

SHAPE OPTIMIZATION IN THREE-DIMENSIONAL CONTACT PROBLEMS WITH COULOMB FRICTION*

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Abstract. We study the discretized problem of the shape optimization of three-dimensional elastic bodies in unilateral contact. The aim is to extend existing results to the case of contact problems obeying the Coulomb friction law. Mathematical modelling of the Coulomb friction problem leads to an implicit variational inequality. It is shown that for small coefficients of friction the discretized problem with Coulomb friction has a unique solution and that this solution is Lipschitzian as a function of a control variable describing the shape of the elastic body.

The two-dimensional case of this problem was studied by the authors in [2]; here we used the so-called implicit programming approach combined with the generalized differential calculus of Clarke. The extension of this technique to the three-dimensional situation is by no means straightforward. The main source of difficulties is the nonpolyhedral character of the second-order (Lorentz) cone, arising in the 3D model. To facilitate the computation of the subgradient information, needed in the used numerical method, we exploit the substantially richer generalized differential calculus of Mordukhovich. Numerical examples illustrate the efficiency and reliability of the suggested approach.

1. Introduction and preliminaries. Contact shape optimization is a special branch of structural optimization whose goal is to find shapes of deformable bodies which are in a mutual contact. A typical problem in many applications is to find shapes guaranteeing a-priori given stress distributions on parts in contact [1]. A specific feature of contact shape optimization is its nonsmooth character due to the fact that the respective state mapping is given by various types of variational inequalities. For contact problems without friction or with the so-called given friction (see [8]), whose mathematical models lead to variational inequalities of the first and the second kind, sensitivity analysis was done in [26] for continuous models and in [9] for discretized models. Assuming a more realistic Coulomb law of friction, the situation becomes much more complicated in view of the fact that the state problem is now represented by a non-trivial implicit variational inequality [5]. For the sake of simplicity we restrict ourselves to structures consisting of one deformable body unilaterally supported by a rigid foundation (the so-called Signorini problem) considering Coulomb friction on a common part.

The two-dimensional case of this problem was studied by the authors in [2]; there we used the so-called implicit programming approach [15, 20] combined with the generalized differential calculus of Clarke [3]. The extension of this technique to the 3D case meets, however, serious hurdles both in the numerical solution of the respective state problem as well as in appropriate stability and sensitivity issues. The main

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source of these hurdles is the *non-polyhedral* character of the *second-order (Lorentz) cone*, arising in the 3D model. To facilitate the computation of the subgradient information, needed in the used numerical method, we have thus invoked the results of [12] and exploited the substantially richer generalized differential calculus of B. Mordukhovich [16, 18]. This means that we compute now a matrix from the Clarke *generalized Jacobian* of the discretized state mapping via the *limiting (Mordukhovich) coderivative* of this mapping. This has enabled us an efficient treatment of coupled multifunctions arising on the right-hand side of the generalized equation defining the state mapping.

The outline of the paper is as follows. Section 2 is devoted to a brief description of the state problem, i.e., the contact problem with Coulomb friction in its original, infinite-dimensional formulation. In Section 3 we describe its finite-element discretization and introduce our shape optimization problem. Thereafter we present various properties of the discretized state mapping and end up with the proof of its *strong regularity*. Section 4 concerns the used implicit-programming method. In particular, it deals with the computation of Clarke's subgradients of the respective composite cost function which have to be supplied to the used algorithm of nonsmooth optimization. In this section, we make use of several sophisticated rules of generalized differentiation. The first part of the last Section 5 is devoted to the numerical solution of the state problem. In the second part we present the test examples.

Our notation is standard: \bar{A} denotes the closure of a set A and, for a multifunction $\Phi : X \rightrightarrows Y$, $\text{Gr } \Phi = \{(x, y) \in X \times Y \mid y \in \Phi(x)\}$ is the graph of Φ . \mathbb{B}_R denotes a ball in \mathbb{R}^n of radius R centered at the origin. For a vector $x \in \mathbb{R}^n$ and an index set $\mathcal{N} \subset \{1, 2, \dots, n\}$, $x_{\mathcal{N}}$ denotes the subvector of x composed from the components x_i , $i \in \mathcal{N}$. For a Lipschitz single-valued mapping $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$, $\bar{\partial}F(x)$ is the *generalized Jacobian* of Clarke, defined by

$$\bar{\partial}F(x) = \text{conv}\left\{\lim_{i \rightarrow \infty} \nabla F(x_i) \mid x_i \xrightarrow{\Omega_F} x\right\},$$

where Ω_F is the set of points at which F is differentiable. If $m = 1$, we speak of the *Clarke subdifferential*.

In Section 4 we extensively use the following notions of the generalized differential calculus of Mordukhovich [17, 18].

Given a closed set $A \subset \mathbb{R}^n$ and a point $\bar{x} \in A$, we denote by $\widehat{N}_A(\bar{x})$ the *Fréchet (regular) normal cone* to A at \bar{x} , defined by

$$\widehat{N}_A(\bar{x}) = \{x^* \in \mathbb{R}^n \mid \limsup_{x \xrightarrow{A} \bar{x}} \frac{\langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \leq 0\}.$$

The *limiting (Mordukhovich) normal cone* to A at \bar{x} , denoted $N_A(\bar{x})$, is defined by

$$N_A(\bar{x}) := \text{Lim sup}_{x \xrightarrow{A} \bar{x}} \widehat{N}_A(x),$$

where “Lim sup” is the Kuratowski-Painlevé outer limit of sets (see [24]). If A is convex, then $N_A(\bar{x}) = \widehat{N}_A(\bar{x})$ amounts to the classic normal cone in the sense of convex analysis.

On the basis of the above notions, we can also describe the local behaviour of multifunctions. Let $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a multifunction with closed graph and $(\bar{x}, \bar{y}) \in \text{Gr } \Phi$. The multifunction $\widehat{D}^*\Phi(\bar{x}, \bar{y}) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$, defined by

$$\widehat{D}^*\Phi(\bar{x}, \bar{y})(y^*) := \{x^* \in \mathbb{R}^n \mid (x^*, -y^*) \in \widehat{N}_{\text{Gr } \Phi}(\bar{x}, \bar{y})\}$$

is called *regular coderivative* of Φ at (\bar{x}, \bar{y}) . Analogously, the multifunction $D^*\Phi(\bar{x}, \bar{y}) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$, defined by

$$D^*\Phi(\bar{x}, \bar{y})(y^*) := \{x^* \in \mathbb{R}^n \mid (x^*, -y^*) \in N_{\text{Gr } \Phi}(\bar{x}, \bar{y})\}$$

is called *limiting (Mordukhovich) coderivative* of Φ at (\bar{x}, \bar{y}) . If Φ happens to be single-valued, we simply write $\widehat{D}^*\Phi(\bar{x})(D^*\Phi(\bar{x}))$. If Φ is continuously differentiable, then $\widehat{D}^*\Phi(\bar{x}) = D^*\Phi(\bar{x})$ amounts to the adjoint Jacobian.

2. Setting of the problem. We start with the definition of the state problem. Let $\hat{\Omega} = \mathcal{R} \times (0, c)$, $\mathcal{R} = (0, a) \times (0, b)$, be a block in \mathbb{R}^3 , $a, b, c > 0$ given. By U_{ad} we denote a family of admissible functions, where

$$U_{ad} = \{\alpha \in C^{0,1}(\bar{\mathcal{R}}) \mid 0 \leq \alpha \leq C_0 \text{ in } \bar{\mathcal{R}}, \|\alpha'\|_{\infty, \mathcal{R}} \leq C_1, \\ C_2 \leq \int_{\mathcal{R}} \alpha \, dx_1 dx_2 \leq C_3\}, \quad (2.1)$$

i.e., the set U_{ad} contains all non-negative, bounded, Lipschitz equi-continuous functions in $\bar{\mathcal{R}}$, satisfying an integral type constraint. Positive numbers C_0, C_1, C_2, C_3 are chosen in such a way that $U_{ad} \neq \emptyset$. With any $\alpha \in U_{ad}$ we associate a subdomain $\Omega(\alpha) \subset \hat{\Omega}$:

$$\Omega(\alpha) = \{(x_1, x_2, x_3) \in \hat{\Omega} \mid x_3 \geq \alpha(x_1, x_2) \quad \forall (x_1, x_2) \in \mathcal{R}\}.$$

Functions $\alpha \in U_{ad}$ will play the role of *control variables* determining the shape of $\Omega(\alpha)$.

Let $\alpha \in U_{ad}$ be *fixed* and consider an elastic body represented by $\Omega(\alpha)$. Its boundary $\partial\Omega(\alpha)$ is split into three non-empty, non-overlapping parts $\Gamma_u(\alpha)$, $\Gamma_p(\alpha)$ and $\Gamma_c(\alpha)$: $\partial\Omega(\alpha) = \bar{\Gamma}_u(\alpha) \cup \bar{\Gamma}_p(\alpha) \cup \bar{\Gamma}_c(\alpha)$, where different boundary conditions will be prescribed. On $\Gamma_u(\alpha)$ the body is fixed, while surface tractions of density $P = (P_1, P_2, P_3)$ act on $\Gamma_p(\alpha)$. The body is *unilaterally* supported along $\Gamma_c(\alpha)$ by a rigid foundation represented by the half-space $\mathbb{R}^2 \times \mathbb{R}_-$, where

$$\Gamma_c(\alpha) = \{(x_1, x_2, x_3) \mid x_3 = \alpha(x_1, x_2) \quad \forall (x_1, x_2) \in \mathcal{R}\}$$

is the graph of α . Finally, $\Omega(\alpha)$ is subject to body forces of density $F = (F_1, F_2, F_3)$. Our aim is to find an equilibrium state taking into account friction on $\Gamma_c(\alpha)$. This state is characterized by a displacement field $u = (u_1, u_2, u_3)$ satisfying the following system of differential equations and boundary conditions:

- (*equilibrium equations*):

$$\frac{\partial \sigma_{ij}}{\partial x_j} + F_i = 0 \text{ in } \Omega(\alpha), \quad i = 1, 2, 3; \quad)^* \quad (2.2)$$

- (*Hooke's law*):

$$\sigma_{ij} := \sigma_{ij}(u) = c_{ijkl} \epsilon_{kl}(u) \text{ in } \Omega(\alpha), \quad \text{where } \epsilon_{kl} = 1/2 \left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} \right), \\ i, j, k, l = 1, 2, 3; \quad (2.3)$$

*Here and in what follows the summation convention is adopted.

- (kinematical boundary conditions):

$$u_i = 0 \text{ on } \Gamma_u(\alpha), \quad i = 1, 2, 3; \quad (2.4)$$

- (prescribed tractions):

$$T_i := \sigma_{ij}\nu_j = P_i \text{ on } \Gamma_p(\alpha), \quad i = 1, 2, 3; \quad (2.5)$$

- (unilateral conditions):

$$\left. \begin{aligned} u_3(x', \alpha(x')) &\geq -\alpha(x') \quad \forall x' = (x_1, x_2) \in \mathcal{R} \\ T_3(x) := \sigma_{33}(x) &\geq 0, \quad T_3(x)(u_3(x) + \alpha(x')) = 0 \quad \forall x \in \Gamma_c(\alpha) \end{aligned} \right\} \quad (2.6)$$

- (Coulomb law of friction):

$$\left. \begin{aligned} \text{if } u_t(x) := (u_1(x), u_2(x), 0) = 0 &\Rightarrow \|T_t(x)\| \leq \mathcal{F}T_3(x), \\ &\text{where } T_t(x) := (T_1(x), T_2(x), 0), \quad x \in \Gamma_c(\alpha); \\ \text{if } u_t(x) \neq 0 &\Rightarrow T_t(x) = -\mathcal{F}T_3(x) \frac{u_t(x)}{\|u_t(x)\|}, \quad x \in \Gamma_c(\alpha). \end{aligned} \right\} \quad (2.7)$$

Remark 2.1. The equations and boundary conditions (2.2)-(2.7) represent the *classical* formulation of a contact problem with Coulomb friction. The meaning of symbols is the following: $\sigma = (\sigma_{ij})_{i,j=1}^3$ stands for a symmetric stress tensor which is related to a linearized strain tensor $\epsilon = (\epsilon_{ij})_{i,j=1}^3$ by means of a linear Hooke's law (2.3), ν is the unit outward normal vector to $\partial\Omega(\alpha)$ and $T = (T_1, T_2, T_3)$ denotes the stress vector on $\partial\Omega(\alpha)$. Finally, $\| \cdot \|$ is the Euclidean norm of a vector and \mathcal{F} is a coefficient of Coulomb friction. In what follows we shall suppose that \mathcal{F} is a *positive constant*.

To give a weak form of the state problem we introduce the following spaces and sets:

$$\begin{aligned} V(\alpha) &= \{v = (v_1, v_2, v_3) \in (H^1(\Omega(\alpha)))^3 \mid v = 0 \text{ on } \Gamma_u(\alpha)\} \\ K(\alpha) &= \{v \in V(\alpha) \mid v_3(x', \alpha(x')) \geq -\alpha(x') \text{ a.e. in } \mathcal{R}\} \\ X &= \{\varphi \in L^2(\mathcal{R}) \mid \exists v \in V(\alpha) : \varphi(x') = v_3(x', \alpha(x')), \quad x' \in \mathcal{R}\} \\ X' &\text{ is the dual of } X, \quad X'_+ \text{ is the cone of positive elements of } X'. \end{aligned}$$

The duality pairing between X' and X will be denoted by $\langle \cdot, \cdot \rangle$.

