In JOURNAL OF ALGORITHMS, 1995.

On Finding Minimal Two-Connected Subgraphs *

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July 30, 1992

Abstract

We present efficient parallel algorithms for the problems of finding a minimal 2-edge-connected spanning subgraph of a 2-edge-connected graph and finding a minimal biconnected spanning subgraph of a biconnected graph. The parallel algorithms run in polylog time using a linear number of PRAM processors. We also give linear time sequential algorithms for minimally augmenting a spanning tree into a 2-edge-connected or biconnected graph.

1 Introduction

In this paper we consider the following two related problems: given a 2-edge-connected (biconnected) graph G, compute a minimal 2-edge-connected (biconnected) spanning subgraph of G, i.e., a 2-edge-connected (biconnected) subgraph in which the deletion of any edge destroys 2-edge-connectivity (biconnectivity). We present efficient parallel algorithms for these problems.

It is known that the corresponding problems of finding *minimum* spanning subgraphs with these properties are NP-hard ([6]). Thus, it is natural to study the simpler problem of computing *minimal* spanning subgraphs with respect to these properties. The problems considered here have applications in the context of network reliability.

Our interest in the parallel complexity of the above problems is prompted by the fact that both problems admit very simple sequential algorithms that seem hard to parallelize. Thus, new techniques are required to obtain an efficient parallel solution for these problems. Several other well-known problems share this property, e.g., the problem of computing a maximal independent set in a graph and the problem of computing a depth-first search tree in a graph (see [11]). To illustrate this, consider the problem of finding a minimal 2-edge-connected spanning subgraph. Let the input graph G have n vertices and m edges. We assume that G is 2-edge-connected. The obvious sequential algorithm finds a minimal 2-edge-connected spanning subgraph of G by examining the edges of G one at a time and removing an edge if the resulting graph is a 2-edge-connected spanning subgraph

*Work supported in part by NSF Grant CCR-89-10707.

of G. The total time is dominated by m calls to the algorithm for testing 2-edge-connectivity ([21], [14], [13], [18]), giving a time bound of $O(m(n+m))$. The time can be brought down to $O(m+n^2)$ by first finding a sparse 2-edge-connected spanning subgraph of G (see section 3). There is a similar sequential algorithm with the same time bound for finding a minimal biconnected spanning subgraph of G. None of these algorithms lends itself to an efficient parallel implementation.

In this paper we present fast parallel algorithms for these problems: they both run in time $O(\log^3 n)$ with a number of CRCW processors close to $(n+m)/\log n$ (for the exact bound, see end of section 3). These algorithms are the first efficient parallel algorithms for these problems. In the outer loop of both parallel algorithms we find a spanning tree in the current subgraph of the input graph that contains the smallest possible number of *redundant edges* (i.e., edges that can be removed without destroying the desired property - 2-edge-connectivity or biconnectivity). A similar step is used in each iteration of an algorithm of [7] to compute a minimal strongly connected spanning subgraph of a strongly connected digraph (*transitive compaction problem*). We augment such a spanning tree with a minimal set of edges restoring the desired property (2-edge-connectivity or biconnectivity). We use tree contraction to construct such a minimal augmentation. This part of our algorithm markedly differs from the corresponding step in the transitive compaction algorithm. As in [7] we show that $O(\log n)$ iterations of this procedure yield the desired spanning subgraph.

We also give linear time algorithms for minimally augmenting a spanning tree into a 2-edge-connected or biconnected graph. If used in the obvious way these procedures yield sequential algorithms for both problems that run in time $O(m+n\log n)$. In recent work ([9]) the basic algorithms have been refined into linear time algorithms for finding a minimal 2-edge-connected spanning subgraph of a 2-edge-connected graph and finding a minimal biconnected spanning subgraph of a biconnected graph. These algorithms use the linear time minimal augmentation procedures described in this paper as subroutines. Similar linear time augmentation procedures have been found independently by Han and Tarjan ([8]).

Our paper is organized as follows. In the next section we introduce the graph-theoretic terminology. In section 3 we present the parallel algorithm for finding a minimal 2-edge-connected spanning subgraph and in section 4 we describe our parallel algorithm for finding a minimal biconnected spanning subgraph. In section 5 we present linear time sequential algorithms for minimally augmenting spanning trees with respect to these properties. In section 6 we summarize our results and mention some related results.

2 Definitions

A *graph* $G = (V, E)$ consists of a set V of *vertices* and a set E of *edges*. We sometimes write $V(G)$ and $E(G)$ for the set of vertices and edges, respectively, and write $n(G)$ and $m(G)$, respectively, for their cardinalities. If the edges are ordered pairs (v, w) of distinct vertices, the graph is *directed*: v is called the *tail* and w is called the *head* of the edge. We also use the term *digraph* for a directed graph. The *indegree* (*outdegree*) of a vertex in a digraph is the number of edges in the digraph whose head (tail) is this node. If the edges are unordered pairs of distinct vertices, also denoted by (v, w), the graph is *undirected*. In that case v and w are *incident* with the edge (v, w). If $v = w$, then the edge (v, w) is a *self-loop*. The *degree* of a vertex is the number of edges incident with it. If E is a multiset, i.e., if edges may have multiple copies, then G is a *multigraph*.

Fix a multigraph $G = (V, E)$. A *path* P in G is a sequence $< v_0 \ldots v_k >$ of vertices of V such that (v_{i-1}, v_i) $(1 \leq i \leq k)$ is an edge in E; the nodes v_0 and v_k are the *endpoints* of P and $v_1 \ldots v_{k-1}$ are the *internal vertices* of P. We say that v_k is *reachable* from v_0. The vertices v_i *lie on the path* P and the edges (v_{i-1}, v_i) are the *edges on the path* P. The *length* of a path is the number of edges on the path. A path is *simple* if $v_0 \ldots v_{k-1}$ are distinct and $v_1 \ldots v_k$ are distinct. A simple path is a *chain* if all of its internal vertices have degree 2 in the graph. A *cycle* in G is a path in G whose endpoints coincide and all of whose edges are distinct. A *simple cycle* is a simple path whose endpoints coincide.

If $G = (V, E)$ and $G' = (V', E')$ are two graphs such that $V' \subseteq V$ and $E' \subseteq E$, then G' is a *subgraph* of G. A subgraph G' of G is a *proper subgraph* of G if it is different from G. A subgraph G' of G having some property is a *maximal* subgraph with this property if it is not a proper subgraph of another subgraph of G with this property. G' is the subgraph of G *induced by the vertices* in V' if E' contains exactly the edges of E between vertices of V'. G' is the subgraph of G *induced by the edges* in E' if $V' = V$. A *spanning subgraph* of G is a subgraph G' with $V' = V$. If G' is a spanning subgraph of G and $E'' \subseteq E$, then $G' + E''$ denotes the graph with vertex set $V'(= V)$ and edge set $E' \cup E''$ and $G' - E''$ denotes the graph with vertex set V' and edge set $E' - E''$.

An undirected graph G is *connected* if every vertex in G is reachable from any other vertex in G. A *connected component* in G is a maximal connected subgraph of G (i.e., it is connected and it is not a proper subgraph of a connected subgraph of G).

A *tree* is a connected (undirected) graph without cycles. A *subtree* of a tree is a subgraph of a tree that is a tree. If T is a tree and u and v are two vertices in T, then the graph $T + (u, v)$ contains a unique (simple) cycle called the *fundamental cycle* of (u, v) in T. A *forest* is a graph without cycles or, equivalently, it is a graph whose connected components are trees. A *leaf* in a forest is a vertex of degree at most 1 in the forest.

A *rooted tree* is a directed graph whose undirected version is a tree, having one vertex, called the *root*, which is the head of no edges, and such that all vertices except the root are the head of exactly one edge. If (v, w) is an edge in the rooted tree, v is the *parent* of w and w is the *child* of v. A *leaf* in the rooted tree is a vertex in the tree that does not have a child. A *descendant* of v is any node reachable from v in the tree (including v). A vertex w is an *ancestor* of v if v is a descendant of w in the tree. A *proper* descendant (ancestor) of v is a descendant (ancestor) of v other than v. The *depth* of a node in a rooted tree is the number of edges on the (unique) simple path from the root to the node in the tree. The *least common ancestor* (lca) of two vertices in a rooted tree is the vertex at maximum depth that is an ancestor of both vertices.

A *spanning tree* of a graph G is a spanning subgraph of G that is a tree. The *tree edges* in G are the edges in G that belong to a given spanning tree; all other edges in G are *nontree edges*. If each edge in G has a real *weight*, the weight of a spanning tree of G is the sum of the weights of its edges. A *minimum spanning tree* in G (with respect to these weights) is a spanning tree in G with minimum weight.

Let $G = (V, E)$ be a graph and let $V_1, \ldots, V_k$ be disjoint nonempty subsets of vertices in G. Let $v_1, \ldots, v_k$ be k new vertices (that do not belong to V). Define a mapping f from V to $V \cup \{v_1, \ldots, v_k\}$ by $f(v) = v_i$ if $v \in V_i$ $(1 \leq i \leq k)$ and $f(v) = v$ if v does not belong to any V_i. The operation of *collapsing the vertices in* $V_1, \ldots, V_k$ consists in replacing the vertices in each V_i by v_i, replacing each edge (v, w) in G by the edge $(f(v), f(w))$ and deleting all self-loops in the resulting graph. Let G' denote the resulting graph. The graph G' is a multigraph. It will often be convenient to *identify* each edge (v, w) in G where $f(v) \neq f(w)$ with a unique edge in G' linking the vertices $f(v)$ and $f(w)$. In these cases we may pick an arbitrary one-to-one mapping between the edges in G' connecting any two distinct vertices x and y and the edges (v, w) in G with $f(v) = x$ and $f(w) = y$ and identify an edge (x, y) with the edge in G to which it is mapped under this one-to-one mapping.

A graph is *k-edge-connected* $(k \geq 1)$ if the removal of at most $k-1$ edges does not disconnect G. An equivalent condition for a graph to be 2-edge-connected is that every edge lies on a cycle. A *2-edge-connected component* of G is a maximal 2-edge-connected subgraph of G (i.e., a 2-edge-connected component is not a proper subgraph of a 2-edge-connected subgraph of G). A *cutedge* in G is an edge in G whose removal disconnects G. Thus, a graph is 2-edge-connected if it is connected and does not contain a cutedge.

A graph is k-vertex-connected $(k \geq 1)$ if at least k vertices have to be removed from the graph in order to disconnect it or reduce it to a single vertex. A vertex in G is a *cutpoint* if removing it together with all incident edges yields a graph that is not connected. A graph G is *biconnected* (or 2-vertex-connected) if it has at least 3 vertices and does not contain a cutpoint. A *block* (or a *biconnected component*) of a graph G is either an isolated vertex in G, or a cutedge in G, or a

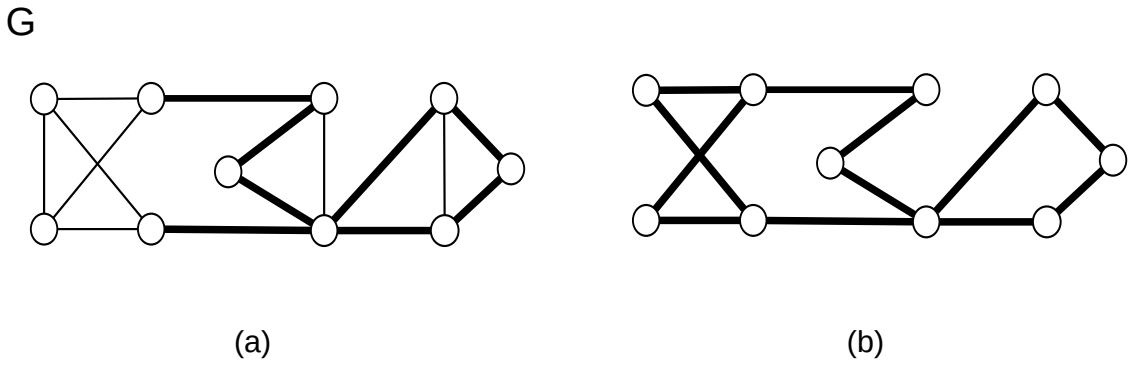


Figure 1: (a) A 2-edge-connected graph G and (b) a minimal 2-edge-connected spanning subgraph of G. Essential and redundant edges are indicated by thick and thin lines, respectively.

maximal biconnected subgraph of G.

