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Perturbative series expansion for the subcritical stationary properties of the contact process

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Abstract

The subcritical stationary properties of the contact process in one dimension are studied by perturbative series expansions in powers of the creation rate. To avoid entering the absorbing state and thus obtaining a nonzero stationary distribution of particles three small modifications on the rules of the contact process are considered. The series expansions for the average number of particles and the spatial correlation length are analysed by Padé approximants from which the values of the critical creation rate and the critical exponents are obtained.

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1. Introduction

The contact process [1–12] is one of the simplest nonequilibrium interacting systems described by a continuous time Markov process. In the usual interpretation, the contact process describes an interacting particle system in which particles are annihilated spontaneously and created catalytically on the sites of a regular lattice. The contact process in an infinite lattice exhibits a continuous phase transition from an absorbing state devoid or particles to an active state with a nonzero density of particles. The contact process is closely related to the Domany–Kinzel model [13]. Their critical behaviour place them in the directed percolation universality class [14, 15].

In the basic contact process, which will concern us here, a particle is created on an empty site with rate λ times the fraction of nearest neighbour occupied sites. If the neighbouring sites are all empty no particle is created. A particle is annihilated with rate 1. If all particles are annihilated the system becomes trapped in the absorbing state devoid of particles. This will always happen in a finite lattice if we wait long enough, no matter what the transition rate is. However, in an infinite system and above a critical value of λ the system will not enter the absorbing state and will remain in the active state.

In the supercritical regime the distribution of particles is such that the particle density is finite. At the critical point the density vanishes but not the average number of particles. In this case the dynamics generates a particle distribution with fractal properties. In the subcritical regime not only the density vanishes but the number of particles vanishes as well. In this regime it is usual to study the time-dependent properties, such as the survival probability and the average number of particles, by starting with a single seed particle [2, 7, 8]. These quantities decay exponentially with time and vanish in the infinite time limit. No particle is then left in this limit.

In this paper we are concerned with the construction of subcritical stationary distribution of particles in an infinite system. To avoid entering the absorbing state and thus obtaining a nonzero subcritical stationary distribution of particles we introduce small modifications on the rules of the contact process. The three modifications considered here are small enough to cause no change in the supercritical properties. They define three variants of the contact process. In the first, we allow a spontaneous creation of particle at a specified site of the lattice. The rate of spontaneous creation is supposed to be very small. In this variant, as we will show below, the subcritical stationary distribution of particles is directly related to that studied by Jensen and Dickman [8, 9]. In the second, a specified site of the lattice is permanently occupied by a particle that is not allowed to be annihilated. In the third, the last particle is forbidden to be annihilated. This variant has been studied by means of Monte Carlo simulation by Tomé and de Oliveira [16]. It is worth mentioning that there are other ways of preventing entering the absorbing state and obtaining a subcritical stationary distribution. We mention the method in which a quasi-stationary state comes out of the surviving trials [17, 18] and that derived from the conservative version of the contact process [19–21].

The main purpose of this paper is the study of the subcritical stationary probability distribution generated by the three variants by means of a perturbative series expansion and Padé analysis. Although the use of the series expansion and Padé analysis has a long and fruitful tradition in the study of phase transitions in equilibrium statistical mechanics [22, 23], its use in nonequilibrium models described by a master equation is more recent but equally successful. Series expansions for the contact process and related models have been used not only for the subcritical but also for the supercritical behaviour [4, 6–9, 24].

The introduction of the modifications allows the use of a perturbative expansion in which the annihilation operator is treated as the unperturbed evolution operator and the creation operator as the perturbation. As is well known, in order to generate a perturbative series expansion, it is necessary to start from a reference state which is the stationary state of the unperturbed evolution operator. Due to the modifications this reference state is no longer the vacuum state as happens to the original contact process. As a result a nontrivial stationary state may now come out of the perturbative expansion.

The approach we use here to get the series expansion is very simple when compared to other methods. Usually, a perturbative series expansion is obtained by considering the Laplace transform $\tilde{\Psi}(s)$ of the time-dependent state vector $\Psi(t)$. After obtaining the perturbative series expansion of the Laplace transform, the desired perturbative expansion is set up by taking the limit $s \to 0$ of $s\tilde{\Psi}(s)$. The approach we used here avoid the Laplace transformation, and therefore the limit procedure, and the series is obtained in a direct and straightforward manner. Another important point of the present method is the use of a representation, which we call sigma representation, associated with the eigenvectors of the annihilation operator instead of the usual occupation representation. These simplifications allowed us to get longer series expansions from which precise values for the critical parameter and the critical exponents may be obtained.