We start with a simpler model of friction in which the unknown component T_3 in (2.7) is replaced by a given slip bound $g \in X'_+$ (model with given friction, see [8]). Its mathematical formulation leads to a *variational inequality* of the 2nd kind:

$$\left. \begin{aligned} \text{Find } u := u(g) &\in K(\alpha) \text{ such that} \\ a_\alpha(u, v - u) + \langle \mathcal{F}g, \|\hat{v}_t\| - \|\hat{u}_t\| \rangle &\geq L_\alpha(v - u) \quad \forall v \in K(\alpha), \end{aligned} \right\} \quad (\mathcal{P}(g))$$

where

$$a_\alpha(u, v) := \int_{\Omega(\alpha)} c_{ijkl} \epsilon_{ij}(u) \epsilon_{kl}(v) \, dx \quad (2.8)$$

$$L_\alpha(v) := \int_{\Omega(\alpha)} F_i v_i \, dx + \int_{\Gamma_p(\alpha)} P_i v_i \, ds \quad (2.9)$$

is the inner energy and the work of external forces, respectively. Further, $\|\hat{v}_t\|$ denotes the Euclidean norm of the vector $\hat{v}_t := v_t \circ \alpha$ given by $(v_1(x', \alpha(x')), v_2(x', \alpha(x')), 0)$, $x' \in \mathcal{R}$. Next we shall suppose that $F \in (L^2(\hat{\Omega}))^3$, $P \in (L^2(\partial\hat{\Omega}))^3$ and the linear elasticity coefficients c_{ijkl} in (2.3) satisfy the usual symmetry and ellipticity conditions [19]. It is well-known that under these assumptions problem $(\mathcal{P}(g))$ has a unique solution $u(g)$ for every $g \in X'_+$. To release the unilateral constraint $u(g) \in K(\alpha)$ we use an alternative formulation of $(\mathcal{P}(g))$ involving Lagrange multipliers:

$$\left. \begin{aligned} \text{Find } (u, \lambda) &:= (u(g), \lambda(g)) \in V(\alpha) \times X'_+ \text{ such that} \\ a_\alpha(u, v - u) + \langle \mathcal{F}g, \|\hat{v}_t\| - \|\hat{u}_t\| \rangle &\geq L_\alpha(v - u) + \langle \lambda, \hat{v}_3 - \hat{u}_3 \rangle \forall v \in V(\alpha) \\ \langle \mu - \lambda, \hat{u}_3 + \alpha \rangle &\geq 0 \quad \forall \mu \in X'_+, \end{aligned} \right\} (\mathcal{M}(g))$$

where $\hat{v}_3(x') := v_3(x', \alpha(x'))$, $x' \in \mathcal{R}$. It is easy to show that problem $(\mathcal{M}(g))$ has a unique solution $(u(g), \lambda(g))$ for every $g \in X'_+$. This makes it possible to define a mapping $\Phi : X'_+ \mapsto X'_+$ by

$$\Phi(g) = \lambda(g) \quad \forall g \in X'_+, \quad (2.10)$$

where $\lambda(g) \in X'_+$ is the second component of the solution to $(\mathcal{M}(g))$.

DEFINITION 2.1. *By a weak solution of a contact problem with Coulomb friction we call any pair $(u, \lambda) \in V(\alpha) \times X'_+$ satisfying*

$$\left. \begin{aligned} a_\alpha(u, v - u) + \langle \mathcal{F}\lambda, \|\hat{v}_t\| - \|\hat{u}_t\| \rangle &\geq L_\alpha(v - u) + \langle \lambda, \hat{v}_3 - \hat{u}_3 \rangle \quad \forall v \in V(\alpha) \\ \langle \mu - \lambda, \hat{u}_3 + \alpha \rangle &\geq 0 \quad \forall \mu \in X'_+. \end{aligned} \right\} (Q(\alpha))$$

From this definition it follows that λ is a fixed-point of Φ . The existence of such fixed-points has been analyzed in [5].

So far the function $\alpha \in U_{ad}$ characterizing the shape of $\Omega(\alpha)$ has been fixed. Now we shall look at $\alpha \in U_{ad}$ as a control variable governing state problem $(Q(\alpha))$. Let \mathcal{S} be the respective control-to-state mapping defined by

$$\mathcal{S}(\alpha) = (u(\alpha), \lambda(\alpha)) \quad \forall \alpha \in U_{ad}, \quad (2.11)$$

where $(u(\alpha), \lambda(\alpha))$ is a solution to $(Q(\alpha))$ and denote by \mathcal{G} the graph of \mathcal{S} . Since $(Q(\alpha))$ may have more than one solution, the mapping \mathcal{S} is set-valued, in general.

Finally, let $J : \mathcal{G} \mapsto \mathbb{R}^1$ be a cost functional. Suppose that there exists a minimizer $(\alpha^*, u(\alpha^*), \lambda(\alpha^*))$ of J on \mathcal{G} . Then $\Omega^* := \Omega(\alpha^*)$ will be called an *optimal domain*. Since we are interested in numerical realization of this problem, its discretization will be necessary.

To discretize the geometry we introduce a system U_{ad}^h containing functions which are uniquely determined by a finite number of degrees of freedom (e.g. Bézier surfaces). Admissible domains will be then given by $\Omega(\alpha_h)$, where $\alpha_h \in U_{ad}^h$. To discretize the state problem we use a finite element method. The classical Galerkin method together with appropriate discretizations $V_h(\alpha_h)$, $K_h(\alpha_h)$, X_h and X'_h of $V(\alpha)$, $K(\alpha)$, X and X' , respectively, represent an efficient tool of solving frictional contact problems. Analogously to the continuous setting we introduce a discrete control-to-state mapping

\mathcal{S}_h defined on U_{ad}^h and its graph \mathcal{G}_h . The discrete shape optimization is then given by the minimization of J on \mathcal{G}_h . In the next section we present an algebraic form of this problem arising from a typical finite element approximation. From this the structure of the problem to be solved will be seen.

3. Algebraic form of contact problems with Coulomb friction. The aim of this section is to present the algebraic form of the discretized state problem and to analyze its basic properties. We will proceed analogously to the continuous setting.

Let the discretization parameter $h > 0$ be fixed. Next we shall use the following notation: $\|\cdot\|_s, \langle \cdot, \cdot \rangle_s$ stand for the Euclidean norm and the scalar product in \mathbb{R}^s , respectively. In the frequent case $s = 2$ the subscript will be omitted. Every $\alpha_h \in U_{ad}^h$ will be uniquely characterized by a discrete design variable $\boldsymbol{\alpha} \in \mathbb{R}^d$ and U_{ad}^h will be identified with a set $\mathcal{U} \subset \mathbb{R}^d$. We shall suppose that \mathcal{U} is a *convex, compact* subset of \mathbb{R}^d . Let $\{\varphi_i\}_{i=1}^n$ be a basis of a finite element space $V_h(\alpha_h)$, $\alpha_h \in U_{ad}^h$, and suppose that its dimension n does not depend on α_h . Then

$$\begin{aligned} a_{\alpha_h}(v_h, z_h) &= \langle \mathbf{A}(\boldsymbol{\alpha})\mathbf{v}, \mathbf{z} \rangle_n \\ L_{\alpha_h}(v_h) &= \langle \mathbf{L}(\boldsymbol{\alpha}), \mathbf{v} \rangle_n \quad \forall v_h, z_h \in V_h(\alpha_h), \end{aligned}$$

where $\mathbf{v} \in \mathbb{R}^n$ is the vector of coordinates of v_h with respect to $\{\varphi_i\}_{i=1}^n$ and $\mathbf{A}(\boldsymbol{\alpha}) \in \mathbb{R}^{n \times n}$, $\mathbf{L}(\boldsymbol{\alpha}) \in \mathbb{R}^n$ is a stiffness matrix and a load vector, respectively, both depending on $\boldsymbol{\alpha} \in \mathcal{U}$. The set of all $\mathbf{A}(\boldsymbol{\alpha}) \in \mathbb{R}^{n \times n}$, $\boldsymbol{\alpha} \in \mathcal{U}$, will be denoted by M . For the sake of consistence we will denote in the sequel all finite-dimensional vectors and all matrices by bold letters. At mappings, however, this convention will not be applied strictly and so most vector-valued and set-valued mappings are denoted by standard (nonbold) letters. We shall suppose that the following assumptions are satisfied:

$$\left. \begin{aligned} &\text{all matrices } \mathbf{A} \in M \text{ are symmetric and uniformly positive definite, i.e.,} \\ &\text{there exists } \gamma > 0 \text{ such that } \langle \mathbf{A}\mathbf{v}, \mathbf{v} \rangle_n \geq \gamma \|\mathbf{v}\|_n^2 \quad \forall \mathbf{v} \in \mathbb{R}^n \quad \forall \mathbf{A} \in M; \end{aligned} \right\} \quad (3.1)$$

$$\left. \begin{aligned} &\text{the mappings } \mathbf{A} : \mathcal{U} \mapsto M, \mathbf{L} : \mathcal{U} \mapsto \mathbb{R}^n \text{ are continuously} \\ &\text{differentiable on an open set containing } \mathcal{U}. \end{aligned} \right\} \quad (3.2)$$

We start with the algebraic form of the contact problem with given friction for *fixed* $\boldsymbol{\alpha} \in \mathcal{U}$. Let $\mathbf{g} \in \mathbb{R}_+^p$ ($p < n$) be a given discrete *slip bound* (p is related to the number of all contact nodes, i.e., the nodes of a used partition of $\widehat{\Omega}(\alpha_h)$ into finite elements which are placed on $\widehat{\Gamma}_c(\alpha_h) \setminus \widehat{\Gamma}_u(\alpha_h)$). The discretized total potential energy has the following form:

$$\mathcal{J}(\mathbf{v}) = \frac{1}{2} \langle \mathbf{v}, \mathbf{A}\mathbf{v} \rangle_n - \langle \mathbf{L}, \mathbf{v} \rangle_n + \mathcal{F}(\mathbf{g}, |\mathbf{T}\mathbf{v}|)_p \quad (3.3)$$

with $\mathbf{A} := \mathbf{A}(\boldsymbol{\alpha})$, $\mathbf{L} := \mathbf{L}(\boldsymbol{\alpha})$ for some $\boldsymbol{\alpha} \in \mathcal{U}$ (since $\boldsymbol{\alpha} \in \mathcal{U}$ is now fixed, it will be omitted in the argument of \mathbf{A} and \mathbf{L}). Further $T : \mathbb{R}^n \mapsto \mathbb{R}^{2p}$ is a linear mapping defined by $T\mathbf{v} := (T_1\mathbf{v}, T_2\mathbf{v}, \dots, T_p\mathbf{v})$, $\mathbf{v} \in \mathbb{R}^n$, where $T_i\mathbf{v} \in \mathbb{R}^2$ is the tangential nodal displacement vector at the i th contact node. The symbol $|\mathbf{T}\mathbf{v}| \in \mathbb{R}^p$ denotes a vector defined by

$$|\mathbf{T}\mathbf{v}| := (\|T_1\mathbf{v}\|, \dots, \|T_p\mathbf{v}\|).$$

Let \mathbb{K} be a closed convex subset of \mathbb{R}^n :

$$\mathbb{K} := \{\mathbf{v} \in \mathbb{R}^n \mid \mathbf{N}\mathbf{v} \geq -\boldsymbol{\alpha}\}, \quad (3.4)$$

where $\mathbf{N} \in \mathbb{R}^{p \times n}$ has the following properties:

- a) It has full row rank.
- b) Each column contains at most one non-zero element.
- c) $\mathbf{N}(\mathbb{R}^n) = \mathbf{N}(\ker T)$.

From a) it follows that

$$\exists \beta = \text{const.} > 0 \text{ such that } \sup_{\substack{\mathbf{v} \in \mathbb{R}^n \\ \mathbf{v} \neq \mathbf{0}}} \frac{\langle \boldsymbol{\mu}, \mathbf{N}\mathbf{v} \rangle_p}{\|\mathbf{v}\|_n} \geq \beta \|\boldsymbol{\mu}\|_p \quad \forall \boldsymbol{\mu} \in \mathbb{R}^p. \quad (3.5)$$

DEFINITION 3.1. *By a solution of a discrete contact problem with given friction we mean a vector $\mathbf{u} \in \mathbb{K}$ satisfying*

$$\langle \mathbf{A}\mathbf{u}, \mathbf{v} - \mathbf{u} \rangle_n + \mathcal{F}\langle \mathbf{g}, |\mathbf{T}\mathbf{v}| - |\mathbf{T}\mathbf{u}| \rangle_p \geq \langle \mathbf{L}, \mathbf{v} - \mathbf{u} \rangle_n \quad \forall \mathbf{v} \in \mathbb{K}. \quad (\mathcal{P}(\mathbf{L}, \mathbf{g}))$$

Next we will analyze the dependence of \mathbf{u} on \mathbf{L} and \mathbf{g} with $\boldsymbol{\alpha} \in \mathcal{U}$ being fixed. This is why we use notation $(\mathcal{P}(\mathbf{L}, \mathbf{g}))$.

To conduct this analysis, we introduce an equivalent formulation of $(\mathcal{P}(\mathbf{L}, \mathbf{g}))$ which involves Lagrange multipliers releasing the constraint $\mathbf{v} \in \mathbb{K}$. It reads as follows:

$$\left. \begin{aligned} \text{Find } (\mathbf{u}, \boldsymbol{\lambda}) \in \mathbb{R}^n \times \mathbb{R}_+^p \text{ such that} \\ \langle \mathbf{A}\mathbf{u}, \mathbf{v} - \mathbf{u} \rangle_n + \mathcal{F}\langle \mathbf{g}, |\mathbf{T}\mathbf{v}| - |\mathbf{T}\mathbf{u}| \rangle_p \\ \geq \langle \mathbf{L}, \mathbf{v} - \mathbf{u} \rangle_n + \langle \boldsymbol{\lambda}, \mathbf{N}\mathbf{v} - \mathbf{N}\mathbf{u} \rangle_p \quad \forall \mathbf{v} \in \mathbb{R}^n \\ \langle \boldsymbol{\mu} - \boldsymbol{\lambda}, \mathbf{N}\mathbf{u} + \boldsymbol{\alpha} \rangle_p \geq 0 \quad \forall \boldsymbol{\mu} \in \mathbb{R}_+^p. \end{aligned} \right\} \quad (\mathcal{M}(\mathbf{L}, \mathbf{g}))$$

The first component \mathbf{u} of the solution to $(\mathcal{M}(\mathbf{L}, \mathbf{g}))$ solves $(\mathcal{P}(\mathbf{L}, \mathbf{g}))$, the second component $\boldsymbol{\lambda}$ represents the discrete normal contact stress. The following result is easy to prove.

PROPOSITION 3.2. *Problems $(\mathcal{P}(\mathbf{L}, \mathbf{g}))$, $(\mathcal{M}(\mathbf{L}, \mathbf{g}))$ have a unique solution \mathbf{u} , $(\mathbf{u}, \boldsymbol{\lambda})$, respectively, for every $\mathbf{L} \in \mathbb{R}^n$ and $\mathbf{g} \in \mathbb{R}_+^p$.*

Proof. Both existence and uniqueness follow directly from (3.1) and (3.5). \square

Next we will establish several useful properties of the solution to $(\mathcal{M}(\mathbf{L}, \mathbf{g}))$.

PROPOSITION 3.3. *Let $(\mathbf{u}, \boldsymbol{\lambda})$ be the solution to $(\mathcal{M}(\mathbf{L}, \mathbf{g}))$. Then*

$$\|\mathbf{u}\|_n \leq \frac{1}{\gamma} \|\mathbf{L}\|_n, \quad (3.6)$$

$$\|\boldsymbol{\lambda}\|_m \leq \frac{1}{\beta} \left(\frac{\|\mathbf{A}\|}{\gamma} + 1 \right) \|\mathbf{L}\|_n, \quad (3.7)$$

where $\gamma, \beta > 0$ are from (3.1) and (3.5).

