On the Existence of Carathèodory Solutions in Mechanical Systems with Friction

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***Abstract*—In this note, it is shown that the differential equation with discontinuous right-hand side, which describes the dynamic behavior of a mechanical system with friction, has a Carathèodory solution. This result supports theoretically the prior work on the control of mechanical systems with friction, which assumes the existence of Carathèodory solutions without proof.**

***Index Terms*—Differential inclusion, friction, mechanical system.**

I. Introduction

Until now, there has been a considerable amount of research work on the control of mechanical systems with friction; see the survey paper [1] and the vast literature therein. Note that the right-hand side of the differential equation, which describes the dynamic behavior of a mechanical system with friction, is discontinuous due to the nature of friction at zero velocity. Hence, the well-known classical result on the existence of solutions in differential equations, which requires local Lipschitzness in state variables and piecewise continuity in time, is not applicable to mechanical systems with friction. Nonetheless, most prior works assume, without proof, the existence of a Carathèodory solution.

In this note, we use differential inclusions [4] to show the existence of a Carathèodory solution in the differential equation with discontinuous right-hand side which describes the dynamic behavior of a mechanical system with friction. Specifically, we first embed the discontinuous right-hand side into a set-valued map. The resulting differential inclusion is then shown to have a solution. Finally, the solution is shown, through some set-theoretic arguments on the basis of the celebrated Cantor–Bendixon's theorem [14], to be in fact the Carathèodory solution of the original differential equation.

The differential inclusion used in our development is different from Filippov's differential inclusion [12], [13] which has been used to study nonsmooth control systems including variable structure systems and adaptive systems with discontinuous switching laws [5]–[9]. In particular, Filippov's differential inclusion does not necessarily characterize the dynamic behavior of a mechanical system with friction on its discontinuity hypersurface, as explained in Section II.

II. Main Results

The motion of a mass which moves on the surface with static friction can be described by

$$m\dot{v}(t) + F(v(t), u(t)) = u(t) \tag{1}$$

where m is the mass, $v \in R$ is the velocity of the mass, $F \in R$ represents the friction force, and $u \in R$ is the external force applied to the mass [3].

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Here, the function F is given by

$$F(v, u) \triangleq \begin{cases} F_s(v), & \text{if } v \neq 0 \\ \lambda(u), & \text{if } v = 0 \end{cases}$$

$$\lambda(u) \triangleq \begin{cases} F_+, & \text{if } u \geq F_+ > 0 \\ u, & \text{if } F_- < u < F_+ \\ F_-, & \text{if } u \leq F_- < 0. \end{cases}$$

In our analysis, we make the following assumptions on the function F_s.

A.1) F_s is continuous on $(-\infty, 0) \cup (0, \infty)$.

A.2) The right limit and the left limit of F_s at zero velocity satisfy the following inequalities.

$$F_+ \geq F_s(0+) \triangleq \lim_{v \to +0} F_s(v)$$
$$F_- \leq F_s(0-) \triangleq \lim_{v \to -0} F_s(v).$$

It should be noted that the above assumptions describe the typical characteristics of static friction. In fact, it can be easily shown that the following friction model which has been extensively studied in the prior literature including [2] and [3] satisfy the above assumptions:

$$F(v, u) \triangleq \begin{cases} F_c \operatorname{sgn}(v), & \text{if } v \neq 0 \\ u, & \text{if } v = 0 \text{ and } |u| \leq F_c + \Delta F_s \\ (F_c + \Delta F_s)\operatorname{sgn}(u), & \text{if } v = 0 \text{ and } |u| > F_c + \Delta F_s \end{cases}$$

where $F_c > 0$ is the magnitude of Coulomb friction, $\Delta F_s > 0$ is the excess of static friction over Coulomb friction, and

$$\operatorname{sgn}(v) \triangleq \begin{cases} 1, & \text{if } v > 0 \\ 0, & \text{if } v = 0 \\ -1, & \text{if } v < 0. \end{cases}$$