2. Occupation representation

In the contact process particles are created catalytically on the sites of a regular lattice with rate λ and annihilated spontaneously with rate 1. Using an occupation variable η_i that takes the value 0 or 1 according to whether the site i is empty or occupied by a particle then a configuration is denoted by $\eta = (\eta_1, \eta_2, \ldots, \eta_N)$, where N is the number of sites of the lattice. The transition rate $w_i(\eta)$ at which the site i changes its state is given by

$$w_i(\eta) = w_i^{\mathrm{a}}(\eta) + w_i^{\mathrm{c}}(\eta), \tag{1}$$

where

$$w_i^{\mathrm{a}}(\eta) = \eta_i \tag{2}$$

is the annihilation transition rate, and

$$w_i^{\mathrm{c}}(\eta) = \frac{\lambda}{2}(1 - \eta_i)(\eta_{i-1} + \eta_{i+1}) \tag{3}$$

is the creation transition rate. We are considering a one-dimensional lattice.

The time evolution of the probability $P(\eta, t)$ of a configuration η at time t is governed by the master equation

$$\frac{\mathrm{d}}{\mathrm{d}t}P(\eta, t) = \sum_i \{w_i(\eta^i)P(\eta^i, t) - w_i(\eta)P(\eta, t)\}, \tag{4}$$

where η^i is that configuration obtained from η by changing η_i to $1 - \eta_i$.

It is convenient to use the vector representation

$$|\psi(t)\rangle = \sum_\eta P(\eta, t)|\eta\rangle, \tag{5}$$

in which the vector

$$|\eta\rangle = |\eta_1, \eta_2, \ldots, \eta_N\rangle \tag{6}$$

represents a state of the system. Is is straightforward to show that the state vector $|\psi(t)\rangle$ evolves in time according to

$$\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle = (W_0 + \lambda V)|\psi(t)\rangle, \tag{7}$$

where W_0 and V are the operators related to the annihilation and creation processes, respectively. They are given by

$$W_0 = \sum_i B_i, \tag{8}$$

and

$$V = \sum_i Q_i(n_{i-1} + n_{i+1}). \tag{9}$$

The local operators B_i, Q_i and the number operator n_i are defined by

$$B_i|\circ\rangle = 0 \qquad \text{and} \qquad B_i|\bullet\rangle = |\circ\rangle - |\bullet\rangle, \tag{10}$$

$$Q_i|\circ\rangle = \tfrac{1}{2}(|\bullet\rangle - |\circ\rangle) \qquad \text{and} \qquad Q_i|\bullet\rangle = 0, \tag{11}$$

and

$$n_i|\circ\rangle = 0 \qquad \text{and} \qquad n_i|\bullet\rangle = |\bullet\rangle. \tag{12}$$

The symbols '$\circ$' and '$\bullet$' represent an empty site ($\eta_i = 0$) and an occupied site ($\eta_i = 1$), respectively.

3. Sigma representation

Since W_0 is a sum of independent operators B_i, its eigenvectors will be a direct product of the eigenvectors of B_i. The right eigenvectors of B_i are $|\circ\rangle$ and $|\bullet\rangle - |\circ\rangle$, with eigenvalues 0 and -1, respectively. The corresponding left eigenvectors are $\langle\bullet| + \langle\circ|$ and $\langle\bullet|$ respectively.

It is convenient to change from the occupation representation spanned by the vectors $|\circ\rangle$ and $|\bullet\rangle$ to a representation spanned by the vectors $|0\rangle$ and $|1\rangle$, which we call sigma representation, defined by

$$|0\rangle = |\circ\rangle \qquad \text{and} \qquad |1\rangle = |\bullet\rangle - |\circ\rangle. \tag{13}$$

The transformation of the left vectors are

$$\langle 0| = \langle\circ| + \langle\bullet| \qquad \text{and} \qquad \langle 1| = \langle\bullet|. \tag{14}$$

In this new representation B_i is diagonal, that is,

$$B_i|0\rangle = 0 \qquad \text{and} \qquad B_i|1\rangle = -|1\rangle. \tag{15}$$

The eigenvectors of W_0 are then

$$|\sigma\rangle = |\sigma_1, \sigma_2, \ldots, \sigma_N\rangle, \tag{16}$$

where $\sigma_i = 0$ or 1, with eigenvalues

$$\Lambda(\sigma) = -\sum_i \sigma_i. \tag{17}$$

Next we need to known how the operator V acts on a vector $|\sigma\rangle$ of the sigma representation. We begin by rearranging the terms in V in the form

$$V = \sum_i (Q_i n_{i+1} + n_i Q_{i+1}). \tag{18}$$

From the definition of Q_i and n_i and using the transformation (13) it follows that

$$Q_i|0\rangle = \tfrac{1}{2}|1\rangle \qquad \text{and} \qquad Q_i|1\rangle = -\tfrac{1}{2}|1\rangle, \tag{19}$$

and that

$$n_i|0\rangle = 0 \qquad \text{and} \qquad n_i|1\rangle = |0\rangle + |1\rangle. \tag{20}$$