Proof. Inserting $\mathbf{v} = \mathbf{0} \in \mathbb{K}$ into $(\mathcal{P}(\mathbf{L}, \mathbf{g}))$ we obtain:

$$\gamma \|\mathbf{u}\|_n^2 \leq \langle \mathbf{A}\mathbf{u}, \mathbf{u} \rangle_n + \mathcal{F}\langle \mathbf{g}, |\mathbf{T}\mathbf{u}| \rangle_p \leq \langle \mathbf{L}, \mathbf{u} \rangle_n$$

from which (3.6) follows. Further, substitutions $\mathbf{v} = \mathbf{0}$ and $\mathbf{v} = 2\mathbf{u}$ into the first inequality in $(\mathcal{M}(\mathbf{L}, \mathbf{g}))$ yield:

$$\langle \mathbf{A}\mathbf{u}, \mathbf{u} \rangle_n + \mathcal{F}\langle \mathbf{g}, |\mathbf{T}\mathbf{u}| \rangle_p = \langle \mathbf{L}, \mathbf{u} \rangle_n + \langle \boldsymbol{\lambda}, \mathbf{N}\mathbf{u} \rangle_p.$$

Therefore,

$$\langle \mathbf{A}\mathbf{u}, \mathbf{v} \rangle_n + \mathcal{F}\langle \mathbf{g}, |\mathbf{T}\mathbf{v}| \rangle_p \geq \langle \mathbf{L}, \mathbf{v} \rangle_n + \langle \boldsymbol{\lambda}, \mathbf{N}\mathbf{v} \rangle_p \quad \forall \mathbf{v} \in \mathbb{R}^n. \quad (3.8)$$

From (3.8) it follows:

$$\langle \mathbf{A}\mathbf{u}, \mathbf{v} \rangle_n = \langle \mathbf{L}, \mathbf{v} \rangle_n + \langle \boldsymbol{\lambda}, \mathbf{N}\mathbf{v} \rangle_p \quad \forall \mathbf{v} \in \ker T. \quad (3.9)$$

From this and (3.5) we obtain (3.7) using that $\mathbf{N}(\mathbb{R}^n) = \mathbf{N}(\ker T)$. \square

Remark 3.1. It is worth noticing that the bounds (3.6) and (3.7) do not depend on $\mathcal{F} > 0$ and $\mathbf{g} \in \mathbb{R}_+^p$.

Assume now that \mathbf{g} is fixed and let $\Psi : \mathbb{R}^n \mapsto \mathbb{K} \times \mathbb{R}_+^p$ be a mapping defined by

$$\Psi(\mathbf{L}) = (\mathbf{u}, \boldsymbol{\lambda}),$$

where $(\mathbf{u}, \boldsymbol{\lambda})$ is a solution of $(\mathcal{M}(\mathbf{L}, \mathbf{g}))$. It is very easy to show that Ψ is Lipschitz on \mathbb{R}^n as follows from the next proposition.

PROPOSITION 3.4. *Let $(\mathbf{u}_i, \boldsymbol{\lambda}_i)$ be the solution of $(\mathcal{M}(\mathbf{L}_i, \mathbf{g}))$, $i = 1, 2$. Then*

$$\|\mathbf{u}_1 - \mathbf{u}_2\|_n \leq \frac{1}{\gamma} \|\mathbf{L}_1 - \mathbf{L}_2\|_n, \quad (3.10)$$

$$\|\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2\|_p \leq \frac{1}{\beta} \left(\frac{\|\mathbf{A}\|}{\gamma} + 1 \right) \|\mathbf{L}_1 - \mathbf{L}_2\|_n. \quad (3.11)$$

Proof. It can be done in the same way as the one of Proposition 3.3. \square

Next, to define a solution of a discrete contact problem with Coulomb friction, we fix \mathbf{L} and introduce the mapping $\Gamma : \mathbb{R}_+^p \mapsto \mathbb{R}_+^p$:

$$\Gamma(\mathbf{g}) = \boldsymbol{\lambda}, \quad (3.12)$$

where $\boldsymbol{\lambda}$ is the second component of a solution to $(\mathcal{M}(\mathbf{L}, \mathbf{g}))$.

DEFINITION 3.5. *As a solution of a discrete contact problem with Coulomb friction we declare any couple $(\mathbf{u}, \boldsymbol{\lambda}) \in \mathbb{R}^n \times \mathbb{R}_+^p$ satisfying*

$$\left. \begin{aligned} \langle \mathbf{A}\mathbf{u}, \mathbf{v} - \mathbf{u} \rangle_n + \mathcal{F} \langle \boldsymbol{\lambda}, |T\mathbf{v}| - |T\mathbf{u}| \rangle_p \\ \geq \langle \mathbf{L}, \mathbf{v} - \mathbf{u} \rangle_n + \langle \boldsymbol{\lambda}, \mathbf{N}\mathbf{v} - \mathbf{N}\mathbf{u} \rangle_p \quad \forall \mathbf{v} \in \mathbb{R}^n \\ \langle \boldsymbol{\mu} - \boldsymbol{\lambda}, \mathbf{N}\mathbf{u} + \boldsymbol{\alpha} \rangle_p \geq 0 \quad \forall \boldsymbol{\mu} \in \mathbb{R}_+^p, \end{aligned} \right\} \quad (\mathcal{P})$$

i.e., $\boldsymbol{\lambda}$ is a fixed point of Γ and $(\mathbf{u}, \boldsymbol{\lambda})$ solves $(\mathcal{M}(\mathbf{L}, \boldsymbol{\lambda}))$.

PROPOSITION 3.6. *For any $\mathbf{L} \in \mathbb{R}^n$ and any $\mathcal{F} > 0$ there exists at least one fixed point $\boldsymbol{\lambda}$ of Γ . All fixed points $\boldsymbol{\lambda}$ belong to $\mathbb{R}_+^p \cap \mathbb{B}_R$, where $R = \frac{1}{\beta} \left(\frac{\|\mathbf{A}\|}{\gamma} + 1 \right) \|\mathbf{L}\|_n$.*

Proof. It is easy to show that Γ is continuous in \mathbb{R}_+^p and maps $\mathbb{R}_+^p \cap \mathbb{B}_R$ into itself as follows from (3.7). The existence of a fixed point follows from Brouwer's fixed point theorem. \square

Now we show that Γ is even contractive in $\mathbb{R}_+^p \cap \mathbb{B}_R$, provided that \mathcal{F} is small enough. Indeed, denote by $\mathbf{u}_i \in \mathbb{K}$ solutions to $(\mathcal{P}(\mathbf{L}, \mathbf{g}_i))$, $i = 1, 2$. Then one has

$$\langle \mathbf{A}\mathbf{u}_i, \mathbf{v} - \mathbf{u}_i \rangle_n + \mathcal{F} \langle \mathbf{g}_i, |T\mathbf{v}| - |T\mathbf{u}_i| \rangle_p \geq \langle \mathbf{L}, \mathbf{v} - \mathbf{u}_i \rangle_n \quad \forall \mathbf{v} \in \mathbb{K}. \quad (3.13)_i$$

Inserting $\mathbf{v} := \mathbf{u}_2$ into (2.13)₁ and $\mathbf{v} := \mathbf{u}_1$ into (2.13)₂ and summing up these inequalities we obtain:

$$\|\mathbf{u}_1 - \mathbf{u}_2\|_n \leq \frac{\mathcal{F}}{\gamma} \|T\| \|\mathbf{g}_1 - \mathbf{g}_2\|_p. \quad (3.14)$$

From (3.9) we know that

$$\langle \mathbf{A}\mathbf{u}_i, \mathbf{v} \rangle_n = \langle \mathbf{L}, \mathbf{v} \rangle_n + \langle \boldsymbol{\lambda}_i, \mathbf{N}\mathbf{v} \rangle_p \quad \forall \mathbf{v} \in \ker T.$$

Hence

$$\|\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2\|_p \leq \frac{1}{\beta} \frac{\mathcal{F}}{\gamma} \|\mathbf{A}\| \|T\| \|\mathbf{g}_1 - \mathbf{g}_2\|_p. \quad (3.15)$$

In this way we proved the following theorem.

THEOREM 3.7. *Let*

$$0 < \mathcal{F} < \frac{\beta\gamma}{\|\mathbf{A}\| \|T\|}. \quad (3.16)$$

Then Γ defined by (3.12) is contractive in $\mathbb{R}_+^p \cap \mathbb{B}_R$ so that the discrete contact problem with Coulomb friction has a unique solution.

COROLLARY 3.8. *Let (3.16) be satisfied. Then the method of successive approximations*

$$\boldsymbol{\lambda}^{(0)} \in \mathbb{R}_+^p \text{ given;}$$

$$\text{for } k = 0, 1, \dots, \text{ set } \boldsymbol{\lambda}^{(k+1)} = \Gamma(\boldsymbol{\lambda}^{(k)}) \text{ until a stopping criterion is fulfilled}$$

is convergent to the unique fixed point of Γ for any choice of $\boldsymbol{\lambda}^{(0)} \in \mathbb{R}_+^p$.

Indeed, it follows from Proposition 3.3 that $\boldsymbol{\lambda}^{(1)} \in \mathbb{B}_R$ independently of the starting value $\boldsymbol{\lambda}^{(0)}$.

Remark 3.2. Let us notice that the bound (3.16) does not depend on $\mathbf{L} \in \mathbb{R}^n$ and it can be also chosen to be independent of $\boldsymbol{\alpha} \in \mathcal{U}$ as follows from (3.1), i.e., there exists an $\mathcal{F}_0 > 0$ such that for any $(\mathcal{F}, \mathbf{L}, \boldsymbol{\alpha}) \in (0, \mathcal{F}_0) \times \mathbb{R}^n \times \mathcal{U}$ there is a unique fixed point $\boldsymbol{\lambda}$ of Γ .

Let $\boldsymbol{\alpha} \in \mathcal{U}$ be fixed, $\mathcal{F} \in (0, \mathcal{F}_0)$ and $(\mathbf{u}, \boldsymbol{\lambda})$ be the unique solution of the contact problem with Coulomb friction in the sense of Definition 3.5. Now, we shall consider the couple $(\mathbf{u}, \boldsymbol{\lambda}) := (\mathbf{u}(\mathbf{L}), \boldsymbol{\lambda}(\mathbf{L}))$ to be a function of the load vector $\mathbf{L} \in \mathbb{R}^n$. We define the mapping $S_L : \mathbb{R}^n \mapsto \mathbb{K} \times \mathbb{R}_+^p$ by

$$S_L(\mathbf{L}) = (\mathbf{u}(\mathbf{L}), \boldsymbol{\lambda}(\mathbf{L})) \quad \forall \mathbf{L} \in \mathbb{R}^n.$$

An easy consequence of (3.11) and (3.14) is the fact that S_L is Lipschitz on \mathbb{R}^n uniformly with respect to $\boldsymbol{\alpha} \in \mathcal{U}$.

PROPOSITION 3.9. *Let $\mathcal{F} \in (0, \mathcal{F}_0)$. Then there exists $\delta > 0$ such that the inequality*

$$\|S_L(\mathbf{L}_1) - S_L(\mathbf{L}_2)\|_{n+p} \leq \delta \|\mathbf{L}_1 - \mathbf{L}_2\|_n$$

holds for all $\mathbf{L}_1, \mathbf{L}_2 \in \mathbb{R}^n$ and all $\boldsymbol{\alpha} \in \mathcal{U}$.

So far, $\boldsymbol{\alpha} \in \mathcal{U}$ has been fixed. Next we will look at $\boldsymbol{\alpha}$ as a control variable parameterizing our state problem. We denote by S the *control-to-state mapping* which assigns $\boldsymbol{\alpha} \in \mathbb{R}^d$ the solutions $(\mathbf{u}, \boldsymbol{\lambda})$ of the contact problem with Coulomb friction in the sense of Definition 3.5. We know that $S(\boldsymbol{\alpha})$ is nonempty for all $\boldsymbol{\alpha} \in \mathcal{U}$ and a singleton if $\mathcal{F} \in (0, \mathcal{F}_0)$. Let $J : \text{Gr } S \mapsto \mathbb{R}$ be an objective.

The *discrete optimal shape design problem* reads as follows:

$$\left. \begin{array}{l} \text{Find } \mathbf{z}^* := (\boldsymbol{\alpha}^*, \mathbf{u}^*, \boldsymbol{\lambda}^*) \in \text{Gr } S \text{ such that} \\ J(\mathbf{z}^*) \leq J(\mathbf{z}) \quad \forall \mathbf{z} \in \text{Gr } S. \end{array} \right\} \quad (\mathbb{P})$$

If $\mathcal{F} \in (0, \mathcal{F}_0)$, then S is single-valued and (\mathbb{P}) takes the form

$$\left. \begin{array}{l} \text{Find } \boldsymbol{\alpha}^* \in \mathcal{U} \text{ such that} \\ \Theta(\boldsymbol{\alpha}^*) \leq \Theta(\boldsymbol{\alpha}) \quad \forall \boldsymbol{\alpha} \in \mathcal{U}, \end{array} \right\} \quad (\tilde{\mathbb{P}})$$

where $\Theta(\boldsymbol{\alpha}) := J(\boldsymbol{\alpha}, S(\boldsymbol{\alpha}))$. The following existence result is easy to prove.

THEOREM 3.10. *Let J be lower semicontinuous on $\text{Gr } S$. Then (\mathbb{P}) has a solution.*

Proof. It is readily seen that $\text{Gr } S$ is a compact subset of $\mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^m$. \square

From now on we will suppose that $\mathcal{F} \in (0, \mathcal{F}_0)$. Our main aim is to show that the single-valued function S is Lipschitz on \mathcal{U} . We start with a reduction of the problem by eliminating all components of $\mathbf{u} \in \mathbb{R}^n$ associated with the non-contact nodes of the finite element partition of $\bar{\Omega}(\alpha_h)$. The resulting problem contains only the tangential components $\mathbf{u}_t \in \mathbb{R}^{2p}$ and the normal components $\mathbf{u}_\nu \in \mathbb{R}^p$ of \mathbf{u} at p contact nodes. In the next step we transform this reduced problem into the following *generalized equation* (GE) for the unknowns \mathbf{u}_t , \mathbf{u}_ν and $\boldsymbol{\lambda}$ (for details see Section 3 in [2]):

$$\left. \begin{array}{l} \mathbf{0} \in \mathbf{A}_{tt}(\boldsymbol{\alpha})\mathbf{u}_t + \mathbf{A}_{t\nu}(\boldsymbol{\alpha})\mathbf{u}_\nu - \mathbf{L}_t(\boldsymbol{\alpha}) + Q_t(\mathbf{u}_t, \boldsymbol{\lambda}) \\ \mathbf{0} = \mathbf{A}_{\nu t}(\boldsymbol{\alpha})\mathbf{u}_t + \mathbf{A}_{\nu\nu}(\boldsymbol{\alpha})\mathbf{u}_\nu - \mathbf{L}_\nu(\boldsymbol{\alpha}) - \boldsymbol{\lambda} \\ \mathbf{0} \in \mathbf{u}_\nu + \boldsymbol{\alpha} + N_{\mathbb{R}_+^p}(\boldsymbol{\lambda}), \end{array} \right\} \quad (3.17)$$

where $\mathbf{A}_{tt}(\boldsymbol{\alpha}) \in \mathbb{R}^{2p \times 2p}$, $\mathbf{A}_{\nu\nu}(\boldsymbol{\alpha}) \in \mathbb{R}^{p \times p}$, $\mathbf{A}_{\nu t}(\boldsymbol{\alpha}) \in \mathbb{R}^{p \times 2p}$, $\mathbf{A}_{t\nu}(\boldsymbol{\alpha}) = \mathbf{A}_{\nu t}^T(\boldsymbol{\alpha})$, $\mathbf{L}_t(\boldsymbol{\alpha}) \in \mathbb{R}^{2p}$, $\mathbf{L}_\nu(\boldsymbol{\alpha}) \in \mathbb{R}^p$.