Suppose that the control force u is governed by the following dynamic feedback controller:

$$\begin{aligned} \dot{\zeta} &= h(t, \zeta, v) \\ \zeta(t) &\in R^n \\ u &= g(t, \zeta, v). \end{aligned} \tag{2}$$

Then, the closed-loop system consisting of the system in (1) and the above controller in (2) can be written as

$$\begin{aligned} \dot{z} &= f(t, z) \\ z(t_0) &= z_0 \in R^{n+1} \end{aligned} \tag{3}$$

where

$$f(t, z) \triangleq \alpha(t, \zeta, v) + cF(v, g(t, \zeta, v))$$
$$z \triangleq \begin{bmatrix} \zeta \\ v \end{bmatrix}$$
$$\alpha(t, \zeta, v) \triangleq \begin{bmatrix} h(t, \zeta, v) \\ \frac{1}{m} g(t, \zeta, v) \end{bmatrix}$$
$$c \triangleq -\begin{bmatrix} 0_{n \times 1} \\ \frac{1}{m} \end{bmatrix}.$$

Here, z_0 is the initial state of the closed-loop system at $t = t_0$. In what follows, we assume that the functions $h: R^{n+2} \to R^n$ and $g: R^{n+2} \to R$ are continuous. Hence, α is a continuous function, and f is discontinuous only when $v = 0$.

It is well known that a differential equation has a Carathèodory solution, provided its right-hand side is locally Lipschitz in state variables and piecewise continuous in time [15]. However, this famous existence theorem does not work for the differential equation in (3) since its right-hand side is discontinuous in z due to the nature of friction at zero velocity. Nonetheless, it has a Carathèodory solution, as shown soon. For readable presentation of our result, we give the precise definition of a Carathèodory solution below.

Definition 1: A function $z: [t_0, t_1) \rightarrow R^{n+1}$ is a Carathèodory solution (C-solution) of the differential equation in (3) if

i) $z(t_0) = z_0$;
ii) z is absolutely continuous on each compact subset of $[t_0, t_1)$;
iii) $\dot{z}(t) = f(t, z(t))$, a.e. on $[t_0, t_1)$. ■

We are now ready to state our main result.

Theorem 1: The differential equation in (3) has a C-solution. ■

To prove the above theorem, we introduce some notational convention about set-valued maps, which can be found in [4], [10], [11], and elsewhere. The set of all subsets of a set X is denoted by 2^X. A set-valued map $F: X \rightarrow 2^Y$ is then a function that associates to any $x \in X$ a subset $F(x)$ of Y. The usual Euclidean norm of a vector $x \in R^m$ is denoted by $\|x\|$. Then, the distance between two points $x_1, x_2 \in R^m$ is defined by $\|x_1 - x_2\|$. The distance between a point $x \in R^m$ and a set $A \subset R^m$ is defined by $d(x, A) \triangleq \inf\{\|x - a\|: a \in A\}$. The separation between two subsets $A, B \subset R^m$ is defined by $d^*(A, B) \triangleq \sup\{d(a, B): a \in A\}$. The distance between two subsets $A, B \subset R^m$ is defined by $d(A, B) = d(B, A) \triangleq \max\{d^*(A, B), d^*(B, A)\}$. The ϵ-neighborhood of a set $A \subset R^m$ is defined by $N_\epsilon(A) \triangleq \{x \in R^m: d(x, A) < \epsilon\}$. In particular, if $A = \{x_0\}$, then $N_\epsilon(\{x_0\})$ is simply denoted by $N_\epsilon(x_0)$. On the other hand, a set-valued map $E: R^m \rightarrow 2^{R^n}$ is said to be upper semicontinuous at x_0 if for any $\epsilon > 0$, there is a $\delta > 0$ such that $d^*(E(x), E(x_0)) < \epsilon$, $\forall x \in N_\delta(x_0)$. Furthermore, it is said to be upper semicontinuous if E is upper semicontinuous at each $x_0 \in R^m$. Finally, a set-valued map $E: R^m \rightarrow 2^{R^n}$ is said to be continuous at x_0 if for any $\epsilon > 0$, there is a $\delta > 0$ such that $d(E(x), E(x_0)) < \epsilon$, $\forall x \in N_\delta(x_0)$. Furthermore, it is said to be continuous if E is continuous at each $x_0 \in R^m$.