From these relations one gets the following important rules:

$$(Q_i n_{i+1} + n_i Q_{i+1})|00\rangle = 0, \tag{21}$$

$$(Q_i n_{i+1} + n_i Q_{i+1})|01\rangle = \tfrac{1}{2}|10\rangle + \tfrac{1}{2}|11\rangle, \tag{22}$$

$$(Q_i n_{i+1} + n_i Q_{i+1})|10\rangle = \tfrac{1}{2}|01\rangle + \tfrac{1}{2}|11\rangle, \tag{23}$$

$$(Q_i n_{i+1} + n_i Q_{i+1})|11\rangle = -\tfrac{1}{2}|10\rangle - \tfrac{1}{2}|01\rangle - |11\rangle. \tag{24}$$

4. Perturbation series expansion

We are interested in obtaining a series expansion in λ of the stationary state $|\psi\rangle$ which obeys the equation

$$(W_0 + \lambda V)|\psi\rangle = 0. \tag{25}$$

Let $|\phi_0\rangle$ be the stationary state of the process defined by W_0, that is,

$$W_0|\phi_0\rangle = 0. \tag{26}$$

Then a perturbative series expansion for $|\psi\rangle$ in powers of λ can be set up. Following Dickman [6], it is straightforward to show that

$$|\psi\rangle = |\phi_0\rangle + \sum_{\ell=1}^{\infty} (-\lambda)^{\ell} (RV)^{\ell} |\phi_0\rangle, \tag{27}$$

where R, the resolvent operator, is the inverse of W_0 within the subspace spanned by the eigenvectors W_0 with nonzero eigenvectors. It is given by

$$R = \sum_{\sigma} |\sigma\rangle \frac{1}{\Lambda(\sigma)} \langle\sigma|, \tag{28}$$

where the summation is over the eigenstates of W_0 with nonzero eigenvalues.

If we write

$$|\psi\rangle = \sum_{\ell=0}^{\infty} \lambda^{\ell} |\psi_{\ell}\rangle, \tag{29}$$

where $|\psi_0\rangle = |\phi_0\rangle$, then one gets from equation (27) the following relation:

$$|\psi_{\ell+1}\rangle = -RV|\psi_{\ell}\rangle, \tag{30}$$

which allows us to obtain the vectors $|\psi_{\ell}\rangle$ recursively by the use of rules (21), (22), (23) and (24) and expression (28) for the operator R.

The stationary state $|\phi_0\rangle$ of the process defined by the evolution operator W_0, given by (8), is simply the state devoid of particles

$$|\phi_0\rangle = |\ldots 000 \ldots\rangle. \tag{31}$$

But no particles can be created out of this state, that is, $V|\phi_0\rangle = 0$. Therefore, by (30), $|\psi_{\ell}\rangle = 0$ for $\ell \neq 0$, and the stationary state $|\psi\rangle$ is the state devoid of particles $|\phi_0\rangle$. To get a nontrivial stationary state we will consider in what follows the three variants of the contact process introduced above.

4.1. First variant

In the first variant, a spontaneous creation of particle at a given site, say at the origin $i = 0$, is added to the original process. This is realized by changing the annihilation part w_0^{a} to

$$w_0^{\mathrm{a}}(\eta) = (1-q)\eta_0 + q(1-\eta_0), \tag{32}$$

where the parameter q is considered to be a small quantity. This modification amounts changing the operator W_0 to

$$W_0 = \sum_{i(\neq 0)} B_i + (1-q)B_0 + q\,Q_0. \tag{33}$$

The stationary state $|\phi_0\rangle$, that is, the eigenvector of W_0 with zero eigenvalue, now reads

$$|\phi_0\rangle = |.0.\rangle + q|.1.\rangle. \tag{34}$$

The right (left) dot in the notation '.x.' means that all sites on the right-hand side (left) are in the state 0. Therefore, up to linear terms in q, the stationary state $|\psi\rangle$ is given by

$$|\psi\rangle = |.0.\rangle + q|.1.\rangle + q \sum_{\ell=1}^{\infty} (-\lambda)^{\ell} (RV)^{\ell} |.1.\rangle. \tag{35}$$

We remark that, although the change in W_0 will cause a change in R, only the terms of zero order in q, given by the right-hand side of equation (28), will enter in the expression of the vector $|\psi\rangle$.