In (3.17) $N_{\mathbb{R}_+^p}(\cdot)$ is the normal cone mapping in the sense of convex analysis and $Q_t : \mathbb{R}^{2p} \times \mathbb{R}^p \mapsto \mathbb{R}^{2p}$ is the set-valued mapping defined by

$$Q_t(\mathbf{u}_t, \boldsymbol{\lambda}) = \partial_{\mathbf{u}_t} j(\mathbf{u}_t, \boldsymbol{\lambda}), \quad j(\mathbf{u}_t, \boldsymbol{\lambda}) = \mathcal{F} \sum_{i=1}^p \lambda_i \|\mathbf{u}_t^i\|,$$

with $\mathbf{u}_t^i \in \mathbb{R}^2$ being the tangential displacement at the i th contact node, $\mathbf{u}_t = (\mathbf{u}_t^1, \dots, \mathbf{u}_t^p)$. Problem (3.17) can be written in a more compact form:

$$\mathbf{0} \in F(\boldsymbol{\alpha})\mathbf{y} - l(\boldsymbol{\alpha}) + Q(\mathbf{y}), \quad (3.18)$$

where $\mathbf{y} = (\mathbf{u}_t, \mathbf{u}_\nu, \boldsymbol{\lambda})$ and

$$F(\boldsymbol{\alpha}) = \begin{bmatrix} \mathbf{A}_{tt}(\boldsymbol{\alpha}) & \mathbf{A}_{t\nu}(\boldsymbol{\alpha}) & \mathbf{0} \\ \mathbf{A}_{\nu t}(\boldsymbol{\alpha}) & \mathbf{A}_{\nu\nu}(\boldsymbol{\alpha}) & -\mathbf{E} \\ \mathbf{0} & \mathbf{E} & \mathbf{0} \end{bmatrix},$$

$$l(\boldsymbol{\alpha}) = (\mathbf{L}_t(\boldsymbol{\alpha}), \mathbf{L}_\nu(\boldsymbol{\alpha}), -\boldsymbol{\alpha})^T,$$

$$Q(\mathbf{y}) = \left(Q_t(\mathbf{u}_t, \boldsymbol{\lambda}), \mathbf{0}, N_{\mathbb{R}_+^p}(\boldsymbol{\lambda}) \right)^T$$

with \mathbf{E} being the identity matrix.

In the rest of this section we prove that the GE (3.18) satisfies the *strong regularity condition* (SRC) of Robinson ([23]) at $(\boldsymbol{\alpha}, \mathbf{y}(\boldsymbol{\alpha}))$ for any $\boldsymbol{\alpha} \in \mathcal{U}$, where $\mathbf{y}(\boldsymbol{\alpha})$ is the unique solution of (3.18). Let $(\bar{\boldsymbol{\alpha}}, \bar{\mathbf{y}})$ be a reference point. According to [23] we associate with (3.18) the following perturbed model at $(\bar{\boldsymbol{\alpha}}, \bar{\mathbf{y}})$:

$$\boldsymbol{\xi} \in F(\bar{\boldsymbol{\alpha}})\mathbf{y} - l(\bar{\boldsymbol{\alpha}}) + Q(\mathbf{y}), \quad (3.19)$$

where $\boldsymbol{\xi} = (\boldsymbol{\xi}_t, \boldsymbol{\xi}_\nu, \boldsymbol{\xi}_\lambda) \in \mathbb{R}^{2p} \times \mathbb{R}^p \times \mathbb{R}^p$ is a canonical perturbation. This problem can be again interpreted as a contact problem with Coulomb friction with a *variable* load vector depending on $\boldsymbol{\xi}$ but formulated on a *given* shape characterized by $\bar{\boldsymbol{\alpha}} \in \mathcal{U}$. Indeed, (3.19) is equivalent to

$$\left. \begin{aligned} \mathbf{0} &\in \mathbf{A}_{tt}(\bar{\boldsymbol{\alpha}})\mathbf{u}_t + \mathbf{A}_{t\nu}(\bar{\boldsymbol{\alpha}})\mathbf{u}_\nu - \mathbf{L}_t(\bar{\boldsymbol{\alpha}}) - \boldsymbol{\xi}_t + Q_t(\mathbf{u}_t, \boldsymbol{\lambda}) \\ \mathbf{0} &= \mathbf{A}_{\nu t}(\bar{\boldsymbol{\alpha}})\mathbf{u}_t + \mathbf{A}_{\nu\nu}(\bar{\boldsymbol{\alpha}})\mathbf{u}_\nu - \mathbf{L}_\nu(\bar{\boldsymbol{\alpha}}) - \boldsymbol{\xi}_\nu - \boldsymbol{\lambda} \\ \mathbf{0} &\in \mathbf{u}_\nu + \bar{\boldsymbol{\alpha}} + \boldsymbol{\xi}_\lambda + N_{\mathbb{R}_+^p}(\boldsymbol{\lambda}). \end{aligned} \right\} \quad (3.20)$$

The last inclusion in (3.20) corresponds to the non-penetration condition $\mathbf{u}_\nu \geq -\bar{\boldsymbol{\alpha}} - \boldsymbol{\xi}_\lambda$. Denote by $\tilde{\boldsymbol{\xi}}_\lambda = (\mathbf{0}, \boldsymbol{\xi}_\lambda) \in \mathbb{R}^{2p}$ the extension of $\boldsymbol{\xi}_\lambda$ by the zero vector and write $\tilde{\mathbf{u}} = (\mathbf{u}_t, \mathbf{u}_\nu)$ in the form

$$\tilde{\mathbf{u}} = \mathbf{w} - \tilde{\boldsymbol{\xi}}_\lambda, \quad (3.21)$$

i.e., $\mathbf{u}_t = \mathbf{w}_t$ and $\mathbf{u}_\nu = \mathbf{w}_\nu - \boldsymbol{\xi}_\lambda$ where $\mathbf{w}_\nu \geq -\bar{\boldsymbol{\alpha}}$. Inserting (3.21) into (3.20) we obtain a new problem for the vector $\mathbf{w} := \mathbf{w}(\boldsymbol{\xi})$:

$$\left. \begin{aligned} \mathbf{0} &\in \mathbf{A}_{tt}(\bar{\boldsymbol{\alpha}})\mathbf{w}_t + \mathbf{A}_{t\nu}(\bar{\boldsymbol{\alpha}})\mathbf{w}_\nu - \mathbf{F}_t(\boldsymbol{\xi}) + Q_t(\mathbf{w}_t, \boldsymbol{\lambda}) \\ \mathbf{0} &= \mathbf{A}_{\nu t}(\bar{\boldsymbol{\alpha}})\mathbf{w}_t + \mathbf{A}_{\nu\nu}(\bar{\boldsymbol{\alpha}})\mathbf{w}_\nu - \mathbf{F}_\nu(\boldsymbol{\xi}) - \boldsymbol{\lambda} \\ \mathbf{0} &\in \mathbf{w}_\nu + N_{\mathbb{R}_+^p}(\boldsymbol{\lambda}), \end{aligned} \right\} \quad (3.22)$$

where

$$\begin{aligned} \mathbf{F}_t(\boldsymbol{\xi}) &:= \mathbf{L}_t(\bar{\boldsymbol{\alpha}}) + \boldsymbol{\xi}_t + \mathbf{A}_{t\nu}(\bar{\boldsymbol{\alpha}})\boldsymbol{\xi}_\lambda \\ \mathbf{F}_\nu(\boldsymbol{\xi}) &:= \mathbf{L}_\nu(\bar{\boldsymbol{\alpha}}) + \boldsymbol{\xi}_\nu + \mathbf{A}_{\nu\nu}(\bar{\boldsymbol{\alpha}})\boldsymbol{\xi}_\lambda \end{aligned}$$

represent a perturbation of the original load vector depending on $\boldsymbol{\xi}$. From Remark 3.2 we know that there exists a unique solution of (3.22) for any $\boldsymbol{\xi}$ and Proposition 3.9 says that this solution as a function of load vectors is Lipschitz on \mathbb{R}^n . Hence, we proved the following result.

THEOREM 3.11. *Let $\mathcal{F} \in (0, \mathcal{F}_0)$ with $\mathcal{F}_0 > 0$ sufficiently small. Then the respective GE (3.18) is strongly regular at each triple $(\boldsymbol{\alpha}, \mathbf{u}, \boldsymbol{\lambda})$ with $\boldsymbol{\alpha} \in \mathcal{U}$ and $(\mathbf{u}, \boldsymbol{\lambda}) = S(\boldsymbol{\alpha})$. Consequently, the control-to-state mapping S is Lipschitz on \mathcal{U} .*

Proof. Follows from Theorem 2.1 in [23]. \square

On the basis of the results of this section the Lipschitz continuity of S could be proved in a direct way, without using the mentioned Robinson's result. The strong regularity of (3.18) will play, however, an important role in the next section.

4. Implicit programming approach (ImP) and sensitivity analysis. In this section the scalar product in \mathbb{R}^n will be denoted by $\langle \cdot, \cdot \rangle$ without any subscript related to the dimension. Consider the problem $(\tilde{\mathbb{P}})$ and assume that the objective J

is continuously differentiable. By virtue of the assumptions and Theorem 3.11, Θ is locally Lipschitz on an open set containing \mathcal{U} . It follows that $(\bar{\mathbb{P}})$ can be solved by a suitable bundle method of nondifferentiable optimization, provided the structure of \mathcal{U} is not too complicated. To this end we have to be able to compute for each admissible design variable $\bar{\alpha}$

- (i) the solution $\bar{\mathbf{y}} = S(\bar{\alpha})$ of the respective contact problem, and
- (ii) an arbitrary subgradient $\boldsymbol{\xi}$ from the Clarke subdifferential $\bar{\partial}\Theta(\bar{\alpha})$.

This section is devoted to task (ii). We start with the observation that

$$\bar{\partial}\Theta(\bar{\alpha}) = \nabla_{\boldsymbol{\alpha}} J(\bar{\alpha}, \bar{\mathbf{y}}) + \text{conv} \{ \mathbf{C}^T \nabla_{\mathbf{y}} J(\bar{\alpha}, \bar{\mathbf{y}}) \mid \mathbf{C} \in \bar{\partial}S(\bar{\alpha}) \}$$

which follows directly from the Chain Rule 2.6.6 in [3]. Since $\emptyset \neq D^*S(\boldsymbol{\alpha})(\mathbf{y}^*) \subset \text{conv} \{ \mathbf{C}^T \mathbf{y}^* \mid \mathbf{C} \in \bar{\partial}S(\bar{\alpha}) \}$ for all \mathbf{y}^* , it suffices for our purposes to compute just one element from the set

$$D^*S(\bar{\alpha})(\nabla_{\mathbf{y}} J(\bar{\alpha}, \bar{\mathbf{y}})),$$

and we are done. To be able to express the coderivative $D^*S(\bar{\alpha})$ in terms of the data of the GE (3.18), we re-order this GE in such a way that the multivalued part

$$Q(\mathbf{y}) = \begin{bmatrix} \Phi(\mathbf{y}^1) \\ \Phi(\mathbf{y}^2) \\ \vdots \\ \Phi(\mathbf{y}^p) \end{bmatrix},$$

where the subvector $\mathbf{y}^i = (\mathbf{u}_i^i, u_\nu^i, \lambda^i) \in \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}_+$ comprises all state variables associated with the i th contact node, and

$$\Phi(\mathbf{y}^i) = \begin{bmatrix} \mathcal{F}\lambda_i \partial \|\mathbf{u}_i^i\|_2 \\ 0 \\ N_{\mathbb{R}_+}(\lambda_i) \end{bmatrix},$$

$i = 1, 2, \dots, p$. Let $\bar{\mathbf{d}} = (\bar{\mathbf{d}}^1, \bar{\mathbf{d}}^2, \dots, \bar{\mathbf{d}}^p) \in (\mathbb{R}^4)^p$ belong to the image set $Q(\bar{\mathbf{y}})$ so that

$$\bar{\mathbf{d}}^i \in \Phi(\bar{\mathbf{y}}^i); \quad i = 1, 2, \dots, p.$$

Due to the above re-ordering, $\mathbf{y}^* = (\mathbf{y}^{*1}, \mathbf{y}^{*2}, \dots, \mathbf{y}^{*p}) \in (\mathbb{R}^4)^p$ belongs to $D^*Q(\bar{\mathbf{y}}, \bar{\mathbf{d}})(\mathbf{d}^*)$ with $\mathbf{d}^* = (\mathbf{d}^{*1}, \mathbf{d}^{*2}, \dots, \mathbf{d}^{*p}) \in (\mathbb{R}^4)^p$ whenever

$$\mathbf{y}^{*i} \in D^*\Phi(\bar{\mathbf{y}}^i, \bar{\mathbf{d}}^i)(\bar{\mathbf{d}}^{*i}), \quad i = 1, 2, \dots, p.$$

This facilitates the computation of the coderivatives \widehat{D}^*Q and D^*Q conducted in the subsequent analysis. Put $m := 4p$.