Based on the above notational convention, we define the set-valued map $E: R \times R^{n+1} \rightarrow 2^{R^{n+1}}$ as

$$
\begin{aligned}
E(t,z) &\triangleq \{\alpha(t,\zeta,v) + cw \in R^{n+1}: w \in \tilde{F}_s(v)\} \\
&= \left\{ \begin{bmatrix} h(t,\zeta,v) \\ \frac{1}{m}\{g(t,\zeta,v) - w\} \end{bmatrix} : w \in \tilde{F}_s(v) \right\}.
\end{aligned} \tag{4}
$$

Here, $\tilde{F}_s$ is also a set-valued map from R into 2^R defined as

$$
\tilde{F}_s(v) \triangleq \begin{cases} \{F_s(v)\}, & \text{if } v \neq 0 \\ [F_-, F_+], & \text{if } v = 0. \end{cases} \tag{5}
$$

Then, the function f in (3) can be embedded into the set-valued map E, i.e.,

$$
f(t, z) \in E(t, z), \qquad \forall (t, z) \in R \times R^{n+1}. \tag{6}
$$

In particular, it holds that

$$
E(t, z) = \{f(t, z)\}, \qquad \text{if } v \neq 0. \tag{7}
$$

The following lemma further clarifies the relationship between the function f in (3) and the set-valued map E in (4).

Lemma 1: At each $(t, z) \in R \times R^{n+1}$, $f(t, z)$ is the unique element of $E(t, z)$ with minimal norm, that is,

$$
\arg \min_{y \in E(t,z)} \|y\| = f(t, z). \tag{8}
$$

Proof: Let $t \in R$ and $z \triangleq (\zeta, v) \in R^{n+1}$. We need to consider two cases: i) $v = 0$, and ii) $v \neq 0$. When $v = 0$, we have

$$
\begin{aligned}
\min_{y \in E(t,z)} \|y\| &= \min_{F_- \le w \le F_+} \|\alpha(t, \zeta, 0) + cw\| \\
&= \min_{F_- \le w \le F_+} \left\| \begin{bmatrix} h(t, \zeta, 0) \\ \frac{1}{m}\{g(t, \zeta, 0) - w\} \end{bmatrix} \right\|.
\end{aligned}
$$

It then follows that

$$
\begin{aligned}
&\arg \min_{y \in E(t,z)} \|y\| \\
&= \begin{cases} \begin{bmatrix} h(t, \zeta, 0) \\ \frac{1}{m}\{g(t, \zeta, 0) - F_+\} \end{bmatrix}, & \text{if } g(t, \zeta, 0) \ge F_+ \\ \begin{bmatrix} h(t, \zeta, 0) \\ 0 \end{bmatrix}, & \text{if } F_- < g(t, \zeta, 0) < F_+ \\ \begin{bmatrix} h(t, \zeta, 0) \\ \frac{1}{m}\{g(t, \zeta, 0) - F_-\} \end{bmatrix}, & \text{if } g(t, \zeta, 0) \le F_- \end{cases} \\
&= \begin{bmatrix} h(t, \zeta, 0) \\ \frac{1}{m}\{g(t, \zeta, 0) - F(0, g(t, \zeta, 0))\} \end{bmatrix} \\
&= f(t, z), \qquad \text{if } v = 0.
\end{aligned} \tag{9}
$$

On the other hand, when $v \neq 0$, (8) is the immediate consequence of (7). ■

We next consider the following differential inclusion:

$$
\dot{z}(t) \in E(t, z(t)), \qquad z(t_0) = z_0 \in R^{n+1}. \tag{10}
$$

The following lemma suggests that the set-valued map E in (4) is regular enough for the above differential inclusion to have a solution.