Using the rules obtained in previous sections we get the following results:

$$|\psi_0\rangle = |.0.\rangle + q|.1.\rangle, \tag{36}$$

$$|\psi_1\rangle = q\left\{|.1.\rangle + \tfrac{1}{2}|.11.\rangle\right\}, \tag{37}$$

$$|\psi_2\rangle = q\left\{\tfrac{1}{2}|.1.\rangle + \tfrac{1}{4}|.11.\rangle + \tfrac{1}{4}|.101.\rangle + \tfrac{1}{6}|.111.\rangle\right\}. \tag{38}$$

Returning to the occupation representation we get

$$|\psi_0\rangle = (1-q)|.\circ.\rangle + q|.\bullet.\rangle, \tag{39}$$

$$|\psi_1\rangle = q\left\{-\tfrac{1}{2}|.\circ.\rangle + \tfrac{1}{2}|.\bullet\bullet.\rangle\right\}, \tag{40}$$

$$|\psi_2\rangle = q\left\{-\tfrac{1}{6}|.\circ.\rangle - \tfrac{1}{12}|.\bullet\bullet.\rangle + \tfrac{1}{12}|.\bullet\circ\bullet.\rangle + \tfrac{1}{6}|.\bullet\bullet\bullet.\rangle\right\}. \tag{41}$$

It is worth mentioning that the stationary state $|\psi\rangle$, given by equation (35), is directly related to the Laplace transform $|\tilde{\Psi}(s)\rangle$ of the time-dependent vector probability $|\Psi(t)\rangle$ of the original contact process. The state $|\Psi(t)\rangle$ is the solution of the evolution equation obtained by starting at time $t = 0$ with a state with a single particle. According to Jensen and Dickman [8, 9], the following series expansion

$$|\tilde{\Psi}(s)\rangle = \sum_{\ell=0}^{\infty} \lambda^{\ell} |\tilde{\Psi}_{\ell}(s)\rangle, \tag{42}$$

can be set up, where for small values of s

$$|\tilde{\Psi}_0(s)\rangle = \frac{1}{s}|.0.\rangle + |.1.\rangle \tag{43}$$

and

$$|\tilde{\Psi}_{\ell+1}(s)\rangle = -RV|\tilde{\Psi}_{\ell}(s)\rangle. \tag{44}$$

These expressions are formally identical to equations (29), (36) and (30), and we may conclude that $|\psi\rangle = q|\tilde{\Psi}(q)\rangle$ for small values of q. In words, the stationary probability of the modified process is formally proportional to the Laplace transform of the time-dependent probability distribution of the original process.

4.2. Second variant

The second variant may be interpreted as a specific type of boundary condition in which the site $i = 0$ is permanently occupied by a particle. In this case it suffices to consider the semi-infinite lattice defined by the sites $i = 1, 2, 3, \ldots$. The creation part $w_1^{\mathrm{c}}(\eta)$ at site $i = 1$ now reads

$$w_1^{\mathrm{c}}(\eta) = \frac{\lambda}{2}(1-\eta_1)(1+\eta_2). \tag{45}$$

The operator W_0 and V are given by

$$W_0 = \sum_{i=1}^{\infty} B_i \tag{46}$$

and

$$V = Q_1 + \sum_{i=1}^{\infty} (Q_i n_{i+1} + n_i Q_{i+1}). \tag{47}$$

The stationary state $|\phi_0\rangle$ defined by W_0 is the state devoid of particles, that is,

$$|\phi_0\rangle = |1.\rangle, \tag{48}$$

where we have included the state of site $i = 0$ in the vector $|1.\rangle$. Again the dot on the right means that all sites on the right-hand side are in the state 0. Due to the presence of the operator Q_1 in V which has the property $Q_1|0\rangle = |1\rangle$, the stationary vector $|\psi\rangle$ will be distinct from the empty state $|1.\rangle$.

Using the rules of the previous sections we get the following results:

$$|\psi_0\rangle = |1.\rangle, \tag{49}$$

$$|\psi_1\rangle = \tfrac{1}{2}|11.\rangle, \tag{50}$$

$$|\psi_2\rangle = -\tfrac{1}{4}|11.\rangle + \tfrac{1}{4}|101.\rangle + \tfrac{1}{8}|111.\rangle. \tag{51}$$

Returning to the occupation representation we get

$$|\psi_0\rangle = |\bullet .\rangle, \tag{52}$$

$$|\psi_1\rangle = -\tfrac{1}{2}|\bullet .\rangle + \tfrac{1}{2}|\bullet\bullet.\rangle, \tag{53}$$

$$|\psi_2\rangle = \tfrac{1}{8}|\bullet .\rangle - \tfrac{3}{8}|\bullet\bullet.\rangle + \tfrac{1}{8}|\bullet\circ\bullet .\rangle + \tfrac{1}{8}|\bullet\bullet\bullet .\rangle. \tag{54}$$