THEOREM 4.1. *Consider the reference pair $(\bar{\alpha}, \bar{\mathbf{y}})$ with $\bar{\alpha} \in \mathcal{U}$, $\bar{\mathbf{y}} = S(\bar{\alpha})$. Then for each $\mathbf{y}^* \in \mathbb{R}^m$*

$$(\nabla_{\boldsymbol{\alpha}}(F(\bar{\alpha})\bar{\mathbf{y}}))^T \mathbf{z} \subset D^*S(\bar{\alpha})(\mathbf{y}^*) \subset (\nabla_{\boldsymbol{\alpha}}(F(\bar{\alpha})\bar{\mathbf{y}}))^T \mathcal{V}, \quad (4.1)$$

provided \mathbf{z} is a solution of the regular adjoint GE (RAGE)

$$0 \in \mathbf{y}^* + (F(\bar{\alpha}))^T \mathbf{z} + \widehat{D}^*Q(\bar{\mathbf{y}}, -F(\bar{\alpha})\bar{\mathbf{y}})(\mathbf{z}), \quad (4.2)$$

and \mathcal{V} is the set of solutions \mathbf{v} to the (limiting) adjoint GE (AGE)

$$0 \in \mathbf{y}^* + (F(\bar{\boldsymbol{\alpha}}))^T \mathbf{v} + D^*Q(\bar{\mathbf{y}}, -F(\bar{\boldsymbol{\alpha}})\bar{\mathbf{y}})(\mathbf{v}). \quad (4.3)$$

Proof. By Theorem 3.11 the GE (3.18) satisfies SRC at $(\bar{\boldsymbol{\alpha}}, \bar{\mathbf{y}})$. It follows from [22, Proposition 3.2] that the qualification condition

$$\left. \begin{aligned} 0 &= (\nabla_{\boldsymbol{\alpha}}(F(\bar{\boldsymbol{\alpha}})\bar{\mathbf{y}}))^T \mathbf{v} \\ 0 &\in (F(\bar{\boldsymbol{\alpha}}))^T \mathbf{v} + D^*Q(\bar{\mathbf{y}}, -F(\bar{\boldsymbol{\alpha}})\bar{\mathbf{y}})(\mathbf{v}) \end{aligned} \right\} \Rightarrow \mathbf{v} = 0,$$

is fulfilled as well. Thus, the result holds true by virtue of [12, Theorem 2]. \square

COROLLARY 4.2. *If $\text{Gr } Q$ is normally regular at $(\bar{\mathbf{y}}, -F(\bar{\boldsymbol{\alpha}})\bar{\mathbf{y}})$, then the second inclusion in (4.1) becomes equality.*

Let \mathbf{y}^* be an arbitrary vector from \mathbb{R}^m . The set $\{\mathbf{C}^T \mathbf{y}^* \mid \mathbf{C} \in \bar{\partial}S(\bar{\boldsymbol{\alpha}})\}$ is nonempty by the Lipschitz continuity of S and amounts to $\text{conv } D^*S(\bar{\boldsymbol{\alpha}})(\mathbf{y}^*)$, see [17, Eq.2.23]. From this we conclude that AGE possesses a solution $\tilde{\mathbf{v}}$ for each $\mathbf{y}^* \in \mathbb{R}^m$. Further, again by the Lipschitz continuity of S , AGE has among its solutions at least one which is the image of \mathbf{y}^* in the linear mapping $(\lim_{i \rightarrow \infty} (\nabla S(\boldsymbol{\alpha}^{(i)}))^T(\cdot))$ for some sequence $\boldsymbol{\alpha}^{(i)} \rightarrow \bar{\boldsymbol{\alpha}}$. Hence, at least one solution of AGE can be computed relatively easily. Under the normal regularity assumption of the corollary, RAGE and AGE coincide and so the above conclusions hold also for RAGE. In the nonregular case, however, RAGE may be difficult to solve or may even not be solvable at all. From these reasons we will use in the computation of the desired subgradient $\boldsymbol{\xi} \in \bar{\partial}\Theta(\bar{\boldsymbol{\alpha}})$ preferably the way via the AGE (4.3) and accept that in the nonregular case the computed vector $\boldsymbol{\xi}$ may be outside of $\bar{\partial}\Theta(\bar{\boldsymbol{\alpha}})$. If this happens, the used bundle method need not inevitably collapse. If it does, however, $\boldsymbol{\xi}$ must be replaced by a *correct* subgradient.

In the next part of this section we will concentrate on the solution of AGE in the case of the GE (3.18) (after the performed re-ordering). Our main task consists in the specification of a linear subspace $\mathcal{L} \subset N_{\text{Gr } Q}(\bar{\mathbf{y}}, -F(\bar{\boldsymbol{\alpha}})\bar{\mathbf{y}})$ such that the linear system

$$0 = \mathbf{y}^* + (F(\bar{\boldsymbol{\alpha}}))^T \mathbf{v} + \mathbf{w}, \quad (\mathbf{w}, -\mathbf{v}) \in \mathcal{L}$$

is as simple as possible. This linear subspace depends clearly on the position of the point $(\bar{\mathbf{y}}, -F(\bar{\boldsymbol{\alpha}})\bar{\mathbf{y}})$ on $\text{Gr } Q$ or, more precisely, on the positions of $(\bar{\mathbf{y}}^i, -F^i(\bar{\boldsymbol{\alpha}})\bar{\mathbf{y}})$ on $\text{Gr } \Phi$, $i = 1, 2, \dots, p$, where $F^i(\bar{\boldsymbol{\alpha}})\bar{\mathbf{y}}$ is the subvector of $F(\bar{\boldsymbol{\alpha}})\bar{\mathbf{y}}$ associated with the i th contact node.

To facilitate the notation, let us consider instead of $(\mathbf{y}^i, -F^i(\boldsymbol{\alpha})\mathbf{y})$ a pair of vectors $(\mathbf{a}, \mathbf{b}) \in \text{Gr } \Phi \subset \mathbb{R}^4 \times \mathbb{R}^4$ (so that, necessarily, $b_3 = 0$) and let us denote by $\mathbf{a}_{12}, \mathbf{b}_{12}$ the two-dimensional vectors $(a_1, a_2)^T, (b_1, b_2)^T$, respectively.

Clearly, $\text{Gr } \Phi$ admits the partition

$$\text{Gr } \Phi = L \cup M_1 \cup M_2 \cup M_3^+ \cup M_3^- \cup M_4,$$

where

$$\begin{aligned}
L &= \{(\mathbf{a}, \mathbf{b}) \in \text{Gr } \Phi \mid b_4 < 0\} \\
M_1 &= \{(\mathbf{a}, \mathbf{b}) \in \text{Gr } \Phi \mid \mathbf{a}_{12} \neq 0, a_4 > 0\} \\
M_2 &= \{(\mathbf{a}, \mathbf{b}) \in \text{Gr } \Phi \mid \mathbf{a}_{12} \neq 0, a_4 = 0, b_4 = 0\} \\
M_3^+ &= \{(\mathbf{a}, \mathbf{b}) \in \text{Gr } \Phi \mid \mathbf{a}_{12} = 0, a_4 > 0, \|\mathbf{b}_{12}\| < \mathcal{F}a_4\} \\
M_3^- &= \{(\mathbf{a}, \mathbf{b}) \in \text{Gr } \Phi \mid \mathbf{a}_{12} = 0, a_4 > 0, \|\mathbf{b}_{12}\| = \mathcal{F}a_4\} \\
M_4 &= \{(\mathbf{a}, \mathbf{b}) \in \text{Gr } \Phi \mid \mathbf{a}_{12} = 0, a_4 = 0, \|\mathbf{b}_{12}\| = \mathcal{F}a_4, b_4 = 0\}.
\end{aligned}$$

The nature of these sets can be explained in mechanical terms. So, L refers to the absence of *contact* (and hence also the absence of friction). When $\mathbf{a}_{12} \neq 0$, we speak about *sliding* in contrast to the case $\mathbf{a}_{12} = 0$ which is called *sticking*. The adjective “weak” is used as in complementarity [20] to indicate an instable situation on a boundary between two stable states. All in all, M_1 amounts to sliding with contact, M_2 to sliding with weak contact, M_3^+ to sticking with contact, M_3^- to weak sticking with contact and M_4 to weak sticking with weak contact. This partition of $\text{Gr } \Phi$ can be presented also in form of a table with “impossible” fields being crossed out (see Table 4.1).

TABLE 4.1

	no contact: $a_4 = 0, b_4 < 0$	weak contact: $a_4 = 0, b_4 = 0$	strong contact: $a_4 > 0, b_4 = 0$
sliding: $\mathbf{a}_{12} \neq 0$	L	M_2	M_1
weak sticking: $\mathbf{a}_{12} = 0$ $\ \mathbf{b}_{12}\ = \mathcal{F}a_4$		M_4	M_3^-
strong sticking: $\mathbf{a}_{12} = 0$ $\ \mathbf{b}_{12}\ < \mathcal{F}a_4$	-----	-----	M_3^+

The sets L, M_1 and M_3^+ exhibit a stable behavior in the sense that the implication

$$\left. \begin{aligned}
&(\bar{\mathbf{a}}, \bar{\mathbf{b}}) \in L \text{ (or } M_1 \text{ or } M_3^+) \\
&(\mathbf{a}, \mathbf{b}) \in \text{Gr } \Phi \\
&(\mathbf{a}, \mathbf{b}) \text{ is sufficiently close to } (\bar{\mathbf{a}}, \bar{\mathbf{b}})
\end{aligned} \right\} \Rightarrow (\mathbf{a}, \mathbf{b}) \in L \text{ (or } M_1 \text{ or } M_3^+)$$

is fulfilled. On the other hand, if $(\bar{\mathbf{a}}, \bar{\mathbf{b}}) \in M_2$, then in each neighborhood \mathcal{O} of $(\bar{\mathbf{a}}, \bar{\mathbf{b}})$ there are points from L and M_1 . Similarly, in each neighborhood \mathcal{V} of $(\bar{\mathbf{a}}, \bar{\mathbf{b}}) \in M_3^-$ there are points from M_1 and M_3^+ and in each neighborhood \mathcal{W} of $(\bar{\mathbf{a}}, \bar{\mathbf{b}}) \in M_4$ there are points from all remaining sets. It follows that for the purpose of providing a subgradient information to the used bundle method it suffices to compute $D^*\Phi(\bar{\mathbf{a}}, \bar{\mathbf{b}})$ in the case of sets L, M_1, M_3^+ , and exploit the outer semicontinuity

$$\left. \begin{aligned}
&\mathbf{a}^{*(i)} \in D^*\Phi(\mathbf{a}^{(i)}, \mathbf{b}^{(i)})(\mathbf{b}^{*(i)}) \\
&(\mathbf{a}^{(i)}, \mathbf{b}^{(i)}) \rightarrow (\bar{\mathbf{a}}, \bar{\mathbf{b}}), \mathbf{b}^{*(i)} \rightarrow \mathbf{b}^*, \mathbf{a}^{*(i)} \rightarrow \mathbf{a}^*
\end{aligned} \right\} \Rightarrow \mathbf{a}^* \in D^*\Phi(\bar{\mathbf{a}}, \bar{\mathbf{b}})(\mathbf{b}^*). \quad (4.4)$$

PROPOSITION 4.3. Let $(\bar{\mathbf{a}}, \bar{\mathbf{b}}) \in L$. Then one has

$$D^*\Phi(\bar{\mathbf{a}}, \bar{\mathbf{b}})(\mathbf{b}^*) = \begin{cases} \{\mathbf{a}^* \in \mathbb{R}^4 \mid a_1^* = a_2^* = a_3^* = 0\} & \text{provided } b_4^* = 0 \\ \emptyset & \text{otherwise.} \end{cases} \quad (4.5)$$

Proof. There is a neighborhood \mathcal{O} of $(\bar{\mathbf{a}}, \bar{\mathbf{b}})$ such that for all $(\mathbf{a}, \mathbf{b}) \in \text{Gr } \Phi \cap \mathcal{O}$ it holds

$$\mathbf{b} \in \begin{bmatrix} 0 \\ 0 \\ 0 \\ N_{\mathbb{R}_+}(0) \end{bmatrix}, \quad b_4 < 0.$$

The rest follows from the formula for the coderivative of the normal cone mapping $N_{\mathbb{R}_+}(\cdot)$ given in [21, Lemma 2.2]. \square

PROPOSITION 4.4. Let $(\bar{\mathbf{a}}, \bar{\mathbf{b}}) \in M_1$. Then one has

$$D^*\Phi(\bar{\mathbf{a}}, \bar{\mathbf{b}})(\mathbf{b}^*) = \begin{bmatrix} & & & 0 & 0 \\ & \mathbf{C}^T & & 0 & 0 \\ 0 & & 0 & 0 & 0 \\ \|\mathbf{a}_{12}\|^{-1}\mathcal{F}a_1 & & \|\mathbf{a}_{12}\|^{-1}\mathcal{F}a_2 & 0 & 0 \end{bmatrix} \mathbf{b}^*, \quad (4.6)$$

where

$$\mathbf{C} = \frac{\mathcal{F}a_4}{\|\mathbf{a}_{12}\|^3} \begin{bmatrix} (a_2)^2 & -a_1a_2 \\ -a_1a_2 & (a_1)^2 \end{bmatrix}.$$

Proof. There is a neighborhood $\tilde{\mathcal{O}}$ of $(\bar{\mathbf{a}}, \bar{\mathbf{b}})$ such that for all $(\mathbf{a}, \mathbf{b}) \in \text{Gr } \Phi \cap \tilde{\mathcal{O}}$ it holds

$$\mathbf{b} = \mathcal{F} \begin{bmatrix} \|\mathbf{a}_{12}\|^{-1}a_4a_1 \\ \|\mathbf{a}_{12}\|^{-1}a_4a_2 \\ 0 \\ 0 \end{bmatrix},$$

and so it suffices to compute the adjoint Jacobian of this operator. \square

In the case of M_3^+ we will make use of the next auxiliary result.