Lemma 2: There exists $t_1 \in (t_0, \infty]$ such that the differential inclusion in (10) has a solution $z: [t_0, t_1) \rightarrow R^{n+1}$, in the sense that

i) $z(t_0) = z$;
ii) z is absolutely continuous on each compact subset of $[t_0, t_1)$;
iii) $\dot{z}(t) \in E(t, z(t))$, a.e. on $[t_0, t_1)$.

Proof: To begin with, we show that

$$
E \text{ is upper semicontinuous.} \tag{11}
$$

For this purpose, we partition R^{n+1} into three regions as follows:

$$
R^{n+1} = M_0 \cup M_+ \cup M_- \tag{12}
$$

where

$$
\begin{aligned}
M_+ &\triangleq \{(\zeta, v) \in R^n \times R: v > 0\} \\
M_- &\triangleq \{(\zeta, v) \in R^n \times R: v < 0\} \\
M_0 &\triangleq \{(\zeta, v) \in R^n \times R: v = 0\}.
\end{aligned}
$$

Recall that a continuous set-valued map is always upper-semicontinuous [4]. The upper-semicontinuity of E on $R \times (M_+ \cup M_-)$ is therefore the direct consequence of (7) and the continuity of f on $R \times (M_+ \cup M_-)$. What remains to do is to show that E is upper semicontinuous on $R \times M_0$.

Now, suppose that $t^* \in R$ and $z^* \triangleq (\zeta^*, v^*) \in M_0$. Then, we have

$$
E(t^*, z^*) = \{\alpha(t^*, \zeta^*, 0) + cw^*: F_- \le w^* \le F_+\}. \tag{13}
$$

Let $\epsilon > 0$ be given. To show the upper-semicontinuity of E at (t^*, z^*), we need to consider the following three cases: i) $z \in M_+$, ii) $z \in M_-$, and iii) $z \in M_0$.

i) $z \in M_+$: In this case, it is clear that

$$
E(t, z) = \{f(t, z)\}. \tag{14}
$$

By (13) and (14) along with the triangular inequality,

$$\begin{aligned}
&d^*(E(t, z), E(t^*, z^*)) \\
&\quad = \sup_{y\in E(t, z)} d(y, E(t^*, z^*)) \\
&\quad = \sup_{y\in E(t, z)} \inf_{y^*\in E(t^*, z^*)} \|y - y^*\| \\
&\quad = \inf_{F_-\le w^*\le F_+} \|\alpha(t, \zeta, v) + cF_s(v) - \alpha(t^*, \zeta^*, 0) - cw^*\| \\
&\quad \le \|\alpha(t, \zeta, v) - \alpha(t^*, \zeta^*, 0)\| + \inf_{F_-\le w^*\le F_+} \|c\{F_s(v) - w^*\}\| \\
&\quad \le \|\alpha(t, \zeta, v) - \alpha(t^*, \zeta^*, 0)\| + \|c\{F_s(v) - F_s(0+)\}\| \\
&\qquad + \inf_{F_-\le w^*\le F_+} \|c\{F_s(0+) - w^*\}\| \\
&\quad = \|\alpha(t, \zeta, v) - \alpha(t^*, \zeta^*, 0)\| + \|c\{F_s(v) - F_s(0+)\}\|. \qquad (15)
\end{aligned}$$

