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THIRD ORDER DIFFERENTIAL SUBORDINATION OF ANALYTIC FUNCTION DEFINED BY FRACTIONAL DERIVATIVE OPERATOR

BY

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Abstract. Third order differential subordination results are obtained for analytic functions in the open disc which are associated with the fractional derivative operator. These results are obtained by investigating suitable classes of admissible functions. Certain examples are also discussed.

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Key words: differential subordination, analytic function, fractional derivative operator.

1. Introduction

Let $\mathcal{H}(U)$ be the class of functions analytic in $U := \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}(U)$ consisting of the functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots$, where $a \in \mathbb{C}$ and $n \in \mathbb{N}$. Also $\mathcal{H}_0 \equiv \mathcal{H}[0, 1]$ and $\mathcal{H} \equiv \mathcal{H}[1, 1]$. Let $\mathcal{A}_p$ denote the class of analytic functions of the form $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$, $(z \in U)$. Let $\mathcal{A}_1 = \mathcal{A}$. Let $f, F \in \mathcal{H}(U)$, then the function f is said to be subordinate to F or F is said to be superordinate to f, if there exists a function w analytic in U with $w(0) = 0$ and $|w(z)| < 1$, $(z \in U)$, such that $f(z) = F(w(z))$.

In such a case we write $f(z) \prec F(z)$. If F is univalent, then $f(z) \prec F(z)$ if and only if $f(0) = F(0)$ and $f(U) \subset F(U)$.

Let α, β and γ be complex numbers with $\gamma \neq 0, -1, -2, \ldots$ Then the Gaussian hypergeometric function is ${}_2F_1(\alpha, \beta, \gamma; z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k} \frac{z^k}{k!}$, where

$(\eta)_k$ is the Pochammer symbol defined in terms of the Gamma function by

$$(\eta)_k = \frac{\Gamma(\eta+k)}{\Gamma(\eta)} = \begin{cases} 1, & \text{if } k=0 \\ \eta(\eta+1)...(\eta+k-1), & \text{if } k \in \mathbb{N}. \end{cases}$$

The hypergeometric function ${}_2F_1(\alpha, \beta, \gamma; z)$ is analytic in U, and if α or β is a negative integer, then it reduces to a polynomial. Among the various number of definitions for fractional calculus operator (see [7] and [8]) we use the fractional derivative operator defined as follows (see [2]).

Definition 1.1. Let $0 \leq \lambda < 1$ and $\mu, \nu \in \mathbb{R}$. Then the generalized fractional derivative operator $I_{0,z}^{\lambda,\mu,\nu}$ of a function $f(z)$ is given by

$$I_{0,z}^{\lambda,\mu,\nu} f(z) = \frac{d}{dz}\left\{ \frac{z^{\lambda-\mu}}{\Gamma(1-\lambda)} \int_0^z (z-\zeta)^{-\lambda} \; {}_2F_1(\mu-\lambda, 1-\nu, 1-\lambda; 1-\frac{\zeta}{z}) f(\zeta) d\zeta \right\}.$$

The function $f(z)$ is analytic in a simply-connected region of the z-plane containing the origin with the order, $f(z) = O(|z|^\epsilon)$, for $\epsilon > \max\{0, \mu - \nu\} - 1$, and multiplicity of $(z-\zeta)^\lambda$ is removed by requiring that $\log(z-\zeta)$ be real when $z - \zeta > 0$.

Definition 1.2. Using the hypothesis of Definition 1.1 the fractional derivative $I_{0,z}^{\lambda+m,\mu+m,\nu+m}$ of a function $f(z)$ is defined by

$$I_{0,z}^{\lambda+m,\mu+m,\nu+m} f(z) = \frac{d^m}{dz^m} I_{0,z}^{\lambda,\mu,\nu} f(z), \quad (z \in U, m \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}). \tag{1.1}$$

Using the above definition, a modification of the fractional derivative operator $\Delta_{z,p}^{\lambda,\mu,\nu}$ by Choi [2]

$$\Delta_{z,p}^{\lambda,\mu,\nu} f(z) = \frac{\Gamma(p+1-\mu)\Gamma(p+1-\lambda+\nu)}{\Gamma(p+1)\Gamma(p+1-\mu+\nu)} z^\mu I_{0,z}^{\lambda,\mu,\nu} f(z), \tag{1.2}$$

for $f(z) \in \mathcal{A}_p$ and $\mu - \nu - p < 1$. Note that $\Delta_{z,p}^{0,0,\nu} f(z) = f(z), \Delta_{z,p}^{1,1,\nu} f(z) = \frac{zf'(z)}{p}$ and $\Delta_{z,p}^{\lambda,\lambda,\nu} f(z) = \Xi_z^{\lambda,p} f(z)$, where $\Xi_z^{\lambda,p}$ is the fractional derivative operator defined by Srivastava and Aouf (see [9] and [10]). Also $\Delta_{z,p}^{\lambda,\mu,\nu}$ maps $\mathcal{A}_p$ onto itself as follows

$$\Delta_{z,p}^{\lambda,\mu,\nu} f(z) = z^p + \sum_{k=1}^{\infty} \frac{(p+1)_k(p+1-\mu+\nu)_k}{(p+1-\mu)_k(p+1-\lambda+\nu)_k} a_{k+p} z^{k+p}, \tag{1.3}$$

$(z \in U; 0 \leq \lambda < 1; \mu - \nu - p < 1; f \in \mathcal{A}_p)$.