LEMMA 4.5. Consider a closed-graph multifunction $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ and define

$$H(\mathbf{x}, v) = vG(\mathbf{x}),$$

where $v \in \mathbb{R}_+$. Let $\bar{v} > 0$, $(\bar{\mathbf{x}}, \bar{v}, \bar{\mathbf{b}}) \in \text{Gr } H$ and put $\bar{\mathbf{d}} = (\bar{v})^{-1}\bar{\mathbf{b}}$. Then for all $\mathbf{b}^* \in \mathbb{R}^m$ it holds

$$\left. \begin{array}{l} \mathbf{x}^* \in \widehat{D}^*G(\bar{\mathbf{x}}, \bar{\mathbf{d}})(\bar{v}\mathbf{b}^*) \\ v^* = \langle \bar{\mathbf{d}}, \mathbf{b}^* \rangle \end{array} \right\} \Rightarrow (\mathbf{x}^*, v^*) \in \widehat{D}^*H(\bar{\mathbf{x}}, \bar{v}, \bar{\mathbf{b}})(\mathbf{b}^*).$$

Proof. By the definition of the Fréchet normal cone we have to show the existence of a function $o : \mathbb{R}_+ \rightarrow \mathbb{R}$ with $\lim_{\lambda \downarrow 0} \lambda^{-1}o(\lambda) = 0$ such that

$$\langle \mathbf{x}^*, \mathbf{x} - \bar{\mathbf{x}} \rangle + \langle \bar{\mathbf{d}}, \mathbf{b}^* \rangle (v - \bar{v}) + \langle -\mathbf{b}^*, \mathbf{b} - \bar{\mathbf{b}} \rangle \leq o(\|\mathbf{x} - \bar{\mathbf{x}}\| + |v - \bar{v}| + \|\mathbf{b} - \bar{\mathbf{b}}\|) \quad (4.7)$$

for all $(\mathbf{x}, v, \mathbf{b}) \in \text{Gr } H$, i.e., for all $(\mathbf{x}, v, v\mathbf{d})$ with $\mathbf{d} \in G(\mathbf{x})$. The expression on the left-hand side of the above inequality amounts thus to

$$\begin{aligned} \langle \mathbf{x}^*, \mathbf{x} - \bar{\mathbf{x}} \rangle + \langle \mathbf{b}^*, (v - \bar{v})\bar{\mathbf{d}} + \bar{v}\bar{\mathbf{d}} - v\mathbf{d} \rangle &= \langle \mathbf{x}^*, \mathbf{x} - \bar{\mathbf{x}} \rangle + \langle \mathbf{b}^*, v(\bar{\mathbf{d}} - \mathbf{d}) \rangle \\ &= \langle \mathbf{x}^*, \mathbf{x} - \bar{\mathbf{x}} \rangle + \langle -\bar{v}\mathbf{b}^*, \mathbf{d} - \bar{\mathbf{d}} \rangle + \langle (\bar{v} - v)\mathbf{b}^*, \mathbf{d} - \bar{\mathbf{d}} \rangle. \end{aligned} \quad (4.8)$$

By the assumptions one has for the first two terms in (4.8) the estimate

$$\langle \mathbf{x}^*, \mathbf{x} - \bar{\mathbf{x}} \rangle + \langle -\bar{v}\mathbf{b}^*, \mathbf{d} - \bar{\mathbf{d}} \rangle \leq o_1(\|\mathbf{x} - \bar{\mathbf{x}}\| + \|\mathbf{d} - \bar{\mathbf{d}}\|) \quad \text{for all } (\mathbf{x}, \mathbf{d}) \in \text{Gr } G,$$

where $o_1 : \mathbb{R}_+ \rightarrow \mathbb{R}$ is a nondecreasing function with $\lim_{\lambda \downarrow 0} \lambda^{-1} o_1(\lambda) = 0$. Since

$$\|\mathbf{d} - \bar{\mathbf{d}}\| = \left\| \frac{\mathbf{b}}{v} - \frac{\bar{\mathbf{b}}}{\bar{v}} \right\| \leq \left\| \frac{\mathbf{b}}{v} - \frac{\bar{\mathbf{b}}}{v} \right\| + \left\| \frac{\bar{\mathbf{b}}}{v} - \frac{\bar{\mathbf{b}}}{\bar{v}} \right\| = \frac{1}{v} \|\mathbf{b} - \bar{\mathbf{b}}\| + \left\| \frac{\bar{\mathbf{b}}}{v\bar{v}} \right\| |v - \bar{v}|,$$

we can find for a given $0 < \varepsilon < |\bar{v}|$ positive constants c_1, c_2 such that

$$\|\mathbf{d} - \bar{\mathbf{d}}\| \leq c_1 \|\mathbf{b} - \bar{\mathbf{b}}\| + c_2 |v - \bar{v}| \quad \text{whenever } |v - \bar{v}| \leq \varepsilon.$$

It follows that

$$\langle \mathbf{x}^*, \mathbf{x} - \bar{\mathbf{x}} \rangle + \langle -\bar{v}\mathbf{b}^*, \mathbf{d} - \bar{\mathbf{d}} \rangle \leq o_1(\|\mathbf{x} - \bar{\mathbf{x}}\| + c_2 |v - \bar{v}| + c_1 \|\mathbf{b} - \bar{\mathbf{b}}\|)$$

whenever $\mathbf{d} = v^{-1}\mathbf{b} \in G(\mathbf{x})$ and $|v - \bar{v}| \leq \varepsilon$.

Using the same idea, for the 3rd term in (4.8) one has

$$|\langle (\bar{v} - v)\mathbf{b}^*, \mathbf{d} - \bar{\mathbf{d}} \rangle| \leq |v - \bar{v}| \|\mathbf{b}^*\| (c_1 \|\mathbf{b} - \bar{\mathbf{b}}\| + c_2 |v - \bar{v}|)$$

whenever $\mathbf{d} = v^{-1}\mathbf{b}$, $|v - \bar{v}| \leq \varepsilon$. So a suitable function o for the inequality (4.7) can easily be constructed and we are done. \square

On the basis of the above result we are now able to handle the set M_3^+ .

PROPOSITION 4.6. *Let $(\bar{\mathbf{a}}, \bar{\mathbf{b}}) \in M_3^+$. Then one has*

$$D^*\Phi(\bar{\mathbf{a}}, \bar{\mathbf{b}})(\mathbf{b}^*) \supset \begin{cases} \mathbb{R}^2 \\ 0 \\ 0 \end{cases} \quad \text{if } b_1^* = b_2^* = 0. \quad (4.9)$$

Proof. It suffices to observe that there is a neighborhood $\widehat{\mathcal{O}}$ of $(\bar{\mathbf{a}}, \bar{\mathbf{b}})$ such that for all $(\mathbf{a}, \mathbf{b}) \in \text{Gr } \Phi \cap \widehat{\mathcal{O}}$ one has $\mathbf{a}_{12} = 0$,

$$\Phi(\mathbf{a}, \mathbf{b}) = \begin{bmatrix} \mathcal{F}a_4 \partial \|\mathbf{a}_{12}\| \\ 0 \\ 0 \end{bmatrix},$$

and that for any $\kappa > 0$

$$\widehat{D}^*(\kappa \partial \|\cdot\|)(\mathbf{a}_{12}, \mathbf{b}_{12})(b_1^*, b_2^*) = \begin{cases} \mathbb{R}^2 & \text{if } b_1^* = b_2^* = 0 \\ \emptyset & \text{otherwise.} \end{cases}$$

The result follows then from the preceding lemma. \square

Since $\text{Gr } \Phi$ is normally regular at all points of $L \cup M_1 \cup M_3^+$, formulas (4.5)–(4.9) lead to a correct subgradient $\boldsymbol{\xi}$ whenever all points $(\bar{\mathbf{y}}^i, -F^i(\bar{\boldsymbol{\alpha}})\bar{\mathbf{y}})$, $i = 1, 2, \dots, p$, belong to this union.

In the sense of the considerations preceding Proposition 4.3, formula (4.5) can be used also in the case of M_2, M_4 and formula (4.9) can be used in the case of M_3^- and M_4 . Since (4.9) leads to the greatest simplification of the respective AGE, we use in our computer program formula (4.5) in the case of L and M_2 and formula (4.9) in the case of M_3^+, M_3^- and M_4 . Nevertheless, since $\text{Gr } \Phi$ is not normally regular at the points from $M_2 \cup M_3^- \cup M_4$, the subgradient $\boldsymbol{\xi}$ computed in this way may be false. This happens, for instance, if there does not exist any sequence $\boldsymbol{\alpha}^{(j)} \rightarrow \bar{\boldsymbol{\alpha}}$ which, together with the respective state variable $\boldsymbol{y}^{(j)}$, would satisfy the conditions

$$\begin{aligned} &(\boldsymbol{y}^{i(j)}, -F^i(\boldsymbol{\alpha}^{(j)})\boldsymbol{y}^{(j)}) \in L \text{ for all } j \\ &\boldsymbol{y}^{(j)} \rightarrow \bar{\boldsymbol{y}} \\ &(\bar{\boldsymbol{y}}^i, -F^i(\bar{\boldsymbol{\alpha}})\bar{\boldsymbol{y}}) \in M_2. \end{aligned}$$

In such a case one can approach the reference point in a different way and the respective formulas come from (4.5), (4.6) and (4.9) by passing to limits in the sense of (4.4). This derivation is not difficult and will be omitted. Instead, we will illustrate in the case of M_2 and M_3^- the potential of the other approach based on the RAGE (4.2). Let $H : \mathbb{R}^4 \rightrightarrows \mathbb{R}^3$ be defined by

$$H(\boldsymbol{a}) := \begin{bmatrix} \mathcal{F}a_4 \partial \|\boldsymbol{a}_{12}\| \\ 0 \end{bmatrix}.$$

From [24, Theorem 10.40] we immediately obtain the inclusion

$$\begin{aligned} \widehat{D}^* \Phi(\bar{\boldsymbol{a}}, \bar{\boldsymbol{b}})(\boldsymbol{b}^*) \supset \left\{ \begin{bmatrix} a_1^* \\ a_2^* \\ 0 \\ \mu + \nu \end{bmatrix} \middle| \begin{bmatrix} a_1^* \\ a_2^* \\ 0 \\ \mu \end{bmatrix} \in \widehat{D}^* H(\bar{\boldsymbol{a}}, \bar{b}_1, \bar{b}_2, 0)(b_1^*, b_2^*, b_3^*), \right. \\ \left. \nu \in \widehat{D}^* N_{\mathbb{R}_+}(\bar{a}_4, \bar{b}_4)(b_4^*) \right\} \quad (4.10) \end{aligned}$$

for all $\boldsymbol{b}^* \in \mathbb{R}^4$. Since $\widehat{D}^* N_{\mathbb{R}_+}$ can easily be computed (see, e.g., [27]), it remains to derive a suitable inner estimate of $\widehat{D}^* H$ in the case when $\bar{\boldsymbol{a}}_{12} \neq 0, \bar{a}_4 = 0, \bar{b}_4 = 0$ (i.e. in the case of M_2) and $\bar{\boldsymbol{a}}_{12} = 0, \bar{a}_4 > 0, \|\bar{\boldsymbol{b}}_{12}\| = \mathcal{F}\bar{a}_4$ (i.e. in the case of M_3^-). In the former case we immediately obtain from (4.6) that

$$\widehat{D}^* H(\bar{\boldsymbol{a}}, \bar{b}_1, \bar{b}_2, 0)(b_1^*, b_2^*, b_3^*) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \|\bar{\boldsymbol{a}}_{12}\|^{-1} \mathcal{F}a_1 & \|\bar{\boldsymbol{a}}_{12}\|^{-1} \mathcal{F}a_2 & 0 \end{bmatrix} \begin{bmatrix} b_1^* \\ b_2^* \\ b_3^* \end{bmatrix} \quad (4.11)$$

and we are done. In the case of M_3^- we make use of the results below.

PROPOSITION 4.7. *Let $(\bar{\boldsymbol{c}}, \bar{\boldsymbol{d}}) \in \mathbb{R}^2 \times \mathbb{R}^2$ with $\bar{\boldsymbol{c}} = 0$ and $\|\bar{\boldsymbol{d}}\| = \mathcal{F}$. Then one has*

$$\widehat{D}^*(\mathcal{F}\partial\|\cdot\|)(\bar{\boldsymbol{c}}, \bar{\boldsymbol{d}})(\boldsymbol{d}^*) = \begin{cases} \{\boldsymbol{c}^* \in \mathbb{R}^2 \mid \langle \boldsymbol{c}^*, \bar{\boldsymbol{d}} \rangle \leq 0\} & \text{provided } \boldsymbol{d}^* \in \mathbb{R}_- \bar{\boldsymbol{d}} \\ \emptyset & \text{otherwise.} \end{cases} \quad (4.12)$$

Proof. Clearly, $\text{Gr}(\mathcal{F}\partial\|\cdot\|) = G_1 + G_2$, where $G_1 = \{0\} \times \mathcal{F}\mathbb{B}_1$ and $G_2 = \{(\boldsymbol{c}, \boldsymbol{d}) \in \mathbb{R}_2 \times \mathbb{R}_2 \mid \boldsymbol{c} \neq 0, \boldsymbol{d} = \|\boldsymbol{c}\|^{-1} \mathcal{F}\boldsymbol{c}\}$. One immediately sees that the contingent cone

$$T_{G_1}(\bar{\boldsymbol{c}}, \bar{\boldsymbol{d}}) = \{(\boldsymbol{h}, \boldsymbol{k}) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid \boldsymbol{h} = 0, \langle \boldsymbol{k}, \bar{\boldsymbol{d}} \rangle \leq 0\}$$

and, consequently,

$$\widehat{N}_{G_1}(\bar{\mathbf{c}}, \bar{\mathbf{d}}) = \{(\mathbf{h}^*, \mathbf{k}^*) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid \mathbf{k}^* \in \mathbb{R}_+ \bar{\mathbf{d}}\}.$$

On the other hand, by the definition of the contingent cone,

$$\begin{aligned} T_{G_2}(\bar{\mathbf{c}}, \bar{\mathbf{d}}) &= \{(\mathbf{h}, \mathbf{k}) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid \exists \mathbf{h}^{(i)} \rightarrow \mathbf{h}, \exists \mathbf{k}^{(i)} \rightarrow \mathbf{k}, \exists \nu^{(i)} \downarrow 0 \text{ such that} \\ &\frac{\mathcal{F}\nu^{(i)}\mathbf{h}^{(i)}}{\|\nu^{(i)}\mathbf{h}^{(i)}\|} = \bar{\mathbf{d}} + \nu^{(i)}\mathbf{k}^{(i)} \forall i\} = \{(\mathbf{h}, \mathbf{k}) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid \mathbf{h} \in \mathbb{R}_+ \bar{\mathbf{d}}, \langle \mathbf{k}, \bar{\mathbf{d}} \rangle = 0\}. \end{aligned}$$

We readily compute that

$$\widehat{N}_{G_2}(\bar{\mathbf{c}}, \bar{\mathbf{d}}) = \{(\mathbf{h}^*, \mathbf{k}^*) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid \langle \mathbf{h}^*, \bar{\mathbf{d}} \rangle \leq 0, \mathbf{k}^* \in \mathbb{R} \bar{\mathbf{d}}\}.$$

Since the Fréchet normal cone to a union of sets is the intersection of Fréchet normal cones to the single sets, we obtain that

$$\widehat{N}_{G_r(\mathcal{F}\partial\|\cdot\|)}(\bar{\mathbf{c}}, \bar{\mathbf{d}}) = \{(\mathbf{h}^*, \mathbf{k}^*) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid \langle \mathbf{h}^*, \bar{\mathbf{d}} \rangle \leq 0, \mathbf{k}^* \in \mathbb{R}_+ \bar{\mathbf{d}}\}$$

and formula (4.12) follows. \square

By a combination of Lemma 4.5 and Proposition 4.7 we arrive at

PROPOSITION 4.8. *Let $\bar{\mathbf{a}}_{12} = 0, a_4 > 0, \|\bar{\mathbf{b}}_{12}\| = \mathcal{F}\bar{a}_4$ and $\bar{\mathbf{d}} = (\bar{a}_4)^{-1}\bar{\mathbf{b}}_{12}$. Then one has*

$$\begin{aligned} \widehat{D}^*H(\bar{\mathbf{a}}, \bar{b}_1, \bar{b}_2, 0)(b_1^*, b_2^*, b_3^*) \supset \{(\mathbf{u}^*, 0, v^*) \in \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \mid \langle \mathbf{u}^*, \bar{\mathbf{d}} \rangle \leq 0, \\ v^* = \langle \bar{\mathbf{d}}, (b_1^*, b_2^*)^T \rangle\} \end{aligned} \quad (4.13)$$

provided $(b_1^*, b_2^*)^T \in \mathbb{R}_- \bar{\mathbf{d}}$.