Note that the last equality follows from the assumption A.2, and further that the second term in its right-hand side satisfies

$$\lim_{v\to+0} \|c\{F_s(v) - F_s(0+)\}\| = 0.$$

This, along with the continuity of α, finally implies that there exist two constants $\delta_1, \delta_2 > 0$ such that

$$\begin{aligned}
&d^*(E(t, z), E(t^*, z^*)) \\
&\quad \le \epsilon, \qquad \forall (t, z) \in [t^* - \delta_1, t^* + \delta_1] \times (N_{\delta_2}(z^*) \cap M_+). \qquad (16)
\end{aligned}$$

ii) $z \in M_-$: Using arguments similar to those used in the above i), we can show that there exist two constants $\delta_3, \delta_4 > 0$ such that

$$\begin{aligned}
&d^*(E(t, z), E(t^*, z^*)) \\
&\quad \le \epsilon, \qquad \forall (t, z) \in [t^* - \delta_3, t^* + \delta_3] \times (N_{\delta_4}(z^*) \cap M_-). \qquad (17)
\end{aligned}$$

iii) $z \in M_0$: In this case, we have

$$E(t, z) = \{\alpha(t, \zeta, v) + cw: F_- \le w \le F_+\}. \qquad (18)$$

By (13) and (18), along with the triangular inequality,

$$\begin{aligned}
&d^*(E(t, z), E(t^*, z^*)) \\
&\quad = \sup_{y\in E(t, z)} d(y, E(t^*, z^*)) \\
&\quad = \sup_{y\in E(t, z)} \inf_{y^*\in E(t^*, z^*)} \|y - y^*\| \\
&\quad = \sup_{F_-\le w\le F_+} \inf_{F_-\le w^*\le F_+} \|\alpha(t, \zeta, v) + cw \\
&\qquad - \alpha(t^*, \zeta^*, v^*) - cw^*\| \\
&\quad \le \sup_{F_-\le w\le F_+} \inf_{F_-\le w^*\le F_+} [\|\alpha(t, \zeta, v) \\
&\qquad - \alpha(t^*, \zeta^*, v^*)\| + \|c(w - w^*)\|] \\
&\quad = \|\alpha(t, \zeta, v) - \alpha(t^*, \zeta^*, v^*)\| \\
&\qquad + \sup_{F_-\le w\le F_+} \inf_{F_-\le w^*\le F_+} \|c(w - w^*)\| \\
&\quad = \|\alpha(t, \zeta, v) - \alpha(t^*, \zeta^*, v^*)\|. \qquad (19)
\end{aligned}$$

By the continuity of α, this implies that there exist two constants δ_5, $\delta_6 > 0$ such that

$$\begin{aligned}
&d^*(E(t, z), E(t^*, z^*)) \\
&\quad \le \epsilon, \qquad \forall (t, z) \in [t^* - \delta_5, t^* + \delta_5] \times (N_{\delta_6}(z^*) \cap M_0). \qquad (20)
\end{aligned}$$

Now, let $\delta \triangleq \min\{\delta_1, \delta_3, \delta_5\}$ and $\delta' \triangleq \min\{\delta_2, \delta_4, \delta_6\}$. Then, it is clear from (12), (16), (17), and (20) that

$$\begin{aligned}
&d^*(E(t, z), E(t^*, z^*)) \\
&\quad \le \epsilon, \qquad \text{whenever } (t, z) \in [t^* - \delta, t^* + \delta] \times N_{\delta'}(z^*). \qquad (21)
\end{aligned}$$

This means that E is upper semicontinuous at (t^*, z^*), for each $(t^*, z^*) \in R \times M_0$, and hence that (11) holds. On the other hand, note from the definition of E in (4) that at each $(t^*, z^*) \in R \times R^{n+1}$, $E(t^*, z^*)$ is compact and convex. We have thus shown that all the hypotheses of Theorem 2.4 in [4], which states necessary conditions for the existence of solutions in differential inclusions, are satisfied. Hence, the assertion in Lemma 2 holds. ■

Before proceeding to the proof of Theorem 1, we introduce some definitions about topological properties of sets. Let A be a subset of a topological space X. A point $x \in A$ is called an isolated point of A if it is not a limit point of A, i.e., if there exists a neighborhood U of x such that $U \cap A = \{x\}$. The set A is said to be perfect if it is closed and has no isolated points, i.e., if A is equal to the set of its own limit points.

We are now ready to prove Theorem 1.

