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Combining Decision Algorithms for Matching in the Union of Disjoint Equational Theories*

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This paper addresses the problem of systematically building a matching algorithm for the union of two disjoint theories $E_1 \cup E_2$ provided that matching algorithms are known in both theories E_1 and E_2. In general, the blind use of combination techniques introduces unification. Two different restrictions are considered in order to reduce this unification to matching. First, we show that combining matching algorithms (with linear constant restriction) is always sufficient for solving a pure fragment of combined matching problems. Second, the investigated method is complete for the largest class of theories where unification is not needed, including regular collapse-free theories and linear theories. Syntactic conditions are given to define this class of theories in which solving the combined matching problem is performed in a modular way.

1. INTRODUCTION

The process of matching is crucial in automated deduction for instance to apply simplification rules, and programming languages based on equational logic also use intensively this mechanism. The operational semantics of such programming languages may be rewriting modulo an equational theory for which a matching algorithm in this theory is required. In this context, efficient matching algorithms have been developed for some meaningful equational theories, including Abelian semigroups (AC) and Abelian monoids ($AC1$).

A match-equation $s \leqslant^? t$ may be viewed (Bürckert, 1989) as an equation $s =^? t$ where t is a ground term (i.e., without variable). Although a unification algorithm can be used for solving such an equation, turning a match-equation into an equation is not always relevant since there exist theories for which matching is decidable whereas unification is not (Szabó, 1982). Moreover, even if this unification algorithm exists, it will be in general less efficient than a specialized matching algorithm. For these two reasons, the specific unification problem called matching has attracted considerable interest. The problem addressed in this paper is the modular construction of matching algorithms.

The combination problem for unification has been extensively studied in (Kirchner, 1985; Herold, 1986; Tidén, 1986; Yelick, 1987) for equational theories built over disjoint signatures. The general case was solved by Schmidt–Schauß (1989) thanks to a non-deterministic algorithm which has then been made more deterministic by Boudet (1990, 1993). Solving a combined unification problem is more than putting together two unification algorithms. The following assumptions must be satisfied: each equational theory has a unification algorithm with arbitrary constant restriction which can be built from a unification algorithm with (free) constants together with a constant elimination algorithm for breaking compound cycles between solved equations that appear during the occur-check process. Recently, Baader and Schulz (1992) have shown an improved method for solving the combined unification problem: *linear* constant restrictions defined thanks to total orderings on variables are sufficient. For instance, a unification algorithm with free symbols can be obtained by applying the combined unification algorithm presented in (Baader and Schulz, 1992) with as input a unification algorithm with *linear* constant restriction. The converse is also true, which means that unification with *linear* constant restriction is equivalent to unification with free symbols. The greatest interest of this new method is that unification algorithms or decision algorithms for unification can be combined in a uniform way.

Based on a similar principle, the combination problem for matching consists of combining two matching algorithms in two (consistent) equational theories E_1 on $\mathcal{T}(\mathcal{F}_1, \mathcal{X})$ and E_2 on $\mathcal{T}(\mathcal{F}_2, \mathcal{X})$ in order to design a matching algorithm for $E_1 \cup E_2$ on $\mathcal{T}(\mathcal{F}_1 \cup \mathcal{F}_2, \mathcal{X})$. The combined matching algorithm transforms a matching problem into sub-problems that are pure in the sense that they can be solved in one component of the combination. However, this problem is more complicated than just plugging a matching algorithm into a combined unification algorithm. For instance, the matching problem $f(s)+f(t) \leqslant^?_{E_1 \cup E_2} 0$ is equivalent to the unification problem $s =^?_{E_1 \cup E_2} t$ if $E_1 = \{x+x=0\}$, E_2 is the free theory over the signature $\{f\}$

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and so cannot be solved by matching only. This example makes clear that additional assumptions on axioms are needed. A first solution was given by Nipkow (1989, 1991) where the axioms of the disjoint theories to be combined were assumed regular; i.e., the left-hand side and the right-hand side of each axiom have the same set of variables.

The techniques initiated in (Baader and Schulz, 1992) for unification are applied in this paper to matching. We present a combined matching algorithm which is complete for solving a large class of problems:

- Conjunction of pure match-equations $s \leqslant^? t$ where only s is pure in one theory. A matching algorithm with linear constant restriction is assumed for each theory.
- Matching combinable problems. Roughly speaking, this means that unification is not required for solving these specific problems. This property is decidable if an algorithm for solving match-equations and solved equations with linear constant restriction is assumed for each theory.
- All matching problems if theories to combine are linear like $AC0$ (AC plus an absorbent element) and $AC1$ (AC plus a unit element), or regular and collapse-free (i.e. a variable cannot be a left-hand side or a right-hand side of an axiom), or "partially linear" which is a strict generalization of regular collapse-free and linear theories. In this context, only a decision algorithm for matching must be provided for each theory. It is remarkable that linear constant restriction is also superfluous for combining decision algorithms for matching.

We also solve here the *general* disjoint case for the combination of matching algorithms. A negative result about the union of non partially linear theories states that a matching algorithm is not the right solver to combine since unification is somehow unavoidable.

The paper is organized as follows: Section 2 recalls notations and the relationship between unification and matching problems. Section 3 describes the different steps of the algorithm and points out that additional restrictions should be made on problems to solve. Section 4 introduces the notions of matching combinable problems and partially linear theories. The combined matching algorithm is given in Section 5 and some examples are developed. Section 6 summarizes and exploits the results proved in this paper. Eventually, we conclude with final remarks and future works.

2. UNIFICATION AND MATCHING PROBLEMS

This section introduces the definitions and notations compatible with (Jouannaud and Kirchner, 1991).

Let $\mathscr{F}$ be a finite set of function symbols, $\mathscr{X}$ an infinite denumerable set of variables. The term algebra $\mathscr{T}(\mathscr{F}, \mathscr{X})$ is the free $\mathscr{F}$-algebra over $\mathscr{X}$. The terms $t_{|\omega}$ and $t[\omega \hookleftarrow s]$ denote respectively the subterm of t at the position ω, and the replacement in t of $t_{|\omega}$ by s. Conversely, t is a superterm of $t_{|\omega}$. The symbol of t occurring at the position ω (resp. the top symbol of t) is written $t(\omega)$ (resp. $t(\varepsilon)$). The set of variables in a term t is denoted by $\mathscr{V}(t)$. A term is *linear* if each of its variables occurs just once.

A substitution σ is an endomorphism of $\mathscr{T}(\mathscr{F}, \mathscr{X})$ denoted by $\{x_1 \mapsto t_1, ..., x_n \mapsto t_n\}$ if there are only finitely many variables $x_1, ..., x_n$ not mapped to themselves. Application of substitutions is written out by postfixed juxtaposition. We call *domain* of the substitution σ the set of variables $\mathscr{D}om(\sigma) = \{x \mid x \in \mathscr{X} \text{ and } x\sigma \neq x\}$, *range* of σ the set of terms $\mathscr{R}an(\sigma) = \bigcup_{x \in \mathscr{D}om(\sigma)} x\sigma$ and *variable range* of σ the set of variables $\mathscr{V}\mathscr{R}an(\sigma) = \cup_{x \in \mathscr{D}om(\sigma)} \mathscr{V}(x\sigma)$. A substitution σ is *idempotent* if $\mathscr{D}om(\sigma) \cap \mathscr{V}\mathscr{R}an(\sigma) = \varnothing$. Substitutions are denoted by letters $\sigma, \mu, \gamma, \phi$, ... Given a set of substitutions S and a substitution σ, $S\sigma$ denotes the instantiated set of substitutions $\{\phi\sigma \mid \phi \in S\}$.

Given a set E of axioms (i.e. pairs of terms of $\mathscr{T}(\mathscr{F}, \mathscr{X})$), the *equational theory* $=_E$ is the congruence closure of E under the law of substitutivity. The equational theory is *regular* if $\mathscr{V}(s) = \mathscr{V}(t)$ for all $s = t$ in E, *linear* if s, t are linear for all $s = t$ in E and *collapse-free* if there is no axiom $s = x$ (with $x \in \mathscr{X}$ and $s \notin \mathscr{X}$) in E. As usual, the equational theory is also improperly denoted by E.

A substitution ϕ is an E-instance on $V \subseteq \mathscr{X}$ of a substitution σ, written $\sigma \leqslant_E^V \phi$ (and read as σ is more general modulo E than ϕ on V), if there exists some substitution μ such that $\forall x \in V$, $x\phi =_E x\sigma\mu$. The equivalence relation $=_E^V$ on substitutions is defined as follows: $\sigma =_E^V \phi$ if $\sigma \leqslant_E^V \phi$ and $\phi \leqslant_E^V \sigma$.

A quantifier-free $\langle \mathscr{F}, \mathscr{X}, E \rangle$-*unification problem* is $\bot, \top$ or a conjunction $\Gamma = \bigwedge_{k \in K} s_k =_E^? t_k$ of equations. There is no solution to $\bot$ and any substitution is a solution of $\top$. A substitution σ is a E-solution of Γ if $\mathscr{T}(\mathscr{F}, X)/=_E \models \Gamma\sigma$, or equivalently $\forall k \in K$, $s_k\sigma =_E t_k\sigma$. The set of all solutions of Γ is denoted by $SU_E(\Gamma)$. An existentially quantified $\langle \mathscr{F}, \mathscr{X}, E \rangle$-unification problem is denoted $\exists \vec{x}: \Gamma$ where $\vec{x}$ is a set of variables included in $\mathscr{X}$ and Γ is a quantifier-free $\langle \mathscr{F}, \mathscr{X}, E \rangle$-unification problem. The set of all solutions of $\exists \vec{x}: \Gamma$ is $SU_E(\exists \vec{x}: \Gamma) = \{\phi \mid \sigma \in SU_E(\Gamma), \phi_{|\mathscr{X} \setminus \vec{x}} = \sigma_{|\mathscr{X} \setminus \vec{x}}\}$. We are dealing with existentially quantified unification problems since some transformation rules used in the following naturally introduce existentially variables. In the rest of the paper, we always solve quantifier-free unification problems and then eliminate existentially quantified variables which have been solved thanks to the following transformation rule

EQE

$$\frac{\exists \{v\} \cup \vec{x}: \Gamma \wedge v =^? t}{\exists \vec{x}: \Gamma} \quad \text{if} \quad v \notin \mathscr{V}(\Gamma) \cup \mathscr{V}(t) \cup \vec{x}$$

If σ denotes the idempotent substitution $\{x_1 \mapsto t_1, ..., x_n \mapsto t_n\}$ on $\mathcal{T}(\mathcal{F}, \mathcal{X})$, then $\hat{\sigma}$ is the $\langle \mathcal{F}, \mathcal{X}, E \rangle$-unification problem $\bigwedge_{k=1}^{n} x_k =_E^? t_k$ called in *solved form*. If σ is the identity (i.e. $\mathcal{D}om(\sigma) = \emptyset$), then $\hat{\sigma}$ is $\top$. A conjunction of equations $\Gamma = (\bigwedge_{k \in K} x_k =_E^? t_k)$ such that for any $k \in K$, $x_k \notin \mathcal{V}(t_k)$ is in *dag solved form* if the repeated application of the transformation rules devoted to the replacement of variables (see Fig. 1) terminates and leads to a solved form.

The set of solutions can be schematized in a more compact form according to the subsumption ordering $\leqslant_E^{\mathcal{V}(\Gamma)}$ on substitutions.

Definition 1. A set of substitutions is a *complete set of E-solutions* of the unification problem Γ, denoted by $CSU_E(\Gamma)$, if

1. $\forall \sigma \in CSU_E(\Gamma)$, $\mathcal{D}om(\sigma) \cap \mathcal{V}\mathcal{R}an(\sigma) = \emptyset$ (idempotency);
2. $CSU_E(\Gamma) \subseteq SU_E(\Gamma)$ (correctness);
3. $\forall \phi \in SU_E(\Gamma)$, $\exists \sigma \in CSU_E(\Gamma)$, $\sigma \leqslant_E^{\mathcal{V}(\Gamma)} \phi$ (completeness).

A *complete set of most general solutions* is a complete set of solutions whose elements cannot be compared with $\leqslant_E^{\mathcal{V}(\Gamma)}$. If this complete set of most general solutions is at most a singleton (resp. a finite set) for all unification problems Γ then E-unification is of type *unitary* (resp. *finitary*). The notion of type extends to some subclasses of unification problems like matching problems. A subclass SC of E-unification problems is *decidable* (resp. *solvable*) if there exists an algorithm such that for each $\Gamma \in SC$ it returns yes or no whether $CSU_E(\Gamma)$ is non-empty or not (resp. computes all elements of a $CSU_E(\Gamma)$).