It is easily verified from (1.2)

$$z(\Delta_{z,p}^{\lambda,\mu,\nu} f(z))' = (p-\mu)\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z) + \mu \Delta_{z,p}^{\lambda,\mu,\nu} f(z). \tag{1.4}$$

Definition 1.3 ([1])**.** Let $\phi : \mathbb{C}^4 \times U \to \mathbb{C}$ and h be univalent in U. If p is analytic in U and satisfies the following (third-order) differential subordination:

$$\phi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \prec h(z), (z \in U), \tag{1.5}$$

then p is called a solution of the differential subordination. The univalent function q, is called a dominant of the solutions of the differential subordination if $p \prec q$, for all p satisfying (1.5). A dominant $\tilde{q}$ that satisfies, $\tilde{q} \prec q$, for all of q of (1.5) is said to be the best dominant.

The theory of differential subordination in $\mathbb{C}$ is the complex analogue of differential inequality in $\mathbb{R}$. Many of the significant works on differential subordination have been pioneered by Miller and Mocanu and their monograph [5] compiled their great efforts in introducing and developing the same. The theory of first and second order differential subordination has been used by many authors to solve problems in this field. There are few articles dealing with third order inequalities and subordination (see [1] and [6]). Jeyaraman et al. [4] have also applied the third order subordination result on Schwarzian derivative. In this work, using the methods of third order differential subordination, sufficient conditions involving the fractional derivative of a normalized analytic function are obtained.

Let Q denote the set of all functions q that are analytic and injective on $\overline{U}/E(q)$, where $E(q) = \{\zeta \in \partial U : \lim_{z\to\zeta} q(z) = \infty\}$, and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial U \backslash E(q)$. Further, let the subclass of Q for $Q(0) \equiv a$ be denoted by $Q(a)$, $Q(0) \equiv Q_0$ and $Q \equiv Q_1$.

Definition 1.4 ([1])**.** Let Ω be a set in $\mathbb{C}$, $q \in Q$ and $n \geq 2$. The class of admissible operators $\Psi_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^4 \times U \to \mathbb{C}$ that satisfy the following admissibility condition: $\psi(u, v, w, x; z) \notin \Omega$ whenever $u = q(\zeta), v = n\zeta q'(\zeta)$,

$$\Re\left\{\frac{w}{v} + 1\right\} \geq n\Re\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right\} \quad \text{and} \quad \Re\left\{\frac{x}{v}\right\} \geq n^2\Re\left\{\frac{\zeta^2 q'''(\zeta)}{q'(\zeta)}\right\}.$$

Lemma 1.1 ([1]). *Let* $p \in \mathcal{H}[a,n]$ *with* $n \geq 2$ *and let* $q \in Q(a)$ *satisfy*

$$\Re\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)}\right\} \geq 0 \text{ and } \left|\frac{zp'(z)}{q'(\zeta)}\right| \leq n,$$

where $z \in U$ *and* $\zeta \in \partial U/E(q)$. *If* Ω *is a set in* $\mathbb{C}$, $\psi \in \Psi_n[\Omega,q]$ *and* $\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \in \Omega$, *then* $p(z) \prec q(z), (z \in U)$.

2. Subordination results obtained by fractional derivative operator

We define the following class of admissible function,

Definition 2.1. Let Ω be a set in $\mathbb{C}$, $q \in Q_0 \cap \mathcal{H}[0,p]$. The class of admissible functions $\Phi_\Delta[\Omega,q]$ consists of those functions $\phi : \mathbb{C}^4 \times U \to \mathbb{C}$ that satisfy the following admissible condition $\phi(a,b,c,d;z) \notin \Omega$ whenever

$$a = q(\zeta), \qquad b = \frac{n\zeta q'(\zeta) - \mu q(\zeta)}{(p-\mu)},$$

$$\Re\left\{\frac{(p-\mu)(p-\mu-1)c - \mu(\mu+1)a}{(p-\mu)b + \mu a} + 2\mu + 1\right\} \geq n\Re\left\{1 + \frac{\zeta q''(\zeta)}{q'(\zeta)}\right\}$$

and

$$\Re\left\{\frac{(p-\mu)(p-\mu-1)(p-\mu-2)d + 3\mu(p-\mu)(p-\mu-1)c - 2\mu a(\mu^2-1)}{(p-\mu)b + \mu a}\right.$$
$$\left. + 3\mu(\mu+1)\right\} \geq n^2\Re\left\{\frac{\zeta^2 q'''(\zeta)}{q'(\zeta)}\right\}, (z \in U; \zeta \in \partial U/E(q); \mu \neq p, p \in \mathbb{N}, n \geq p).$$

Theorem 2.1. *Let* $\phi \in \Phi_\Delta[\Omega,q]$. *If* $f \in \mathcal{A}_p$ *and* $q \in Q_0 \cap \mathcal{H}[0,p]$ *with*

$$\Re\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)}\right\} \geq 0 \text{ and } |(p-\mu)\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z) + \mu\Delta_{z,p}^{\lambda,\mu,\nu} f(z)| \leq n|q'(\zeta)|,$$

$(0 \leq \lambda < 1, \mu \notin \{p, p-1\}, p \in \mathbb{N}; z \in U; \zeta \in \partial U \backslash E(q), n \geq p$ *and* $n \geq 2)$,

$$\left\{\phi(\Delta_{z,p}^{\lambda,\mu,\nu} f(z), \Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z),\right.$$
$$\left.\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z), \Delta_{z,p}^{\lambda+3,\mu+3,\nu+3} f(z); z) : z \in U\right\} \subset \Omega, \tag{2.1}$$

then $\Delta_{z,p}^{\lambda,\mu,\nu} f(z) \prec q(z), (z \in U)$.