An *ear decomposition* $D = [P_0, P_1, \ldots, P_{r-1}]$ of an undirected graph $G = (V, E)$ is a partition of E into an ordered collection of edge disjoint simple paths $P_0, \ldots, P_{r-1}$ such that P_0 is an edge, $P_0 \cup P_1$ is a simple cycle, and each endpoint of P_i, for $i > 1$, is contained in some P_j, $j < i$, and none of the internal vertices of P_i are contained in any P_j, $j < i$. The paths in D are called *ears*. D is an *open ear decomposition* if none of the P_i is a simple cycle. A *trivial ear* is an ear other than P_0 containing a single edge.

3 Finding a Minimal 2-Edge-Connected Spanning Subgraph

In this section we consider the following problem: given a 2-edge-connected graph G, find a minimal 2-edge-connected spanning subgraph of G, i.e., a 2-edge-connected spanning subgraph of G that does not have a 2-edge-connected spanning subgraph of G as a proper subgraph. Figure 1 shows a 2-edge-connected graph G and a minimal 2-edge-connected spanning subgraph of G.

3.1 The High-Level Algorithm

Our parallel algorithm for computing a minimal 2-edge-connected spanning subgraph makes use of a partition of the edges of a 2-edge-connected graph into two classes: those that can be removed without destroying 2-edge-connectivity and those whose removal destroys 2-edge-connectivity. Formally, for an arbitrary 2-edge-connected graph H, we say that an edge e is *redundant in* H if $H - e$ is 2-edge-connected; edges that are not redundant in H are *essential in* H. We sometimes shall not mention the graph H if it is clear from the context. Note that an edge that is essential in H is also

essential in any 2-edge-connected spanning subgraph of H. Similarly an edge that is redundant in a 2-edge-connected spanning subgraph of H is also redundant in H. In figure 1 essential and redundant edges are indicated by thick and thin lines, respectively. Since the graph in figure 1(b) is a minimal 2-edge-connected spanning subgraph of the graph in figure 1(a), all of its edges are essential.

Let $G = (V, E)$ be a 2-edge-connected graph with n vertices and m edges. We present our algorithm for finding a minimal 2-edge-connected spanning subgraph of G in a top-down fashion. At the highest level it has the following structure:

Algorithm 1 Finding a minimal 2-edge-connected spanning subgraph of G.
Input 2-edge-connected graph G.
Output Minimal 2-edge-connected spanning subgraph H of G.

(0) Find a 2-edge-connected spanning subgraph H of G with fewer than $2n$ edges.

(1) While H contains redundant edges, repeat the following two steps:

(1.1) Find a spanning tree in H that contains the smallest possible number of redundant edges in H. Call this tree T_H.

(1.2) Determine a minimal subset B of edges in H such that the graph $T_H + B$ is 2-edge-connected. Let $H = T_H + B$.

The purpose of step (0) is to speed up subsequent iterations of the while-loop by computing a sparse subgraph of the input graph. In this step we compute an ear decomposition of G ([14], [13], [18]) and eliminate all trivial ears. Let H be the resulting graph. H is clearly a 2-edge-connected spanning subgraph of G. Let m' denote the number of edges of H and let q be the number of (nontrivial) ears in the above ear decomposition. A proof by induction over q establishes $m' = n + q - 2$. Since H contains no trivial ears, we have $q \leq n - 1$; hence, $m' \leq 2n - 3$ as required in step (0).

For steps (1) and (1.1) we need to determine the redundant edges in H. A *separating pair of edges for* H is a pair (e_1, e_2) of edges of H such that the graph $H - e_1 - e_2$ is not connected. Note that an edge of H is redundant if and only if it does not occur in any separating pair of edges for H. Hence, we identify the redundant edges in H by finding all separating pairs of edges in H. For this we modify the vertex triconnectivity algorithm in [3], [18] (see also [19]). We now assign weight 0 to essential edges and weight 1 to redundant edges and choose for T_H a minimum spanning tree in this graph. The implementation of step (1.2) will be discussed in the next section.

Note: An alternative (sequential) method developed independently by Han and Tarjan ([8]) for computing a minimal 2-edge-connected spanning subgraph can be parallelized; the parallel imple-

mentation is similar to our method but it does not require redundant and essential edges to be computed explicitly.

The correctness of algorithm 1 follows from these two observations: (1) at the start of any iteration of the while-loop (step (1) of algorithm 1) the graph H is a spanning 2-edge-connected subgraph of G; (2) the number of edges in H strictly decreases in each iteration of the while-loop.

The following lemma implies that the number of redundant edges in H decreases quite rapidly. This results in a $O(\log n)$ upper bound on the number of iterations of algorithm 1.

Lemma 1 *There is a spanning tree in H that contains at most $2/3$ of the redundant edges in H.*

Proof. Let Ess denote the set of essential edges in H. Let r denote the number of redundant edges in H and c the number of connected components of the graph $(V(H), Ess)$. Fix a spanning tree T in H with the smallest possible number of redundant edges of H. The tree T contains exactly $c-1$ redundant edges of H. Furthermore, if $c > 1$, then each connected component of $(V(H), Ess)$ is incident with at least 3 redundant edges in H and hence, $c \leq 2/3 \cdot r$. The claim follows. □

Corollary 1 *Algorithm 1 terminates after $O(\log n)$ iterations of the while-loop.*

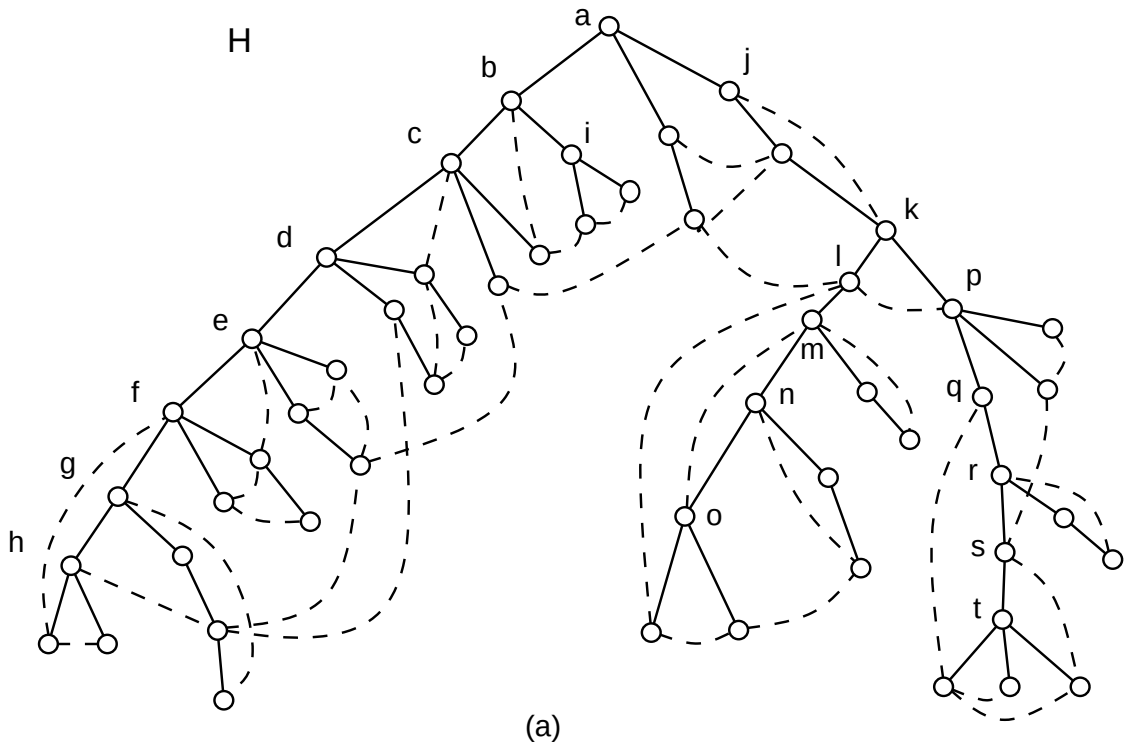
Proof. Let H' and H'' represent the graph H at the start of two consecutive iterations of the while-loop. Note that all edges in H'' added to the spanning tree $T_{H'}$ in step (1.2) of the current iteration are essential in H''. Thus, any redundant edge in H'' is redundant in H' and belongs to $T_{H'}$. The previous lemma then implies that the number of redundant edges in H'' is at most $2/3$ times the number of redundant edges in H'. The claim follows. □

The main work in algorithm 1 is done in step (1.2) in which we minimally augment the spanning tree T_H into a 2-edge-connected graph. We call the (minimal) set B of edges in H that will be added to T_H a *minimal augmentation* for T_H in H. In the next section we describe how such a minimal augmentation can be computed efficiently in parallel.

3.2 Computing a Minimal Augmentation in Parallel

For this section we fix an iteration of the while-loop of algorithm 1. We describe how to compute a minimal augmentation for the tree T_H computed in step (1.1) of the current iteration.

Our method does not use the fact that T_H contains a minimum number of redundant edges in H. We may therefore assume that T_H is an arbitrary spanning tree in H. We shall illustrate our minimal augmentation procedure with the example in figure 2(a). Figure 2(a) shows a graph H with a spanning tree T_H; the edges of T_H are indicated by solid lines and the nontree edges are



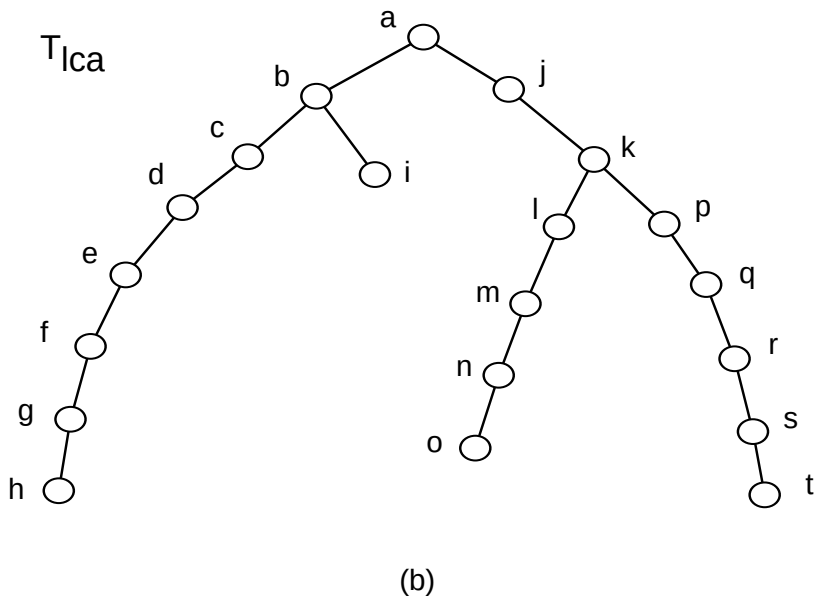


Figure 2: (a) Graph H with the edges of T_H (rooted at a) indicated by solid lines and the nontree edges indicated by dashed lines. Vertices with a letter are lca's of nontree edges. (b) The corresponding tree T_{lca}.

indicated by dashed lines. We shall make use of *parallel tree contraction* to compute such a minimal augmentation. Tree contraction was first introduced by Miller and Reif ([15]). We use a variant of tree contraction proposed in [17, 16]. This method is based on the operation *Shrink* which we shall now describe.