A $\langle \mathcal{F} \cup C, \mathcal{X}, E \rangle$-unification problem, where C is a set of additional constants, is a unification problem with free constants. Given a $\langle \mathcal{F}, \mathcal{X}, E \rangle$-unification problem P and $C \subset \mathcal{X}$ a set of variables, (P, C) denotes the $\langle \mathcal{F} \cup C, \mathcal{X} \backslash C, E \rangle$-unification problem Γ where variables in C are considered as free constants. The set of *skolemized* variables C occurring in Γ is denoted by $\mathcal{G}\mathcal{V}(\Gamma)$. The set of *solved* variables occurring in Γ is denoted by $\mathcal{S}\mathcal{V}(\Gamma)$ and contains any variable $x \notin C$ occurring once in Γ and which is left-hand side of an equation in Γ. A $\langle \mathcal{F}, \mathcal{X}, E \rangle$-matching problem $\bigwedge_{k \in K} s_k \leqslant_E^? t_k$ is $(\bigwedge_{k \in K} s_k =_E^? t_k, C)$ such that $\bigcup_{k \in K} \mathcal{V}(t_k) \subseteq C$. A match-equation is $s \leqslant_E^? t$ where s is the left-hand side and t is the right-hand side. By this definition, a matching problem is a special case of a unification problem with free constants where right-hand sides are ground and this enables us to deal with the unification framework. Some other particular subsets of $\mathcal{G}\mathcal{V}(\Gamma)$ are used in the rest of the paper:

- $\mathcal{R}\mathcal{V}(\Gamma) = \bigcup_{k \in K} \mathcal{V}(t_k)$, the set of variables occurring in the right-hand sides of match-equations.
- $\overline{\mathcal{R}\mathcal{V}}(\Gamma) = \mathcal{G}\mathcal{V}(\Gamma) \backslash \mathcal{R}\mathcal{V}(\Gamma)$, the set of skolemized variables which do not occur in right-hand sides of match-equations.
- $\mathcal{C}\mathcal{V}(\Gamma)$, the set of skolemized variables which are right-hand sides of match-equations in Γ, i.e., variables x such that $s \leqslant_E^? x$ is a match-equation in Γ.

3. COMBINATION PROBLEM FOR MATCHING

Let E_1, E_2 be two equational theories built over disjoint signatures $\mathcal{F}_1, \mathcal{F}_2$ which means $\mathcal{F}_1 \cap \mathcal{F}_2 = \emptyset$. We are interested in the set of axioms $E = E_1 \cup E_2$ built over $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$. Notice that the equational theory $=_{E_1 \cup E_2}$ is not equal to $=_{E_1} \cup =_{E_2}$. An E-matching algorithm works as follows for a given input matching problem Γ.

1. Γ is transformed to an *equivalent* conjunction of two unification problems $\Gamma_1 \wedge \Gamma_2$, one for each component (i.e., Γ and $\Gamma_1 \wedge \Gamma_2$ have the same set of solutions).
2. Γ_1 (resp. Γ_2) is solved w.r.t. E_1 (resp. E_2).
3. Solutions from each component are recombined.

3.1. Purification

The first step of our combination algorithm transforms a matching problem Γ into a conjunction $\Gamma_1 \wedge \Gamma_2$ of almost pure unification problems such that Γ and $\Gamma_1 \wedge \Gamma_2$ are equivalent. Purification is achieved by replacing some subterms in Γ with new variables and adding related solved equations (see Fig. 2). This transformation is first performed in the left-hand side of each match-equation since a variable

VarRep

$$\frac{\Gamma \wedge x =^? y}{\Gamma\{x \mapsto y\} \wedge x =^? y} \quad \text{if } x, y \in \mathcal{V}(\Gamma)$$

Rep

$$\frac{\Gamma \wedge x =^? t}{\Gamma\{x \mapsto t\} \wedge x =^? t} \quad \text{if } x \in \mathcal{V}(\Gamma), t \notin \mathcal{X}$$

FIG. 1. Merging.

Left Purification

$$\frac{\exists \vec{x} : (\Gamma \wedge p(s,t))}{\exists \vec{x} \cup \{v\} : (\Gamma \wedge p(s[\omega \hookleftarrow v], t) \wedge v =^? s_{|\omega})}$$

where

- $\omega \in \mathcal{A}Pos(s)$,
- v is a new variable,
- $p \in \{=^?, \leq^?\}$.

FIG. 2. Left Purification.

cannot be introduced in the right-hand side unless destroying the groundness hypothesis on this right-hand side.

DEFINITION 2. Let $i, j \in \{1, 2\}$ and $i \neq j$. A term in $\mathcal{T}(\mathcal{F}_i, \mathcal{X})$ is *i-pure*. An *i-pure* unification problem is built on i-pure terms. A match-equation $(s \leqslant^? t)$ is *i-(left pure)* if s is i-pure. A matching problem is *i-(left pure)* if its match-equations are i-left pure.

A term with its top symbol in $\mathcal{F}_i$ is called *i-term*. An *alien* subterm of an i-term is a j-(sub)term $(j \neq i)$ such that all its superterms are i-terms. $\mathcal{A}Pos(t)$ is the set of positions of alien subterms in t.

In the rest of the paper, we always use i and j as two arbitrary distinct theory indexes among 1 and 2.

We make precise now the unification problem obtained after purification.

DEFINITION 3. An *E-(extended matching problem)* Γ is

$$(P \wedge \hat{\sigma}, C),$$

where (P, C) is an E-matching problem and σ is an idempotent substitution such that $\mathcal{D}om(\sigma) \cap (\mathcal{V}(P) \cup C) = \emptyset$ and $\forall x \in \mathcal{D}om(\sigma)$, $\mathcal{V}(x\sigma) \nsubseteq C$. The extended matching problem is *i-(left pure)* if Γ is i-left pure and $\hat{\sigma}$ is i-pure. An (E_1, E_2)-*extended matching problem* Γ is a conjunction

$$((P_1 \wedge \hat{\sigma}_1) \wedge (P_2 \wedge \hat{\sigma}_2), C)$$

where $(P_i \wedge \hat{\sigma}_i, C)$ is an i-left pure extended matching problem such that $\hat{\sigma}_1 \wedge \hat{\sigma}_2$ is in dag solved form. $\mathcal{S}_i(\Gamma)$ denotes σ_i and $\mathcal{S}\mathcal{V}_i(\Gamma)$ denotes $\mathcal{D}om(\sigma_i)$ for $i = 1, 2$.

An (E_1, E_2)-*matching problem* is an (E_1, E_2)-extended matching problem such that σ_1 and σ_2 are the identity substitution.

When E is clear from context, we simply use extended matching problem instead of E-extended matching problem. The decomposition of an extended matching problem into a conjunction $(P \wedge \hat{\sigma}, C)$ is unique since equations in $\hat{\sigma}$ cannot be viewed as match-equations.

PROPOSITION 1. *$E_1 \cup E_2$-matching is decidable (resp. finitary, solvable) iff $E_1 \cup E_2$-extended matching problem is decidable (resp. finitary, solvable) iff (E_1, E_2)-extended matching is decidable (resp. finitary, solvable).*

The three different problems mentioned in the previous proposition are in some sense equivalent but only the last one can be solved in a modular way thanks to matching algorithms known for theories E_1 and E_2.

Indeed, the second step of the combination algorithm consists in solving the i-left-pure match-equations $s \leqslant^?_E t$ with respect to the equational theory E_i. The right-hand side is purified by using the concept of variable abstraction which takes into account E-equality. For this purpose, we introduce the ordered rewriting system R obtained by unfailing completion of E (Boudet, 1990; Baader and Schulz, 1992) which is convergent on $\mathcal{T}(\mathcal{F} \cup V)$ for some finite set of variables V strictly included in $\mathcal{X}$.

DEFINITION 4. A *variable abstraction* is a one-to-one mapping π from the set of normalized terms $T\downarrow_R = \{u\downarrow_R \mid u \in \mathcal{T}(\mathcal{F} \cup V)$ and $u\downarrow_R \in \mathcal{T}(\mathcal{F} \cup V)\backslash V\}$ to a set of variables from $\mathcal{X}\backslash V$. π^{-1} denotes the converse of π.

The term t^{π_i}, called *i-abstraction* of the term t, is inductively defined as follows:

- if $t = v \in V$ then $t^{\pi_i} = v$;
- if $t = f(s_1, ..., s_p)$ and $f \in \mathcal{F}_i$ then $t^{\pi_i} = f(s_1^{\pi_i}, ..., s_p^{\pi_i})$;

else if $t\downarrow_R \notin V$ then $t^{\pi_i} = \pi(t\downarrow_R)$ else $t^{\pi_i} = t\downarrow_R$.

The substitution $\sigma^{\pi_i} = \{x \mapsto (x\sigma)^{\pi_i}\}_{x \in \mathcal{D}om(\sigma)}$ is the *i-abstraction* of the substitution σ.

By this definition, E-equal alien subterms are abstracted by the same variable. Note also that π^{-1} is not strictly speaking a substitution since its domain is infinite. However, π^{-1} is viewed in the following as the substitution of variables abstracting some subterms of the given unification problem.

EXAMPLE 1. Assume $E_1 = \{x + x = x\}$ and E_2 is the empty theory with $\mathcal{F}_2 = \{f\}$. $(f(x) + f(x + x))^{\pi_1} = X + X$ if $X = \pi(f(x)\downarrow_R) = \pi(f(x + x)\downarrow_R)$.

The right-hand side of an i-left pure match-equation $s \leqslant^? t$ is replaced by its i-abstraction t^{π_i} and the related i-pure match-equation $s \leqslant^? t^{\pi_i}$ is considered.

3.2. Solving in One Component

Solutions of $s \leqslant^?_E t$ may be found by solving $s \leqslant^?_{E_i} t^{\pi_i}$ but the i-abstraction of the right-hand side t must be performed carefully because $t =_E t'$ does not imply $t^{\pi_i} =_{E_i} t'^{\pi_i}$.

EXAMPLE 2. Assume $E_1 = \{x + x = x\}$ and E_2 is the empty theory with $\mathcal{F}_2 = \{f\}$. The match-equation $f(f(x)) \leqslant^?_E f(f(a) + f(a))$ is equivalent to $f(f(x)) \leqslant^?_E f(f(a))$. But $f(f(x)) \leqslant^?_{E_2} f(C)$, where $C = \pi(f(a)\downarrow_R)$, has no solution whereas $f(f(x)) \leqslant^?_{E_2} f(f(a))$ has a unique solution $\{x \mapsto a\}$.

We show in this section that solving $s \leqslant^?_E t^{\pi_i}$ in one component is correct and complete if t is in layer-reduced form, which means that t "looks like" its normal form $t\downarrow_R$. This result was already necessary for the matching in the union of regular theories. However, the transformation of the right-hand side used by Nipkow (1991) is here related to the normal form w.r.t. the rewrite system obtained after unfailing completion. This rewriting approach for describing heterogeneous equational proofs has been introduced later on (Boudet, 1990).

DEFINITION 5. A term t is in *layer-reduced form* if t is a variable or if $t(\varepsilon)$ and $t\downarrow_R(\varepsilon)$ are symbols in the same theory and alien subterms of t are in layer-reduced form.

The notion of layer-reduced form can be also applied for deciding the word-problem (Schmidt–Schauß, 1989) as shown next.

LEMMA 1. *If s is in layer-reduced form and $s \to_R t$ then t is in layer-reduced form and $s^{\pi_i} =_{E_i} t^{\pi_i}$.*

COROLLARY 1. *Let s, t be two terms in layer-reduced form. Then $s =_E t \Leftrightarrow s^{\pi_i} =_{E_i} t^{\pi_i}$.*

Proof. $(\Leftarrow)$ Obvious.

$$(\Rightarrow)\ s^{\pi_i} =_{E_i} (s\downarrow_R)^{\pi_i} = (t\downarrow_R)^{\pi_i} =_{E_i} t^{\pi_i}. \quad \blacksquare$$

Consequently, two i-terms s and t in layer-reduced form are E-equal if and only if

$$s[\omega_1 \hookleftarrow v_1]\cdots[\omega_m \hookleftarrow v_m] =_{E_i} t[\omega_{m+1} \hookleftarrow v_{m+1}]\cdots[\omega_n \hookleftarrow v_n],$$

where $\omega_1, ..., \omega_m, \omega_{m+1}, ..., \omega_n$ are alien positions of s and t and $v_1, ..., v_m, v_{m+1}, ..., v_n$ are new variables which abstract alien subterms of s and t such that for any k, k' in $\{1, ..., n\}$, v_k and $v_{k'}$ are identical if and only if the related abstracted subterms are E-equal. Note that these subterms are also in layer-reduced form.