Proof. Define the analytic function in U by

$$g(z):\ = \Delta_{z,p}^{\lambda,\mu,\nu} f(z). \tag{2.2}$$

Making use of (1.4) and (2.2) we have

$$\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z) = \frac{zg'(z) - \mu g(z)}{(p-\mu)}. \tag{2.3}$$

Further computations shows that

$$\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z) = \frac{z^2 g''(z) - 2\mu z g'(z) + \mu(\mu+1)g(z)}{(p-\mu)(p-\mu-1)} \tag{2.4}$$

and

$$\begin{aligned}\Delta_{z,p}^{\lambda+3,\mu+3,\nu+3} f(z) &= \frac{z^3 g'''(z) - 3\mu z^2 g''(z) + 3\mu(\mu+1) z g'(z)}{(p-\mu)(p-\mu-1)(p-\mu-2)} \\ &\quad - \frac{\mu(\mu^2+3\mu+2)g(z)}{(p-\mu)(p-\mu-1)(p-\mu-2)}.\end{aligned} \tag{2.5}$$

We now define the transformation from $\mathbb{C}^4$ to $\mathbb{C}$ by

$$\begin{aligned}&a = u, b = \frac{v - \mu u}{p-\mu}, c = \frac{w - 2\mu v + \mu(\mu+1)u}{(p-\mu)(p-\mu-1)}, \\ &d = \frac{x - 3\mu w + 3\mu(\mu+1)v - \mu(\mu^2+3\mu+2)u}{(p-\mu)(p-\mu-1)(p-\mu-2)},\end{aligned} \tag{2.6}$$

$$\begin{aligned}\psi(u,v,w,x;z) = \phi(a,b,c,d;z) = \Bigg(u, \frac{v-\mu u}{p-\mu}, \frac{w-2\mu v+\mu(\mu+1)u}{(p-\mu)(p-\mu-1)}, \\ \frac{x - 3\mu w + 3\mu(\mu+1)v - \mu(\mu^2+3\mu+2)u}{(p-\mu)(p-\mu-1)(p-\mu-2)}; z\Bigg).\end{aligned} \tag{2.7}$$

Using Lemma 1.1, (2.2)-(2.5) and (2.6) from (2.7) we obtain

$$\begin{aligned}\psi(g(z), zg'(z), z^2g''(z), z^3g'''(z); z) = \phi\Big(\Delta_{z,p}^{\lambda,\mu,\nu} f(z), \\ \Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z), \Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z), \Delta_{z,p}^{\lambda+3,\mu+3,\nu+3} f(z); z\Big).\end{aligned} \tag{2.8}$$

Hence (2.1) becomes

$$\psi(g(z), zg'(z), z^2g''(z), z^3g'''(z); z) \in \Omega. \tag{2.9}$$

A computation using (2.6) yields

$$\frac{w}{v}+1=\frac{(p-\mu)(p-\mu-1)c-\mu(\mu+1)a}{(p-\mu)b+\mu a}+2\mu+1,$$

$$\frac{x}{v}=\frac{(p-\mu)(p-\mu-1)(p-\mu-2)d+3\mu(p-\mu)(p-\mu-1)c-2\mu a(\mu^2-1)}{(p-\mu)b+\mu a}$$
$$+3\mu(\mu+1).$$

Thus, the admissibility condition for $\phi\in\Phi_\Delta[\Omega,q]$ in the Definition 2.1 is equivalent to the admissibility for ψ given in Definition 1.4. Hence, $\psi\in\Psi[\Omega,q]$ and by Lemma 1.1, $g(z)\prec q(z),(z\in U)$ or equivalently $\Delta_{z,p}^{\lambda,\mu,\nu}f(z)\prec q(z),(z\in U)$.

If $\Omega\neq\mathbb{C}$ is a simply connected domain, then $\Omega=h(U)$ for some conformal mapping h of U onto Ω. In this case, $\Phi_\Delta[h(U),q]$ is written as $\Phi_\Delta[h,q]$.

Corollary 2.1. *Let* $q(z)=Mz,(M>0)$ *and* $f\in\mathcal{A}_p$,

$$|(p-\mu)\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1}f(z)+\mu\Delta_{z,p}^{\lambda,\mu,\nu}f(z)|\le nM$$

$(0\le\lambda<1,\mu\notin\{p,p-1\},p\in\mathbb{N};z\in U,n\ge p$ *and* $n\ge 2)$. *If* Ω *is a set in* $\mathbb{C}$ *and* $\phi\in\Phi_\Delta[\Omega,M]$ *satisfies*

$$\left\{\begin{aligned}&\phi(\Delta_{z,p}^{\lambda,\mu,\nu}f(z),\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1}f(z),\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2}f(z),\\&\qquad\Delta_{z,p}^{\lambda+3,\mu+3,\nu+3}f(z);z)\end{aligned}\right\}\in\Omega, \tag{2.10}$$

then $\Delta_{z,p}^{\lambda,\mu,\nu}f(z)\prec Mz$, $(z\in U)$.

The following result is an immediate consequence of Theorem 2.1.

