpha_2\gamma_k\varepsilon_k + (c-2)(1-\alpha_2)\gamma_k\varepsilon^k \\
&\geq -c(1-\alpha_2\gamma_k)\varepsilon^k \\
&= -c\varepsilon^{k+1},
\end{aligned}
$$

where the third inequality follows from that Ψ_{n+2} is Lipschitz continuous with the Lipschitz constant 2 and $c \geq 2$. (See (2.2))

Therefore we have

$$c\varepsilon^{k+1} \geq H_i(z^{k+1}, \varepsilon^{k+1}) \geq -c\varepsilon^{k+1}, \quad \text{for} \quad i = n+3, \dots, 2(n+2).$$

By the definition of H, the parameter ε is not involved in the first $n+2$ components of H, i.e.,

$$H_i(z^{k+1}, \varepsilon^k) = H_i(z^{k+1}, \varepsilon^{k+1}), \text{ for } i = 1, 2, \dots n+2.$$

Hence (3.5) holds. □

THEOREM 3.3. *Suppose that M is a P_0 matrix. Then Algorithm 3.1 is well defined. Let $\{(z^k, \varepsilon^k)\}$ be a sequence generated by Algorithm 3.1.*

1. *$\{z^k\}$ remains in the bounded set $D(\varepsilon^0)$, and $\{\varepsilon^k\}$ decreases monotonically in R_{++}.*
2. *If an accumulation point $(\bar{z}, \bar{\varepsilon})$ of $\{(z^k, \varepsilon^k)\}$ satisfies $\bar{x}_{n+2} - \bar{y}_{n+2} \neq -2\bar{\varepsilon}$ or $\bar{\varepsilon} = 0$ then*

$$\lim_{k\to\infty} \varepsilon^k = 0 \quad \text{and} \quad \lim_{k\to\infty} H(z^k, \varepsilon^k) = 0, \tag{3.7}$$

and for all accumulation points $\{\hat{z}, \hat{\varepsilon}\}$ *of* $\{z^k, \varepsilon^k\}$

$$\hat{\varepsilon} = 0 \quad \text{and} \quad H(\hat{z}, \hat{\varepsilon}) = H_0(\hat{z}) = 0. \tag{3.8}$$

Proof. First we show that

$$D(\varepsilon^k) \subseteq D(\varepsilon^{k-1}), \quad \text{for} \quad \varepsilon^k \le \varepsilon^{k-1}.$$

Suppose that $z \in D(\varepsilon^k)$. Then

$$\begin{aligned}\|H(z, \varepsilon^{k-1})\| &\le \|H(z, \varepsilon^{k-1}) - H(z, \varepsilon^k)\| + \|H(z, \varepsilon^k)\| \\ &\le 2(\varepsilon^{k-1} - \varepsilon^k) + c\varepsilon^k \\ &= c\varepsilon^{k-1} - (c-2)(\varepsilon^{k-1} - \varepsilon^k) \\ &\le c\varepsilon^{k-1},\end{aligned}$$

where we use $c \ge 2$ and (2.2). Hence $z \in D(\varepsilon^{k-1})$. This, together with Theorem 3.1, implies that the sequence generated by Algorithm 3.1 remains in $D(\varepsilon^0)$.

By Theorem 3.2, Algorithm 3.1 is well defined. Furthermore, by construction of Algorithm 3.1,

$$0 < \varepsilon^{k+1} \le (1 - \alpha_2 \gamma_k)\varepsilon^k < \varepsilon^k.$$

Hence $\{\varepsilon^k\}$ is a monotonically decreasing sequence, and there is $\bar{\varepsilon}$ such that

$$\lim_{k \to \infty} \varepsilon^k = \bar{\varepsilon}.$$

If $\bar{\varepsilon} = 0$, then from Theorem 3.2, we have (3.7). Moreover, since $\{\varepsilon^k\}$ is a monotonically decreasing sequence, (3.5) and (3.7) imply (3.8).

Suppose on the contrary that $\bar{\varepsilon} > 0$. Then this implies $\gamma_k \to 0$.