Let T be an arbitrary rooted tree. A *leaf chain* in T is a sequence $< v_0, v_1, \ldots, v_k >$ of vertices in T such that v_k is a leaf, v_i is the unique child of v_{i-1} for $i > 0$ and the parent v of v_0 is either the root of T or has at least 2 children. For example consider the tree in figure 2(b). This tree has one leaf chain with 6 nodes ($< c, d, e, f, g, h >$), one leaf chain with 5 nodes ($< p, q, r, s, t >$), one leaf chain with 4 nodes ($< l, m, n, o >$) and one leaf chain with 1 node ($< i >$).

The operation Shrink reduces a tree by removing the vertices in the leaf chains. Tree contraction reduces an arbitrary rooted tree to a single vertex by repeatedly applying the Shrink operation to it. It can be shown ([17, 16]) that $O(\log n)$ applications of Shrink contract any tree on n nodes to a single vertex. Applying Shrink to the tree in figure 2(b) removes vertices $c \ldots h, i, l \ldots o, p \ldots t$ and applying it one more time removes vertices b, j, k, thus leaving a tree with the single vertex a.

In the sequel we assume that the spanning tree T_H, computed in step (1.1) of algorithm 1, is rooted at an arbitrary vertex. The tree contraction will be performed on a tree T_{lca} closely related to the tree T_H. The tree is defined as follows: the vertices of T_{lca} are the least common ancestors (*lca's*) in T_H of nontree edges in H (i.e., the lca's of the endpoints of those edges). The root of T_{lca} is the root of T_H. A vertex v is a child of u in T_{lca} if and only if v is a descendant of u in T_H and none of the internal vertices on the path from u to v in T_H is an lca of a nontree edge. For instance, consider the graph H depicted in figure 2(a) with the spanning tree T_H denoted by the solid edges. The corresponding tree T_{lca} is the tree in figure 2(b).

Let F denote the set of nontree edges in H, i.e., those edges in H that do not belong to the spanning tree T_H. Let T denote the tree T_{lca} at the current stage of tree contraction. The algorithm for computing a minimal augmentation for T_H works in stages. In each stage the algorithm maintains the following three sets of nontree edges (subsets of F): the set IN of edges that have been processed at an earlier stage and have been chosen for the minimal augmentation; the set C of those nontree edges whose lca lies on a leaf chain of T; and the set R of nontree edges that are still to be processed. At the current stage we add to the augmentation a subset A of C that is minimal with respect to the property that $T_H + IN + R + A$ is 2-edge-connected. Thus, we add a subset of edges from C to the augmentation while taking into account those edges that have already been added as well as those that will be examined at later stages. This will ensure that the resulting set of edges for the augmentation is indeed minimal. This will be proven formally in lemma 2.

For the example in figure 2(a) nontree edges with an lca other than a, b, j, k will be processed at the first stage of our algorithm (algorithm 2), those whose lca is b, k or j are processed at the second

stage, and those with lca a are processed at the third and last stage.

For technical reasons we view the tree consisting of a single vertex as having a leaf chain with no root vertex; one more iteration of tree contraction will yield the empty tree. The fine-structure of step (2.2) will be developed following lemma 2. Note that a stage as explained earlier corresponds to an iteration of the while-loop.

Algorithm 2 Finding a minimal augmentation for T_H.
Input Tree T_H, set F of nontree edges.
Output Minimal augmentation IN for T_H from edges in F.

(1) Initialize : $IN := \emptyset$, $R := F$, and $T := T_{lca}$.

(2) While T is non-empty, perform the following actions for all leaf chains of T in parallel:

(2.1) Let C be the subset of edges in R whose lca is a vertex on some leaf chain of T. Let $R := R - C$.

(2.2) Find a minimal subset $A \subseteq C$ such that $T_H + IN + R + A$ is 2-edge-connected.

(2.3) Let $IN := IN \cup A$.

(2.4) Remove the leaf chains from T.

Lemma 2 *Upon completion of algorithm 2, IN is a minimal augmentation for T_H.*

Proof. We claim that the following statement holds before any iteration of step (2):

(*) $T_H + IN + R$ is 2-edge-connected and every edge of IN is essential in $T_H + IN + R$.

Upon termination of algorithm 2, R is empty and (*) implies that IN is a minimal augmentation for T_H.

We prove that (*) holds before each iteration of step (2) by induction on the iteration number. Note that (*) holds before the first iteration where $R = F$ and $IN = \emptyset$ since $H = T_H + F$ is 2-edge-connected. Assume that (*) holds before the ith iteration of step (2) ($i \geq 1$). Denote by IN_j and R_j the sets IN and R, respectively, before the jth iteration for any $j \geq 1$. By step (2.2) in the ith iteration $T_H + IN_i + R_{i+1} + A$ is 2-edge-connected, and hence, by step (2.3), $T_H + IN_{i+1} + R_{i+1}$ is 2-edge-connected. Note that $IN_{i+1} \cup R_{i+1} \subseteq IN_i \cup R_i$. Thus, by the inductive assumption , the edges of IN_i are essential in $T_H + IN_{i+1} + R_{i+1}$. By the minimality of the set A constructed in step (2.2) each edge of $IN_{i+1} - IN_i$ is essential in $T_H + IN_{i+1} + R_{i+1}$. This shows that (*) holds before iteration $i + 1$ of step (2). □

In the sequel we consider a fixed iteration of the while-loop in algorithm 2. We have reduced the problem of computing a minimal augmentation for T_H to the problem of computing a minimal augmentation for $T_H + IN + R$ from the edges in C, i.e., a minimal set $A \subseteq C$ such that $T_H + IN + R + A$ is 2-edge-connected. At this point it is not clear that this problem is easier than the original problem of computing a minimal augmentation for T_H. We shall see, however, that the special structure of C, i.e., the fact that all edges in C have their lca on a leaf chain of T, allows for a fast parallel solution of the modified augmentation problem. Intuitively, this is because nontree edges whose lca's lie on different leaf chains can be processed independently.

It is helpful to view the minimal augmentation problem in terms of edges in the spanning tree T_H being *covered* by nontree edges. To make this precise, note that any nontree edge e in H produces a unique cycle in $T_H + e$, the *fundamental cycle* of e. We say that edge e *covers* the edges of T_H that lie on its fundamental cycle. We say that a set D of nontree edges covers a set D' of tree edges if each tree edge in D' is covered by some edge in D. The definition of 2-edge-connectivity implies that A is a minimal augmentation for $T_H + IN + R$ if A is a minimal set of nontree edges in C covering the edges in T_H not already covered by an edge in $IN \cup R$.

Since we are only interested in the edges of T_H not covered by edges in $IN \cup R$, we would like to somehow "remove" those edges in T_H that are covered by an edge in $IN \cup R$. This motivates the following definition. Let H' be a subgraph of H containing tree T_H as a subgraph. The *condensation of* graph H' is a graph H'' obtained from H' by collapsing the vertex sets of the 2-edge-connected components of H'. (The operation of "collapsing" is defined formally in section 2.) Note that H'' is a tree. We root this tree at the vertex into which the root of T_H is collapsed.

The following lemma shows that we can reduce the problem of minimally augmenting H' to the problem of minimally augmenting its condensation H''.

Lemma 3 *Let H' be a subgraph of H containing T_H and let H'' be the condensation of H'. For any $B \subseteq E(H)$, $H' + B$ is 2-edge-connected iff $H'' + B$ is 2-edge-connected.*

Proof. The edges of H'' are the cutedges of H'. Moreover, a cycle in $H' + B$ containing a cutedge of H' maps to a cycle in $H" + B$ containing the corresponding edge of $H"$. If $H' + B$ is 2-edge-connected, then every cutedge of H' is on a cycle in $H' + B$ and hence, every edge of $H" + B$ is on a cycle. Therefore, $H" + B$ is 2-edge-connected.

Conversely, we note that a cycle in $H" + B$ yields a cycle in $H' + B$ containing all the cutedges of H' that lie on the cycle of $H" + B$. Thus, if $H" + B$ is 2-edge-connected, every edge of $H"$ is on a cycle in $H'' + B$ and hence, every cutedge of H' is on a cycle in $H' + B$. Therefore, $H' + B$ is 2-edge-connected. □

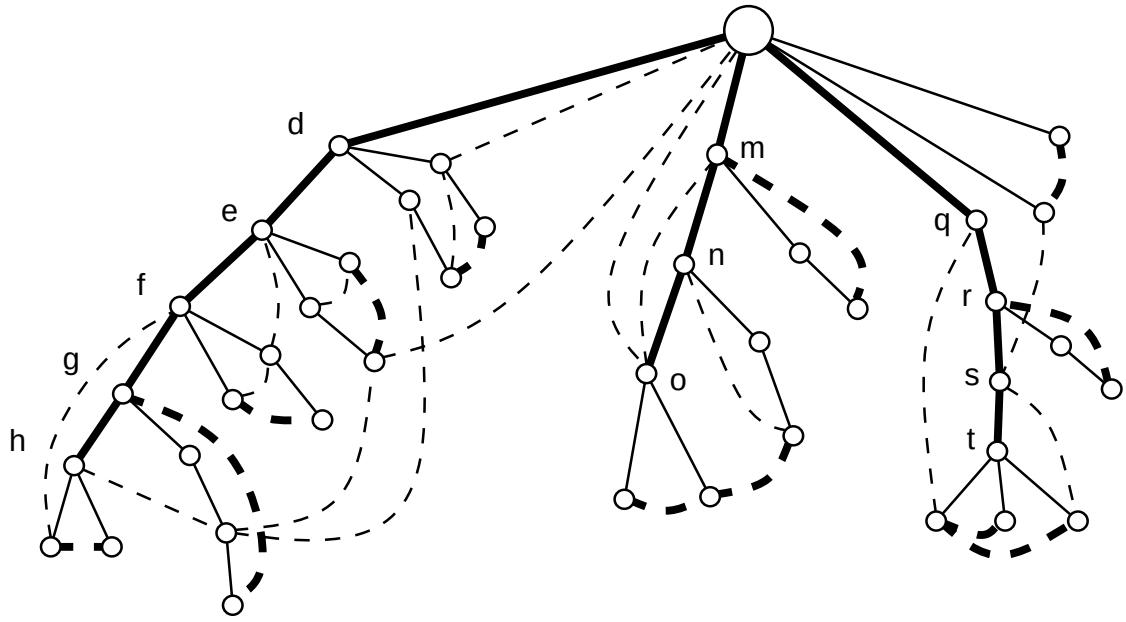


Figure 3: Condensation T_0 of $T_H + IN + R$. The edges of T_0 are indicated by solid lines. Thick solid lines indicate stem edges. Dashed lines correspond to nontree edges in H. Thick dashed lines represent nontree edges that are put into A_1.

Let us return to our original problem. We want to compute a minimal augmentation $A \subseteq C$ for $T_H + IN + R$. Let T_0 denote the condensation of $T_H + IN + R$. By lemma 3 we may choose for A a minimal augmentation for T_0. We shall illustrate our method to compute such a minimal augmentation with the example in figure 2. Consider the first iteration of algorithm 2 on the graph H depicted in figure 2(a). In this case $IN = \emptyset$ and R is the set of nontree edges whose lca is in the set $\{a, b, j, k\}$. The condensation T_0 of $T_H + IN + R$ is indicated in figure 3 by solid lines. The dashed lines represent nontree edges in C.