Theorem 2.2. *Let* $\phi\in\Phi_\Delta[h,q]$. *If* $f\in\mathcal{A}_p$, $q\in\mathcal{H}[0,p]\cap Q_0$, *with*

$$\Re\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}\right)\ge 0 \text{ and } |(p-\mu)\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1}f(z)+\mu\Delta_{z,p}^{\lambda,\mu,\nu}f(z)|\le n|q'(\zeta)|$$

$(0\le\lambda<1,\mu\notin\{p,p-1\},p\in\mathbb{N};z\in U,\zeta\in\partial U/E(q),n\ge p$ *and* $n\ge 2)$ *and if* $\phi(\Delta_{z,p}^{\lambda,\mu,\nu}f(z),\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1}f(z),\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2}f(z),\Delta_{z,p}^{\lambda+3,\mu+3,\nu+3}f(z);z)$ *is analytic in* U, *then* $\phi(\Delta_{z,p}^{\lambda,\mu,\nu}f(z),\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1}f(z),\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2}f(z),\Delta_{z,p}^{\lambda+3,\mu+3,\nu+3}f(z);z)\prec h(z)$, *implies* $\Delta_{z,p}^{\lambda,\mu,\nu}f(z)\prec q(z),(z\in U)$.

When the behaviour of q is not known on ∂U, we obtain the similar arguments as in [5, Corollary 1.1].

Corollary 2.2. *Let q be univalent in U, with $q(0)=0$ and for $\varrho \in (0,1)$ set $q_\varrho(z) = q(\varrho z)$. Let $\phi \in \Phi_\Delta[h, q_\varrho]$. If $f \in \mathcal{A}_p$ and q_ϱ satisfy*

$$\Re\left\{\frac{\zeta q_\varrho''(\zeta)}{q_\varrho'(\zeta)}\right\} \geq 0 \text{ and } |(p-\mu)\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z) + \mu\Delta_{z,p}^{\lambda,\mu,\nu} f(z)| \leq n|q_\varrho'(\zeta)|,$$

$(0 \leq \lambda < 1, \mu \notin \{p, p-1\}, p \in \mathbb{N}; z \in U, \zeta \in \partial U/E(q), n \geq p$ and $n \geq 2)$. If $\phi(\Delta_{z,p}^{\lambda,\mu,\nu} f(z), \Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z), \Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z), \Delta_{z,p}^{\lambda+3,\mu+3,\nu+3} f(z); z)$ is analytic in U, then $\phi(\Delta_{z,p}^{\lambda,\mu,\nu} f(z), \Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z), \Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z), \Delta_{z,p}^{\lambda+3,\mu+3,\nu+3} f(z); z) \prec h(z)$, implies $\Delta_{z,p}^{\lambda,\mu,\nu} f(z) \prec q(z)$, $(z \in U)$.

The following theorem gives a relation between the best dominant of the differential subordination and the solution of the corresponding differential equation.

Theorem 2.3. *Let $\phi \in \Phi_\Delta[h, q_\varrho]$, $\phi : \mathbb{C}^4 \times U \to \mathbb{C}$ and $\phi(q(z), T_1(z), T_2(z), T_3(z); z)$ is analytic in U where*

$$T_1(z) = \frac{zq'(z) - \mu q(z)}{(p-\mu)},$$
$$T_2(z) = \frac{z^2 q''(z) - 2\mu z q'(z) + \mu(\mu+1)q(z)}{(p-\mu)(p-\mu-1)},$$
$$T_3(z) = \frac{z^3 q'''(z) - 3\mu z^2 q''(z) + 3\mu(\mu+1) z q'(z) - \mu(\mu^2+3\mu+2)q(z)}{(p-\mu)(p-\mu-1)(p-\mu-2)}.$$

Let h be univalent in U and suppose the differential equation

$$\phi(q(z), T_1(z), T_2(z), T_3(z); z) = h(z) \tag{2.11}$$

has a solution $q \in Q_0 \cap \mathcal{H}[0,p]$. If $f \in \mathcal{A}_p$ satisfies

$$\Re\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)}\right\} \geq 0 \text{ and } |(p-\mu)\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z) + \mu\Delta_{z,p}^{\lambda,\mu,\nu} f(z)| \leq n|q'(\zeta)|$$

$(0 \leq \lambda < 1, \mu \notin \{p, p-1\}, p \in \mathbb{N}; z \in U, \zeta \in \partial U/E(q), n \geq p$ and $n \geq 2)$ then

$$\phi\Big(\Delta_{z,p}^{\lambda,\mu,\nu} f(z), \Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z), \Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z), \Delta_{z,p}^{\lambda+3,\mu+3,\nu+3} f(z); z\Big) \prec h(z), \tag{2.12}$$

implies $\Delta_{z,p}^{\lambda,\mu,\nu} f(z) \prec q(z)$, $(z \in U)$ and $q(z)$ is the best dominant.

Proof. By applying Theorem 2.1, we see that q is a dominant of (2.12). Since q also satisfies (2.11), it is also a solution of the differential subordination (2.12) and q will be dominated by all the dominants of (2.12). Hence, q is the best dominant of (2.11).

We next introduce a new admissible class, $\Phi_{\Delta,1}[\Omega, q]$.