Since $\{z^k\}$ remains in the bounded set $D(\varepsilon^0)$, taking a subsequence if necessary, we may assume that the sequence $\{z^k\}$ converges to some $\bar{z}$. Based on the ε reduction step, we have

$$\|H(z^k, (1 - \frac{1}{\alpha_2^2}\gamma_{k-1})\varepsilon^{k-1})\| > (1 - \frac{1}{\alpha_2^2}\gamma_{k-1})c\varepsilon^{k-1}.$$

Since $\gamma_k \to 0$, by passing to limits, we have

$$\|H(\bar{z}, \bar{\varepsilon})\| \ge c\bar{\varepsilon} > 0. \tag{3.9}$$

Since $\bar{x}_{n+2} - \bar{y}_{n+2} \ne -2\bar{\varepsilon}$, $H'(\bar{z}, \bar{\varepsilon})$ is nonsingular by Lemma 2.4. Hence we can find a unique solution $\Delta\bar{z}$ of the linear equations in Step 1. Furthermore, from (3.9) it is a strictly decent direction for $\|H(\cdot, \bar{\varepsilon})\|$ at $\bar{z}$. As a result, the corresponding linear search step length $\bar{\lambda}$ and ε reduction step length $\bar{\gamma}$ are both bounded below by a positive constant. Notice that the function H and its Jacobian H' are continuous in a neighborhood of $(\bar{z}, \bar{\varepsilon})$. It follows that Δz^k converges to $\Delta\bar{z}$ and therefore γ_k must be uniformly bounded below by some positive constant for all large k. This contradicts to the assumption that $\gamma_k \to 0$. Hence we must have $\varepsilon^k \to 0$, and so (3.7) and (3.8) hold. □

COROLLARY 3.4. *Suppose that the solution set* $S_0(M, q)$ *of the* P_0 *matrix* $LCP(M, q)$ *is nonempty and bounded. Then there exist* $\Gamma > 0$ *and* $\varepsilon^0 > 0$ *such that the sequence*

$\{z^k\}$ generated by Algorithm 3.1 is bounded and its limiting points are solutions of LCP(N,p).

Proof. By Lemma 2.6, there exists $\Gamma > 0$ and $\varepsilon^0 > 0$ such that $H'(z,\varepsilon)$ is nonsingular for all $z \in D(\varepsilon^0)$.

Hence at any accumulation point $(\bar{z},\bar{\varepsilon})$ generated by Algorithm 3.1, $\bar{x}_{n+2}-\bar{y}_{n+2} \neq -2\bar{\varepsilon}$. By Theorem 3.3, we complete the proof. □

COROLLARY 3.5. *Under Assumption 2.1, the conclusion of Corollary 3.4 holds.*

Proof. The proof is similar to that of Theorem 2.8. It is sufficient to show that any limit point of the sequence $\{t_2^k, s_2^k\}$ generated by Algorithm 3.1 for solving LCP(M_2,q_2) is a strictly complementarity solution.

Let

$$u^k = \Psi(t_2^k, s_2^k, \varepsilon^k).$$

By Corollary 3.4 and (ii) of Assumption 2.1, any limit point of $\{t_2^k, s_2^k\}$ is a solution of LCP(M_2,q_2), and

$$\|u^k\| \le c\varepsilon^k \to 0.$$

Notice that we have

$$s_2 - u^k = M_2(t_2^k - u^k) + \varepsilon^k r_2 + q_2 + (M_2 - I)u^k \tag{3.10}$$

$$t_2^k - u^k > 0, \quad s_2^k - u^k > 0, \quad (s_2^k - u^k)_i(t_2 - u^k)_i = (\varepsilon^k)^2. \tag{3.11}$$

Let

$$q_2(\varepsilon^k) = q_2 + (M_2 - I)u^k + \varepsilon^k r_2.$$

The boundedness of $S_0(M_2,q_2)$ implies there is $k_0 \ge 0$, such that for all $k \ge k_0$, $S_0(M_2,q_2(\varepsilon^k))$ is nonempty [18]. Since $(t_2^k - u^k, s_2^k - u^k)$ is an interior point of LCP$(M_2,q_2(\varepsilon^k))$, the monotone property of M_2 implies $S_0(M_2,q_2(\varepsilon^k))$ is bounded. Moreover, since $S_0(M_2,q_2)$ has a strictly complementarity solution and $\|q_2(\varepsilon^k)-q_2\| \le (\|r_2\| + c\|M-I\|)\varepsilon^k \to 0$, there is $k_1 \ge k_0$ such that for all $k \ge k_1$, $S_0(M_2,q_2(\varepsilon^k))$ has a strictly complementarity solution.

Therefore, by Theorem 2.8, there is ε^{k_1} such that for $\varepsilon \in (0,\varepsilon^{k_1})$ the smooth path for LCP$(M_2,q_2(\varepsilon^k))$ exists and leads to a strictly complementarity solution.