The computation of a layer-reduced form is possible and an algorithm can be derived from the definitions below since matching (and thus the word-problem) is decidable in each theory.

DEFINITION 6. An alien subterm or a variable u of an i-term t is *collapsing* for t if $t^{\pi_i} =_{E_i} u^{\pi_i}$.

The related E_i-equality is collapsing since u^{π_i} is a variable. In a more constructive way, an alien subterm or a variable u is collapsing for a term t such that its alien subterms are in layer-reduced form if and only if

$$t[\omega_1 \hookleftarrow v_1]\cdots[\omega_m \hookleftarrow v_m] =_{E_i} x,$$

where $\omega_1, ..., \omega_m$ are alien positions, $v_1, ..., v_m$ are new variables such that

$$\forall k, k' \in \{1, ..., m\}, \qquad v_k = v_{k'} \Leftrightarrow t_{|\omega_k} =_E t_{|\omega_{k'}}$$

and if u is a variable then $x = u$, else $x = v_n$ provided $u = t_{|\omega_n}$ is the corresponding alien subterm in layer-reduced form.

EXAMPLE 3. Assume $E_1 = \{x + x = x\}$ and E_2 is the empty theory with $\mathscr{F}_2 = \{f\}$. $f(a)$ is collapsing for $f(a) + f(a)$.

DEFINITION 7. $t\Downarrow$ denotes the term defined from t as follows:

- $t\Downarrow = t$ if t is a variable.
- $t\Downarrow = u$ if there exists a term u collapsing for $t' = t[\omega_k \hookleftarrow (t_{|\omega_k})\Downarrow]_{\omega_k \in \mathscr{A}Pos(t)}$. Else $t\Downarrow = t'$.

Due to the second point of Definition 7, syntactically different terms $t\Downarrow$ can be constructed: one of them is arbitrarily chosen.

PROPOSITION 2. *$t\Downarrow$ is a term in layer-reduced form which is E-equal to t and computable provided the word-problem is decidable in E_i for $i = 1, 2$.*

Proof. By induction on the theory height of t defined as $ht(t) = 1 + \max_{\omega_k \in \mathscr{A}Pos(t)} ht(t_{|\omega_k})$.

- If $ht(t) = 1$ then t is i-pure and $t\downarrow_R$ is a variable x if and only if x is collapsing for t or equivalently if and only if $t =_{E_i} x$.
- Otherwise, according to the induction hypothesis, there exists an algorithm for computing layer-reduced forms of alien subterms of t. We are then able to decide if there exists a term u collapsing for $t' = t[\omega_k \hookleftarrow (t_{|\omega_k})\Downarrow]_{\omega_k \in \mathscr{A}Pos(t)}$ and $t\Downarrow = u$ is in layer-reduced form. Otherwise, such a term u does not exist, $t'\downarrow_R(\varepsilon)$ and $t'(\varepsilon)$ are symbols in the same theory and so $t\Downarrow = t'$ is in layer-reduced form. $\blacksquare$

COROLLARY 2 (Schmidt-Schauß, 1989). *If E_1 and E_2 are two disjoint theories, then the word-problem in $E_1 \cup E_2$ is decidable if the word-problem in E_i is decidable for $i = 1, 2$.*

EXAMPLE 4. Assume $E_1 = \{x + x = x\}$ and E_2 is the empty theory with $\mathscr{F}_2 = \{f\}$. Let s be the term $f(a, y) + (f(a, y) + f(a, y))$ and t be the term $f(a + a, y)$. We have $s =_E t$ since $(s\Downarrow)^{\pi_2} = s\Downarrow = f(a, y) = t\Downarrow = (t\Downarrow)^{\pi_2}$.

THEOREM 1. *Let $(s \leqslant^? t)$ be an i-left pure match-equation, σ a substitution normalized w.r.t. R and t a term in layer-reduced form. Then $s\sigma =_E t \Leftrightarrow (s\sigma)^{\pi_i} =_{E_i} t^{\pi_i}$.*

Note that $(s\sigma)^{\pi_i}$ is identical to $s\sigma^{\pi_i}$ and $\sigma^{\pi_i} \leqslant_E^{\mathscr{X}} \sigma$.

Proof. Assume the substitution σ R-normalized, it is easy to prove (Baader and Schulz, 1992; Ringeissen, 1993) that $(s\sigma)^{\pi_i} =_{E_i} ((s\sigma)\downarrow_R)^{\pi_i}$. Since t and $t\downarrow_R$ are in layer-reduced form, $t^{\pi_i} =_{E_i} (t\downarrow_R)^{\pi_i}$ with $(s\sigma)\downarrow_R = t\downarrow_R$. $\blacksquare$

This theorem is similar to the one given in (Baader and Schulz, 1992) for unification.

COROLLARY 3. *Let*

$$\Gamma = \left(\bigwedge_{i=1}^{2}\left(\bigwedge_{k_i \in K_i} s_{i,k_i} \leqslant^? t_{i,k_i}\right) \wedge \hat{\sigma}_i, C\right)$$

be an (E_1, E_2)-*extended matching problem. Then, a*

$$CSU_{E_1 \cup E_2}\left(\bigwedge_{i=1}^{2}\left(\bigwedge_{k_i \in K_i} s_{i,k_i} \leqslant^? (t_{i,k_i}\Downarrow)^{\pi_i}\right) \wedge \hat{\sigma}_i, C\right)\pi^{-1}$$

provides a $CSU_{E_1 \cup E_2}(\Gamma)$.

We are now mainly interested in solving a conjunction $\Gamma_1 \wedge \Gamma_2$ of two pure extended matching problems.

3.3. Combining Solutions from Each Equational Theory

The main difficulty is now to combine solutions of each pure extended matching problem. We first remind how this question has been solved for unification and then apply techniques developed in this more general case to extended matching. In the context of unification, a same variable may be instantiated in both theories. The method initiated by Schmidt-Schauß (1989) consists of choosing nondeterministically for each variable the theory in which it will be instantiated and skolemize the variable in the alien theory, so that there is no more conflict of theories. However, the conjunction of two solutions does not give a solved form since a compound cycle could appear, for example $x =^? t_1[y] \wedge y =^? t_2[x]$. For breaking such a cycle, the idea is to choose (Baader and Schulz, 1992), again in a nondeterministic way, a linear ordering on variables, for example $x < y$ (or $y < x$). In each theory, pure problems are solved according to this linear restriction where alien variables are considered as free constants and thus unification with *linear constant restriction* is needed. Let us briefly recall this notion introduced in (Baader and Schulz, 1992).

Consider that terms are built over the signature $\mathscr{F} \cup C$, where C denotes a set of additional free constants. Any constant $c \in C$ is equipped with a set V_c of variables. Let Γ be a unification problem with occurrences of free constants. An E-solution σ of Γ with *constant restriction* is an E-solution such that for any $c \in C$ and any $x \in V_c$, c does not occur in $x\sigma$. It is enough to deal with *linear* constant restriction, which means: for a given linear ordering $<$ on $\mathscr{X} \cup C$, the sets V_c are defined as $V_c = \{x \mid x \in \mathscr{X} \text{ and } x < c\}$. The set of E-solutions (resp. a complete set of E-solutions) with linear constant restriction is denoted $SU_E^{<}(\Gamma)$ (resp. $CSU_E^{<}(\Gamma)$).

Coming back to the conjunction of two pure unification problems $\Gamma_1 \wedge \Gamma_2$, we have to consider all possible linear orderings $<$ on[1] $V_1 \oplus V_2$ where V_1 denotes the variables instantiated in E_1 and V_2 the set of variables instantiated in E_2. Variables in V_1 (resp. V_2) are then skolemized in the E_2-unification problem Γ_2 (resp. E_1-unification problem Γ_1) and we say improperly that they are *skolemized* in E_2 (resp. E_1). Notice that variables in $V_1 \cap V_2$ are skolemized in E_1 and E_2. Two solutions $\sigma_1 \in SU_{E_1}^{<}(\Gamma_1, V_2)$ and $\sigma_2 \in SU_{E_2}^{<}(\Gamma_2, V_1)$ with respect to the same linear restriction $<$ are easily combined since the conjunction $\hat{\sigma}_1 \wedge \hat{\sigma}_2$ is in dag solved form.

[1] $\oplus$ is the disjoint union operator; that is, $V_1 \oplus V_2 = (V_1 \cup V_2) \backslash (V_1 \cap V_2)$.

DEFINITION 8. Let $<$ be a linear ordering on an arbitrary disjoint union $V_1 \oplus V_2 = \mathscr{V}(\Gamma_1 \wedge \Gamma_2)$. The *combined solution* $\sigma_1 \odot \sigma_2$ of $\Gamma_1 \wedge \Gamma_2$ w.r.t. $<$ obtained from $\sigma_1 \in SU_{E_1}^{<}(\Gamma_1, V_2)$ and $\sigma_2 \in SU_{E_2}^{<}(\Gamma_2, V_1)$ is inductively defined as follows: let x be a variable in V_i and $\{y_k\}_{k \in K}$ be the set of (smaller) variables in V_j, $j \neq i$. Then $x\sigma = x\sigma_i\{y_k \mapsto y_k\sigma\}_{k \in K}$. The set of combined solutions is denoted $SU_{E_1}^{<}(\Gamma_1, V_2) \odot SU_{E_2}^{<}(\Gamma_2, V_1)$.

PROPOSITION 3 (Baader and Schulz, 1992). *A combined solution is a solution, i.e.,*

$$SU_{E_1}^{<}(\Gamma_1, V_2) \odot SU_{E_2}^{<}(\Gamma_2, V_1) \subseteq SU_{E_1 \cup E_2}(\Gamma_1 \wedge \Gamma_2).$$

For the completeness part, we need the fact that combining solutions in complete sets of solutions provides a complete set of combined solutions.

PROPOSITION 4 (Baader and Schulz, 1992). *The set of combined solutions*

$$\{\sigma_1 \odot \sigma_2 \mid \sigma_1 \in CSU_{E_1}^{<}(\Gamma_1, V_2), \sigma_2 \in CSU_{E_2}^{<}(\Gamma_2, V_1)\}$$

is a complete set of solutions of $SU_{E_1}^{<}(\Gamma_1, V_2) \odot SU_{E_2}^{<}(\Gamma_2, V_1)$.

Care must be taken that two variables instantiated identically by a solution in one theory should be considered as the same skolemized variable in the other theory.

EXAMPLE 5. Solving $(x + c =^? c', \{c, c'\})$ yields no solution if $+$ is idempotent but $x + c \leqslant^? c$ has the unique solution $\{x \mapsto c\}$. The identification $\{c' \mapsto c\}$ is necessary in order to obtain a solution.

As a consequence of skolemization, we have to consider each unification problem $\Gamma_1 \wedge \Gamma_2 \wedge \hat{\xi}$ where $\hat{\xi}$ is a unification problem pure in both theories such that ξ is a substitution which ranges over variables.

DEFINITION 9. Let V and W be two sets of variables. An identification ξ on V to W is an idempotent substitution such that $\mathscr{D}om(\xi) \subseteq V$ and $\mathscr{R}an(\xi) \subseteq W$. The set of identifications on V to W is denoted by ID_V^W or ID_V if $W = V$.

Let $<$ be a linear ordering on $V_1 \oplus V_2$. An identification $\xi \in ID_{V_1 \oplus V_2}^{V_1 \cup V_2}$ is compatible with $<$ if

- $\forall x, y \in \mathscr{D}om(\xi)$, $x\xi = y\xi \Rightarrow \exists i \in \{1, 2\}$, $x, y \in V_i$;
- $\forall x, y \in V_1 \oplus V_2$, $x\xi < y\xi \Rightarrow x < y$.

The identification $\xi_{|V_i}$ is denoted by ξ_i.

Contrary to (Baader and Schulz, 1992), we choose first a theory for each variable, second a linear ordering and finally an identification. The aim is to delay as much as possible the identification of variables which is the difficult point in preserving matching problems. Delaying identifications as long as possible allows to reduce their number since some identifications may be finally even not necessary for computing solutions with a linear constant restriction. We can easily adapt the result of (Baader and Schulz, 1992) to the case where $\Gamma_1 \wedge \Gamma_2$ already contains skolemized variables.