Definition 2.2. Let Ω be a set in $\mathbb{C}$, $q(z) \in Q_0 \cap \mathcal{H}[0,p]$. The class of admissible functions $\Phi_{\Delta,1}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^4 \times U \to \mathbb{C}$ that satisfy the following admissible condition $\phi(a,b,c,d;z) \notin \Omega$ whenever

$$a = q(\zeta), b = \frac{n\zeta q'(\zeta) + (p-\mu-1)q(\zeta)}{(p-\mu)},$$

$$\Re\left\{\frac{(p-\mu)(p-\mu-1)c - (p-\mu-1)(p-\mu-2)a}{(p-\mu)b - (p-\mu-1)a} - 2(p-\mu) + 3\right\} \geq n\ \Re\left\{1 + \frac{\zeta q''(\zeta)}{q'(\zeta)}\right\}$$

and

$$\Re\left\{\frac{(p-\mu-2)(p-\mu-1)(p-\mu)d}{(p-\mu)b - (p-\mu-1)a} - \frac{3(p-\mu-1)[(p-\mu)(p-\mu-1)c - (p-\mu-1)(p-\mu-2)a]}{(p-\mu)b - (p-\mu-1)a} - \frac{(p-\mu-1)(p-\mu-2)(p-\mu-3)a}{(p-\mu)b - (p-\mu-1)a} - 3(p-\mu-1)(p-\mu-2) + 6(p-\mu-1)^2\right\} \geq n^2\ \Re\left\{\frac{\zeta^2 q'''(\zeta)}{q'(\zeta)}\right\},$$

where $z \in U, \zeta \in \partial U \backslash E(q), \mu \neq p, p \in \mathbb{N}$ and $n \geq 2$.

Theorem 2.4. *Let $\phi \in \Phi_{\Delta,1}[\Omega, q]$. If $f \in \mathcal{A}_p$, with $q \in Q_0 \cap \mathcal{H}[0,p]$*

$$\Re\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)}\right\} \geq 0, \left|(p-\mu)\frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{z^{p-1}} + (\mu-p-1)\frac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{z^{p-1}}\right| \leq n|q'(\zeta)|,$$

$(0 \leq \lambda < 1, \mu \notin \{p, p-1\}, p \in \mathbb{N}; z \in U, \zeta \in \partial U/E(q), n \geq p$ *and* $n \geq 2)$

satisfies

$$\left\{\phi\left(\frac{\Delta_{z,p}^{\lambda,\mu,\nu}f(z)}{z^{p-1}},\frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1}f(z)}{z^{p-1}},\frac{\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2}f(z)}{z^{p-1}},\frac{\Delta_{z,p}^{\lambda+3,\mu+3,\nu+3}f(z)}{z^{p-1}};z\right)\right\}\subset\Omega, \tag{2.13}$$

then $\frac{\Delta_{z,p}^{\lambda,\mu,\nu}f(z)}{z^{p-1}} \prec q(z), (z\in U)$.

Proof. Define the analytic function $g(z)$ in U by

$$g(z) := \frac{\Delta_{z,p}^{\lambda,\mu,\nu}f(z)}{z^{p-1}}. \tag{2.14}$$

A simple computation yields

$$\frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1}f(z)}{z^{p-1}} = \frac{zg'(z)+(p-\mu-1)g(z)}{(p-\mu)}, \tag{2.15}$$

$$\frac{\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2}f(z)}{z^{p-1}} = \frac{z^2g''(z)+2(p-\mu-1)zg'(z)+(p-\mu-1)(p-\mu-2)g(z)}{(p-\mu-1)(p-\mu-2)} \tag{2.16}$$

and

$$\frac{\Delta_{z,p}^{\lambda+3,\mu+3,\nu+2}f(z)}{z^{p-1}} = \frac{z^3g'''(z)+3(p-\mu-1)z^2g''(z)}{(p-\mu)(p-\mu-1)(p-\mu-2)} + 3\frac{(p-\mu-2)(p-\mu-1)zg'(z)}{(p-\mu)(p-\mu-1)(p-\mu-2)} - \frac{(p-\mu-2)(p-\mu-1)(p-\mu)g(z)}{(p-\mu)(p-\mu-1)(p-\mu-2)}. \tag{2.17}$$

Define the transformation $\mathbb{C}^4$ to $\mathbb{C}$ by

$$\begin{aligned} a &= u, b = \frac{v+(p-\mu-1)u}{p-\mu},\\ c &= \frac{w+2(p-\mu-1)v+(p-\mu-1)(p-\mu-2)u}{(p-\mu)(p-\mu-1)},\\ d &= \frac{x+3(p-\mu-1)w}{(p-\mu)(p-\mu-1)(p-\mu-2)} + \frac{3(p-\mu-2)(p-\mu-1)v}{(p-\mu)(p-\mu-1)(p-\mu-2)}\\ &\quad + \frac{(p-\mu-1)(p-\mu-2)(p-\mu-2)u}{(p-\mu)(p-\mu-1)(p-\mu-2)}. \end{aligned} \tag{2.18}$$

Let

$$
\begin{aligned}
\psi(u,v,w,x;z) = \phi(a,b,c,d;z) = \phi\Bigg(& u, \frac{v+(p-\mu-1)u}{p-\mu}, \\
& \frac{w+2(p-\mu-1)v+(p-\mu-1)(p-\mu-2)u}{(p-\mu)(p-\mu-1)}, \\
& \frac{x+3(p-\mu-1)w}{(p-\mu)(p-\mu-1)(p-\mu-2)} + \frac{3(p-\mu-2)(p-\mu-1)v}{(p-\mu)(p-\mu-1)(p-\mu-2)} \\
& + \frac{(p-\mu-1)(p-\mu-2)(p-\mu-2)u}{(p-\mu)(p-\mu-1)(p-\mu-2)}; z\Bigg).
\end{aligned} \tag{2.19}
$$