By (3.10) and (3.11), $(t_2^k - u^k, s_2^k - u^k)$ is on the path. Therefore, using $\|u^k\| \to c\varepsilon^k \to 0$ again, we complete this proof. □

We can restart Algorithm 3.1 when $\bar{x}_{n+2} - \bar{y}_{n+2} = -2\bar{\varepsilon}$. In particular, we have the following proposition.

PROPOSITION 3.6. *Suppose $H'(z,\varepsilon)$ is singular. Then for $\hat{\Gamma} = \Gamma + \varepsilon$ and*

$$\hat{z}_i = z_i,\ i \neq n+2, 2n+2, \quad \hat{x}_{n+2} = x_{n+2} - \frac{1}{2}\varepsilon, \quad \hat{y}_{n+2} = y_{n+2} + \frac{3}{2}\varepsilon,$$

$\hat{H}'(\hat{z},\varepsilon)$ is nonsingular and

$$\hat{H}(\hat{z},\varepsilon) = H(z,\varepsilon), \tag{3.12}$$

where $\hat{H}$ is the function using $\hat{\Gamma}$.

Proof. At the new point

$$\hat{z} = (x_1, \ldots, x_{n+1}, \hat{x}_{n+2}, y_1, \ldots, y_{n+1}, \hat{y}_{n+2})$$

the new LCP$(N, \hat{p})$ with the new $\hat{\Gamma}$ satisfies

$$\hat{H}_i(\hat{z}, \varepsilon) = H_i(z, \varepsilon), \quad i = 1, 2, \ldots, 2(n+1)+1$$

and

$$\hat{H}_{2(n+2)}(\hat{z}, \varepsilon) = \hat{x}_{n+2} = x_{n+2} - \frac{1}{2}\varepsilon = x_{n+2} + \frac{\varepsilon}{4}\int_{-4}^{-2}(2+\mu)d\mu = H_{2(n+2)}(z, \varepsilon),$$

where we use $\hat{x}_{n+2} - \hat{y}_{n+2} \leq -4\varepsilon$ and $\rho_2(\mu) = 0$ for $\mu \leq -4$. Hence we have (3.12). Furthermore, $\hat{H}'(\hat{z}, \varepsilon)$ is nonsingular since

$$\hat{x}_{n+2} - \hat{y}_{n+2} = x_{n+2} - y_{n+2} - 2\varepsilon = -4\varepsilon < -2\varepsilon.$$

□

4. Numerical results. In this section, we report numerical results for testing Algorithm 3.1. These test problems are P_0 matrix LCP with unbounded solution set, which include a random test problem and a Murty-type problem with an unbounded solution set.

Problem 1. Example 1.1

Problem 2. Example 2.1 with $\delta = \frac{1}{2}$.

Notice that this problem has no an interior point.

Problem 3. A Murty-type problem with an unbounded solution set.

$$M = \begin{pmatrix} 1 & 2 & \ldots & 2 & 2 \\ 0 & 1 & \ldots & 2 & 2 \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & 1 & 2 \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}, \qquad q = -\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 0 \end{pmatrix}.$$

The solution set of this problem contains the unbounded line $t = (0, \ldots 0, t_n)^T$ for all $t_n \geq 0.5$. Moreover, the set of interior points is empty.

Problem 4 A random problem [15].

$$M = \begin{pmatrix} P + D_1 & P + D_2 \\ -I & 0 \end{pmatrix} \qquad q = -\begin{pmatrix} e \\ 0 \end{pmatrix},$$

where $P = A^T A$, $A \in R^{\frac{n}{2} \times \frac{n}{2}}$ with $0 < a_{ij} < 1$, D_1 and D_2 are diagonal matrices with $1 \leq (D_1)_{ii}, (D_2)_{ii} \leq 3$. This problem is a P_0 matrix LCP and has an engineering application. The solution set of this problem contains the unbounded set $\{t = (0, \ldots 0, t_{\frac{n}{2}+1}, \ldots, t_n)^T : t_i \geq 1, i = \frac{n}{2}+1, \ldots, n\}$. Moreover, the set of interior points is empty.