THEOREM 2. *Let Γ_1 and Γ_2 be two (respectively 1-pure and 2-pure) unification problems with skolemized variables. A $CSU_{E_1 \cup E_2}(\Gamma_1 \wedge \Gamma_2)$ is provided by the union of*

$$CSU^{<}_{E_1}(\Gamma_1 \xi_2 \wedge \hat{\xi}_1, V_2) \odot CSU^{<}_{E_2}(\Gamma_2 \xi_1 \wedge \hat{\xi}_2, V_1)$$

for each

1. $V_1 \supseteq \mathscr{GV}(\Gamma_1 \wedge \Gamma_2)$ *and* $V_2 \supseteq \mathscr{GV}(\Gamma_1 \wedge \Gamma_2)$ *s.t.* $V_1 \oplus V_2 = \mathscr{V}(\Gamma_1 \wedge \Gamma_2) \backslash \mathscr{GV}(\Gamma_1 \wedge \Gamma_2)$;
2. *linear ordering* $<$ *on* $V_1 \oplus V_2$;
3. *identification* $\xi \in ID^{V_1 \cup V_2}_{V_1 \oplus V_2}$ *compatible with* $<$.

Proof. Consider the set of constants $C = \{c_x \mid x \in \mathscr{GV}(\Gamma_1 \wedge \Gamma_2)\}$ and the empty theory $\varnothing$ which has C as signature. The problem may be seen as an $E_1 \cup E_2 \cup \varnothing$-unification problem where variables in $\mathscr{GV}(\Gamma_1 \wedge \Gamma_2)$ abstract constants in C: purification has introduced the equation $x =^? c_x$ for each $x \in \mathscr{GV}(\Gamma_1 \wedge \Gamma_2)$. Then we can apply the combined algorithm due to Baader and Schulz in this particular case:

- Identification of two variables x and y in $\mathscr{GV}(\Gamma_1 \wedge \Gamma_2)$ leads to a failure in the empty theory $\varnothing$ since $c_x \neq c_y$. So identifications are only taken from $ID^{V_1 \cup V_2}_{V_1 \oplus V_2}$ where $V_1 \oplus V_2$ does not contain $\mathscr{GV}(\Gamma_1 \wedge \Gamma_2)$.
- A variable $x \in \mathscr{GV}(\Gamma_1 \wedge \Gamma_2)$ is necessarily instantiated in $\varnothing$ since $x =^? c_x$. Consequently, x is skolemized in E_1 and in E_2, and so occurs in $V_1 \cap V_2$.
- Let $\sigma \in SU_{E_1 \cup E_2}(\Gamma_1 \wedge \Gamma_2)$ be a R-normalized substitution. A variable instantiated in E_1 or in E_2 cannot occur in the term $x\sigma$ for $x \in \mathscr{GV}(\Gamma_1 \wedge \Gamma_2)$ since $x\sigma = c_x$. So it is sufficient to consider linear orderings on $V_1 \oplus V_2 = (V_1 \cup V_2) \backslash (V_1 \cap V_2) = \mathscr{V}(\Gamma_1 \wedge \Gamma_2) \backslash \mathscr{GV}(\Gamma_1 \wedge \Gamma_2)$. ∎

Corollary 3 states that an impure matching problem Γ is equivalent to solve a conjunction of two pure extended matching problems $\Gamma_1 \wedge \Gamma_2$. Since a matching problem is also a unification problem, the combination of solutions initiated for unification can be reused as well for matching. The question that must now be studied is the following one: are problems to consider in Theorem 2 solvable given a matching algorithm with linear constant restriction?

3.3.1. *Skolemization of Variables*

Considering additional skolemized variables in an extended matching problem does not create a more complicated unification problem.

PROPOSITION 5. *Let $i, j \in \{1, 2\}$ such that $i \neq j$. Let Γ_i be an E_i-extended matching problem and V_j be a set of variables instantiated in E_j. Then (Γ_i, V_j) is an E_i-extended matching problem.*

Proof. Since variables in V_j are skolemized in Γ_i, a term with variables in V_j is viewed as a ground term and a solved equation $x =^? t$ can be seen as a match-equation $t \leqslant^? x$ if $x \in V_j$. More precisely, $(x =^? t, V_j)$ is identical to $(x \leqslant^? t, V_j)$ if $\mathscr{V}(t) \subseteq V_j$ and $(x =^? t, V_j)$ is equivalent to $(t \leqslant^? x, V_j)$ if $x \in V_j$. ∎

3.3.2. *Linear Constant Restriction*

In the general case, an impure matching problem is equivalent after purification to a conjunction of pure extended matching problems. Again, we would like to be able to solve this kind of problems with a linear constant restriction. As for unification, a constant elimination algorithm is of greatest interest for taking into account the linear constant restriction.

DEFINITION 10. The unification problem $\bigwedge_{k=1}^{n} (x_k =^? t_k[c_k])$ with constant restriction such that $\forall k \in \{1, ..., n\}$, $V_{c_k} = \{x_k\}$ and $\forall k, l \in \{1, ..., n\}$, $x_l \notin \mathscr{V}(t_k)$ and $(k \neq l \Rightarrow x_k \neq x_l)$ is called *constant elimination problem.*

THEOREM 3. *Extended matching (resp. unification) with constant restriction is finitary iff*

- *matching (resp. unification) is finitary,*
- *and constant elimination problem is finitary.*

Proof. Baader and Schulz (1992) show how to construct a unification algorithm with constant restriction by combining a unification algorithm together with a constant elimination algorithm. This can be done as well for matching. The first algorithm is devoted to the solving process and the second one is used for taking into account the constant restriction. ∎

Remark. This result does not hold for linear constant restriction (Baader and Schulz, 1992). Boudet (1990) and Schmidt–Schauß (1989) compose also both unification and constant elimination algorithms in each theory for solving the combined unification.

3.3.3. *Identification of Variables*

The property to be an extended matching problem may not be preserved after identification of variables.

Let us first consider the simple case where $\Gamma_1 \wedge \Gamma_2$ is only a conjunction of two pure matching problems sharing possibly some variables. Then, for each identification ξ, $\Gamma_i \xi$ remains an i-left pure matching problem and $\Gamma_i \xi \wedge \hat{\xi}$ is obviously solvable thanks to a matching algorithm since $\mathscr{Dom}(\xi) \cap \mathscr{V}(\Gamma_i \xi) = \varnothing$. Consequently, we have the following result:

THEOREM 4. *(E_1, E_2)-matching is decidable (resp. finitary, solvable) if E_i-matching $(i=1, 2)$ with linear constant restriction is decidable (resp. finitary, solvable).*

In general, solving an extended pure matching problem with linear constant restriction is not sufficient since an i-pure extended matching problem Γ_i which is identified with ξ, namely $(\Gamma_i \xi_j \wedge \hat{\xi}_i)$, may be equivalent to a proper unification problem as shown next:

$$(\Gamma_i \wedge x =^? s \wedge y =^? t \wedge x =^? y) \Leftrightarrow (\Gamma_i \wedge s =^? t \wedge x =^? y)$$

if $x, y \in V_i$ are assumed instantiated in E_i.

The idea of the combined matching algorithm is to perform only identifications which do not lead to a unification problem. Roughly speaking, it means that each variable is identified with a variable bound to a ground term. Consequently, if $t\Downarrow$ denotes now a ground term, the previous problem becomes

$$(\Gamma_i \wedge x =^? s \wedge y =^? t\Downarrow \wedge x =^? y)$$
$$\Leftrightarrow (\Gamma_i \wedge s \leqslant^? t\Downarrow \wedge y \leqslant^? t\Downarrow \wedge x =^? y).$$

The problem to solve is still an extended pure matching problem. But this restriction does not always preserve the completeness of the computed set of solutions.

For regular collapse-free theories, this restriction on identifications can be assumed without loss of completeness. Let us consider again the proof of Theorem 1 (see also Fig. 4): if a variable abstracts an alien subterm in the left-hand side, then this variable occurs necessarily in the right-hand side of the E_i-equality and abstracts also a ground term. Thus, we can use for this class of theories a special transformation rule for purification which introduces only match-equations. By applying repeatedly the `Left Purification (RCF)` rule given in Fig. 3, we obtain an (E_1, E_2)-matching problem which can be solved thanks to Theorem 4.

THEOREM 5. *If E_1 and E_2 are two regular collapse-free theories, then $E_1 \cup E_2$-matching is decidable (resp. finitary, solvable) iff E_i-matching $(i=1, 2)$ is decidable (resp. finitary, solvable).*

Proof. Let $i, j \in \{1, 2\}$ such that $j \neq i$. For any $\sigma_i \in CSU_{E_i}(\Gamma_i, V_j)$, we have $\mathscr{V}\mathscr{R}an(\sigma_i) \subseteq \mathscr{R}\mathscr{V}(\Gamma_i)$. Therefore, $\sigma_i \in CSU^{<}_{E_i}(\Gamma_i, V_j)$ for any linear ordering $<$ on $V_1 \oplus V_2 = \mathscr{V}(\Gamma_1 \wedge \Gamma_2) \backslash \mathscr{G}\mathscr{V}(\Gamma_1 \wedge \Gamma_2)$ and so linear constant restrictions are superfluous for regular collapse-free theories. ▮

Nipkow (1991) assumes for the regular case that an algorithm is provided for computing the finitely many substitutions in $CSU_{E_i}(\Gamma_i, V_j)$. In our approach, we only need algorithms for deciding whether the set $CSU_{E_i}(\Gamma_i, V_j)$ is empty or not. But on the other hand, a stronger assumption on theories is needed since they must be also collapse-free. In the rest of the paper, we will generalize this first modularity result to the largest possible class of theories.

The class of linear theories is presented in the conclusion of (Nipkow, 1991) as another good candidate for combining matching algorithms. Again, by looking at Theorem 1 (see also Fig. 5), we can observe that the identification of two variables in one term is not necessary to retrieve the left-hand side of an E_i-equality. In the following we will show that for linear theories, it is useless to perform such identifications introducing unification problems too complicated for a matching algorithm. The problem related to the identification step is extensively investigated in the next section.

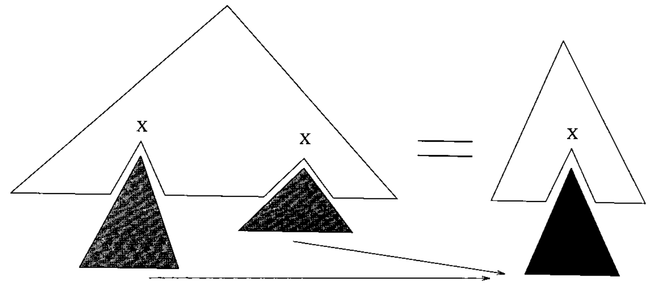


FIG. 4. E_i-`Equality` in regular collapse-free theories.

Left Purification(RCF)

$$\frac{\exists \vec{x} : (\Gamma \wedge s \leq^? t))}{\bigvee_{\omega' \in \mathcal{A}Pos(t)} \exists \vec{x} \cup \{v\} : (\Gamma \wedge s[\omega \hookleftarrow v] \leq^? t \wedge s_{|\omega} \leq^? t_{|\omega'} \wedge v \leq^? t_{|\omega'})}$$

where

- $\omega \in \mathcal{A}Pos(s)$,
- v is a new variable.

FIG. 3. `Left Purification` for regular collapse-free theories.

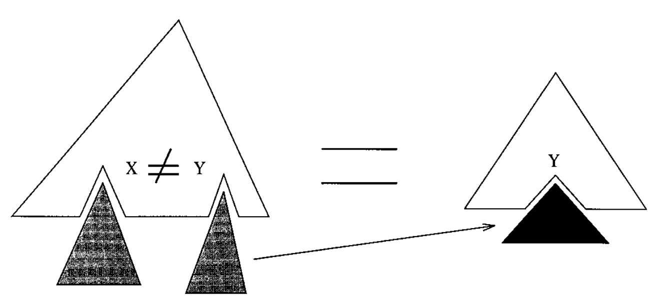


FIG. 5. E_i-`Equality` in linear theories.

4. MATCHING COMBINABLE THEORIES

The aim of this section is to generalize the property of regular collapse-free theories and linear theories to drop some identifications without loss of completeness. Let Γ be an extended matching problem. An identification of two skolemized variables c and c' can correspond to the unification of two terms $t =^? t'$ where t and t' are alien subterms (with variables) bound respectively to c and c'. This is possible if c and c' are skolemized variables which do not occur in right-hand sides of Γ or if $l[c, c'] \leqslant^? c$ is a match-equation of Γ. Such an identification, say $\xi = \{c' \mapsto c\}$, can be avoided if the solutions of the identified problem $\Gamma\xi$ are obtained by applying the identification ξ on the solutions of Γ.