Our method for computing a minimal augmentation for T_0 is based on partitioning the edges in T_0 into two classes. Recall that tree T represents T_{lca} at the current stage of tree contraction (it will change in step (2.4) at the end of this iteration of the while-loop in algorithm 2). Let $L =< v_0, \ldots, v_k >$ be a leaf chain of T. The path in T_H from v_0 to v_k is termed the *stem of* L. We call those edges in T_0 that belong to a stem of a leaf chain *stem edges* and refer to the remaining edges as *non-stem edges*. In figure 3 the stem edges are indicated by thick solid lines and nonstem edges are indicated by thin solid lines.

We build a minimal augmentation for T_0 in three stages: in stage 1 we select a minimal subset $A_1 \subseteq C$ covering the non-stem edges in T_0. In stage 2 we choose a minimal subset $A_2 \subseteq C$ whose edges cover all stem edges in T_0 not covered in stage 1. The graph $T_0 + A_1 + A_2$ is 2-edge-connected.

Furthermore, by the minimality of A_2, all edges of A_2 are essential in $T_0 + A_1 + A_2$. To obtain a minimal augmentation for T_0, we shall remove certain edges of A_1 from $T_0 + A_1 + A_2$ so that the resulting subgraph of $T_0 + A_1 + A_2$ is 2-edge-connected and all remaining nontree edges in $A_1 \cup A_2$ are essential in this subgraph. This is done in the third and final stage.

One can show (lemma 4) that a set of nontree edges $A_1 \subseteq C$ covers all non-stem edges of T_0 if and only if each leaf of T_0 is incident with at least one edge of A_1. Thus, in stage 1 we need to compute a minimal set $A_1 \subseteq C$ such that each leaf of T_0 is incident with at least one edge of A_1. This is exactly what algorithm 3 does. In figure 3 the thick dashed lines represent nontree edges that are put into A_1. Note that the set of these edges is minimal with respect to the property that each leaf of T_0 is incident with an edge in this set.

Algorithm 3 Covering non-stem edges of T_0.
Input Tree T_0, set C of nontree edges.
Output Minimal subset $A_1 \subseteq C$ covering the non-stem edges of T_0.

(1) Let G_l be the graph whose vertices are the leaves of T_0 and whose edges are the edges of C between them. Find a spanning forest in G_l.

(2) For each tree T_l in this forest do the following:

  (2.1) Root the tree at an arbitrary vertex. Determine the depth of each node in T_l. Mark all edges in T_l that connect a vertex at even depth with a child (at odd depth). Also, for each leaf in T_l mark the unique incident tree edge (if it is not already marked).

  (2.2) A marked edge in the tree is *bad* if both of its endpoints have at least two marked edges (including this edge) incident with them. Each node (at even depth) eliminates any bad edges to its children.

  (2.3) If a node at even depth loses all marked edges to its children in the previous step, it marks a single edge to one of its children in T_l.

(3) Any leaf of T_0 that is not yet incident with a marked edge (i.e., it is an isolated vertex of G_l) marks an arbitrary edge of C incident with it.

Let A_1 be the subset of edges in C that are marked.

Lemma 4 *A_1 covers all non-stem edges of T_0 if and only if each leaf of T_0 is incident with an edge of A_1.*

Proof. Suppose that A_1 covers the non-stem edges of T_0. Let v be a leaf of T_0 and let e be the edge from v to its parent in T_0. To show that an edge of A_1 is incident with v, it suffices to show

that e is a non-stem edge in T_0. Suppose that e links a node v' in T_H to its parent in T_H. By the definition of T_0 and R, no edge of R is incident with a descendant of v' in T_H (otherwise the endpoints of e would have been collapsed into a single vertex of T_0). Furthermore, if an lca of an edge in IN is a descendant of v', it must be a proper descendant. Hence $v = v'$ and v is a leaf of T_H. It follows that v is not an lca of an edge in C. Hence, e is a non-stem edge.

Now let each leaf of T_0 be incident with an edge of A_1. Let e be a non-stem edge of T_0 connecting some vertex v to its parent in T_0. Let D be the set of edges in A_1 incident with leaves in T_0 that are descendants of v. By our assumption D is nonempty. Moreover, any edge in D has its lca on a stem in T_H; hence its fundamental cycle must contain edge e. We conclude that each non-stem edge in T_0 is covered by an edge of A_1. □

Lemma 5 *A_1 is a minimal subset of edges in C covering all non-stem edges of T_0.*

Proof. By lemma 4 we only need to show that A_1 is a minimal set of edges such that each leaf of T_0 is incident with an edge in A_1. We first show that each leaf of T_0 is incident with some edge in A_1. If v is a leaf in tree T_l of the forest, then its unique incident tree edge marked in step (2.1) will never be unmarked. Now assume that v is not a leaf. If its depth is even, it will certainly be incident with a marked edge after step (2.3). If its depth is odd, the edge to its parent in T_l can be unmarked only if one of its children is a leaf. The edge to that child is marked in step (2.1) and will never be unmarked. Finally, any isolated vertices in G_e (all of whose incident edges of C connect to nodes that are not leaves of T_0) will be incident with a marked edge after step (3). Hence, every leaf in T_0 is incident with an edge of A_1.

To establish the minimality of A_1, fix an edge in A_1. If it was marked in step (3), then only one endpoint of this edge is a leaf of T_0 and this endpoint will lie on a single marked edge. For the remaining edges of A_1, observe that edges that were bad after step (2.1) were removed during step (2.2) and none of the edges marked in step (2.3) are bad. Hence, every edge of A_1 has at least one endpoint that is a leaf in T_0 and that is not incident with another edge of A_1. Therefore A_1 is minimal. □

In stage 1 we covered all non-stem edges of T_0 and possibly some stem edges. In the second stage we proceed to cover stem edges in T_0 that have not been covered in stage 1. The algorithm we use here is similar to an algorithm in [17, 16] for finding a minimum feedback-vertex-set in a reducible flow graph.

To compute a minimal subset $A_2 \subseteq C$ covering the stem edges of T_0 not covered by edges of A_1, we appeal again to lemma 3. According to this lemma we may choose for A_2 a minimal set of edges in C covering the edges in the condensation T_1 of $T_0 + A_1$. Note that T_1 is also the condensation of graph $T_H + IN + R + A_1$. The tree T_1 has a rather simple structure: any vertex other than the

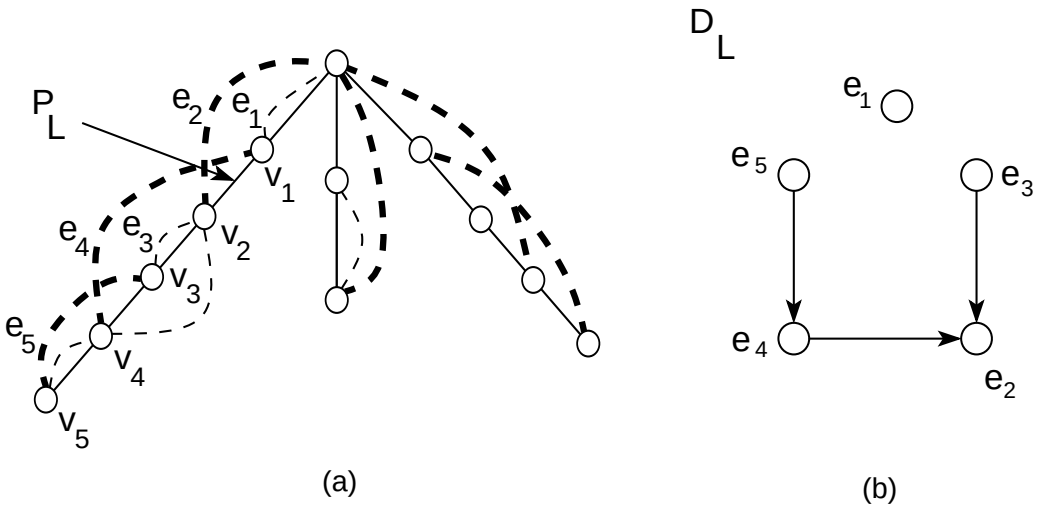


Figure 4: (a) Condensation T_1 of $T_H + IN + R + A_1$. Solid lines correspond to edges in T_1. Dashed lines correspond to nontree edges in H. Thick dashed lines represent edges in A_2. On path P_L we have $up(v_i) = e_i$. (b) Graph D_L corresponding to root-to-leaf path P_L in T_1. Edges e_5, e_4 and e_2 belong to A_2 because they represent vertices on a maximal path in D_L starting at e_5.

root of T_1 has at most one child. Moreover, the edges on any root-to-leaf path in T_1 belong to the stem of a single leaf chain of T. The tree T_1 corresponding to the example in figure 3 is shown in figure 4(a).

Let P_L denote the root-to-leaf path in T_1 containing edges on the stem of leaf chain L of T. We note that all the nontree edges in C connect vertices on the same path P_L for some leaf chain L (this follows from the definition of C). Let C_L be the set of edges in C that connect two distinct vertices on P_L. It suffices to show how to compute a minimal subset of edges in C_L that covers the edges on path P_L. For any edge of C_L, let the *upper endpoint* refer to the endpoint that is closer to the root of T_1 and let the *lower endpoint* be that endpoint that is further away from the root of T_1. The following greedy procedure picks a minimal subset of edges covering the edges on P_L: pick as first edge an edge of C_L incident with the unique leaf on P_L and whose endpoint is closest to the root of T_1. At any point pick as the next edge an edge in C_L whose lower endpoint lies between the endpoints of the edge that was picked last and whose upper point is closest to the root of T_1. We are done once we pick an edge whose upper endpoint is the root of T_1. The following algorithm is a parallel version of this greedy strategy.

Algorithm 4 Covering T_1.
Input Tree T_1, edge set $C - A_1$.
Output Minimal subset $A_2 \subseteq C - A_1$ covering the edges in T_1.

In parallel for each leaf chain L, do:

(1) Let C_L be the subset of edges in $C - A_1$ that connect two distinct vertices on P_L. For each vertex v on P_L let E_v be the set of edges in C_L incident with v and whose other endpoint lies strictly above v on P_L (i.e., closer to the root of T_1). Each vertex v on P_L with $E_v \neq \emptyset$ selects a single edge in E_v whose upper endpoint is closest to the root of T_1. Denote this edge by $up(v)$ (see figure 4(a)).

(2) Construct the auxiliary digraph D_L defined as follows (see figure 4(b)): the vertex set of D_L is the set $\{up(v) : v \text{ is a vertex on } P_L \text{ and } E_v \neq \emptyset\}$. There is a directed edge in D_L from e to e' if and only if the lower endpoint of e' lies between the endpoints of e, possibly coinciding with the upper endpoint of e, and the upper endpoint of e' is strictly above that of e (i.e, it is closer to the root of T_1); moreover, among all edges whose lower endpoint lies between the endpoints of e, the upper endpoint of e' is closest to the root of T_1. (Note: There may be several such edges e'; in this case pick an arbitrary such edge e'. Hence, each vertex in D_L has outdegree at most 1.)

(3) Starting at $up(v)$, where v is the unique leaf on P_L, construct a maximal path in D_L, i.e., a path that ends at a vertex that has outdegree 0 in D_L. Let A_L be the set of edges in C_L that are represented by vertices on this path.

Let $A_2 = \bigcup_L A_2^L$

Observation 1 *The lower endpoints on path P_L of edges in A_2^L are all distinct. Furthermore, the upper endpoints on P_L of the edges in A_2^L are all distinct.*

Lemma 6 *A_2 is a minimal set of edges covering the edges of T_1.*

Proof. As mentioned earlier, edges in $C - A_1$ have their endpoints in T_1 on the same path P_L from the root to some leaf of T_1. Therefore, it suffices to show that the edges in A_2^L minimally cover the edges of P_L for an arbitrary leaf chain L of T. We shall first prove that each edge in P_L is covered by some edge in A_2^L. Assume for a contradiction that e is the lowest edge on P_L not covered by any edge in A_2^L. Consider the subset S of edges of $C - A_1$ that cover e. Since H is 2-edge-connected, S is nonempty. Let e' be an edge of S. Since e is the lowest uncovered edge on P_L, the lower endpoint of e' lies between the endpoints of some edge $e'' \in A_2^L$ (may coincide with its upper endpoint). The path in D_L found in step (3) is maximal. Therefore, e'' has a successor edge on the path whose upper endpoint is at least as high on the stem as that of e'. Hence, that edge covers e, contradicting the assumption that e is not covered.