4.3. Third variant

In the third variant the last particle is not allowed to be annihilated. This prohibition is accomplished by changing the annihilation transition rate $w_i^{\mathrm{a}}(\eta)$ to

$$w_i^{\mathrm{a}}(\eta) = \eta_i \gamma(\eta), \tag{55}$$

where the quantity $\gamma(\eta)$ vanishes whenever η is a state with just one particle and equals 1 otherwise. The operator W_0 is now given by

$$W_0 = \sum_i B_i \Gamma, \tag{56}$$

where the operator Γ is defined by $\Gamma|\eta\rangle = \gamma(\eta)|\eta\rangle$ so that it gives zero when acting on a state $|.\bullet .\rangle$ with just one particle. The operator V is unchanged.

To set up the eigenvalues of W_0 we begin by noting that $W_0|.1.\rangle = 0$ so that $|.1.\rangle$ is the stationary state of the process defined by W_0. Moreover, whenever $|\sigma\rangle$ is a state distinct from $|.1.\rangle$ then it follows that $\Gamma|\sigma\rangle = |\sigma\rangle$. Using these properties and the transformation from the occupation representation to the sigma representation, it is straightforward to show that the right eigenvectors of W_0 with nonzero eigenvalues are

$$|\phi_\sigma\rangle = |\sigma\rangle + I(\sigma)|.1.\rangle, \tag{57}$$

where $|\sigma\rangle$ is a vector distinct from $|.1.\rangle$ and $I(\sigma) = 1$ or -1 according to whether $|\Lambda(\sigma)|$ is even or odd, respectively. The corresponding left eigenvectors and eigenvalues are merely $\langle\sigma|$ and $\Lambda(\sigma)$, respectively. From these results it follows that the operator R is now given by

$$R = \sum_\sigma |\phi_\sigma\rangle \frac{1}{\Lambda(\sigma)} \langle\sigma|, \tag{58}$$

where the summation is over the eigenstates of W_0 with nonzero eigenvalues.

Using the rules of the previous sections we get the following results:

$$|\psi_0\rangle = |.1.\rangle, \tag{59}$$

$$|\psi_1\rangle = \tfrac{1}{2}|.1.\rangle + \tfrac{1}{2}|.11.\rangle, \tag{60}$$

$$|\psi_2\rangle = \tfrac{1}{12}|.1.\rangle + \tfrac{1}{4}|.101.\rangle + \tfrac{1}{6}|.111.\rangle. \tag{61}$$

Returning to the occupation representation we get

$$|\psi_0\rangle = |.\bullet.\rangle, \tag{62}$$

$$|\psi_1\rangle = -\tfrac{1}{2}|.\bullet.\rangle + \tfrac{1}{2}|.\bullet\bullet.\rangle, \tag{63}$$

$$|\psi_2\rangle = \tfrac{1}{12}|.\bullet.\rangle - \tfrac{1}{3}|.\bullet\bullet.\rangle + \tfrac{1}{12}|.\bullet\circ\bullet.\rangle + \tfrac{1}{6}|.\bullet\bullet\bullet.\rangle. \tag{64}$$

5. Critical behaviour

5.1. First variant

From the series expansion of the stationary probability vector $|\psi\rangle$ we may determine three quantities. The first quantity is the average number of particles, given by

$$n = \langle .0.| \sum_i n_i |\psi\rangle. \tag{65}$$

The second quantity is the survival probability, given by

$$P = 1 - \langle .\circ.|\psi\rangle. \tag{66}$$

The third quantity is the correlation length ξ, defined in one dimension by

$$\xi = \langle .0.|\Xi|\psi\rangle, \tag{67}$$

where Ξ is the operator that acting on a vector $|\eta\rangle$ gives the size of the cluster of particles in $|\eta\rangle$, that is, the distance between the first and last particles. For example, $\Xi|.\bullet\circ\bullet\bullet.\rangle = 4|.\bullet\circ\bullet\bullet.\rangle$.