DEFINITION 11. Given an E-extended matching problem Γ, $NONLIN(\Gamma)$ denotes the set of identifications $\xi \in ID^{\mathscr{GV}(\Gamma)}_{\mathscr{RV}(\Gamma)}$ such that for each $x \in \mathscr{D}om(\xi)$, $x\xi \in \mathscr{V}(l)$ if $l \leqslant^?_E x\xi$ is in Γ and $x \in \mathscr{V}(l)$. An E-extended matching problem Γ with a linear constant restriction $<$ is *matching combinable* if for each identification ξ compatible with $<$ such that $\xi \in ID_{\overline{\mathscr{RV}}(\Gamma)} \circ NONLIN(\Gamma)$, then we have $SU^<_E(\Gamma\xi) = SU^<_E(\Gamma)\xi$. An equational theory E is *matching combinable* if any E-extended matching problem is matching combinable.

EXAMPLE 6. Let us consider the Boolean theory B built over the signature $\{\neg, +, \cdot, 0, 1\}$ where $\neg, +, \cdot$ denote respectively *not*, *or*, *and*. The Boolean theory B is not matching combinable since $\Gamma = (c + \neg c' \leqslant^? 1, \{c, c'\})$ falsifies Definition 11. However, it is possible to decide if a given B-extended matching problem Γ is matching combinable. B-unification with constant restriction is unitary (Ringeissen, 1992) and it is sufficient to check, according to Lemma 2, that the most general unifier of $\Gamma\xi$ w.r.t. $<$ is an instance of the most general unifier of Γ w.r.t. $<$ identified with ξ.

4.1. Partially Linear Theories

We give a condition on theorems of the equational theory in order to express both ideas that a variable can be eliminated from a term without identification and that a variable can be collapsing for a term without identification. This is possible for instance in a theory that contains the following theorems

- $x \star x = 0$ and $x \star y = 0$
- $f(x, x) = x$ and $f(x, y) = x$

since the second theorem generalizes in both cases the first one and may replace its application.

DEFINITION 12. A term r E-eliminates a set of variables V of l if $l =_E r$ and for each $x \in V$, $x \in \mathscr{V}(l)$ and $x \notin \mathscr{V}(r)$.

An equational theory E is *partially linear* if for each linear term l, and each identification $\xi \in ID_{\mathscr{V}(l)}$, we have

- for each term r such that r E-eliminates $V\xi$ of $l\xi$, then there exists d such that d E-eliminates V of l;
- $l\xi =_E x$ implies $l =_E x$ if $x\xi = x$.

Regular collapse-free theories satisfy the above definition. For a linear theory E, if l is a linear term and $\xi \in ID_{\mathscr{V}(l)}$, then $l\xi =_E r\xi$ iff $l =_E r$. Hence, linear theories are partially linear. But there are also other partially linear theories.

EXAMPLE 7.

$$DA = \begin{cases} x + (y + z) = (x + y) + z \\ x * (y + z) = x * y + x * z \\ (x + y) * z = x * z + y * z \end{cases}$$

$$Z = \begin{cases} x + 0 = 0 \\ x * 0 = 0 \end{cases}$$

If 0 occurs in $l\xi$ then 0 occurs also in l and $l =_{DAZ} 0$. Otherwise $l\xi$ is equal to another term modulo DA which is a regular collapse-free theory. Therefore, DAZ is partially linear but neither regular nor linear. A DAZ-matching algorithm is easily derived from a DA-matching algorithm. The constant elimination problem is equivalent to match on 0. Then, DAZ-extended matching with linear constant restriction is finitary. However, it is easy to show that $CSU_{DAZ}(\Gamma)$ includes $CSU_{DA}(\Gamma)$ for any unification problem Γ involving $+, *$ only and so DAZ-unification is infinitary since DA-unification is infinitary (Szabó, 1982).

LEMMA 2. *If ξ is an identification on skolemized variables in $\mathscr{GV}(\Gamma)$ and compatible with $<$, then a $CSU^<_E(\Gamma)\xi$ is a $CSU^<_E(\Gamma\xi)$ if and only if $SU^<_E(\Gamma\xi) = SU^<_E(\Gamma)\xi$.*

Proof. ($\Leftarrow$) For the correctness, we simply use the fact that $CSU^<_E(\Gamma)\xi \subseteq SU^<_E(\Gamma)\xi = SU^<_E(\Gamma\xi)$ and for the completeness,

$$\forall \psi \in SU^<_E(\Gamma\xi) = SU^<_E(\Gamma)\xi \; \exists \sigma \in CSU^<_E(\Gamma)\xi, \sigma \leqslant^{\mathscr{V}(\Gamma)}_E \psi.$$

So $CSU^<_E(\Gamma)\xi$ is a $CSU^<_E(\Gamma\xi)$.

($\Rightarrow$) According to the hypothesis,

$$\forall \psi \in SU^<_E(\Gamma\xi) \; \exists \sigma \in CSU^<_E(\Gamma)\xi, \sigma \leqslant^{\mathscr{V}(\Gamma)}_E \psi.$$

For any substitution ϕ satisfying the linear restriction $<$ such that $\sigma \leqslant^{\mathscr{V}(\Gamma)}_E \phi$ and $\sigma \in CSU^<_E(\Gamma)\xi$, we have $\phi \in SU^<_E(\Gamma)\xi$. Therefore, $SU^<_E(\Gamma\xi) \subseteq SU^<_E(\Gamma)\xi$. Since the inclusion $SU^<_E(\Gamma)\xi \subseteq SU^<_E(\Gamma\xi)$ is always satisfied, we can conclude that $SU^<_E(\Gamma)\xi = SU^<_E(\Gamma\xi)$. ∎

PROPOSITION 6. *An equational theory E is partially linear if and only if E is matching combinable.*

Proof. ($\Leftarrow$) For the first point in Definition 12, just consider $(x =^? l, C)$ w.r.t. $<$ on $V \oplus C$ such that $V = \{x\}$ and $C = \mathcal{V}(l)$ and for the second one, the ground equation $(x =^? l, C)$ with $C = \mathcal{V}(l)$ and $x \in C$.

($\Rightarrow$) According to Lemma 2, it is sufficient to prove that

$$\forall \psi \in SU_E^<(\Gamma\xi)\ \exists \phi \in CSU_E^<(\Gamma),\ \phi\xi \leqslant_E^{\mathcal{V}(\Gamma)} \psi$$

since $SU_E^<(\Gamma)\xi \subseteq SU_E^<(\Gamma\xi)$. Let ψ be a substitution such that $(\psi\xi) \in CSU_E^<(\Gamma\xi)$. There are two kinds of equations:

1. Let $s \leqslant^? t$ be a match-equation in Γ. Since $\mathcal{D}om(\xi) \subseteq \overline{\mathcal{RV}}(\Gamma)$ we have $s(\psi\xi) =_E t\xi = t$. If t is not a skolemized variable then all skolemized variables occurring in $(s\psi)$ and identified with ξ are eliminated by t. Definition 12 implies that there exists u such that $s\psi =_E u$ with $u =_E u\xi =_E (s\psi)\xi =_E t$.

If t is a skolemized variable x then $s(\psi)\xi =_E x$ and Definition 12 implies $s\psi =_E x$.

2. Let $x =^? t$ be a solved equation in Γ and $x(\psi\xi) =_E t(\psi\xi)$. If $x(\psi\xi)$ eliminates a skolemized variable of $t(\psi\xi)$ then Definition 12 implies that there exists ϕ such that $x\phi =_E t\phi$ with $\phi\xi = (\psi\xi)$ on $\mathcal{V}(t)$ and $(x\phi)\xi =_E x(\psi\xi)$. Otherwise $\{x \mapsto t\}$ is a solution of $x = t$ and $\{x \mapsto t\}\xi \leqslant_E^{\{x\}} \psi\xi$. The same reasoning can be applied again since x does not appear elsewhere in Γ. ∎

4.2. Modularity of Partial Linearity

In this section, we show that the class of *partially linear* theories is closed under disjoint union.

PROPOSITION 7. *If E_1, E_2 are two disjoint partially linear theories, then $E = E_1 \cup E_2$ is also a partially linear theory.*

Proof. Consider an equality $l\xi =_E r$ such that r eliminates $V\xi$ of $l\xi$. Since collapse E_i-equalities can be assumed linear without loss of generality, we have $(l\xi)\Downarrow = (l\Downarrow)\xi$ and so

$$((l\Downarrow)\xi)^{\pi_i} =_{E_i} (r\Downarrow)^{\pi_i}$$

There exists an identification ξ_i on $\mathcal{V}((l\Downarrow)^{\pi_i})$ such that $(l\Downarrow)^{\pi_i}\xi_i =_{E_i} (r\Downarrow)^{\pi_i}$ and $(r\Downarrow)^{\pi_i}$ eliminates a set of variables $V_i\xi_i$ which abstract terms with occurrences of V:

$$\forall x \in V \quad \exists x_i \in V_i, \qquad x \in \mathcal{V}(x_i\pi^{-1}).$$

Since E_i is partially linear, there exists d such that $(l\Downarrow)^{\pi_i} =_{E_i} d$ and d eliminates V_i. Then $l =_E (l\Downarrow)^{\pi_i}\pi^{-1} =_E d\pi^{-1}$ with $V \cap \mathcal{V}(d\pi^{-1}) = \emptyset$ since $\mathcal{V}(d) \cap V_i = \emptyset$.

In the same way, $l\xi =_E x$ implies successively $((l\xi)\Downarrow)^{\pi_i} =_{E_i} x$, $((l\Downarrow)\xi)^{\pi_i} =_{E_i} x$ then $(l\Downarrow)^{\pi_i}\xi_i =_{E_i} x$ and $(l\Downarrow)^{\pi_i} =_{E_i} x$ since E_i is partially linear. We build up again an E-equality by applying π^{-1}, $l =_E ((l\Downarrow)^{\pi_i})\pi^{-1} =_E x\pi^{-1} = x$. ∎

Conversely, if one theory E_i is not partially linear, then the union $E_1 \cup E_2$ is not partially linear since two i-pure terms are E-equal if and only if they are E_i-equal.

4.3. Extended Matching vs Unification

We are mostly interested in extended matching with linear constant restriction which is strongly related to extended matching with free symbols. In the following, $\emptyset$ denotes the empty theory.

DEFINITION 13. An *E-freely extended matching problem* is a unification problem

$$\Gamma \wedge \hat{\sigma}_\emptyset,$$

where Γ is an E-extended matching problem and $\sigma_\emptyset$ a substitution of terms built over free symbols such that

- $\mathcal{D}om(\sigma_\emptyset) \cap \mathcal{VR}an(\sigma_\emptyset) = \emptyset$;
- $\mathcal{R}an(\sigma_\emptyset) \cap \mathcal{X} = \emptyset$;
- $\forall x, y \in \mathcal{D}om(\sigma_\emptyset),\ x \neq y \Rightarrow CSU_\emptyset(x\sigma_\emptyset =^? y\sigma_\emptyset) = \emptyset$.

The last point of this definition means that terms in the range of $\sigma_\emptyset$ are not unifiable. Consequently, variables in the domain of $\sigma_\emptyset$ are necessarily instantiated in $\emptyset$ and cannot be identified.

PROPOSITION 8. *If E-freely extended matching is decidable (resp. finitary, solvable) then E-extended matching with linear constant restriction is decidable (resp. finitary, solvable).*

Proof. According to (Baader and Schulz, 1992), just consider

$$\sigma_\emptyset = \{c \mapsto f_c(x_1, ..., x_m) \mid \{x_1, ..., x_m\} = V_c\},$$

where $V_c = \{x \in V \mid x < c \text{ and } c \in C\}$. ∎

This result holds only for linear constant restriction but not for arbitrary constant restriction.

The next result states that there exist theories for which the introduction of unification problems cannot be avoided.

PROPOSITION 9. *If E is not partially linear then $E \cup \emptyset$-freely extended matching is decidable (resp. finitary, solvable) iff $E \cup \emptyset$-unification is decidable (resp. finitary, solvable).*

Proof. We show that any unification problem is equivalent to an extended matching problem. Three cases must be considered according to three different reasons for which an identification is needed. The first one is due to an identification of eliminated variables. The second one is due to an identification of persistent variables which helps to eliminate another variable. The last one is due to the identification of a collapsing variable.

1. There exists a linear term l, an identification $\xi \in ID_{\mathcal{V}(l)}$ such that $l\xi$ contains a variable x at positions $\Omega = \{\omega_1, ..., \omega_n\}$ for $n > 1$ and an equality $l\xi =_E r$ with $x \notin \mathcal{V}(r)$ for which the matching problem

$$l[\omega_1 \hookleftarrow h(t_1)][\cdots][\omega_n \hookleftarrow h(t_n)] \leqslant^? r$$

is equivalent to the unification problem

$$t_1 =^? \cdots =^? t_n$$

if h is a free symbol and variables in l are skolemized.