We observe that in both cases of the sets M_2 and M_3^- the respective RAGEs (4.2) amount to systems of linear equations and *inequalities*. In the case of M_2 the inequalities arise from the Fréchet normal cone to the graph of $N_{\mathbb{R}_+}$ at zero, whereas in the case of M_3^- they come directly from formula (4.13). This is a typical phenomenon arising in connection with RAGEs and it causes a substantial complication in comparison with AGEs (4.3) derived on the basis of Propositions 4.3, 4.4 and 4.6. Therefore we recommend the usage of this approach only if all other available formulas do not lead to a correct subgradient.

5. Numerical results. The results of the previous sections will now be used for construction of a numerical method for the solution of $(\tilde{\mathbb{P}})$. We assume that the friction coefficient \mathcal{F} is small enough in the sense of Theorem 3.7 so that the solution of the contact problem with Coulomb friction is unique. Further, as in Section 4, we assume that the cost functional J is continuously differentiable. For the minimization of Θ we use the BT code [28] based on the bundle-truss algorithm of Schramm and Zowe [25]. In every step of the iteration process, this code needs the function value $\Theta(\boldsymbol{\alpha})$ and one (arbitrary) Clarke's subgradient of Θ at $\boldsymbol{\alpha}$.

5.1. Solving the state problem. To compute a function value $J(\boldsymbol{\alpha}, S(\boldsymbol{\alpha}))$, we have to evaluate $S(\boldsymbol{\alpha})$, i.e., to solve the fixed-point problem (\mathcal{P}) . For that, we use the method of successive approximations introduced in Corollary 3.8. Each iterative step requires to solve the contact problem with given friction $(\mathcal{M}(\mathbf{L}, \mathbf{g}))$, in which the slip bound \mathbf{g} is updated by the result of the previous iteration, i.e., $\mathbf{g} \equiv \boldsymbol{\lambda}^{(k)}$. The

problem $(\mathcal{M}(\mathbf{L}, \mathbf{g}))$ is solved using the so-called *reciprocal variational formulation* (see [7, 10, 11]). As in the previous sections, we denote by $\boldsymbol{\lambda} \in \mathbb{R}^p$ the vector of normal contact stresses. Further, let $\boldsymbol{\tau} \in \mathbb{R}^{2p}$ be the vector of tangential contact stresses. Notice that with each contact node we associate one component λ_i of $\boldsymbol{\lambda}$ and two components τ_{2i-1}, τ_{2i} of $\boldsymbol{\tau}$. After eliminating the first variable \mathbf{u} from $(\mathcal{M}(\mathbf{L}, \mathbf{g}))$, we arrive at the problem formulated in terms of contact stresses:

$$\min_{\boldsymbol{\sigma} \in \mathbb{S}} f(\boldsymbol{\sigma}) := \frac{1}{2} \boldsymbol{\sigma}^T \mathbf{Q} \boldsymbol{\sigma} - \boldsymbol{\sigma}^T \mathbf{h} \quad (5.1)$$

with

$$\mathbb{S} = \{\boldsymbol{\sigma} = (\boldsymbol{\lambda}, \boldsymbol{\tau}) \in \mathbb{R}^{3p} \mid \lambda_i \geq 0, \tau_{2i-1}^2 + \tau_{2i}^2 \leq g_i, \quad i = 1, 2, \dots, p\},$$

where $\mathbf{Q} = \mathbf{B} \mathbf{A}^{-1} \mathbf{B}^T$, $\mathbf{h} = \mathbf{B} \mathbf{A}^{-1} \mathbf{L} + \mathbf{c}$, $\mathbf{B} = (\mathbf{N}^T, \mathbf{T}^T)^T$, $\mathbf{c} = (\boldsymbol{\alpha}^T, 0^T)^T$ and $\mathbf{T} \in \mathbb{R}^{2p \times n}$ stands for a matrix representation of the linear mapping T used in (3.3). After computing $\boldsymbol{\lambda}, \boldsymbol{\tau}$ from (5.1), one obtains the eliminated variable \mathbf{u} by

$$\mathbf{u} = \mathbf{A}^{-1}(\mathbf{L} - \mathbf{N} \boldsymbol{\lambda} - \mathbf{T} \boldsymbol{\tau}).$$

As (5.1) is a strictly convex problem with quadratic objective and separable quadratic constraints, it can be solved by the algorithm proposed by Kučera in [13] and analyzed in [14]. The algorithm generalizes ideas of Dostál and Schöberl [4] originally proposed for convex quadratic problems with simple bounds. Because an efficient solution procedure for (5.1) is essential for the overall efficiency of our numerical approach, we give a brief description of the algorithm.

Let $\mathcal{N} = \{1, \dots, 3p\}$ be the set of all indices. At a point $\boldsymbol{\sigma} \in \mathbb{S}$, we denote the gradient of f by $\mathbf{r} = \mathbf{r}(\boldsymbol{\sigma}) = \mathbf{Q} \boldsymbol{\sigma} - \mathbf{h}$ and introduce an active set $\mathcal{A} \subseteq \mathcal{N}$ by

$$\begin{aligned} \mathcal{A} := \mathcal{A}(\boldsymbol{\sigma}) &= \{i \mid \lambda_i = 0\} \cup \{j \mid j = 2i - 1 + p : \tau_{2i-1}^2 + \tau_{2i}^2 = g_i\} \\ &\quad \cup \{j \mid j = 2i + p : \tau_{2i-1}^2 + \tau_{2i}^2 = g_i\}. \end{aligned}$$

Using the orthogonal projection $P_{\mathbb{S}} : \mathbb{R}^{3p} \mapsto \mathbb{S}$, we define the *projected gradient* for a fixed $\tilde{\alpha} \geq 0$ as

$$\tilde{\mathbf{r}} = \tilde{\mathbf{r}}(\boldsymbol{\sigma}) = \frac{1}{\tilde{\alpha}} (\boldsymbol{\sigma} - P_{\mathbb{S}}(\boldsymbol{\sigma} - \tilde{\alpha} \mathbf{r}(\boldsymbol{\sigma}))).$$

Notice that the projected gradient enables us to write down the optimality criterion characterizing the solution $\boldsymbol{\sigma}^*$ of (5.1) in the form $\tilde{\mathbf{r}}(\boldsymbol{\sigma}^*) = 0$. Our algorithm is based on the observation that non-zero components of $\tilde{\mathbf{r}}(\boldsymbol{\sigma})$ at $\boldsymbol{\sigma} \neq \boldsymbol{\sigma}^*$ determine descent directions changing appropriately the active set. To this end, we introduce components of $\tilde{\mathbf{r}}(\boldsymbol{\sigma})$ and $\mathbf{r}(\boldsymbol{\sigma})$ called the *projected free gradient* $\tilde{\boldsymbol{\varphi}} = \tilde{\boldsymbol{\varphi}}(\boldsymbol{\sigma})$, the *projected boundary gradient* $\tilde{\boldsymbol{\beta}} = \tilde{\boldsymbol{\beta}}(\boldsymbol{\sigma})$, and the *free gradient* $\boldsymbol{\varphi} = \boldsymbol{\varphi}(\boldsymbol{\sigma})$, respectively, defined by

$$\begin{aligned} \tilde{\boldsymbol{\varphi}}_{\mathcal{A}} &= 0, & \tilde{\boldsymbol{\varphi}}_{\mathcal{N} \setminus \mathcal{A}} &= \tilde{\mathbf{r}}_{\mathcal{N} \setminus \mathcal{A}}, \\ \tilde{\boldsymbol{\beta}}_{\mathcal{A}} &= \tilde{\mathbf{r}}_{\mathcal{A}}, & \tilde{\boldsymbol{\beta}}_{\mathcal{N} \setminus \mathcal{A}} &= 0, \\ \boldsymbol{\varphi}_{\mathcal{A}} &= 0, & \boldsymbol{\varphi}_{\mathcal{N} \setminus \mathcal{A}} &= \mathbf{r}_{\mathcal{N} \setminus \mathcal{A}}. \end{aligned}$$

We combine three steps to generate a sequence of iterates $\{\boldsymbol{\sigma}^{(l)}\}$ that approximates the solution to (5.1):

- the *expansion step*: $\boldsymbol{\sigma}^{(l+1)} = \boldsymbol{\sigma}^{(l)} - \tilde{\alpha} \tilde{\boldsymbol{\varphi}}(\boldsymbol{\sigma}^{(l)})$,
- the *proportioning step*: $\boldsymbol{\sigma}^{(l+1)} = \boldsymbol{\sigma}^{(l)} - \tilde{\alpha} \tilde{\boldsymbol{\beta}}(\boldsymbol{\sigma}^{(l)})$,
- the *conjugate gradient step*: $\boldsymbol{\sigma}^{(l+1)} = \boldsymbol{\sigma}^{(l)} - \alpha_{cg}^{(l)} \boldsymbol{p}^{(l)}$, where the stepsize $\alpha_{cg}^{(l)}$ and the conjugate gradient directions $\boldsymbol{p}^{(l)}$ are computed recurrently (see [6]) so that the recurrence starts from $\boldsymbol{\sigma}^{(s)}$ generated by the last expansion or proportioning step and $\mathcal{A}(\boldsymbol{\sigma}^{(l+1)}) = \mathcal{A}(\boldsymbol{\sigma}^{(s)})$.

The expansion step may add indices while the proportioning step may release indices to/from the current active set. The conjugate gradient step is used to carry out efficiently minimization of the objective f on the interior of the set $W(\boldsymbol{\sigma}^{(s)}) = \{\boldsymbol{\sigma} \in \mathbb{S} \mid \boldsymbol{\sigma}_{\mathcal{A}} = \boldsymbol{\sigma}_{\mathcal{A}}^{(s)}, \mathcal{A} = \mathcal{A}(\boldsymbol{\sigma}^{(s)})\}$. Moreover, the algorithm exploits a given constant $\tilde{\Gamma} > 0$ and the *releasing criterion*

$$\tilde{\boldsymbol{\beta}}(\boldsymbol{\sigma}^{(l)})^T \boldsymbol{r}(\boldsymbol{\sigma}^{(l)}) \leq \tilde{\Gamma} \tilde{\boldsymbol{\varphi}}(\boldsymbol{\sigma}^{(l)})^T \boldsymbol{r}(\boldsymbol{\sigma}^{(l)}) \quad (5.2)$$

to decide which of the steps will be performed.

ALGORITHM 5.1 Let $\boldsymbol{\sigma}^{(0)} \in \mathbb{S}$, $\tilde{\Gamma} > 0$, $\tilde{\alpha} \in (0, \|\boldsymbol{Q}\|^{-1}]$ and $\varepsilon \geq 0$ be given. For $\boldsymbol{\sigma}^{(l)}$, $\boldsymbol{\sigma}^{(s)}$ known, $0 \leq s \leq l$, where $\boldsymbol{\sigma}^{(s)}$ is computed by the last step expansion or proportioning, choose $\boldsymbol{\sigma}^{(l+1)}$ by the following rules:

- If $\|\tilde{\boldsymbol{r}}(\boldsymbol{\sigma}^{(l)})\| \leq \varepsilon$, return $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{(l)}$.
- If $\boldsymbol{\sigma}^{(l)}$ fulfills (5.2), try to generate $\boldsymbol{\sigma}^{(l+1)}$ by the conjugate gradient step. If $\boldsymbol{\sigma}^{(l+1)} \in \text{Int } W(\boldsymbol{\sigma}^{(s)})$, accept it, else generate $\boldsymbol{\sigma}^{(l+1)}$ by the expansion step.
- If $\boldsymbol{\sigma}^{(l)}$ does not fulfil (5.2), generate $\boldsymbol{\sigma}^{(l+1)}$ by the proportioning step.

Contrary to the simple bound problem analyzed in [4], the algorithm does not exhibit the finite terminating property; the same convergence rate is, however, achieved. In [14] one can find the following statement.

THEOREM 5.1. Let $\boldsymbol{\sigma}^* \in \mathbb{S}$ denote the solution to (5.1), α_{min} denote the smallest eigenvalue of \boldsymbol{Q} and $\hat{\Gamma} = \max\{\tilde{\Gamma}, \tilde{\Gamma}^{-1}\}$. Let $\{\boldsymbol{\sigma}^{(l)}\}$ be the sequence generated by Algorithm 5.1 with $\varepsilon = 0$. Then

$$f(\boldsymbol{\sigma}^{(l+1)}) - f(\boldsymbol{\sigma}^*) \leq \eta \left(f(\boldsymbol{\sigma}^{(l)}) - f(\boldsymbol{\sigma}^*) \right),$$

where

$$\eta = 1 - \frac{\tilde{\alpha} \alpha_{min}}{2 + 2\hat{\Gamma}} < 1.$$

The error in the \boldsymbol{Q} -energy norm is bounded by

$$\|\boldsymbol{\sigma}^{(l)} - \boldsymbol{\sigma}^*\|_{\boldsymbol{Q}}^2 \leq 2\eta^l \left(f(\boldsymbol{\sigma}^{(0)}) - f(\boldsymbol{\sigma}^*) \right).$$

Theorem 5.1 yields the best value of the convergence rate factor η for the choice $\Gamma = \hat{\Gamma} = 1$ and $\tilde{\alpha} = \|\boldsymbol{Q}\|^{-1}$. Then

$$\eta = 1 - \frac{1}{4} \kappa(\boldsymbol{Q}),$$

where $\kappa(\boldsymbol{Q})$ is the spectral condition number of \boldsymbol{Q} .

5.2. Numerical examples. In order to work with a relatively small number of control variables and, at the same time, to get a smooth shape of the optimal boundary, we will model the contact boundary Γ_c by a Bézier surface of order d . The design variable $\boldsymbol{\alpha}$ is thus a vector of its control points. The Bézier surface ϑ_α of order (d_1, d_2) in $\overline{\mathcal{R}} (= [0, a] \times [0, b])$ is generated by a $d_1 \times d_2$ matrix $\boldsymbol{\alpha}$ as

$$\vartheta_\alpha(x_1, x_2) = \sum_{i=0}^{d_1} \sum_{j=0}^{d_2} \alpha^{(i,j)} \beta_{d_1}^i(x_1) \beta_{d_2}^j(x_2), \quad (x_1, x_2) \in \overline{\mathcal{R}},$$

where

$$\beta_{d_1}^\ell(x_1) = \frac{1}{a^{d_1}} \binom{d_1}{\ell} x_1^\ell (a - x_1)^{d_1 - \ell}, \quad \beta_{d_2}^\ell(x_2) = \frac{1}{b^{d_2}} \binom{d_2}{\ell} x_2^\ell (b - x_2)^{d_2 - \ell}.$$

The corner points of ϑ_α are identical with the ‘‘corner elements’’ of the control matrix. The surface itself lies in the convex hull of the control points. This means that any upper and lower bounds on the control points hold for the whole surface in $\overline{\mathcal{R}}$, too.