Proof of Theorem 1: As will be shown soon, the solution z: $[t_0, t_1) \to R^{n+1}$ of the differential inclusion in (10), whose existence is guaranteed by Lemma 2, satisfies that

$$\dot{z}(t) = \arg \min_{y\in E(t, z(t))} \|y\|, \qquad \text{a.e. on } [t_0, t_1). \qquad (22)$$

Then, this along with Lemma 1, will lead to the assertion of Theorem 1. We now show that (22) holds. Note that this is equivalent to showing that for any $\bar{t}$, $t_0 \le \bar{t} \le t_1$,

$$\dot{z}(t) = \arg \min_{y\in E(t, z(t))} \|y\|, \qquad \text{a.e. on } [t_0, \bar{t}]. \qquad (23)$$

Choose any point $\bar{t} \in (t_0, t_1)$. Then, partition the closed interval $[t_0, \bar{t}\,]$ as follows.

$$[t_0, \bar{t}\,] = E_n \cup E_0 \qquad (24)$$

where E_n and E_0 are the subsets of the closed interval $[t_0, \bar{t}\,]$ defined by

$$\begin{aligned}
E_n &\triangleq \{t \in [t_0, \bar{t}\,]P: v(t) \ne 0\} \\
E_0 &\triangleq \{t \in [t_0, \bar{t}]: v(t) = 0\}.
\end{aligned}$$

Here, note that the set E_0 is closed since v is continuous. Then, the Cantor–Bendixon's theorem [14] assures that E_0 contains a perfect subset E_p and a countable subset E_c such that

$$E_0 = E_p \cup E_c, \qquad E_p \cap E_c = \emptyset. \qquad (25)$$

On the other hand, it follows from iii) in Lemma 2 that there exist two subsets $A, B \subset [t_0, \bar{t}\,]$ such that z always has a derivative on A; B is a measure-zero set; and

$$[t_0, \bar{t}\,] = A \cup B, \qquad A \cap B = \emptyset \qquad (26)$$

$$\dot{z}(t) \in E(t, z(t)), \qquad \forall t \in A. \qquad (27)$$

It then follows from (24) to (26) that

$$\begin{aligned}
[t_0, \bar{t}\,] &= E_n \cup E_0 \\
&= (E_n \cap A) \cup (E_n \cap B) \cup (E_0 \cap A) \cup (E_0 \cap B) \\
&= (E_n \cap A) \cup (E_n \cap B) \cup (E_p \cap A) \cup (E_c \cap A) \\
&\quad \cup (E_0 \cap B).
\end{aligned}$$

Here, observe that

$$E_n \cap B, \qquad E_c \cap A, \quad \text{and} \quad E_0 \cap B \text{ are sets of measure zero.} \qquad (28)$$

Furthermore, it holds that

$$\dot{z}(t) = \arg \min_{y\in E(t,z(t))} \|y\|, \qquad \forall t \in (E_n \cap A) \cup (E_p \cap A) \qquad (29)$$

as shown below. This along with (28) then implies that (23) holds.

Finally, we show that (29) holds. First, suppose that $t \in E_n \cap A$. Then, $\dot{z}(t)$ exists and $E(t, z(t))$ is a singleton. It is thus clear from (27) that

$$\dot{z}(t) = \arg \min_{y \in E(t, z(t))} \|y\|, \quad \forall t \in E_n \cap A. \tag{30}$$

Next, suppose that $t \in E_p \cap A$. By the definition of the perfect set, there then exists a sequence of real numbers $t_n \in E_p$ satisfying $t_n \neq t$, $\forall n \in N$ and $\lim_{n\to\infty} t_n = t$. Hence, $\dot{v}(t)$ exists and

$$\dot{v}(t) = \lim_{n\to\infty} \frac{v(t_n) - v(t)}{t_n - t} = 0.$$

This, along with (27), implies that

$$\dot{z}(t) = \begin{bmatrix} \dot{\zeta}(t) \\ \dot{v}(t) \end{bmatrix} = \begin{bmatrix} h(t, \zeta(t), 0) \\ 0 \end{bmatrix}, \qquad \forall t \in E_p \cap A. \tag{31}$$