2. There exists a linear term l and an identification $\xi \in ID_{\mathcal{V}(l)}$ such that

- a variable x occurs at positions $\Omega = \{\omega_1, ..., \omega_n\}$ of $l\xi$ for $n > 1$,
- a variable y occurring at a position ω' of l can be eliminated from $l\xi$,

for which the compound cycle problem

$$\exists y, z\colon z =^? l[\omega_1 \hookleftarrow h(t_1)][\cdots][\omega_n \hookleftarrow h(t_n)][y]_{\omega'} \wedge y =^? f(z)$$

is again equivalent to the unification problem

$$t_1 =^? \cdots =^? t_n$$

if h, f are free and variables in l are skolemized.

3. There exists a linear term l, an identification $\xi \in ID_{\mathcal{V}(l)}$ such that $l\xi$ contains a variable x at positions $\Omega = \{\omega_1, ..., \omega_n\}$ $(n > 1)$ for which the freely extended matching problem

$$\exists x\colon l[\omega_1 \hookleftarrow h(t_1)][\cdots][\omega_n \hookleftarrow h(t_n)] =^? x \wedge x =^? h(t_n)$$

is equivalent to the unification problem

$$t_1 =^? \cdots =^? t_n$$

if h is a free symbol and variables in l are skolemized. ∎

When solving freely extended matching problem is possible for a non partially linear theory E together with the empty theory (the simplest one) then it is also possible to solve the combined unification problem. The reason is that unification with free symbols is equivalent to unification with linear constant restriction (Baader and Schulz, 1992).

In the following, the combined matching problem is solved for partially linear theories.

5. COMBINED MATCHING ALGORITHM

The matching algorithm for the combination transforms the input heterogeneous matching problem into some pure extended matching problems (Γ_1, V_2) and (Γ_2, V_1) which are then solved according to the same linear restriction $<$ on $V_1 \oplus V_2$ after possible identification of variables.

In Section 3.1, we have seen that purification of the right-hand sides of i-left pure match-equations $s \leqslant^? t$ is performed according to the notion of variable abstraction. For sake of convenience, our algorithm introduces explicitly new variables v_k which abstract alien subterms $t{\Downarrow}_{|\omega_k}$. The related solved equations $v_k \leqslant^? t{\Downarrow}_{|\omega_k}$ such that $t{\Downarrow}(\omega_k) \in \mathcal{F}_j$ are added (see Fig. 6). Then, variables v_k are necessarily instantiated in E_j and skolemized in E_i, which means that $s =^? t{\Downarrow}[\omega_k \hookleftarrow v_k]_{\omega_k \in \mathcal{A}Pos(t\Downarrow)}$ is viewed after possible identification of v_k's as a match-equation $s \leqslant^? t{\Downarrow}[\omega_k \hookleftarrow v_k]_{\omega_k \in \mathcal{A}Pos(t\Downarrow)}$ in E_i. Thus, we have to consider an E_1-extended matching problem (Γ_1, V_2) and an E_2-extended matching problem (Γ_2, V_1). The next step consists of applying admissible identifications ξ_i for which we prove that $(\Gamma_i \wedge \hat{\xi}_i, V_j)$ is again equivalent to an E_i-extended matching problem and so ξ_i does not introduce an arbitrary unification problem. The justification for this restriction on considered identifications being complete is partial linearity of theories.

5.1. Informal Description

We first give here a brief overview of our combined matching algorithm.

1. Input is a conjunction of match-equations $s_k \leqslant^?_{E_1 \cup E_2} t_k$. Variables occurring in t_k's are marked ground.

2. Purify left-hand sides (`Left Purification`; see Fig. 2).

This generates equations. Two left pure extended matching problems are obtained.

Right Purification

$$\frac{\exists \vec{x} : (\Gamma \wedge s \leq^? t)}{\exists \vec{x} \cup \vec{v} : (\Gamma \wedge s =^? (t{\Downarrow})[\omega_k \hookleftarrow v_k]_{k \in K} \wedge \bigwedge_{k \in K} v_k \leq^? t{\Downarrow}_{|\omega_k})}$$

where

- $s \in \mathcal{T}(\mathcal{F}_i, \mathcal{X})$,
- $\{\omega_k\}_{k\in K} = \mathcal{A}Pos(t{\Downarrow})$ if $t{\Downarrow}(\epsilon) \in \mathcal{F}_i$ else $\{\omega_k\} = \{\epsilon\}$,
- $\vec{v} = \{v_k\}_{k \in K}$ is a set of new variables.

FIG. 6. `Right purification.`

3. The right-hand sides of equations should be made layer-reduced. Purify right-hand sides (`Right Purification`; see Fig. 6). The new variables have to be marked ground.

Computation of layer-reduced forms are possible since a matching algorithm is assumed for each theory. Two pure problems are obtained.

4. Divide as follows the set of all variables into the non-disjoint sets V_1 and V_2. If a variable x is marked ground at Step 1, then $x \in V_1 \cap V_2$. If x is marked ground at Step 3 and is equal to an i-pure term, then $x \in V_i$. Otherwise, x is in a unique set chosen non-deterministically among V_1 and V_2.

Variables in V_i are called i-variables. An i-variable is skolemized in the j-pure problem ($i \neq j$). A variable marked ground is a 1-variable and a 2-variable and so is skolemized in both pure problems.

5. For each i-variable x, proceed as follows:

• If there exists a j-pure equation ($i \neq j$) $x =^? s$, then (possibly) identify x with another i-variable y such that there exists a j-pure equation $y =^? s'$ or with an i-variable y occurring in s or with a ground marked i-variable y.

• Otherwise, (possibly) identify x with a ground marked i-variable y. Add $x =^? y$ in the i-pure problem and apply $\{x \mapsto y\}$ in the j-pure problem.

Only some identifications are considered (Definition 14). The completeness is preserved due to the fact that theories are partially linear or equivalently matching combinable (Lemma 4 and Lemma 5).

6. Merge pure problems thanks to replacement rules (`Merging`).

Pure extended matching problems are obtained (Lemma 3).

7. Choose a linear ordering on all the variables in the range of the identification that are not marked ground at Step 1.

8. Solve each pure extended matching problem using linear constant restriction.

Combining solutions leads to a complete set of solutions of the input matching problem (Theorem 6).

Example 8. Assume $E_1 = \{x * y = y * x, x * 1 = x\}$ and E_2 is the free theory over the unary function symbol f. The match-equation

$$f(f(x) * y) * f(y) \leqslant^?_{E_1 \cup E_2} f(1) * f(f(a))$$

has a unique solution $\{x \mapsto a, y \mapsto 1\}$ which can be retrieved thanks to the following extended matching problem

$$\Gamma = (u_1 * u_2 \leqslant^? f(1) * f(f(a)) \wedge u_3 =^? u_4 * y) \wedge (u_1 =^? f(u_3) \wedge u_2 =^? f(y) \wedge u_4 =^? f(x))$$

and the related pure problems:

$$(\Gamma_1 \xi_2 \wedge \hat{\xi}_1, V_2) = ((u_1 * u_2 =^? g_1 * g_2 \wedge u_4 * y =^? u_3 \wedge g_3 =^? 1)\, \xi_2 \wedge \hat{\xi}_1, V_2)$$

$$(\Gamma_2 \xi_1 \wedge \hat{\xi}_2, V_1) = ((u_1 =^? f(u_3) \wedge u_2 =^? f(y) \wedge u_4 =^? f(x) \wedge g_1 =^? f(g_3) \wedge g_2 =^? f(f(a)))\, \xi_1 \wedge \hat{\xi}_2, V_1)$$

where the set V_1 of 1-variables is $\{y, g_3, a\}$, the identification ξ_1 of 1-variables is $\{y \mapsto g_3\}$, the set V_2 of 2-variables is $\{x, u_1, u_2, u_3, u_4, g_1, g_2, a\}$ and the identification ξ_2 of 2-variables is $\{u_1 \mapsto g_2, u_2 \mapsto g_1, u_3 \mapsto u_4\}$. The identification ξ_1 maps y onto the ground marked 1-variable g_3 which is equal to the ground term 1. Analogously, the identification ξ_2 maps respectively the 2-variables u_1, u_2 onto the ground marked 2-variables g_2, g_1 and maps u_3 onto the 2-variable u_4 occurring in the 1-equation $u_3 =^? u_4 * y$. After the replacement of variables, we solve the following pure matching problems:

$$g_2 * g_1 \leqslant^? g_1 * g_2 \wedge u_4 * g_3 \leqslant^? u_4 \wedge g_3 \leqslant^? 1 \Leftrightarrow g_3 =^? 1$$

$$f(f(x)) \leqslant^? f(f(a)) \wedge f(g_3) \leqslant^? f(g_3) \Leftrightarrow x =^? a.$$

Since y is identified with g_3, we finally get the expected solution $\{x \mapsto a, y \mapsto 1\}$.

5.2. Correctness and Completeness of the Algorithm

The correctness of the combined matching algorithm is quite obvious since the pure problems to solve at the final step are also considered by the standard combined unification algorithm (Baader and Schulz, 1992). Conversely, the completeness is much more complicated to prove since many branches have been pruned in order to generate only pure extended matching problems.

We define now formally which identifications are considered in the algorithm.

Definition 14. Let Γ_i be the i-pure unification problem obtained from an (E_1, E_2)-extended matching problem Γ thanks to `Right Purification`. Let $\mathcal{GSV}(\Gamma_i)$ be $\mathcal{GV}(\Gamma) \cup (\mathcal{SV}(\Gamma_i) \backslash \mathcal{SV}_i(\Gamma))$ and let V_1 and V_2 be two sets of variables such that $V_1 \supseteq \mathcal{GSV}(\Gamma_1)$, $V_2 \supseteq \mathcal{GSV}(\Gamma_2)$ and $V_1 \oplus V_2 = \mathcal{V}(\Gamma_1 \wedge \Gamma_2) \backslash \mathcal{GV}(\Gamma)$. The set of *admissible* identifications of variables V_i in Γ_i is

$$ADM_{V_i}(\Gamma_i) = \{\mu_i \in ID^{V_i}_{\mathcal{SV}_j(\Gamma)} \circ ID^{\mathcal{GSV}(\Gamma_i)}_{V_i \backslash \mathcal{SV}_j(\Gamma)} \text{ s.t. } \forall x \in \mathcal{SV}_j(\Gamma), x\mu_i \in \mathcal{V}(x\mathcal{S}_j(\Gamma)) \cup \mathcal{SV}_j(\Gamma) \cup \mathcal{GSV}(\Gamma_i)\}.$$

We must prove that admissible identifications introduce only extended matching problems.

LEMMA 3. *If* $\mu_1 \in ADM_{V_1}(\Gamma_1)$ *and* $\mu_2 \in ADM_{V_2}(\Gamma_2)$, *then*

- $(\Gamma_1\mu_2 \wedge \hat{\mu}_1, V_2)$ *and* $(\Gamma_2\mu_1 \wedge \hat{\mu}_2, V_1)$ *are respectively equivalent to an* E_1*-extended matching problem* (Ω_1, V_2) *and to an* E_2*-extended matching problem* (Ω_2, V_1) *thanks to* `Merging`.
- $\Omega_1 \wedge \Omega_2$ *is equivalent to an* (E_1, E_2)*-extended matching problem thanks to* `Merging`.

Proof. Let us first check that (Γ_i, V_j) is an E_i-extended matching problem. `Right Purification` introduces new variables in right-hand sides of Γ_i which are necessarily in V_j. Hence, an equation $s =^?_{E_i} (t\Downarrow)[\omega_k \hookleftarrow v_k]_{k \in K}$ in Γ_i is a match-equation $s \leqslant^?_{E_i} (t\Downarrow)[\omega_k \hookleftarrow v_k]_{k \in K}$ in (Γ_i, V_j) since v_k's are in V_j.

Since μ_i is idempotent, we can now consider independently each $\Gamma_i \wedge x =^? y$ such that $x\mu_i = y$ with $y \neq x$.

- Let $x \in V_i$ and $y \in \mathcal{GSV}(\Gamma_i)$. Then, the occurrences of the variable x are replaced by a ground term with `Merging` and this obviously leads to an extended matching problem.
- Let $x, y \in \mathcal{SV}_j(\Gamma)$. Then x occurs in Γ_i necessarily in left-hand sides of match-equations $(s[x] \leqslant^?_{E_i} t)$ or in right-hand sides of solved equations $(z =^?_{E_i} t[x])$. Hence, `Merging` only consists of replacing x by y in Γ_i and this leads to an extended matching problem due to the positions of x in Γ_i.
- Let $x \in \mathcal{SV}_j(\Gamma)$ and $y \in \mathcal{V}(x\mathcal{S}_j(\Gamma))$. If $y \notin \mathcal{D}om(\mathcal{S}_i(\Gamma))$, then see the previous case. Otherwise, we can distinguish two subcases:

  — If $s[x] \leqslant^?_{E_i} t \wedge y =^?_{E_i} u$ is in Γ_i, then it is replaced by $s[u] \leqslant^?_{E_i} t \wedge y =^?_{E_i} u$ thanks to `Merging`.