To see why A_2^L is minimal, let $e_0 \ldots e_k$ be the edges in A_2^L in the order they occur as vertices on the maximal path in D_L. By observation 1 and the definition of D_L, the edge from the lowest vertex (leaf) on P_L to its parent in T_1 is covered only by e_0. Furthermore, the lower endpoint of e_{i+1} lies strictly above the upper endpoint of e_{i-1}. It follows that the edges on P_L between those endpoints are covered only by e_i. Finally, again by observation 1, the highest edge on P_L is covered only by e_k. Hence, the edges of A_2^L cover the edges of P_L minimally. □

Corollary 2 *The graph $T_0 + A_1 + A_2$ is 2-edge-connected and the edges in A_2 are essential in this graph.*

Proof. By lemma 5 each non-stem edge of T_0 is covered by some edge in A_1. By lemma 6 each stem edge of T_0 is covered either by an edge of A_1 or an edge of A_2. Hence, $T_0 + A_1 + A_2$ is 2-edge-connected. By lemma 6 the set A_2 is a minimal set such that $T_1 + A_2$ is 2-edge-connected. By lemma 3 A_2 is also a minimal set such that $T_0 + A_1 + A_2$ is 2-edge-connected. We conclude that the edges of A_2 are essential in $T_0 + A_1 + A_2$. □

Observation 2 *Every vertex of P_L is incident with at most 2 edges of A_2^L.*

The set $A_1 \cup A_2$ may not be a minimal augmentation for T_0. Indeed, an edge of A_1 is redundant in $T_0 + A_1 + A_2$ if all the tree edges on its fundamental cycle in T_0 are covered by other edges in $A_1 \cup A_2$.

To obtain a minimal augmentation for T_0, we shall remove a subset of the edges in A_1 from $T_0 + A_1 + A_2$ such that the resulting subgraph of $T_0 + A_1 + A_2$ is 2-edge-connected and all edges of $A_1 \cup A_2$ contained in this subgraph are essential in this subgraph.

We note that an edge e in A_1 can be removed from $T_0 + A_1 + A_2$ (without destroying 2-edge-connectivity) if no edge on the fundamental cycle of e in T_0 is a cutedge in the resulting graph. We may extend this idea to a subset $B \subseteq A_1$ with the property that the fundamental cycles of the edges in B are edge-disjoint: we may remove exactly those edges e in B from $T_0 + A_1 + A_2$ with the property that none of the edges on the fundamental cycle of e in T_0 is a cutedge in $T_0 + A_1 + A_2 - B$.

Note that edges in A_1 whose lca's lie on different leaf chains of T (the tree representing T_{lca} at the current stage of tree contraction) have edge-disjoint fundamental cycles in T_0. Thus, we may apply the idea from the previous paragraph to process edges of C whose lca's belong to different leaf chains independently of each other. Let A_1^L contain those edges in A_1 whose lca lies on leaf chain L. It suffices to show how to remove a maximal subset of edges in A_1^L so that the remaining edges are essential. Note that edges in A_1^L may have overlapping fundamental cycles in T_0. Thus, it is not sufficient (in general) to remove all the edges in A_1^L from $T_0 + A_1 + A_2$ at one time and check the resulting graph for cutedges. Here is where observation 2 comes in: since each vertex on

P_L is incident with at most 2 edges of A_2^L, we only need to look at at most two vertices in each 2-edge-connected component of $T_0 + A_1$. We also note that each edge in A_1^L has at least one endpoint that is a leaf of T_0 and that is not incident with another edge of A_1^L. We call such an endpoint a *critical endpoint* of an edge of A_1^L. We observe that an edge in A_1^L can only be redundant in $T_0 + A_1 + A_2$ if its critical endpoints are incident with edges in A_2^L. Thus, we only need to examine an edge of A_1^L if it is the unique edge in A_1^L incident with a leaf in T_0 that is an endpoint of an edge of A_2^L. It follows that we only need to consider at most two edges of A_1^L in each 2-edge-connected component of $T_0 + A_1$. Thus, we may process the edges in A_1^L in two phases. In each phase we look at edges that lie in different 2-edge-connected components of $T_0 + A_1$. These edges certainly have edge-disjoint cycles in T_0. Hence, we may apply the idea described in the previous paragraph to remove a maximal set of edges in each phase.

The approach we have just outlined is essentially that taken by the following algorithm.

Algorithm 5 Making A_1 minimal.
Input Graph T_0, edge sets A_1 and A_2.
Output Subset $A_1' \subseteq A_1$ such that $A_1' \cup A_2$ is a minimal augmentation for T_0.

(0) Let $H^* = T_0 + A_1 + A_2$.

(1) In parallel for each leaf chain L, do:

(1.1) In parallel for each vertex w on P_L incident with at least one edge of A_2^L, do:

(1.1.1) Let B_w be the 2-edge-connected component of $T_0 + A_1$ that corresponds to w (possibly, B_w is a single vertex). Let V_w be the set of endpoints in B_w of the edges of A_2^L. (By observation 2, we have $|V_w| \leq 2$.)

(1.1.2) Process the vertices v of V_w sequentially as follows:

(1.1.2.1) If there is a unique edge e in A_1 incident with v, remove e from H^*.

(1.1.2.2) If an edge on the fundamental cycle of e in T_0 is a cutedge in H^*, put e back into H^*.

(2) Let A_1' be the subset of edges in A_1 that are contained in H^*.

Lemma 7 *$T_0 + A_1' + A_2$ is 2-edge-connected and each edge of A_1' is essential in $T_0 + A_1' + A_2$.*

Proof. The graph $T_0 + A_1' + A_2$ is a spanning subgraph of $T_0 + A_1 + A_2$. To see that $T_0 + A_1' + A_2$ is 2-edge-connected, note that an edge of A_1 removed in step (1.1.2.1) is not put back in step (1.1.2.2) only if in H^* the edges of T_0 that were covered by e are still covered (by some other edges). It

follows that all edges of T_0 are covered by edges of $A_1' \cup A_2$ and therefore $T_0 + A_1' + A_2$ is indeed 2-edge-connected.

We now show that each edge of A_1' is essential in $T_0 + A_1' + A_2$. Fix $e \in A_1$. A *critical endpoint of* $e \in A_1$ is an endpoint of e in T_0 that is a leaf in T_0 and that is not incident with another edge of A_1. By lemma 4 and lemma 5 each edge in A_1 has at least one critical endpoint in T_0. Edge e can only be redundant in $T_0 + A_1 + A_2$ if its critical endpoints are incident with edges of A_2. In that case e will be removed in step (1.1.2.1). It is put back only if its removal produces a cutedge in H^* among the edges of T_0 covered by it. Since different edges removed at one time in step (1.2.1) cover disjoint sets of edges of T_0, e is put back only if it is indeed essential at this point. Since we do not add edges to H^* (the current subgraph of $T_0 + A_1 + A_2$) after this point, an edge that is put back in step (1.1.2.2) is essential in $T_0 + A_1' + A_2$. □

Theorem 1 *The set $A = A_1' \cup A_2$ is a minimal augmentation for $T_H + IN + R$.*

Proof. Since $T_0 + A_1' + A_2$ is a spanning subgraph of $T_0 + A_1 + A_2$ and each edge of A_2 is essential in $T_0 + A_1 + A_2$ (by corollary 2), the edges in A_2 are also essential in $T_0 + A_1' + A_2$. With lemma 7 we conclude that $A_1' \cup A_2$ is a minimal augmentation for T_0. By lemma 3 (with $H' = T_H + IN + R$ and $H'' = T_0$) it follows that $A_1' \cup A_2$ is also a minimal augmentation for $T_H + IN + R$. □

3.3 Analysis

We now analyze the processor and time requirements of our algorithm on a PRAM. For the definitions of the various PRAM models we refer the reader to the survey paper by Karp and Ramachandran ([11]). One iteration of the while-loop of algorithm 2 can be implemented to run almost optimally in time $O(\log n)$ on an ARBITRARY CRCW PRAM. Only the fact that no optimal algorithms are currently known for bucket sort and for computing connected components prevents us from achieving optimal performance. For sorting we use the algorithm of Bhatt et al. ([1]) which sorts n integers in the range $0 \ldots n^{O(1)}$ in time $O(\log n / \log\log n)$ with $n(\log\log n)^2 / \log n$ processors on an ARBITRARY CRCW PRAM. We need to compute connected components at several places in our algorithm (see below). The most efficient connectivity algorithm ([2]) computes the connected components of a graph with n nodes and m edges represented by adjacency lists in $O(\log n)$ time with $(m+n)\alpha(m,n)/\log n$ processors of an ARBITRARY PRAM; if the graph is represented by an unordered list of its edges, we can construct its adjacency list by sorting the edges lexicographically (using bucket sort in the range $0 \ldots n^{O(1)}$); this can be done in time $O(\log n)$ with $m \log\log n / \log n$ processors of an ARBITRARY PRAM.

Since more processor-efficient algorithms for these problems may be developed in the future, we shall adopt the following conventions: $B(n)$ denotes the number of processors required to sort n

integers in the range $0 \ldots n^{O(1)}$ in time $O(\log n)$; $C(n,m)$ denotes the number of processors needed to compute connected components of a graph with n nodes and m edges represented by adjacency lists in time $O(\log n)$; finally, $A(n,m)$ stands for $max\{B(n), C(n,m)\}$. As mentioned above we have currently $B(n) = n \log\log n / \log n$ (on ARBITRARY) and $C(n,m) = (n+m)\alpha(m,n)/\log n$ (on ARBITRARY). A step in our algorithm is *A-optimal* if it runs in time $O(\log n)$ with $A(n,2n)$ processors and is *C-optimal* if it runs in time $O(\log n)$ with $C(n,2n)$ processors. In these definitions we have replaced m by $2n$ since step (0) of algorithm 1 reduces the number of edges that need to be processed to less than $2n$.

In the following analysis the time and processor bounds that do not specify a particular PRAM model all hold for the ARBITRARY CRCW PRAM model. We assume that the input graph G is represented by its adjacency lists. Step (0) of algorithm 1 is executed only once. Its complexity is dominated by that for finding an ear decomposition in G. This can be done in time $O(\log n)$ using $C(n,m)$ processors ([14, 13, 18]). As shown earlier (corollary 1), $O(\log n)$ iterations of algorithm 1 yield a minimal 2-edge-connected spanning subgraph of G. We analyze the work done in one such iteration.

We identify redundant edges in H by finding separating pairs of edges in H. For this we modify the algorithm for finding triconnected components given in [3, 18] (see also [19]). Thus, we can compute separating pairs of edges A-optimally. The complexity of computing T_H is the same as that of computing connected components on a graph with n nodes and at most $2n$ edges; we thus compute T_H C-optimally.