Using Lemma(1.1), (2.14)-(2.17), (2.18) and (2.19) we obtain

$$
\begin{aligned}
&\psi(g(z), zg'(z), z^2g''(z), z^3g'''(z); z) \\
=&\phi\left(\frac{\Delta_{z,p}^{\lambda,\mu,\nu}f(z)}{z^{p-1}}, \frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1}f(z)}{z^{p-1}}, \frac{\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2}f(z)}{z^{p-1}}, \frac{\Delta_{z,p}^{\lambda+3,\mu+3,\nu+3}f(z)}{z^{p-1}}; z\right).
\end{aligned}
$$

Hence using (2.13)

$$
\psi(g(z), zg'(z), z^2g''(z), z^3g'''(z); z) \in \Omega. \tag{2.20}
$$

Also we get

$$
\frac{w}{v} + 1 = \frac{(p-\mu)(p-\mu-1)c-(p-\mu-1)(p-\mu-2)a}{(p-\mu)b-(p-\mu-1)a} - 2(p-\mu)+3
$$

and

$$
\begin{aligned}
\frac{x}{v} = & \frac{(p-\mu-2)(p-\mu-1)(p-\mu)d}{(p-\mu)b-(p-\mu-1)a} \\
& - 3\frac{(p-\mu-1)[(p-\mu)(p-\mu-1)c-(p-\mu-1)(p-\mu-2)a]}{(p-\mu)b-(p-\mu-1)a} \\
& - \frac{(p-\mu-1)(p-\mu-2)(p-\mu-3)a}{(p-\mu)b-(p-\mu-1)a} \\
& - 3(p-\mu-1)(p-\mu-2)+6(p-\mu-1)^2.
\end{aligned}
$$

Thus, the admissibility condition for $\phi \in \Phi_{\Delta,1}[\Omega, q]$ in the Definition 2.2 to the admissibility condition for ψ given in Definition 1.4. Hence, $\psi \in \Psi[\Omega, q]$ and Lemma 1.1, $g(z) \prec q(z)$, $(z \in U)$. or equivalently $\frac{\Delta_{z,p}^{\lambda,\mu,\nu}f(z)}{z^{p-1}} \prec q(z)$, $(z \in U)$. Proceeding similarly as in Theorem 2.2, the following result is an immediate consequence of the above Theorem

Theorem 2.5. *Let* $\phi \in \Phi_{\Delta,1}[h,q]$. *If* $f \in \mathcal{A}_p$ *and* $q \in \mathcal{H}[0,p] \cap Q_0$, *with*

$$\Re\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}\right) \geq 0, \ \left|(p-\mu)\frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{z^{p-1}} + (\mu-p-1)\frac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{z^{p-1}}\right| \leq n\left|q'(\zeta)\right|,$$

$(0 \leq \lambda < 1, \mu \notin \{p, p-1\}, p \in \mathbb{N}; z \in U; \zeta \in \partial U/E(q), n \geq p$ *and* $n \geq 2)$. *If*

$$\phi\left(\frac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{z^{p-1}}, \frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{z^{p-1}}, \frac{\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z)}{z^{p-1}}, \frac{\Delta_{z,p}^{\lambda+3,\mu+3,\nu+3} f(z)}{z^{p-1}}; z\right)$$

is analytic in U, *then*

$$\phi\left(\frac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{z^{p-1}}, \frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{z^{p-1}}, \frac{\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z)}{z^{p-1}}, \frac{\Delta_{z,p}^{\lambda+3,\mu+3,\nu+3} f(z)}{z^{p-1}}; z\right) \prec h(z),$$

implies $\frac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{z^{p-1}} \prec q(z)$, $(z \in U)$.

3. Examples

Consider the following function

$$q(z) = M\frac{Mz + \alpha}{M + \bar{\alpha}z}, \qquad (z \in U, M \geq 0), \tag{3.1}$$

with $|\alpha| < M$, q is univalent in $\bar{U}$ and satisfies $q(U) = U_M = \{w : |w| < M\}$, $q(0) = \alpha, q \in Q(\alpha)$ and $E(q) = \emptyset$.

Lemma 3.1 ([1])**.** *Let* q *be given by* (3.1) *and* $p(z) = \alpha + \alpha_n z^n + \alpha_{n+1} z^{n+1} + \dots$ *be analytic in* U *with* $p(z) \neq \alpha$ *and* $n \geq 2$. *If there exists points* $z_0 = r_0 e^{i\theta} \in U_M$ *and* $w_0 \in \partial U$ *such that* $p(z_0) = q(w_0), p(U_{r_0}) \subset U_M$ *and*

$$|zp'(z)||M + \bar{\alpha}e^{i\theta}|^2 \leq Mn[M^2 - |\alpha|^2] \tag{3.2}$$

when $z \in \bar{U}_{r_0}$ *and* $\theta \in [0, 2\pi]$, *then*

$$z_0 p'(z_0) = nq(w_0)\frac{|q(w_0) - \alpha|^2}{|q(w_0)|^2 - |\alpha|^2}, \quad \Re\frac{z_0 p''(z_0)}{p'(z_0)} + 1 \geq n\frac{|q(w_0) - \alpha|^2}{|q(w_0)|^2 - |\alpha|^2}$$

and

$$\Re\frac{z_0^2 p'''(z_0)}{p'(z_0)} \geq 6n^2 \Re\frac{[\bar{\alpha}q(w_0) - |\alpha|^2]^2}{[|q(w_0)|^2 - |\alpha|^2]^2}.$$

By using Lemma 3.1, the conditions of admissibility given in Definition 2.1 changes as given by the following definition, $\phi \in \Phi_\Delta[\Omega, q]$ is denoted by $\phi \in \Phi_\Delta[\Omega, M, \alpha]$. Since $q(z) = Me^{i\theta}$, with $\theta \in [0, 2\pi]$ when $|z| = 1$.