  — If $z =^?_{E_i} s[x] \wedge y =^?_{E_i} u$ is in Γ_i, then it is replaced by $z =^?_{E_i} s[u] \wedge y =^?_{E_i} u$ thanks to `Merging`. Note that $z \notin \mathcal{V}(u)$. Otherwise, we have

$$z =^?_{E_i} s[x] \wedge x =^?_{E_j} (x\mathcal{S}_j(\Gamma))[y] \wedge y =^?_{E_i} u[z]$$

is a compound cycle (i.e. is not a dag solved form) and this contradicts Γ is an (E_1, E_2)-extended matching problem.

Let Ω_i be the i-pure problem obtained from $(\Gamma_i\mu_j \wedge \hat{\mu}_i)$ thanks to `Merging`. Applying now `Merging` on $\Omega_1 \wedge \Omega_2$ with variables occurring in $\mathcal{GSV}(\Gamma_1) \cup \mathcal{GSV}(\Gamma_2)$ leads to an (E_1, E_2)-extended matching problem Ω where match-equations are left-pure. This step is of course the reverse of `Right Purification`. ∎

A less operational definition of admissible identifications would consist of choosing ξ such that `Merging` applied on $\Gamma_1 \wedge \Gamma_2 \wedge \hat{\xi}$ terminates and leads to an $E_1 \cup E_2$-extended matching problem.

We now prove that any identification can be decomposed into some admissible identifications and identifications satisfying Definition 11.

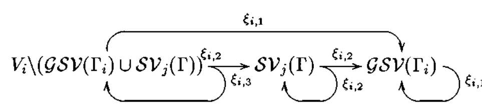
LEMMA 4. *Let* $i, j \in \{1, 2\}$ *such that* $i \neq j$. *For any* $\xi'_i \in ID_{V_i}$ *and any maximal identification* $\mu_i \in ADM_{V_i}(\Gamma_i)$ *less than* ξ'_i *w.r.t.* $\leqslant^{V_i}$, *there exists* $\xi_i \in ID_{\overline{\mathcal{RV}((\Gamma_j\mu_i, V_i))}} \circ NONLIN((\Gamma_j\mu_i, V_i))$ *such that* $\xi'_i =^{V_i} \mu_i\xi_i$.

Proof. The set of variables V_i is divided into three sets $V_i \backslash (\mathcal{GSV}(\Gamma_i) \cup \mathcal{SV}_j(\Gamma))$, $\mathcal{SV}_j(\Gamma)$ and $\mathcal{GSV}(\Gamma_i)$. Variables taken from two different sets are identified as follows:

- Variables from $V_i \backslash (\mathcal{GSV}(\Gamma_i) \cup \mathcal{SV}_j(\Gamma))$ are identified with $\mathcal{GSV}(\Gamma_i)$.
- Variables from $V_i \backslash (\mathcal{GSV}(\Gamma_i) \cup \mathcal{SV}_j(\Gamma))$ are identified with $\mathcal{SV}_j(\Gamma)$.
- Variables from $\mathcal{SV}_j(\Gamma)$ are identified with $\mathcal{GSV}(\Gamma_i)$.

Hence, an identification $\xi'_i \in ID_{V_i}$ can be decomposed into $\xi'_i =^{V_i} \xi_{i,3} \circ \xi_{i,2} \circ \xi_{i,1}$ where

1. $\xi_{i,1} = \mu_{i \mid V_i \backslash \mathcal{SV}_j(\Gamma)} \in ID^{\mathcal{GSV}(\Gamma_i)}_{V_i \backslash (\mathcal{GSV}(\Gamma_i) \cup \mathcal{SV}_j(\Gamma))} \circ ID_{\mathcal{GSV}(\Gamma_i)}$,
2. $\xi_{i,2} \in ID^{\mathcal{SV}_j(\Gamma)}_{V_i \backslash (\mathcal{GSV}(\Gamma_i) \cup \mathcal{SV}_j(\Gamma))} \circ ID_{\mathcal{SV}_j(\Gamma)} \circ ID^{\mathcal{GSV}(\Gamma_i)}_{\mathcal{SV}_j(\Gamma)}$,
3. $\xi_{i,3} \in ID_{V_i \backslash (\mathcal{GSV}(\Gamma_i) \cup \mathcal{SV}_j(\Gamma))}$.



Remark that $\mathcal{GSV}(\Gamma_i) \subseteq \mathcal{RV}((\Gamma_j, V_i))$ and $V_i \cap \mathcal{SV}_j(\Gamma) \subseteq \mathcal{CV}((\Gamma_j, V_i))$. Hence, $\xi_{i,2}$ can be decomposed into $\xi_{i,2} =^{V_i} \xi_{i,22} \circ \xi_{i,21}$ such that

1. $\xi_{i,21} = \mu_{i \mid \mathcal{SV}_j(\Gamma)}$,
2. $\xi_{i,22} \in NONLIN((\Gamma_j(\xi_{i,21} \circ \xi_{i,1}), V_i))$.

Moreover, we have $\xi_{i,21} \circ \xi_{i,1} = \mu_i \in ADM_{V_i}(\Gamma_i)$ and so $\Gamma_j(\xi_{i,21} \circ \xi_{i,1}) = \Gamma_j\mu_i$. Consequently,

$$\xi'_i =^{V_i} (\xi_{i,3} \circ \xi_{i,22}) \circ \mu_i$$

where $\xi_i = \xi_{i,3} \circ \xi_{i,22} \in ID_{\overline{\mathcal{RV}((\Gamma_j\mu_i, V_i))}} \circ NONLIN((\Gamma_j\mu_i, V_i))$. ∎

It is now pointed out why identifications satisfying Definition 11 are useless for combining solutions of matching combinable problems.

LEMMA 5. *Let* $<$ *be a linear ordering on* $V_1 \oplus V_2$ *and* ξ *an identification compatible with* $<$. *Then*

$$SU^{<}_{E_1}(\Gamma_1\xi_2 \wedge \hat{\xi}_1, V_2) \odot SU^{<}_{E_2}(\Gamma_2\xi_1 \wedge \hat{\xi}_2, V_1) \subseteq SU^{<}_{E_1}(\Gamma_1, V_2) \odot SU^{<}_{E_2}(\Gamma_2, V_1)$$

if $SU^{<}_{E_1}(\Gamma_1\xi_2, V_2) = SU^{<}_{E_1}(\Gamma_1, V_2)\xi_2$ *and* $SU^{<}_{E_2}(\Gamma_2\xi_1, V_1) = SU^{<}_{E_2}(\Gamma_2, V_1)\xi_1$.

Proof. Let $\sigma_1 \in SU^{<}_{E_1}(\Gamma_1\xi_2 \wedge \hat{\xi}_1, V_2) \subseteq SU^{<}_{E_1}(\Gamma_1\xi_2, V_2)$, $\sigma_2 \in SU^{<}_{E_2}(\Gamma_2\xi_1 \wedge \hat{\xi}_2, V_1) \subseteq SU^{<}_{E_1}(\Gamma_2\xi_1, V_1)$. By assumption, there exist $\phi_1 \in SU^{<}_{E_1}(\Gamma_1, V_1)$ and $\phi_2 \in SU^{<}_{E_2}(\Gamma_2, V_2)$ such that $\sigma_1 = \phi_1\xi_2$ and $\sigma_2 = \phi_2\xi_1$.

We prove by nœtherian induction on $<$ that

$$\sigma_1 \odot \sigma_2 = \phi_1 \odot \phi_2.$$

Let $z \in V_i$ be the minimal variable w.r.t. $<$. We have $z(\phi_1 \odot \phi_2) = z\phi_i = z\phi_i\xi_j$ since $\mathcal{D}om(\xi_j) \cap \mathcal{V}\mathcal{R}an(\phi_i) = \emptyset$. Then $z\phi_i\xi_j = z\sigma_i = z(\sigma_1 \odot \sigma_2)$.

Assume $y(\sigma_1 \odot \sigma_2) = y(\phi_1 \odot \phi_2)$ for $y < x$ and $x \in V_i$. By definition,

$$\begin{aligned} x(\sigma_1 \odot \sigma_2) &= (x\sigma_i)\{y_k \mapsto y_k(\sigma_1 \odot \sigma_2)\}_{k \in K} \\ &= (x\phi_i\xi_j)\{y_k \mapsto y_k(\sigma_1 \odot \sigma_2)\}_{k \in K}. \end{aligned}$$

Two identified variables have obviously the same solution:

$$\forall k, k' \in K, y_k\xi_j = y_{k'}\xi_j \Rightarrow y_k(\sigma_1 \odot \sigma_2) = y_{k'}(\sigma_1 \odot \sigma_2).$$

Then, it is not necessary to apply ξ_j before applying $\{y_k \mapsto y_k(\sigma_1 \odot \sigma_2)\}_{k \in K}$:

$$\begin{aligned} x(\sigma_1 \odot \sigma_2) &= (x\phi_i\xi_j)\{y_k \mapsto y_k(\sigma_1 \odot \sigma_2)\}_{k \in K} \\ &= x\phi_i\{y_k \mapsto y_k(\sigma_1 \odot \sigma_2)\}_{k \in K} \\ &= x\phi_i\{y_k \mapsto y_k(\phi_1 \odot \phi_2)\}_{k \in K} \\ &= x(\phi_1 \odot \phi_2). \quad \blacksquare \end{aligned}$$

We are now ready to summarize the different steps of the algorithm in the following theorem.

THEOREM 6 (`Match-Combi` Algorithm). *Let* Γ_1 *and* Γ_2 *be two pure unification problems such that* $\Gamma_1 \wedge \Gamma_2$ *is obtained from an* (E_1, E_2)*-extended matching problem* Γ *thanks to* `Right Purification`. *A* $CSU_{E_1 \cup E_2}(\Gamma)$ *is provided by the union of*

$$CSU^{<}_{E_1}(\Gamma_1\mu_2 \wedge \hat{\mu}_1, V_2) \odot CSU^{<}_{E_2}(\Gamma_2\mu_1 \wedge \hat{\mu}_2, V_1)$$

if $(\Gamma_1\mu_2 \wedge \hat{\mu}_1, V_2)$ *and* $(\Gamma_2\mu_1 \wedge \hat{\mu}_2, V_1)$ *w.r.t.* $<$ *are equivalent to matching combinable problems for each*

1. $V_1 \supseteq \mathcal{GSV}(\Gamma_1)$ *and* $V_2 \supseteq \mathcal{GSV}(\Gamma_2)$ *s.t.* $V_1 \oplus V_2 = \mathcal{V}(\Gamma_1 \wedge \Gamma_2) \backslash \mathcal{GV}(\Gamma)$,
2. *linear ordering* $<$ *on* $V_1 \oplus V_2$,
3. *identification* $\mu \in ID^{V_1 \cup V_2}_{V_1 \oplus V_2}$ *compatible with* $<$ *s.t.* $\mu_1 \in ADM_{V_1}(\Gamma_1)$ *and* $\mu_2 \in ADM_{V_2}(\Gamma_2)$.