From the series for $|\psi\rangle$ we obtain the following results up to third order in λ:

$$n = q\left\{1 + \lambda + \tfrac{1}{2}\lambda^2 + \tfrac{1}{4}\lambda^3\right\}, \tag{68}$$

$$P = q\left\{1 + \tfrac{1}{2}\lambda + \tfrac{1}{6}\lambda^2 + \tfrac{5}{72}\lambda^3\right\}, \tag{69}$$

$$\xi = q\left\{1 + \tfrac{1}{2}\lambda + \tfrac{5}{12}\lambda^2 + \tfrac{2}{9}\lambda^3\right\}. \tag{70}$$

At the critical point these quantities have a singular behaviour governed by critical exponents. We assume that

$$n \sim \varepsilon^{-a}, \tag{71}$$

$$P \sim \varepsilon^{-b}, \tag{72}$$

$$\xi^d \sim \varepsilon^{-c}, \tag{73}$$

where $\varepsilon = \lambda_c - \lambda$ is the deviation of the creation rate λ from its critical value λ_c. Due to the relation $|\psi\rangle = q|\tilde{\Psi}(q)\rangle$ it follows that the critical behaviour of these quantities are the same as those of the Laplace transforms of the related quantities in the original contact process. From the critical behaviour of the Laplace transforms [8], we may conclude that the exponents a and b are given by

$$a = \nu_{\parallel} + d\nu_{\perp} - 2\beta, \tag{74}$$

$$b = \nu_{\parallel} - \beta, \tag{75}$$

$$c = \nu_{\parallel} + d\nu_{\perp} - \beta, \tag{76}$$

where β is the order parameter exponent, $\nu_{\parallel}$ is the time correlation length exponent and $\nu_{\perp}$ is the spatial correlation length exponent.

In the limit $q \to 0$, the three quantities n, P and ξ vanish as expected. However, their ratios attain finite values. For instance, we may consider the ratio $n_s = n/P$, which is the number of particles in surviving trials [11], and the ratio $\xi_s = \xi/P$, which is the size of a cluster of particles in surviving trials. Up to third order in λ they are given by

$$n_s = 1 + \tfrac{1}{2}\lambda + \tfrac{1}{12}\lambda^2 + \tfrac{5}{9}\lambda^3, \tag{77}$$

$$\xi_s = 1 + \tfrac{1}{4}\lambda^2 + \tfrac{1}{36}\lambda^3. \tag{78}$$

From the critical behaviour of n, P and ξ it follows that their critical behaviour is given by

$$n_s \sim \varepsilon^{-(d\nu_{\perp}-\beta)}, \tag{79}$$

$$\xi_s \sim \varepsilon^{-\nu_{\perp}}. \tag{80}$$

Since the series expansion for this first variant is formally identical to that obtained by Jensen and Dickman [8], we will not present further results for this variant. The coefficients of the series expansion as well as the Padé analysis can be found in the cited work.

5.2. Second and third variants

In these two variants the quantities of interest are the average number of particles n, defined by (65), and the correlation length ξ, defined by (67). For the second variant we get from the series for $|\psi\rangle$ the following results up to third order in λ:

$$n = 1 + \tfrac{1}{2}\lambda + \tfrac{1}{8}\lambda^3, \tag{81}$$

$$\xi = 1 + \tfrac{1}{2}\lambda + \tfrac{1}{8}\lambda^2 + \tfrac{1}{12}\lambda^3. \tag{82}$$

For the third variant we get, up to third order in λ,

$$n = 1 + \tfrac{1}{2}\lambda + \tfrac{1}{12}\lambda^2 + \tfrac{1}{18}\lambda^3, \tag{83}$$

$$\xi = 1 + \tfrac{1}{2}\lambda + \tfrac{1}{6}\lambda^2 + \tfrac{1}{18}\lambda^3. \tag{84}$$

The other coefficients for n and ξ, for both the second and third variants, are shown in tables 1 and 2.

Around the critical point we assume that the number of particles diverges as [16]

$$n \sim \varepsilon^{-(d\nu_{\perp}-\beta)}, \tag{85}$$

and that the spatial correlation length diverges as

$$\xi \sim \varepsilon^{-\nu_{\perp}}. \tag{86}$$

Note that the critical behaviour of n and ξ are the same as those of n_s and ξ_s of the first variant.

Table 1. Coefficients of λ^ℓ for the average number of particle n for the second and third variants in the subcritical series expansion.