Algorithm 2 finds a minimal augmentation for T_H using $O(\log n)$ levels of tree contraction. We prepare algorithm 2 by computing the lca's of nontree edges in H and identifying the vertices in T_H that are lca's of nontree edges; this can be done optimally in time $O(\log n)$ using the algorithm of [20]. Next we construct the tree T_{lca}. We shall not compute an explicit representation of the tree (in terms of adjacency lists) but rather compute enough information to identify the leaf chains quickly. Using the Euler tour technique ([22]) on tree T_H we compute the following three quantities for each lca v in T_H: the number of lca's in T_H that precede v in preorder; the number of lca's on the unique path from the root of T_H to v; the number of descendants of v in T_H (including v) that are lca's of nontree edges. We denote these three quantities by $num(v)$, $d(v)$ and $nd(v)$, respectively. We call an lca v that has at least two children in T_{lca} a *split vertex*. Note that an lca v (other than the root of T_H) belongs to a leaf chain of T_{lca} iff none of its descendants in T_H is a split vertex. Once we have determined the split vertices in T_H, we can check the latter property using one more application of the Euler tour technique (on tree T_H). We compute the split vertices with the help of the three quantities $num(v)$, $d(v)$ and $nd(v)$ as follows: using the quantities $num(v)$ we write the lca's in T_H in preorder into an auxiliary array A. Note that a vertex v at position $n_v = num(v)$ in A is a split vertex iff the vertex w at position $n_v + 1$ in A satisfies $d(w) = d(v) + 1$ and it has a

sibling in T_{lca}. Note that w has a sibling in T_{lca} iff the vertex z at position $n_v + 1 + nd(w)$ in A satisfies $d(z) = d(w)(= d(v) + 1)$. We conclude that the asymptotic complexity of computing the leaf chains in T_{lca} is the same as that of applying the Euler tour technique to T_H, i.e., $O(\log n)$ time with $n/\log n$ processors ([22]).

We shall now examine the complexity of one iteration of the while-loop in algorithm 2. In one such iteration we shall need to compute connected or 2-edge-connected components in various graphs. Note that we can compute (2-edge-)connected components C-optimally provided we have the adjacency lists for the graphs. Since many of the graphs are obtained by collapsing subsets of vertices, it is not clear that we can obtain the adjacency list for the graphs without resorting to sorting. For that reason we shall only claim that we can compute (2-edge-)connected components A-optimally (instead of C-optimally). (This will not affect the overall complexity of one iteration of the while-loop in algorithm 2 since we require sorting in algorithm 4.)

To prepare for stage 1 (algorithm 3) we compute the condensation of $T_H + IN + R$. This amounts to computing the 2-edge-connected components of $T_H + IN + R$, which can be done using an ear decomposition algorithm ([14], [13], [18]) that is A-optimal. For step (1) assume that each vertex in V_l is assigned a unique number in the range $1 \ldots n$. By making each vertex of V_l choose a single edge of C to a lower numbered vertex in V_l (if there is such an edge) we obtain a spanning forest in G_l. We compute the adjacency lists for this forest B-optimally. For step (2.1) we apply the Euler tour technique to each tree in the forest. All remaining steps of algorithm 3 take time $O(\log n)$ with $n/\log n$ processors. Thus, stage 1 can be done A-optimally.

We now come to stage 2 (algorithm 4). We compute T_1 (condensation of $T_0 + A_1$) by determining the 2-edge-connected components in $T_0 + A_1$; this can be done C-optimally. The complexity of the remaining steps is dominated by the construction of D_L in step (2). We can reduce the problem of determining the edges of D_L to the following problem: given a sequence of $\leq n$ numbers determine the maxima for $\leq 2n$ (possibly overlapping) intervals of this sequence. This can be done in time $O(\log n)$ on $n/\log n$ processors as follows. Suppose a sequence of n numbers is given in an array $A[0 \ldots n-1]$. Processor i is assigned the *segment* $A[(i-1) \cdot \log n, \ldots, i \cdot \log n - 1]$. It computes the maximum of its segment as well as the maxima of subintervals of its segment of the form $A[(i-1) \cdot \log n, \ldots, r]$ or $A[s, \ldots, i \cdot \log n - 1]$ where $(i-1) \cdot \log n \leq r, s \leq i \cdot \log n - 1$. Note that we can compute these maxima in time $O(\log n)$ with $n/\log n$ processors. We store the $n/\log n$ maxima of the segments in a second array $B[0 \ldots n/\log n - 1]$. For simplicity we assume that $n/\log n$ is of the form 2^k for some integer k. We compute the maxima of all intervals in B of the form $B[j \cdot 2^i, (j+1) \cdot 2^i - 1]$ for $0 \leq i \leq k$ and $0 \leq j < \frac{n}{2^i \log n}$. The total number of such intervals is $O(n)$. It is straightforward to compute the maxima of these intervals in time $O(\log n)$ with $n/\log n$ processors. With the above maxima we can answer in constant time each maximum query of the form $max\{A[i], A[i+1], \ldots, A[j-1], A[j]\}$ where i and j belong to different segments. In order to

answer maximum queries whose endpoints lie in the same segment of A, each processor constructs in time $O(\log n)$ a *Cartesian* tree for its segment (see [4]) and preprocesses the tree in $O(\log n)$ time so that *lca* queries can be answered in constant time (using the algorithm of [10]). As shown in [4] each maximum query whose endpoints lie in the same segment is answered by computing the *lca* of the endpoints in the Cartesian tree for this segment; this takes constant time. Altogether we see that we can answer each maximum query in constant time after a preprocessing phase that requires $O(\log n)$ time with $n/\log n$ processors.

To compute a maximal path in D_L (step (3) of algorithm 4), transform D_L into an undirected forest rooted at the sinks in D_L. We compute the adjacency lists for this forest B-optimally using bucket sort. We can then compute the maximal path in D_L optimally in time $O(\log n)$ by applying the Euler tour technique ([22]) to each tree in the forest. We conclude that one execution of stage 2 can be done B-optimally.

In stage 3 (algorithm 5) we compute the cutedges in H^* by determining the 2-edge-connected components of H^*. This can be done A-optimally. Hence algorithm 5 can be implemented A-optimally.

In summary we see that one iteration of the while-loop of algorithm 2 runs in time $O(\log n)$ with $A(n,2n)$ ARBITRARY processors (or in time $O(\log^2 n)$ with $A(n,2n)$ EREW processors). Since algorithm 2 terminates after $O(\log n)$ iterations (stages of tree contraction) and algorithm 1 makes $O(\log n)$ calls to algorithm 2, algorithm 1 runs in time $O(\log^3 n)$ with $C(n,m)/\log^2 n + A(n,2n)$ processors on an ARBITRARY PRAM (or $O(\log^4 n)$ time with $C(n,m)/\log^2 n + A(n,2n)$ processors on an EREW PRAM). Thus, the *work* done by our algorithms (time-processor product) is $C(n,m)\log n + A(n,2n)\log^3 n$ on an ARBITRARY PRAM. In [9] a linear time sequential algorithm for computing a minimal 2-edge-connected spanning subgraph is given. That algorithm can be parallelized. The parallel version uses our parallel minimal augmentation procedure as a subroutine. The work done by this modified algorithm is $C(n,m)\log n + A(n,2n)\log^2 n$ on ARBITRARY (i.e., the work is roughly improved by a factor of $\log n$). The same improvement applies to the related problem of computing a minimal biconnected spanning subgraph (see next section).

4 Finding a Minimal Biconnected Spanning Subgraph in Parallel

The problem considered in this section is: given a biconnected graph G, find a minimal biconnected spanning subgraph of G , i.e., a biconnected spanning subgraph of G that does not have a biconnected spanning subgraph of G as a proper subgraph. There is an obvious similarity between this problem and the one discussed in the last section. Indeed, some techniques used in the last section will be applicable here. Several new problems, however, will arise in the context of biconnectivity

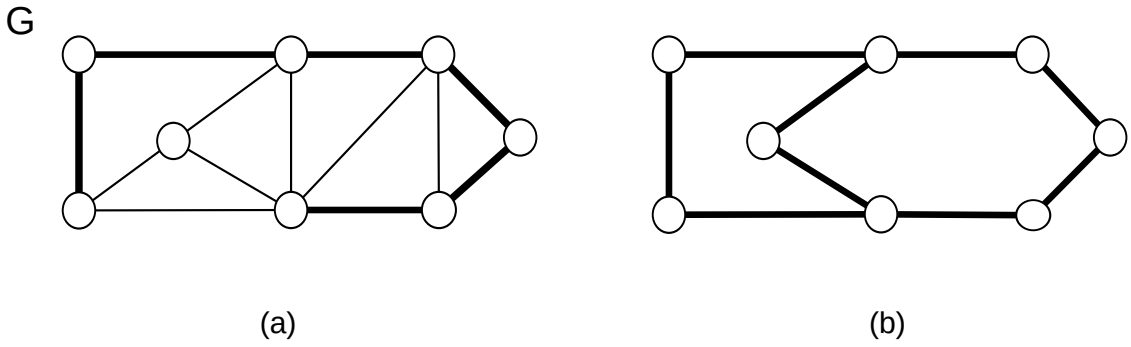


Figure 5: (a) A biconnected graph G and (b) a minimal biconnected spanning subgraph of G. Essential and redundant edges are indicated by thick and thin lines, respectively.

and will require new techniques. Figure 5 shows a biconnected graph G and a minimal biconnected spanning subgraph of G.

4.1 The High-Level Algorithm

As in the previous section we start out by defining the notions of redundant and essential edges. Given a biconnected graph H, we say that an edge e of H is *redundant in* H if $H - e$ is biconnected; an edge of H is *essential in* H if it is not redundant in H. As in the last section we shall sometimes omit H if it is clear from the context. In figure 5 essential and redundant edges are indicated by thick and thin lines, respectively. Since the graph in figure 5(b) is a minimal biconnected spanning subgraph of the graph in figure 5(a), all of its edges are essential.

We use the high-level strategy given by algorithm 1 (replace "2-edge-connected" by "biconnected"). We compute a biconnected spanning subgraph H of G with fewer than $2n$ edges by finding an open ear decomposition for G ([3], [18]) and removing all trivial ears. In one iteration of algorithm 1 we do the following: first, we find a spanning tree T_H in H (the current biconnected spanning subgraph of G) that includes a minimum number of redundant edges in H; next, we determine a *minimal augmentation* for T_H, i.e., a minimal set B of nontree edges in H such that $T_H + B$ is biconnected; finally, we update H to be the graph $T_H + B$.

To identify the redundant edges in H at each iteration, we construct the graph H' from H by adding a new vertex v_e for each edge e in H and replacing e by two edges (u, v_e) and (v_e, v) where $e = (u, v)$. A *separating pair (of vertices) in* H' is a pair of vertices in H' whose removal disconnects H'. We note that an edge e of H is redundant if and only if it does not occur in any separating pair of H' of the form $\{v_e, u\}$ for some vertex u of H. An efficient parallel algorithm for finding all separating pairs in H' ([3], [18]) can thus be modified to identify all redundant edges efficiently.

As for 2-edge-connectivity we remark that an alternative (sequential) method of Han and Tarjan ([8]) for computing a minimal biconnected spanning subgraph can be parallelized; the parallel implementation is similar to our method but it avoids the explicit computation of redundant and essential edges.

The proof of lemma 1 carries over to show that there is a spanning tree in H that contains at most 2/3 of the redundant edges in H. Hence, $O(\log n)$ iterations of the while-loop of algorithm 1 (suitably modified) will yield a minimal biconnected spanning subgraph of G.

4.2 Computing a Minimal Augmentation in Parallel

We shall now describe a parallel algorithm for finding a minimal augmentation for T_H (with respect to biconnectivity). Again, the high-level structure is the same as that given in the previous section (algorithm 2; replace "2-edge-connected" by "biconnected"). The proof of lemma 2 carries over to show that the modified version of algorithm 2 does indeed compute a minimal augmentation for T_H (with respect to biconnectivity).

Recall that during the execution of the algorithm the set of nontree edges is partitioned into disjoint subsets IN, C, and R: IN contains the edges that have already been committed to the minimal augmentation, C is the set of edges examined during this iteration of tree contraction (i.e., whose lca lies on a leaf chain of T), and R contains those edges that are to be considered at future iterations.