On the other hand, note that

$$\begin{aligned} \min_{y \in E(t, z(t))} \|y\| &= \min_{F_- \le w \le F_+} \left\| \begin{bmatrix} h(t, \zeta(t), 0) \\ \frac{1}{m}\{g(t, \zeta(t), 0) - w\} \end{bmatrix} \right\| \\ &\ge \left\| \begin{bmatrix} h(t, \zeta(t), 0) \\ 0 \end{bmatrix} \right\|, \qquad \forall t \in E_p \cap A. \end{aligned} \tag{32}$$

By (27), (31), and (32), we have

$$\arg \min_{y \in E(t, z(t))} \|y\| = \dot{z}(t), \qquad \forall t \in E_p \cap A. \tag{33}$$

Now, (29) follows immediately from (30) and (33). ■

Finally, we explain the reason why Filippov's differential inclusion [12], [13] does not necessarily characterize the dynamic behavior of mechanical systems with friction described by (1), (2) along with A.1) and A.2). To this aim, we compute Filippov's differential inclusion corresponding to the differential equation in (3) explicitly using the calculus for computing Filippov's differential inclusion.

$$\dot{z}(t) \in K[f](t, z(t)), \tag{34}$$

where the set-valued map $K[f]: R \times R^{n+1} \to 2^{R^{n+1}}$ is given by

$$\begin{aligned} K[f](t, z) &\triangleq \{\alpha(t, \zeta, v) + cw \in R^{n+1}: w \in \hat{F}_s(v)\} \\ &= \left\{ \begin{bmatrix} h(t, \zeta, v) \\ \frac{1}{m}\{g(t, \zeta, v) - w\} \end{bmatrix} : w \in \hat{F}_s(v) \right\}. \end{aligned} \tag{35}$$

Here, $\hat{F}_s$ is also a set-valued map from R into 2^R and is defined by

$$\hat{F}_s(v) \triangleq \begin{cases} \{F_s(v)\}, & \text{if } v \neq 0 \\ [F_s(0-), F_s(0+)], & \text{if } v = 0. \end{cases} \tag{36}$$

Through some arguments similar to those used in the proof of Theorem 1, we can then show that the solution of the differential inclusion in (34) is indeed the Carathèodory solution of the following differential equation:

$$\dot{z} = \tilde{f}(t, z), \qquad z(t_0) = z_0 \in R^{n+1} \tag{37}$$

where

$$\begin{aligned} \tilde{f}(t, z) &\triangleq \alpha(t, \zeta, v) + c\tilde{F}(v, g(t, \zeta, v)) \\ \tilde{F}(v, u) &\triangleq \begin{cases} F_s(v), & \text{if } v \neq 0 \\ \tilde{\lambda}(u), & \text{if } v = 0 \end{cases} \\ \tilde{\lambda}(u) &\triangleq \begin{cases} F_s(0+), & \text{if } u \ge F_s(0+) > 0 \\ u, & \text{if } F_s(0-) < u < F_s(0+) \\ F_s(0-), & \text{if } u \le F_s(0-) < 0. \end{cases} \end{aligned}$$

Observe that in the case of $F_+ > F_s(0+)$ or $F_- < F_s(0-)$, the right-hand side of the above differential equation in (37) does not agree with that of the differential equation in (3) on the hypersurface $\{(\zeta, v) \in R^n \times R: v = 0\}$. Hence, Filippov's differential inclusion does not necessarily characterize the dynamic behavior of a mechanical system with friction on its discontinuity hypersurface $\{(\zeta, v) \in R^n \times R: v = 0\}$.

III. Conclusion

We have shown that the differential equation with discontinuous right-hand side, which describes the dynamic behavior of a mechanical system with friction, has a Carathèodory solution. Nonetheless, further research is necessary to show whether or not it is unique.

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