ℓ	Second	Third
0	1.000 000 000 000 000	1.000 000 000 000 0000
1	0.500 000 000 000 000	0.500 000 000 000 0000
2	0.000 000 000 000 000	0.083 333 333 333 3333
3	0.125 000 000 000 000	0.055 555 555 555 5556
4	−0.093 750 000 000 000	−0.008 333 333 333 3333
5	0.111 979 166 666 667	0.018 387 345 679 0124
6	−0.116 753 472 222 222	−0.013 402 116 402 1164
7	0.124 385 127 314 815	0.013 517 832 997 3965
8	−0.129 084 080 825 617	−0.012 457 084 635 5885
9	0.132 276 797 196 502	0.011 889 129 387 7448
10	−0.133 815 792 800 631	−0.011 233 961 703 3972
11	0.134 350 973 173 751	0.010 626 814 450 3707
12	−0.134 385 866 153 155	−0.010 017 907 756 8664
13	0.134 359 669 467 490	0.009 426 685 263 0381
14	−0.134 512 665 155 863	−0.008 857 789 634 4389
15	0.134 912 975 608 337	0.008 320 792 570 2731
16	−0.135 506 586 874 224	−0.007 821 265 897 1593
17	0.136 189 646 637 980	0.007 362 178 549 5991
18	−0.136 864 742 672 810	−0.006 943 476 690 8879
19	0.137 470 871 166 280	0.006 562 978 439 8898
20	−0.137 987 574 408 428	−0.006 217 193 722 5142
21	0.138 423 450 233 117	0.005 902 139 909 8014
22	−0.138 800 189 698 668	−0.005 613 923 762 5574
23	0.139 139 882 266 804	0.005 349 082 570 2793
24	−0.139 458 441 755 729	−0.005 104 710 136 9433
25	0.139 764 377 494 116	0.004 878 438 179 9974
26	−0.140 060 583 585 506	−0.004 668 343 684 9256
27	0.140 346 941 466 963	0.004 472 838 033 0116
28	−0.140 622 416 820 138	−0.004 290 570 855 9361
29	0.140 886 230 616 994	0.004 120 361 311 8942
30	−0.141 138 240 791 685	
31	0.141 378 864 861 045	
32	−0.141 608 839 415 536	
33	0.141 828 994 883 551	
34	−0.142 040 114 152 729	
35	0.142 242 875 174 728	

5.3. Padé approximants

To obtain the exponents as well as the critical parameter λ_c from the coefficients of the series for n and ξ we have used an analysis by Padé approximants [22, 23]. If a quantity Q behaves as $Q \sim (\lambda - \lambda_c)^\theta$, then its logarithmic derivative behaves as

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \ln Q \sim \frac{\theta}{\lambda - \lambda_c}. \tag{87}$$

Therefore, the critical exponent θ and the critical parameter λ_c are identified as the residue and pole, respectively, of a given Padé approximant of the logarithm derivative.

Tables 3 and 4 show the estimates of λ_c and the exponents obtained from the coefficients of tables 1 and 2 by the use of Padé approximants of the logarithmic derivative. The determination of the deviations of the results given by the Padé approximants from the true values represents

Table 2. Coefficients of λ^ℓ for the spatial correlation length ξ for the second and third variants in the subcritical series expansion.

ℓ	Second	Third
0	1.000 000 000 000 000	1.000 000 000 000 0000
1	0.500 000 000 000 000	0.500 000 000 000 0000
2	0.125 000 000 000 000	0.166 666 666 666 6667
3	0.083 333 333 333 333	0.055 555 555 555 5556
4	−0.024 305 555 555 556	0.014 120 370 370 3705
5	0.050 245 949 074 074	0.007 646 604 938 2716
6	−0.050 118 513 695 988	−0.000 923 340 130 8054
7	0.057 850 842 995 618	0.002 987 686 985 1593
8	−0.062 559 206 523 165	−0.002 323 549 072 0257
9	0.067 393 613 526 403	0.002 475 279 487 9555
10	−0.071 384 381 697 677	−0.002 353 458 864 9680
11	0.074 982 356 264 358	0.002 294 559 404 4978
12	−0.078 276 583 001 871	−0.002 199 729 374 2912
13	0.081 457 024 149 850	0.002 104 492 285 2996
14	−0.084 614 333 790 258	−0.002 003 388 287 2320
15	0.087 781 399 900 321	0.001 902 932 112 4057
16	−0.090 935 696 363 601	−0.001 804 998 167 5254
17	0.094 032 672 324 418	0.001 711 735 916 5925
18	−0.097 030 597 799 896	−0.001 624 121 091 3964
19	0.099 906 288 141 423	0.001 542 586 815 5175
20	−0.102 657 905 786 795	−0.001 467 078 542 2359
21	0.105 299 292 561 241	0.001 397 279 992 8762
22	−0.107 851 100 207 176	−0.001 332 742 707 6196
23	0.110 333 215 617 021	0.001 272 985 412 2679
24	−0.112 760 549 958 613	−0.001 217 546 093 2682
25	0.115 142 151 000 704	0.001 166 005 244 0016
26	−0.117 482 422 596 543	−0.001 117 990 544 4874
27	0.119 783 063 174 871	0.001 073 173 357 2414
28	−0.122 044 754 480 048	−0.001 031 262 888 5921
29	0.124 268 197 198 081	0.000 992 000 864 6124

a difficult problem. We may look at the convergence of the results as one increases the order of the Padé approximants. However, the convergence may not be clear. We have alternatively adopt our results as averages of results coming from distinct estimates. Using this point of view, our result for the critical value of the creation rate is taken to be $\lambda_c = 3.2979(1)$ which is in good agreement with the values $\lambda_c = 3.297\,85(2)$, obtained for the one-dimensional contact process [8]. Using the same approach we find the following results for the exponent related to the average number or particles $\nu_\perp - \beta = 0.821(1)$ and for the exponent related to the spatial correlation length $\nu_\perp = 1.097(2)$. These results should be compared with the results $\nu_\perp - \beta = 0.820\,35(10)$ and $\nu_\perp = 1.096\,84(6)$ obtained for the directed percolation in one dimension [10].