Definition 3.1. Let Ω be a set in $\mathbb{C}$. Let q be given by (3.1) and let $n \geq 2$. The class of admissible functions $\phi \in \Phi_\Delta[\Omega, M, \alpha]$ consists $\phi : \mathbb{C}^4 \times U \to \mathbb{C}$ that satisfy the admissible condition $\phi(a, b, c, d; z) \notin \Omega$ whenever

$$a = Me^{i\theta}, b = \frac{\left[n\frac{|M-\bar{\alpha}e^{i\theta}|^2}{M^2-|\alpha|^2} - \mu\right]Me^{i\theta}}{(p-\mu)},$$

$$\Re\left\{\frac{(p-\mu)(p-\mu-1)c - \mu(\mu+1)a}{(p-\mu)b+\mu a} + 2\mu + 1\right\} \geq n\frac{|M-\bar{\alpha}e^{i\theta}|^2}{M^2-|\alpha|^2},$$

and

$$\Re\Bigg\{\frac{(p-\mu)(p-\mu-1)(p-\mu-2)d + 3\mu(p-\mu)(p-\mu-1)c - 2\mu(\mu^2-1)a}{(p-\mu)b+\mu a} + 3\mu(\mu+1)\Bigg\} \geq \frac{6n^2}{[M^2-|\alpha|^2]^2}\Re[\bar{\alpha}Me^{i\theta} - |\alpha|^2]^2$$

$(z \in U; \theta \in [0, 2\pi]; \zeta \in \partial U \backslash E(q); \mu \neq p, p \in \mathbb{N}, n \geq p$ and $n \geq 2)$.

In the special case when $\alpha = 0$, we obtain the following class of admissible functions in $\Phi_\Delta[\Omega, M]$.

Definition 3.2. Let Ω be a set in $\mathbb{C}$ and $M > 0$. The class of admissible functions $\phi \in \Phi_\Delta[\Omega, M]$ consists of those function $\phi : \mathbb{C}^4 \times U \to \mathbb{C}$ such that

$$\phi\Bigg(Me^{i\theta}, \frac{n-\mu}{p-\mu}Me^{i\theta}, \frac{L + (\mu(\mu+1) - 2\mu n)Me^{i\theta})}{(p-\mu)(p-\mu-1)}, \frac{K - 3\mu L + [3\mu(\mu+1)n - \mu(\mu^2+3\mu+2)]Me^{i\theta}}{(p-\mu)(p-\mu-1)(p-\mu-2)}; z\Bigg) \notin \Omega \tag{3.3}$$

whenever $z \in U, \theta \in [0, 2\pi], \Re[Le^{-i\theta}] \geq n(n-1)M, \Re[Ke^{-i\theta}] \geq 0$ and $n \geq 2$.

We use Theorem 2.1, to obtain the following differential subordination

Theorem 3.1. *Let q be given by* (3.1) *and let $f \in \mathcal{A}_p$ satisfy*

$$|(p-\mu)\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1}f(z) + \mu\Delta_{z,p}^{\lambda,\mu,\nu}f(z)||M + \bar{\alpha}e^{i\theta}|^2 \leq nM[M^2 - |\alpha|^2],$$

$(z \in U; \zeta \in \partial U \backslash E(q); \mu \notin p, p \in \mathbb{N}, n \geq p$ *and* $n \geq 2)$. *If* Ω *be a set in* $\mathbb{C}$ *and* $\phi \in \Phi_{\Delta}[\Omega, M, \alpha]$, *then* $\{\phi(\Delta_{z,p}^{\lambda,\mu,\nu} f(z),\ \Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z),\ \Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z),$ $\Delta_{z,p}^{\lambda+3,\mu+3,\nu+3} f(z); z) : z \in U\} \subset \Omega$, *implies* $\Delta_{z,p}^{\lambda,\mu,\nu} f(z) \prec q(z)$, $(z \in U)$.

Corollary 3.1. *Let* $\phi \in \Phi_{\Delta}[\Omega, M]$, $M \geq 0$. *If* $f \in \mathcal{A}_p$ *with*

$$|(p-\mu)\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z) + \mu \Delta_{z,p}^{\lambda,\mu,\nu} f(z)| \leq nM$$

satisfies $(z \in U; \mu \notin p, p \in \mathbb{N}, n \geq p$ *and* $n \geq 2)$, $\{\phi(\Delta_{z,p}^{\lambda,\mu,\nu} f(z),\ \Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z),$ $\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z), \Delta_{z,p}^{\lambda+3,\mu+3,\nu+3}; z)\} \in \Omega$, *then* $|\Delta_{z,p}^{\lambda,\mu,\nu} f(z)| < M$, $(z \in U)$. *For the case* $\Omega = q(U) = \{w : |w| < M\}$, $\Phi_{\Delta}[\Omega, M]$ *is denoted by* $\Phi_{\Delta}[M]$.