Consider one iteration of the while-loop of algorithm 2. Let H_0 denote the graph $T_H + IN + R$ after step (2.1) of this iteration. We are left with the problem of finding a minimal augmentation for H_0, i.e., a minimal subset of edges in C such that its addition to H_0 will result in a biconnected graph (step (2.2) of algorithm 2). We shall define an operation on graphs called *block condensation*. It is reminiscent of the condensation of a graph defined in the last section. The concept of a block condensation will be helpful in explaining various steps of our minimal augmentation procedure.

We need the following result.

Lemma 8 *Let H' be a subgraph of H containing tree T_H. The intersection of a block of H' with T_H forms a tree.*

Proof. It suffices to show that any two vertices v, w in a block B of H' are connected by a path containing only tree edges (edges of T_H) that belong to B. Since B is a connected subgraph of H', v and w are connected by some path P in B, possibly containing nontree edges. Note that if a nontree edge belongs to B then all the tree edges on its fundamental cycle in T_H belong to B as well. Hence we may replace on path P each nontree edge by a path of tree edges in B, obtaining a path in B consisting only of tree edges. □

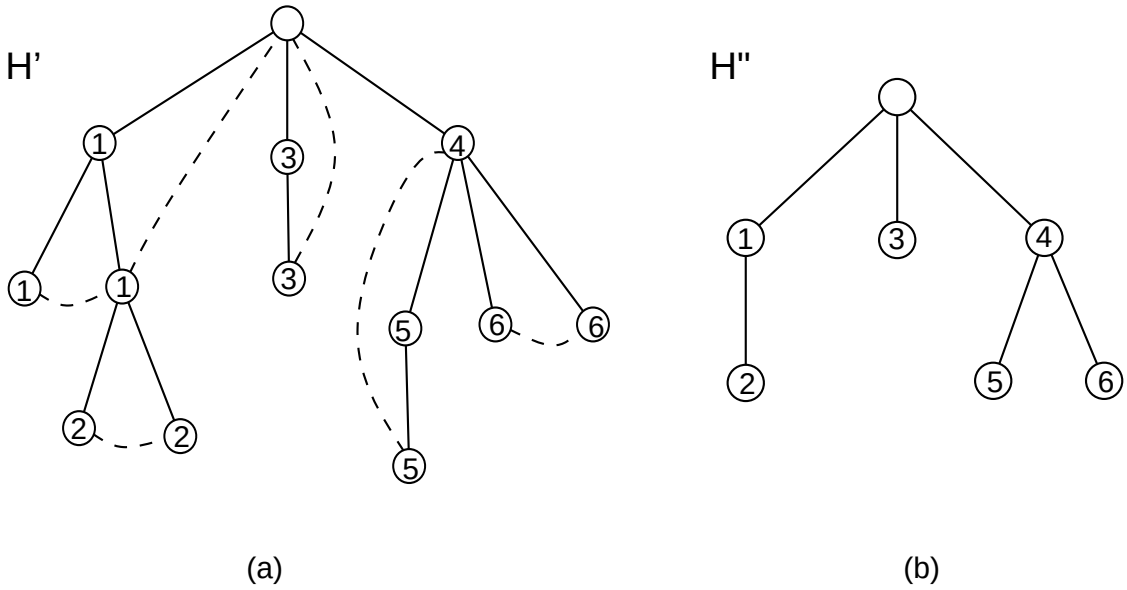


Figure 6: (a) The graph H'; the edges in T_H are indicated by solid lines. Vertices numbered i belong to the core of the block collapsed into the vertex numbered i in H''. (b) The block condensation H'' of H'.

Let H' be a subgraph of H containing tree T_H and let B be a block of H'. By lemma 8 the intersection of B with T_H is a subtree of T_H. We call the root of that subtree (i.e., the vertex closest to the root of T_H) the *root of* B. We call the subset of vertices of B that are different from the root of B the *core of* B. As an easy consequence of lemma 8 we see that the cores of two distinct blocks of H' have an empty intersection. The *block condensation* of H' is obtained by collapsing the core of each block of H' into a single vertex (see section 2 for the definition of collapsing) and replacing in the resulting multigraph multiple edges by single edges. Note that the block condensation of H' is a tree. We root the tree at the root of T_H. A simple example of a graph and its block condensation is given in figure 6.

There is a natural correspondence between an edge e of H that does not belong to H' and the edge e' linking those vertices in the block condensation of H' into which the endpoints of e in H' have been collapsed. We shall usually not differentiate between an edge connecting vertices of H' and the corresponding edge connecting vertices in the block condensation of H'. For instance, we shall say that an edge of H is *incident* with a vertex v in the block condensation of H' if the corresponding edge connecting vertices in the block condensation is incident with this vertex.

We shall now describe how to compute a minimal augmentation $A \subseteq C$ for $H_0 = T_H + IN + R$ (step (2.2) of algorithm 2), where C denotes the set of nontree edges examined at the current iteration of algorithm 2. As in the previous section, T denotes the tree representing T_{lca} at the current stage

of tree contraction and the path in T_H corresponding to leaf chain L of T is called the *stem of L (in T_H)*. We shall compute a minimal augmentation for H_0 in five stages. At each of the first three stages we shall add a set of edges to achieve a certain property. In the final two "cleanup stages" we shall get rid of excess edges to obtain a minimal augmentation. Overall our method is somewhat similar to that used for 2-edge-connectivity but it is more complicated.

An example is given in figure 7. Figure 7(a) shows the graph H with the tree T_H; the corresponding tree T_{lca} is depicted in figure 7(b). In this example we consider the second iteration of algorithm 2: the lca's of nontree edges processed at this iteration are represented by the gray vertices in figure 7(a) and 7(b). Note that the gray vertices are exactly the vertices on the stems of leaf chains in T_H. The edges in IN, R and C are accordingly marked in figure 7(a).

In the first stage we select a minimal set of edges $A_1 \subseteq C$ whose addition to H_0 results in a graph all of whose cutpoints lie on stems of leaf chains. The following algorithm accomplishes this. It makes use of algorithm 3.

Algorithm 6 Eliminating cutpoints that do not lie on a stem.
Input Graph $H_0 = T_H + IN + R$, edge set C.
Output Minimal subset A_1 of C such that $H_0 + A_1$ has all its cutpoints on stems in T_H.

(1) Compute the block condensation T_0 of H_0 (see figure 7(c)).

(2) Let V_l be the set of leaves of T_0 that correspond to the cores of blocks whose root (in T_H) is not a vertex on a stem (in T_H). Using the method of algorithm 3 compute a minimal set $A_1 \subseteq C$ such that each leaf in V_l (i.e., the core collapsed into this leaf) is incident with at least one edge in A_1.

In figure 7(c) the square vertices represent the leaves in V_l. The thick dashed lines in figure 7(a) represent a minimal set of nontree edges in C such that each leaf in V_l (i.e., a vertex in the corresponding core) is incident with an edge in this set.

Lemma 9 *For any $F \subseteq C$, $H_0 + F$ has all its cutpoints on stems iff each vertex in V_l is incident with at least one edge in F.*

Proof. Assume that each vertex in V_l is incident with an edge in F. Consider a block B in H_0 whose root v is a cutpoint of H_0 that does not lie on a stem of T_H. All leaves of T_0 (block condensation of H_0) that are descendants (in T_0) of the vertex in T_0 corresponding to the core of B belong to V_l since their roots do not lie on a stem of a leaf chain. Thus these leaves are incident with edges of F. The fundamental cycles of these edges in F contain v and a vertex on the stem of some leaf chain. Since this holds for any block of H_0 whose root is v, vertex v is not a cutpoint in $H_0 + F$.

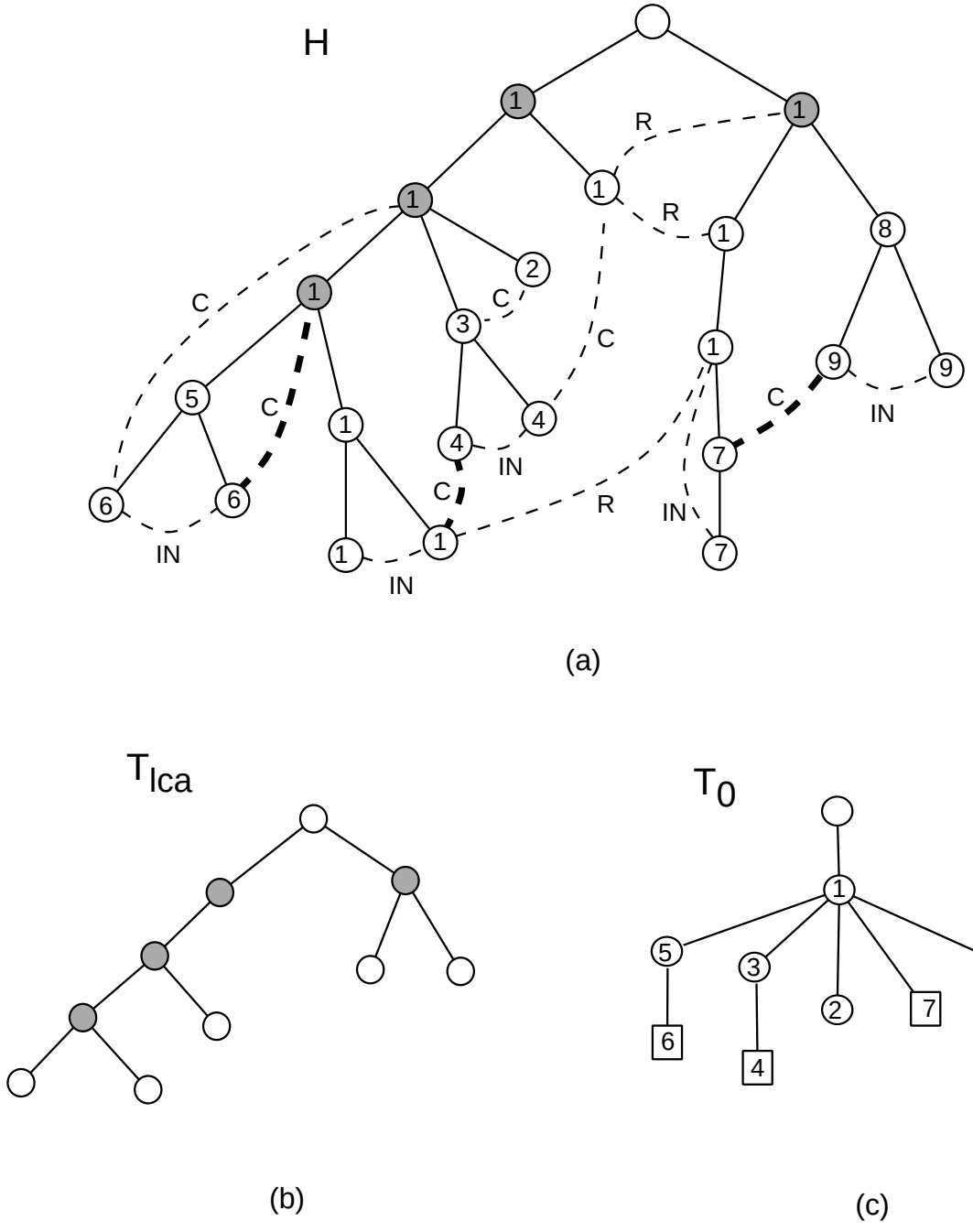


Figure 7: (a) Graph H with spanning tree T_H indicated by solid lines. Thick dashed lines represent edges of C put into A_1. The gray vertices are the lca's of nontree edges in C. Vertices with the same number belong to the core of the same block of $T_H + IN + R$. (b) Tree T_{lca}. Gray vertices represent those lca's currently processed. (c) Block condensation T_0 of $T_H + IN + R$. The square vertices are the leaves in V_l. Vertex i represents the core of a block of $T_H + IN + R$ whose vertices are numbered i in (a) ($i = 1 \ldots 9$).