From equations (85) and (86) we see that the correlation length behaves as a function of the average number or particles as

$$\xi \sim n^{1/d_F}, \tag{88}$$

for sufficient large n, where

$$d_F = \frac{d\nu_\perp - \beta}{\nu_\perp} \tag{89}$$

Table 3. Estimates for the critical point λ_c and the value of the critical exponent $\nu_\perp - \beta$ obtained from the Padé approximants to the logarithm derivative of n.

Variant	Approximant	λ_c	$\nu_\perp - \beta$
Second	[11/11]	3.297 962	0.819 91
	[12/12]	3.297 963	0.819 92
	[13/13]	3.297 878	0.819 67
	[14/14]	3.297 903	0.819 75
	[15/15]	3.297 895	0.819 73
	[16/16]	3.297 895	0.819 73
	[16/17]	3.297 880	0.819 67
	[17/16]	3.297 883	0.819 69
	[17/17]	3.297 894	0.819 72
Third	[10/10]	3.298 511	0.822 92
	[11/11]	3.298 444	0.822 71
	[12/12]	3.298 524	0.822 94
	[13/13]	3.298 264	0.822 09
	[13/14]	3.298 148	0.821 60
	[14/13]	3.298 183	0.821 75
	[14/14]	3.298 224	0.821 93

Table 4. Estimates for the critical point λ_c and the value of the critical exponent $\nu_\perp$ obtained from the Padé approximants to the logarithm derivative of ξ.

Variant	Approximant	λ_c	$\nu_\perp$
Second	[10/10]	3.300 118	1.103 59
	[11/11]	3.298 758	1.100 90
	[12/12]	3.298 290	1.099 32
	[13/13]	3.298 179	1.098 87
	[13/14]	3.298 175	1.098 85
	[14/13]	3.298 174	1.098 84
	[14/14]	3.298 179	1.098 87
Third	[10/10]	3.297 686	1.094 61
	[11/11]	3.296 861	1.094 62
	[12/12]	3.297 713	1.094 72
	[13/13]	3.297 707	1.094 69
	[13/14]	3.297 714	1.094 72
	[14/13]	3.297 740	1.094 84
	[14/14]	3.297 696	1.094 66

is the fractal dimension of the critical cluster. If a series expansion of ξ in powers of n is known then an estimate of d_F can be obtained from a Padé analysis, with no need of the critical parameter λ_c. From the results of tables 1 and 2 we can indeed set up a series expansion of ξ in powers of $n - 1$ by eliminating λ from both series. From the Padé analysis of the logarithm derivative of ξ with respect to n, we find the results shown in table 5. The fractal dimension coming from this analysis is $d_F = 0.7477(2)$ which is in good agreement with $d_F = 0.747\,92(2)$ that follows by inserting the cited results coming from direct percolation [10] in equation (89).

Table 5. Estimates of the critical exponent d_F obtained from the Padé approximants to the logarithm derivative of ξ as a function of $n - 1$.

Variant	Approximant	d_F
Second	[8/9]	0.748 95
	[9/10]	0.748 95
	[10/11]	0.748 29
	[11/12]	0.747 66
	[12/13]	0.747 89
	[13/14]	0.747 77
Third	[8/9]	0.747 39
	[9/10]	0.747 59
	[10/11]	0.747 58
	[11/12]	0.747 59
	[12/13]	0.747 60
	[13/14]	0.747 59

6. Conclusion

We have studied the subcritical stationary properties of three variants of the contact process in one dimension. The modifications that define the three variants are small enough to cause no change in the supercritical properties. The stationary distribution generated by these variants of the contact process are studied by a perturbative series expansion. For the second and third variants we have obtained the coefficients of the average number of particles and the spatial correlation length. The two series for both variants were then analysed by the Padé approximants for their logarithm derivatives. Our estimates for the critical creation rate as well as for the critical exponents are in good agreement with the previous results for the contact process and the directed percolation. Although the stationary states coming from three variants of the basic contact process studied here give distinct results, their critical behaviour are described by the same critical exponents. We remark finally that the stationary states of the first variant has a straight relationship with the quasi-stationary states coming from the surviving trials.

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