We implemented Algorithm 3.1 in MATLAB using the following parameters:

$$\sigma = 0.125, \quad \alpha_1 = 0.625, \quad \alpha_2 = 0.925, \quad \Gamma = n+5, \quad c = 2\sqrt{2(n+2)} + 1.$$

TABLE 4.1
Numerical results of Algorithm 3.1

prob	n	k	k_m	$\|t\|+\|s\|$	$\|H_0\|$
1	3	5	0	1	2.0837e-9
2	2	9	0	0.5	7.2783e-9
3	100	10	2	2	1.9674e-9
4	100	10	7	0.1928	3.9348e-9

Based on Lemma 3.1, we chose the initial point as

$$x^0 = (e, 1, 0), \quad y^0 = (e, 1, \Gamma - (n+1)), \quad \varepsilon^0 = \frac{c+2}{2(c+1)}.$$

We terminate the iteration if $\|H_0(z^k)\| \le 1.0^{-8}$ or $k \ge 200$.

In Table 4.1, we report our results for these 4 problems. The columns in Table 4.1 have the following meanings:

prob: number of test problem,
n : dimension of test example,
k: number of iterations,
k_m the total iteration number of line search steps
$\|t\|+\|s\|$: the value $\|t\|+\|s\|$ at the final iterate
$\|H_0\|$: the value $\|H_0(z)\|$ at the final iterate

Remark 4.1 Regularization methods [7, 16, 17, 19] have been used successfully to solve ill-posed problems. However, if the original problem has a solution with $t = 0$, then the regularization method only can generate this solution. In many cases we need a nonzero solution or a strictly complementarity solution if it exists. In such cases, the Big-Γ smoothing method may give a satisfactory solution. Our numerical results show this advantage. In particular, Algorithm 3.1 generated the following final iterates.

Problem 1: $t = (0, 0.5, 0)$, $s = (0.5, 0, 0.5)$.
Problem 2: $t = (0, 0.5)$, $s = (0, 0)$.
Problem 3: $t = (0, \dots, 0, 1, 0)$, $s = (1, \dots, 1, 0, 0)$.
Problem 4: number of components with $t_i, s_i > 1.0^{-8}$ is n

All final iterates approach to maximal complementarity solutions. We guess such numerical results are due to the properties of the path. Let us use Example 1.1 to compare the Big-Γ smooth path with the regularization path [7]:

$$\{t(\varepsilon) \ : \ t(\varepsilon) \text{ is a solution of the LCP}(M + \varepsilon I, q), \quad \varepsilon > 0\}.$$

Example 1.1 contains a solution with $t = 0$.

Since $M + \varepsilon I$ is P matrix for every $\varepsilon > 0$, the regularization path $t(\varepsilon) \equiv (0, 0, 0)$ for every $\varepsilon > 0$.

In the Big-Γ smoothing model for Example 1.1, $r = e - Me - q = 0$. We choose $\Gamma = n + 5 = 8$. At the Big-Γ smooth path,

$$x_i > 0, y_i > 0 \quad y_i = (Nx + p)_i, \quad x_i y_i = \varepsilon^2, \quad i = 1, 2, 3, 4$$

and

$$x_5 = \int_{-\infty}^{(x_5 - y_5)/\varepsilon} (x_5 - y_5 - \varepsilon\mu)\rho_2(\mu)d\mu, \quad y_5 = \Gamma - x_1 - x_2 - x_3 - x_4 - x_5.$$

We can calculate this point for $\varepsilon \in (0,1]$:

$$x_1 = \frac{4\varepsilon^2}{\sqrt{1+8\varepsilon^2}+1} \le 1, \quad x_2 = \frac{\sqrt{1+8\varepsilon^2}+1}{4} \le 1, \quad x_3 = \frac{4\varepsilon^2}{\sqrt{1+8\varepsilon^2}+1} \le 1,$$

$$x_4 = \varepsilon \le 1, \quad \text{and} \quad x_5 = 0, \text{ or } x_5 = \Gamma - x_1 - x_2 - x_3 - x_4 - 2\varepsilon,$$

where we use $\rho_2(\mu) = 0$, for $\mu \notin [-4,0]$ to calculate x_5. Two Big-Γ smooth paths never cross each other and converge to two solutions

$$z^{*,1} = (0, \frac{1}{2}, 0, 0, \Gamma - \frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}, 0, 0),$$

and

$$z^{*,2} = (0, \frac{1}{2}, 0, 0, 0, \frac{1}{2}, 0, \frac{1}{2}, 0, \Gamma - \frac{1}{2}),$$

respectively, as $\varepsilon \to 0$. Both of them contain a strictly complementarity solution of the original LCP(M,q)

$$(t^*, s^*) = (0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}).$$

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