The discrete shape optimization problem is defined as follows:

$$\left. \begin{array}{l} \text{minimize } J(\boldsymbol{\alpha}, S(\boldsymbol{\alpha})) \\ \text{subject to } \boldsymbol{\alpha} \in \mathcal{U}, \end{array} \right\} \quad (\tilde{\mathbb{P}}_{\mathcal{B}})$$

where \mathcal{U} is given by

$$\mathcal{U} = \left\{ \boldsymbol{\alpha} \in \mathbb{R}^{d_1 \times d_2} \mid \begin{array}{l} 0 \leq \alpha^{(i,j)} \leq C_0, \quad i = 0, 1, \dots, d_1, \quad j = 0, 1, \dots, d_2; \\ |\alpha^{(i+1,j)} - \alpha^{(i,j)}| \leq C_1 \frac{a}{d_1}, \quad i = 0, 1, \dots, d_1 - 1, \quad j = 0, 1, \dots, d_2; \\ |\alpha^{(i,j+1)} - \alpha^{(i,j)}| \leq C_1 \frac{b}{d_2}, \quad i = 0, 1, \dots, d_1, \quad j = 0, 1, \dots, d_2 - 1; \\ \sum_{i=0}^{d_1} \sum_{j=0}^{d_2} \alpha^{(i,j)} = C_2 (d_1 + 1)(d_2 + 1) \end{array} \right\}$$

and C_0, C_1, C_2 are given positive constants. The first set of constraints guarantees that $|F_\alpha(x)| \leq C_0$ for all $x \in \overline{\mathcal{R}}$. The second and third constraint sets take care of the slopes of ϑ_α in the direction of axes x_1, x_2 . It is well known that if the control points satisfy these conditions, then $|\frac{\partial}{\partial x_k} \vartheta_\alpha(x_1, x_2)| \leq C_1$ for all $(x_1, x_2) \in \overline{\mathcal{R}}, k = 1, 2$.

The equality constraint is added to control the volume of the domain by the control points of the Bézier surface. The number $(c - C_2)ab$ equals the volume of $\Omega(\boldsymbol{\alpha})$ defined by

$$\Omega(\boldsymbol{\alpha}) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid (x_1, x_2) \in (0, a) \times (0, b), F_\alpha(x_1, x_2) < x_3 < c\}; \quad (5.3)$$

see Figure 5.1 which shows the body in 3D and 2D view. Thus the equality constraint has a physical meaning of preserving the weight of the structure.

Both test examples solved below differ only in the cost function. The shape of the elastic body $\Omega(\boldsymbol{\alpha}), \boldsymbol{\alpha} \in \mathcal{U}$, is given by (5.3) with $a = 2, b = 1, c = 1$. The set of admissible designs \mathcal{U} is determined by the choice $C_0 = 0.75, C_1 = 0.5, C_2 = 0.01$.

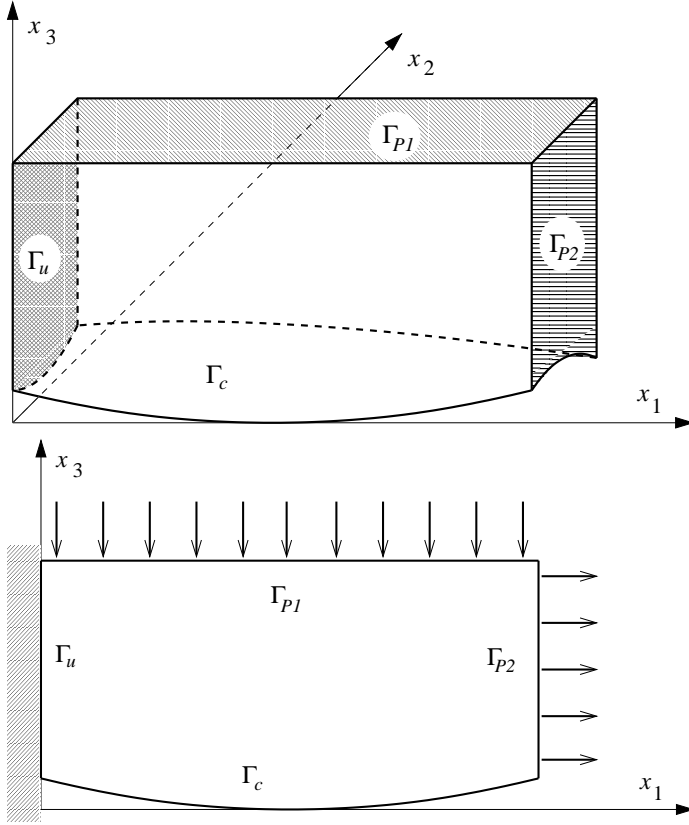


FIG. 5.1. The elastic body and applied loads.

The left-hand face $\Gamma_u = \{x \in \Omega(\boldsymbol{\alpha}) \mid x_1 = 0\}$ is the part of the boundary with the prescribed Dirichlet condition where all displacements are fixed to zero. The non-zero external loads are defined as follows. The top face $\Gamma_{P1} = \{x \in \Omega(\boldsymbol{\alpha}) \mid x_3 = 1\}$ is subjected to the constant pressure $P_1 = -8 \cdot 10^{-2} \frac{\text{N}}{\text{m}^2}$. The right-hand face $\Gamma_{P2} = \{x \in \Omega(\boldsymbol{\alpha}) \mid x_1 = 1\}$ is subjected to the constant pressure $P_2 = 5 \cdot 10^{-2} \frac{\text{N}}{\text{m}^2}$. The bottom face Γ_c represented by the graph of $\vartheta(\boldsymbol{\alpha})$ is supported by a rigid half-space $\mathbb{R}^2 \times \mathbb{R}_-$. The examples were solved with the Young modulus $E = 21.19 \text{ Pa}$, Poisson's constant $\sigma = 0.277$, and the friction coefficient $\mathcal{F} = 0.3$.

The reference body, a prism of size $2 \times 1 \times 1$, was uniformly carved into $24 \times 11 \times 11 = 2904$ bricks. The finite element discretization was constructed by using tri-linear elements. The total number of nodal displacements was 10398 including 864 contact displacements. The partition of the each $\Omega(\boldsymbol{\alpha})$ was constructed from the partition of the reference body $\hat{\Omega}$ by a suitable coordinate transformation in the x_3 direction. The total number of design variables (control points of the Bézier surface) was 32 ($d_1 = 8, d_2 = 4$).

Example 5.1. We try to find a shape of the contact surface for which the normal stress is as close as possible to a prescribed function. The corresponding problem ($\tilde{\mathbb{P}}$)

can be formulated as

$$\begin{aligned} & \text{minimize } \|\bar{\lambda}_\nu - \lambda_\nu\|_2^2 \\ & \text{subject to } \alpha \in \mathcal{U}, \end{aligned}$$

where $\bar{\lambda}_\nu$ is a vector of desired normal stresses. This target function is a step function, depicted in Figure 5.2 (left). Figure 5.2 (right) shows the distribution of the contact normal stresses for the initial shape, given by the constant vector $\alpha_0 = [0.01, \dots, 0.01]$.

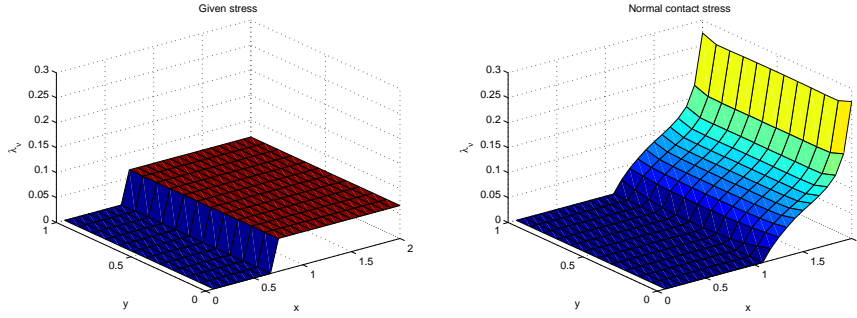


FIG. 5.2. Example 5.2, target normal stress (left) and normal stress for initial design (right).

The objective function value for the initial design was equal to $J(\alpha_0) = 3.1583304 \cdot 10^{-5}$. The stopping parameter for the code BT was set to $\varepsilon = 1 \cdot 10^{-4}$. This required precision was reached after 114 iterations. (With a stopping tolerance decreased to $\varepsilon = 1 \cdot 10^{-3}$, the code finished already after 21 iterations.) Figure 5.3 presents the optimal solution, i.e., the optimal shape of the contact boundary, while Figure 5.4 compares the contact normal stresses with the prescribed values. We see that the stresses for the optimal shape follow the step function rather closely.

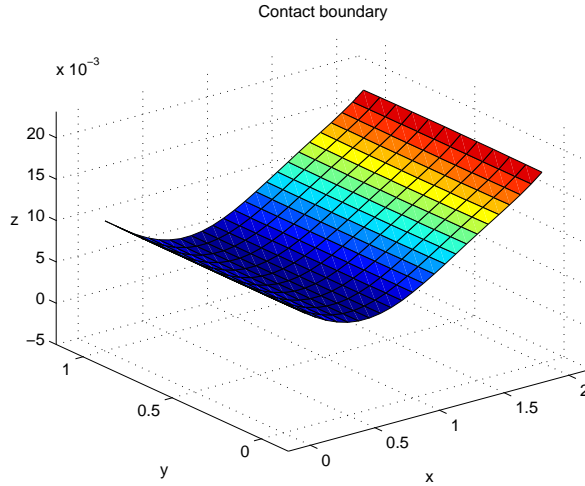


FIG. 5.3. Example 5.2, optimal design.

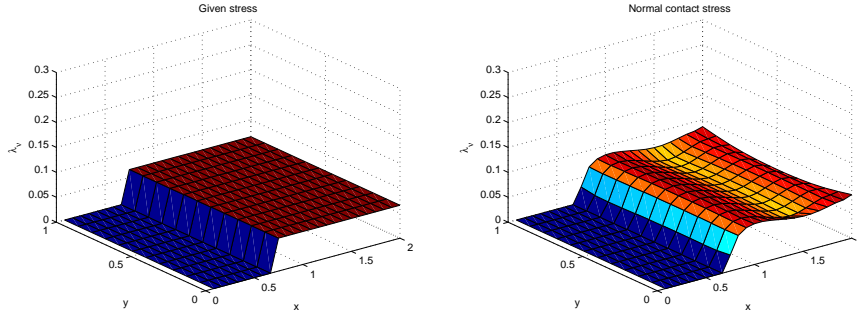


FIG. 5.4. *Example 5.2, target normal stress (left) and normal stress for optimal design (right).*

Finally, we present the optimal shape in terms of the matrix of the respective control points:

$$\boldsymbol{\alpha}_{\text{opt}} = \begin{bmatrix} 0.0107 & 0.0107 & 0.0107 & 0.0107 \\ 0.0074 & 0.0073 & 0.0073 & 0.0074 \\ 0.0036 & 0.0035 & 0.0036 & 0.0036 \\ 0.0042 & 0.0042 & 0.0043 & 0.0042 \\ 0.0093 & 0.0093 & 0.0093 & 0.0092 \\ 0.0128 & 0.0127 & 0.0127 & 0.0129 \\ 0.0141 & 0.0138 & 0.0138 & 0.0141 \\ 0.0181 & 0.0182 & 0.0182 & 0.0181 \end{bmatrix}.$$

The optimal value of the objective function was $J_{\text{opt}} = 1.6045224 \cdot 10^{-6}$.

Example 5.2. We will now minimize the Euclidean norm of the contact stresses. The shape optimization problem reads as follows:

$$\begin{aligned} & \text{minimize } \|\boldsymbol{\lambda}_\nu\|_2^2 \\ & \text{subject to } \boldsymbol{\alpha} \in \mathcal{U}. \end{aligned}$$

The initial shape is given as in the previous example. The stopping parameter for the code BT was set to $\varepsilon = 8 \cdot 10^{-4}$. This criterion was reached after 640 iterations. The objective function value for the initial shape was equal to $J(\boldsymbol{\alpha}_0) = 0.61121816$ and for the optimal shape $J_{\text{opt}} = 0.34117444$. Below we present the optimal shape in terms of the matrix of the respective control points:

$$\boldsymbol{\alpha}_{\text{opt}} = \begin{bmatrix} 0.0002 & 0.0003 & 0.0000 & 0.0003 \\ 0.0012 & 0.0017 & 0.0023 & 0.0009 \\ 0.0060 & 0.0063 & 0.0053 & 0.0063 \\ 0.0067 & 0.0071 & 0.0084 & 0.0061 \\ 0.0141 & 0.0144 & 0.0133 & 0.0146 \\ 0.0120 & 0.0129 & 0.0135 & 0.0118 \\ 0.0182 & 0.0181 & 0.0181 & 0.0183 \\ 0.0199 & 0.0208 & 0.0208 & 0.0199 \end{bmatrix}.$$

Figure 5.5 shows the optimal design of the contact surface. In Figure 5.6 we present the normal stress for the initial design (left) and for the optimal design (right).

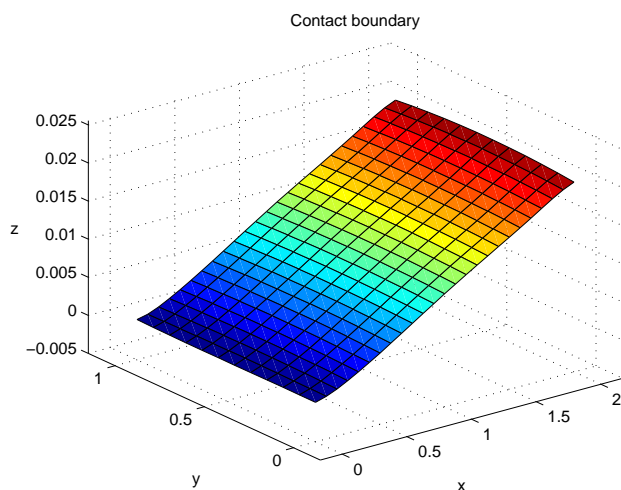


FIG. 5.5. Example 5.2, optimal design.

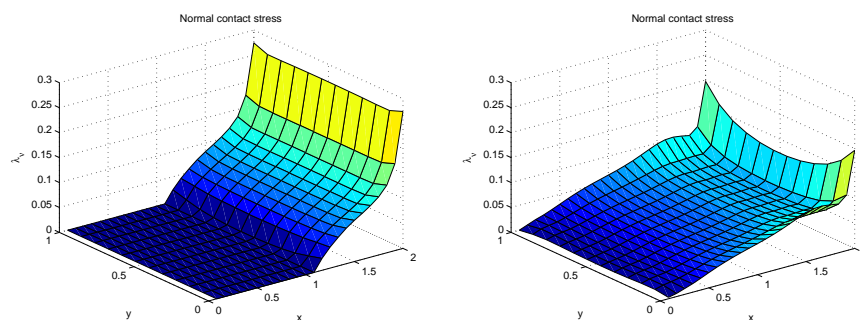


FIG. 5.6. Example 5.2, normal stress for the initial (left) and optimal (right) design.

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