Corollary 3.2. *Let* $\phi \in \Phi_{\Delta}[M]$. *If* $f \in \mathcal{A}_p$ *with* $|(p-\mu)\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z) +$ $\mu \Delta_{z,p}^{\lambda,\mu,\nu} f(z)| \leq nM$ *satisfies* $(z \in U; p \in \mathbb{N}, \mu \notin \{p, p-1\} n \geq p$ *and* $n \geq 2)$,

$$\left|\phi(\Delta_{z,p}^{\lambda,\mu,\nu} f(z), \Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z), \Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z), \Delta_{z,p}^{\lambda+3,\mu+3,\nu+3} f(z); z)\right| < M,$$

then $|\Delta_{z,p}^{\lambda,\mu,\nu} f(z)| < M, (z \in U)$.

Example 1. Let $\phi_1(a, b, c, d; z) = (1-\delta)\frac{c}{a} + \delta b$, $(0 \leq \delta \leq 1$ and $z \in U)$, satisfy the admissibility condition (3.1) and hence the Corollary 3.2 yields

$$\left|(1-\delta)\frac{\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)} + \delta \Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)\right| < M, (M > 0),$$

implies $|\Delta_{z,p}^{\lambda,\mu,\nu} f(z)| < M, (M>0)$. When $\delta = 1$, we have $|\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)|$ $< M, (M > 0)$, $|\Delta_{z,p}^{\lambda,\mu,\nu} f(z)| \leq M, (M > 0)$. The above result was obtained in [3, Corollary 2.10]. When $\delta = 0$, we have $|\frac{\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}| < M$, $(M > 0)$, implies $|\Delta_{z,p}^{\lambda,\mu,\nu} f(z)| < M, (M > 0)$.

Example 2. Let $\phi_2(a, b, c, d; z) = \alpha(z)a + \beta(z)b$, with $\Re(\alpha(z)) \geq 1$ and $\Re(\beta(z)) \geq 0$ for $(z \in U)$ satisfy the admissibility condition (3.1) and hence Corollary 3.2 yields, $|\alpha(z)\Delta_{z,p}^{\lambda,\mu,\nu} f(z) + \beta(z)\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)| < M$, $(M > 0)$, implies $|\Delta_{z,p}^{\lambda,\mu,\nu} f(z)| < M, (M > 0)$. Consider the case when $q(z) = Mz$, $M > 0$ the class of admissible functions $\phi \in \Phi_{\Delta,1}[\Omega, q]$ is denoted by $\phi \in \Phi_{\Delta,1}[\Omega, M]$ is defined as follows:

Definition 3.3. Let Ω be a set in $\mathbb{C}$. The class of admissible functions $\phi \in \Phi_{\Delta,1}[\Omega, M]$ consists of those functions $\phi\colon \mathbb{C}^4 \times U \to \mathbb{C}$ such that

$$\begin{aligned}\phi\Bigg(&Me^{i\theta}, \frac{n+(p-\mu-1)Me^{i\theta}}{(p-\mu)},\\ &\frac{L+(p-\mu-1)[2n+(p-\mu-2)]Me^{i\theta}}{(p-\mu)(p-\mu-1)},\\ &\frac{K+(p-\mu-1)[3L+(p-\mu-2)[n+(p-\mu-3)]Me^{i\theta}]}{(p-\mu)(p-\mu-1)(p-\mu-2)}; z\Bigg) \notin \Omega\end{aligned} \tag{3.4}$$

whenever $z \in U, \theta \in [0, 2\pi], \Re(Le^{-i\theta}) \geq n(n-1)M, \Re(Ke^{-i\theta}) \geq 0, p \in \mathbb{N}$ and $n \geq 2$.

From the above Definition and Theorem 2.4, we have the following Corollary.

Corollary 3.3. *If* $\phi \in \Phi_{\Delta,1}[\Omega, M]$. *If* $f \in \mathcal{A}_p$ *and*

$$\left|(p-\mu)\frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{z^{p-1}} + (\mu-p-1)\frac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{z^{p-1}}\right| \leq nM,$$

$(0 \leq \lambda < 1, \mu \neq p, p \in \mathbb{N}, z \in U, n \geq p$ *and* $n \geq 2)$. *If*

$$\begin{aligned}\Bigg\{\phi\Bigg(&\frac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{z^{p-1}}, \frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{z^{p-1}}, \frac{\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z)}{z^{p-1}},\\ &\frac{\Delta_{z,p}^{\lambda+3,\mu+3,\nu+3} f(z)}{z^{p-1}}; z\Bigg)\Bigg\} \in \Omega,\end{aligned} \tag{3.5}$$

then $\left|\frac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{z^{p-1}}\right| < M, (z \in U)$.

In case $\Omega = q(U) = \{w : |w| < M\}$, the class $\phi \in \Phi_{\Delta,1}[\Omega, M]$ is simply denoted by $\phi \in \Phi_{\Delta,1}[M]$, then the above Corollary takes the following form

Corollary 3.4. *If* $\phi \in \Phi_{\Delta,1}[M]$. *If* $f \in \mathcal{A}_p$ *and*

$$\left|(p-\mu)\frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{z^{p-1}} + (\mu-p-1)\frac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{z^{p-1}}\right| \leq nM,$$

$(0 \le \lambda < 1, \mu \ne p, p \in \mathbb{N}, z \in U, n \ge p$ *and* $n \ge 2)$. *If*

$$\left|\left\{\phi\left(\frac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{z^{p-1}}, \frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{z^{p-1}}, \frac{\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z)}{z^{p-1}}, \frac{\Delta_{z,p}^{\lambda+3,\mu+3,\nu+3} f(z)}{z^{p-1}}; z\right)\right\}\right| < M, \tag{3.6}$$

then $\left|\frac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{z^{p-1}}\right| < M$.

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