Proof. The idea is to prove that a combined solution obtained through an arbitrary identification can be also retrieved thanks to an admissible one. Consider an identification $\xi' \in ID^{V_1 \cup V_2}_{V_1 \oplus V_2}$ compatible with $<$. According to Lemma 4,

$$SU^{<}_{E_1}(\Gamma_1\xi'_2 \wedge \hat{\xi}'_1, V_2) \odot SU^{<}_{E_2}(\Gamma_2\xi'_1 \wedge \hat{\xi}'_2, V_1)$$

is included in the union of

$$\begin{aligned} &SU^{<}_{E_1}((\Gamma_1\mu_2 \wedge \hat{\mu}_1)\,\xi_2 \wedge \hat{\xi}_1, V_2) \\ &\quad \odot SU^{<}_{E_2}((\Gamma_2\mu_1 \wedge \hat{\mu}_2)\,\xi_1 \wedge \hat{\xi}_2, V_1) \end{aligned}$$

for all $\mu_i \in ADM_{V_i}(\Gamma_i)$, $\xi_i \in ID_{\overline{\mathcal{RV}((\Gamma_j\mu_i, V_i))}} \circ NONLIN((\Gamma_j\mu_i, V_i))$ such that $\xi'_i =^{V_i} \mu_i\xi_i$. Unification problems $(\Gamma_1\mu_2 \wedge \hat{\mu}_1, V_2)$ and $(\Gamma_2\mu_1 \wedge \hat{\mu}_2, V_1)$ are equivalent to extended matching problems thanks to Lemma 3. These problems are moreover assumed matching combinable. By definition, we have

$$\begin{aligned} SU^{<}_{E_1}((\Gamma_1\mu_2 \wedge \hat{\mu}_1)\,\xi_2, V_2) &= SU^{<}_{E_1}(\Gamma_1\mu_2 \wedge \hat{\mu}_1, V_2)\,\xi_2, \\ SU^{<}_{E_2}((\Gamma_2\mu_1 \wedge \hat{\mu}_2)\,\xi_1, V_1) &= SU^{<}_{E_2}(\Gamma_2\mu_1 \wedge \hat{\mu}_2, V_1)\,\xi_1, \end{aligned}$$

and so

$$\begin{aligned} &SU^{<}_{E_1}((\Gamma_1\mu_2 \wedge \hat{\mu}_1)\,\xi_2 \wedge \hat{\xi}_1, V_2) \\ &\quad \odot SU^{<}_{E_2}((\Gamma_2\mu_1 \wedge \hat{\mu}_2)\,\xi_1 \wedge \hat{\xi}_2, V_1) \end{aligned}$$

is included in

$$SU^{<}_{E_1}(\Gamma_1\mu_2 \wedge \hat{\mu}_1, V_2) \odot SU^{<}_{E_2}(\Gamma_2\mu_1 \wedge \hat{\mu}_2, V_1),$$

according to Lemma 5. Therefore,

$$SU^{<}_{E_1}(\Gamma_1\xi'_2 \wedge \hat{\xi}'_1, V_2) \odot SU^{<}_{E_2}(\Gamma_2\xi'_1 \wedge \hat{\xi}'_2, V_1)$$

is included in the union of

$$SU^{<}_{E_1}(\Gamma_1\mu_2 \wedge \hat{\mu}_1, V_2) \odot SU^{<}_{E_2}(\Gamma_2\mu_1 \wedge \hat{\mu}_2, V_1)$$

for all $\mu_i \in ADM_{V_i}(\Gamma_i)$. $\blacksquare$

The reader is invited to compare Theorem 2 with the previous one. Improvements lie in the non-deterministic choice of a theory for each variable and in the non-deterministic choice of an identification.

EXAMPLE 9. Let us consider again the Boolean theory B built over the signature $\{\neg, +, \cdot, 0, 1\}$, f a free symbol and g a commutative symbol.

1. Solving $x+f(x) \leqslant^? 1$. After `Purification` Step, we have $x+c \leqslant^? 1 \wedge c =^? f(x)$. We may assume w.l.o.g. that c is instantiated in the empty theory since it is collapse-free. $x+c \leqslant^? 1$ is equivalent in Booleans to $x =^? \neg c + c \cdot y$. There is a compound cycle. We have to choose $x < c$ to break it. This leads to $x =^? 1 \wedge y =^? \neg c \cdot z + c$. Hence $\{x \mapsto 1\}$ is a solution. It is a complete set of solutions since the unique solvable unification problem is matching combinable: if x is also skolemized in Booleans then it yields no solution whether x and c are identified or not.

2. Solving $x + g(y, z) \leqslant^? g(a, b) + g(b, a)$. After `Purification` Step, we get $x + c_1 \leqslant^? c_2 \wedge c_1 =^? g(y, z) \wedge c_2 =^? g(a, b)$. The unique way which leads to a solution is to identify c_1 and c_2. In B, the match-equation $x + c_1 \leqslant^? c_1$ is solved: $x =^? c_1 \cdot v$. In the commutative theory, we have to solve $g(y, z) \leqslant^? g(a, b)$. Finally, there are two solutions:

$$\{x \mapsto g(a, b) \cdot v, y \mapsto a, z \mapsto b\}$$
$$\{x \mapsto g(b, a) \cdot v, y \mapsto b, z \mapsto a\}.$$

These solutions define a complete set of solutions: the unique solvable unification problem is obviously matching combinable.

6. MODULARITY RESULTS ON MATCHING

The combined matching algorithm seen in the previous section is complete for all theories for which extended matching problems are matching combinable, i.e. partially linear theories.

THEOREM 7. *If E_1 and E_2 are two partially linear theories then $E_1 \cup E_2$-extended matching is decidable (resp. finitary, solvable) if E_i-extended matching with linear constant restriction $(i = 1, 2)$ is decidable (resp. finitary, solvable).*

This theorem can be easily lifted since the combined matching algorithm is able to solve freely extended matching.

THEOREM 8. *If E_1 and E_2 are two partially linear theories then $E_1 \cup E_2$-freely extended matching is decidable (resp. finitary, solvable) if and only if E_i-freely extended matching $(i = 1, 2)$ is decidable (resp. finitary, solvable).*

Proof. The combined matching problem can be applied by a straightforward generalization to the union of three partially linear theories E_1, E_2 and $\varnothing$. Then, solvable problems after `Merging` are matching combinable since variables instantiated in $\varnothing$ cannot be identified. ∎

This result is now applied to the combination of a partially linear theory with the empty theory $\varnothing$ which is also a partially linear theory. The second point comes from Proposition 9.

COROLLARY 4.

- *if E is a partially linear theory then $E \cup \varnothing$-freely extended matching is decidable (resp. finitary, solvable) iff E-freely extended matching is decidable (resp. finitary, solvable).*
- *if E is not a partially linear theory then $E \cup \varnothing$-freely extended matching is decidable (resp. finitary, solvable) iff $E \cup \varnothing$-unification is decidable (resp. finitary, solvable).*

Theorem 8 allows solving some $E_1 \cup E_2$-unification problems involving free symbols which are more general than $E_1 \cup E_2$-matching problems. If we restrict to this last kind of problems, we can get rid of the linear constant restriction since solving matching problems cannot create a compound cycle and partially linear theories generalize in this sense regular theories.

LEMMA 6. *Let E be a partially linear theory and Γ a matching problem. Then, there exists a $CSU_E(\Gamma)$ such that*

$$\forall \sigma \in CSU_E(\Gamma)\ \forall x \in \mathscr{D}om(\sigma)\ \forall c \in \overline{\mathscr{RV}}(\Gamma), \qquad c \notin \mathscr{V}(x\sigma).$$

Proof. Let σ be a solution of the matching problem Γ and σ' be the substitution obtained from σ by renaming each skolemized variable $c \in \mathscr{VR}an(\sigma)$ into $c' \notin \mathscr{GV}(\Gamma)$. Hence, terms in $\mathscr{R}an(\sigma')$ do not contain any skolemized variable in $\overline{\mathscr{RV}}(\Gamma)$. For any match-equation $s \leqslant^?_E t$ in Γ, we have $s\sigma = t$ and three possibilities:

- If there are many occurrences of $c \in \overline{\mathscr{RV}}(\Gamma)$ in $s\sigma$, then $s\sigma' =_E t$ since E is partially linear.
- If there is one and only one occurrence of $c \in \overline{\mathscr{RV}}(\Gamma)$ in $s\sigma$, then $s\sigma' =_E t$ is a renaming of $s\sigma =_E t$.
- Otherwise, $s\sigma' = s\sigma =_E t$.

Consequently, $\sigma' \in SU_E(\Gamma)$. Moreover, if $\sigma \leqslant_E^{\mathscr{V}(\Gamma)} \phi$ then $\sigma' \leqslant_E^{\mathscr{V}(\Gamma)} \phi$ and so $\sigma \in CSU_E(\Gamma)$ implies $\sigma' \in CSU_E(\Gamma)$. ∎

The choice of a linear ordering is thus useless for partially linear theories.

THEOREM 9. *If E_1 and E_2 are two partially linear theories then $E_1 \cup E_2$-matching is decidable (resp. finitary, solvable) if and only if E_i-matching $(i = 1, 2)$ is decidable (resp. finitary, solvable).*

Proof. Consider a conjunction of pure extended matching problems $(\Gamma_1 \wedge \hat{\sigma}_1, V_2)$ and $(\Gamma_2 \wedge \hat{\sigma}_2, V_1)$ obtained after `Merging`. The unification problem $\hat{\sigma}_1 \wedge \hat{\sigma}_2$ is necessarily in dag solved form thanks to performed identifications. According to Lemma 6, for each $\phi_i \in CSU_{E_i}(\Gamma_i, V_j)$, the range of ϕ_i does not contain a variable in the domain of σ_j for $i \neq j$. Therefore, solving

(Γ_1, V_2) and (Γ_2, V_1) does not introduce a compound cycle. We have that

$$\bigcup_{\phi_i \in CSU_{E_i}(\Gamma_i, V_j)} CSU_{E_i}(\hat{\phi}_i \wedge \hat{\sigma}_i, V_j)$$

is a $CSU_{E_i}(\Gamma_i \wedge \hat{\sigma}_i, V_j)$ where $(\hat{\phi}_1 \wedge \hat{\sigma}_1) \wedge (\hat{\phi}_2 \wedge \hat{\sigma}_2)$ is in dag solved form for each $\phi_1 \in CSU_{E_1}(\Gamma_1, V_2)$ and $\phi_2 \in CSU_{E_2}(\Gamma_2, V_1)$. So, there exist a substitution σ and a linear ordering $<$ such that $\{\sigma\}$ is a complete set of combined solutions of $(\hat{\phi}_1 \wedge \hat{\sigma}_1) \wedge (\hat{\phi}_2 \wedge \hat{\sigma}_2)$ w.r.t. $<$. Combined solutions w.r.t. other linear orderings than $<$ are obviously $E_1 \cup E_2$-instances of σ. ∎

This result extends Theorem 5 and can be applied to the combination of a partially linear theory with the empty theory.

COROLLARY 5. *If E is a partially linear theory then E-matching with free symbols is decidable* (*resp. finitary, solvable*) *iff E-matching is decidable* (*resp. finitary, solvable*).

Recall that partially linear theories include regular collapse-free theories and linear theories. A previous result was already known for all regular theories (Nipkow, 1991) and not only collapse-free ones but it cannot be used for combining decision algorithms.

EXAMPLE 10. Assume $E_1 = \{f(x, y) = y\}$ and $E_2 = \{x \star y = y \star x\}$ the commutativity. The heterogeneous match-equation $f(y \star y, y \star z) \leqslant^? a \star b$ is equivalent to $(f(v, w) \leqslant^? c) \wedge (v =^? y \star y \wedge w =^? y \star z \wedge c \leqslant^? a \star b)$ by purification, where variables v, w, c are skolemized in E_1 since E_2 is collapse-free. The identification $\{v \mapsto c\}$ leads to a failure in E_1. Nevertheless, the match-equation $f(v, w) \leqslant^? c$ becomes true with the identification $\{w \mapsto c\}$. This identification leads to $y \star z \leqslant^? a \star b$ in E_2, which is then solved and yields a complete set of solutions: $\{y \mapsto a, z \mapsto b\}$ and $\{y \mapsto b, z \mapsto a\}$.

7. CONCLUSION

We have considered two different issues of the combination problem for matching. First, we have shown how to combine matching algorithms with linear constant restriction in order to solve only conjunction of left pure match-equations. Then, the general case has been solved for partially linear theories which include linear theories and regular collapse-free theories. In this context, combining extended matching problems involving free symbols needs linear constant restriction, whilst the linear constant restriction is useless for combining matching problems.

There is no hope to extend this result to non partially linear theories. Corollary 4 states this fact. Let $E_1, ..., E_n$ be n theories including the empty theory and a non partially linear one. Then, $E_1 \cup \cdots \cup E_n$-freely extended matching is equivalent to $E_1 \cup \cdots \cup E_n$-unification and the combined unification algorithm given in (Baader and Schulz, 1992) can be used as well. One may argue that extended matching is sometimes a too strong extension of matching. However, this notion seems to be crucial for solving the combined matching problem in a modular way by using the purification paradigm. Another solution would be to decompose directly a heterogeneous matching problem into a conjunction of pure matching problems without introducing new solved equations. But this seems to be possible only for regular theories.

In this paper, we have only studied the case where signatures of equational theories are disjoint. A more general problem could be: Is it possible to reuse combination techniques for non-disjoint equational theories? In (Domenjoud *et al.*, 1994), we extend the result due to Nipkow (1991) about combination of matching algorithms in regular theories to the case where shared symbols satisfy an appropriate notion of constructors. By applying techniques introduced in (Ringeissen, 1992; Kirchner and Ringeissen, 1994), we are also able to combine non-disjoint partially linear theories provided that shared symbols are only constants like for instance in the union $E_1 \cup E_2$ where $E_1 = AC(+) \cup \{x + 0 = x\}$ and $E_2 = AC(*) \cup \{x * 0 = 0\}$. Further works on the non-disjoint case are still necessary in order to allow more than shared constants for some specific partially linear theories.

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