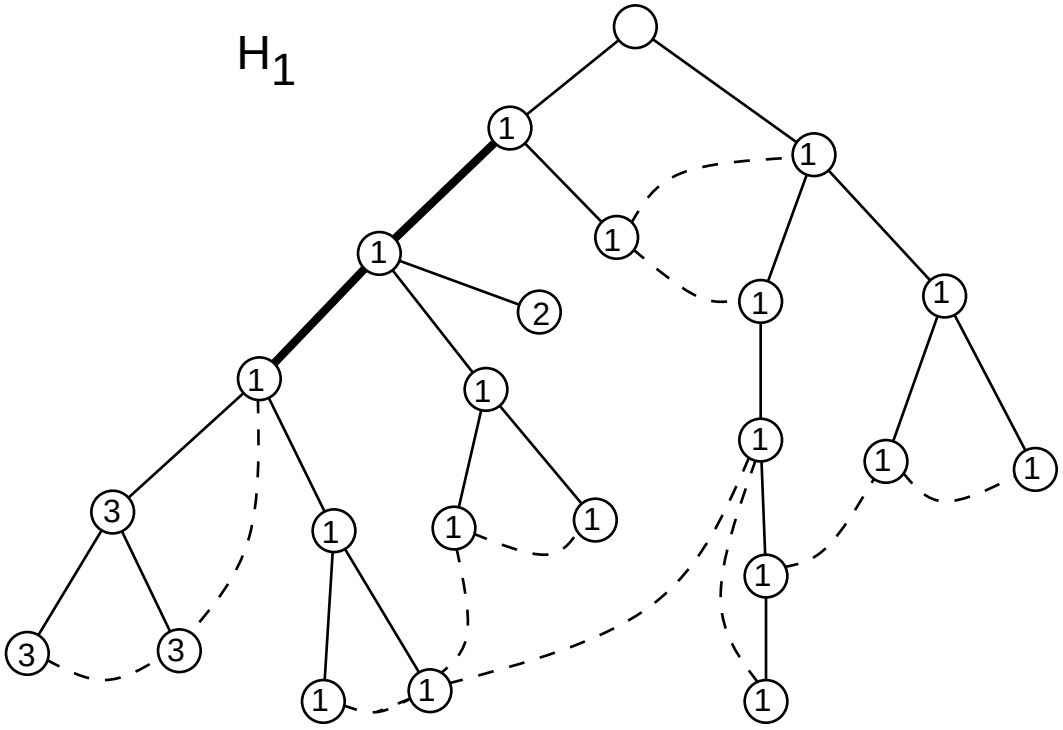


Figure 8: Graph $H_1 = T_H + IN + R + A_1$. Stem edges are indicated by thick solid lines. The graph contains three blocks whose core vertices are numbered $1, 2, 3$. The core vertices numbered 1 belong to an internal block of H_1; the two other blocks are external.

Conversely, if some vertex in V_l is not incident with an edge in F, then this vertex represents the core of a block in $H_0 + F$ whose root is a cutpoint in $H_0 + F$ that does not lie on a stem in T_H. □

The following result is immediate with lemma 9.

Corollary 3 *The set A_1 is minimal with the property that all cutpoints of $H_0 + A_1$ lie on stems in T_H.* □

Let H_1 denote the graph $H_0 + A_1$ ($= T_H + IN + R + A_1$). We classify the blocks of H_1 into 2 classes. An *external block* of H_1 is a block whose root lies on some stem in T_H but does not include any edge of this stem; a block is *internal* if it is not external. Figure 8 shows the graph $H_1 = T_0 + IN + R + A_1$ for the example in figure 7. The graph H_1 in figure 8 consists of 3 blocks whose core vertices are numbered 1, 2, and 3. The block whose core vertices are numbered 1 is the only internal block; it is rooted at the root of T_H. The other two blocks are external.

In stage 2 of our minimal augmentation procedure we construct a minimal set of edges $A_2 \subseteq C - A_1$ such that $H_1 + A_2$ has no external blocks. We eliminate external blocks for the following reason: we want to find a (minimal) set of edges whose addition to H_1 will yield a biconnected graph. If

all blocks in H_1 are internal, then the removal of a cutpoint v will break up the graph in exactly two components: one component contains all vertices other than v that lie in blocks whose root is v or a vertex below v on the stem containing v while the other component contains all remaining vertices (other than v). In this case augmenting H_1 into a biconnected graph amounts to finding a set of edges connecting these two components for any cutpoint v – a relatively easy task as we shall see later. The following algorithm removes external blocks.

Algorithm 7 Eliminating external blocks in H_1.
Input Graph H_1, edge set $C - A_1$.
Output Minimal set $A_2 \subseteq C - A_1$ such that all the blocks of $H_1 + A_2$ are internal.

(1) Compute the block condensation of H_1. Let V_l denote the set of leaves in the block condensation representing the cores of external blocks of H_1.

(2) Call an edge of $C - A_1$ *external* if it links a vertex in the core of some external block B of H_1 with a vertex that lies outside of all the external blocks of H_1 that have the same root as B. Let V_e be the subset of nodes in V_l incident with an external edge. Using the method of algorithm 3 compute a set $A_2^{(1)}$ of *external* edges in $C - A_1$ that is minimal with respect to the property that each vertex in V_e is incident with at least one edge in $A_2^{(1)}$.

(3) Consider the graph Q whose vertices are the vertices in V_l and whose edges are the edges of $C - A_1$ between them.
If $V_e = \emptyset$, compute a minimal set $A_2^{(2)} \subseteq C - A_1$ such that the subgraph of Q induced by the edges in $A_2^{(2)}$ is connected.
If $V_e \neq \emptyset$, compute a minimal set of edges $A_2^{(2)} \subseteq C - A_1$ such that each vertex in Q is connected to some vertex in V_e through a path in Q consisting of edges in $A_2^{(2)}$. We compute such a set of edges by collapsing the nodes of V_e into a single vertex and letting $A_2^{(2)}$ be the set of edges in a spanning tree of the resulting graph.

Let $A_2 = A_2^{(1)} \cup A_2^{(2)}$.

Let H_2 be the graph $H_1 + A_2$. Recall that T represents T_{lca} at the current stage of tree contraction.

Lemma 10 *If T is not reduced to a single node, then all the blocks of H_2 are internal.*

Proof. Let B be an external block of H_1 and let v_B denote the vertex of V_l representing the core of B in the block condensation of H_1. Let r_B denote the root of block B in T_H. By the definition of an external block r_B belongs to a stem of a leaf chain. We shall show that B is contained in an internal block of H_2. We consider 2 cases.

Case 1: $v_B \in V_e$, i.e., v_B is incident with an external edge in the set $A_2^{(1)}$ computed in step (2) of algorithm 7. Let w be the endpoint of e in H_1 that does not belong to an external block in H_1 having r_B as its root. If the fundamental cycle of e in T_H contains an edge of a stem, B will be contained in an internal block of H_2. Otherwise the fundamental cycle of e in T_H intersects a stem in T_H only in r_B. By the definition of an external edge this is only possible if w belongs to an internal block B' with root r_B. Since $w \neq r_B$, the edge in T_H from w to its parent belongs to B' and also lies on the fundamental cycle of e. Again, B will be contained in an internal block of H_2.

Case 2: $v_B \in V_l - V_e$, i.e., v_B is not incident with an external edge. First, we claim that there is a path in Q from v_B to a vertex in V_e (and hence $V_e \neq \emptyset$). Assume for a contradiction that there is no such path. Then there exists a connected component in Q that does not contain a vertex in V_e. The nodes in this component represent the cores of external blocks sharing the same root node r_B (since none of them is incident with an external edge). Furthermore, the union of these cores forms the core of a block in $H_2 + C$ rooted at r_B. If T is not reduced to a single node, there is at least one more block in $H_2 + C$ (whose root is the root of T_H), contradicting the biconnectivity of $H_2 + C$. Thus, in the subgraph of Q induced by the edges in $A_2^{(2)}$ there is a simple path from v_B to a vertex $v \in V_e$ such that none of the internal vertices on this path belongs to V_e. This path yields a simple path in H_2, not intersecting a stem, from a vertex in the core of B to a vertex in the core of a block B' rooted at r_B with the property that some vertex in the core of B' is incident with an (external) edge in $A_2^{(1)}$. Hence, B and B' are contained in a larger block B'' of H_2. From the analysis for case 1 we know that B'' is an internal block of H_2. □

In the third stage of the minimal augmentation procedure we shall compute a minimal augmentation A_3 for H_2. The following result says something about the interplay between the edges of A_2 and those of A_3. It will used in the proof of lemma 15. A *critical endpoint* of an edge of $A_2^{(1)}$ is an endpoint of that edge in V_e that is not incident with another edge of $A_2^{(1)}$. Note that each edge of $A_2^{(1)}$ has at least one critical endpoint. Recall that an edge of C is *external* if it links a vertex in the core of some external block B of H_1 with a vertex that lies outside of all the external blocks of H_1 that have the same root as B.

Lemma 11 *Let $F \subseteq C$ be a minimal augmentation for H_2. All edges of $A_2^{(2)}$ are essential in $H_2 + F$. Moreover, the critical endpoints of an edge in $A_2^{(1)}$ that is redundant in $H_2 + F$ are incident with edges of F.*

Proof. First, we show that an arbitrary edge $e \in A_2^{(2)}$ is essential in $H_2 + F$. Consider the spanning subgraph of Q induced by the edges in $A_2^{(2)}$. By the minimality of $A_2^{(2)}$ the removal of e from this subgraph yields a connected component in the subgraph that does not contain a vertex of V_e and hence no core of a block in H_1 represented by a vertex in this component is incident with an external edge in $A_1^{(1)}$. Let us call a block of H_1 whose core is represented by a vertex in this

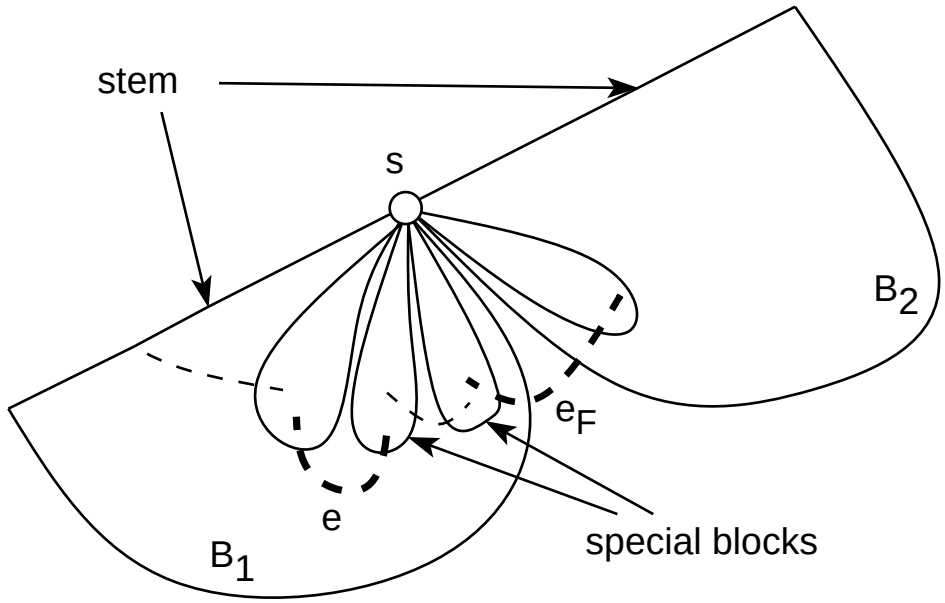


Figure 9: B_1 and B_2 are internal blocks of H_2. The two special blocks (rooted at s) are blocks of H_1 contained in B_1. Edge e belongs to $A_2^{(2)}$ and edge e_F is contained in F.

connected component *special*. Let s denote the common root in T_H of all special blocks. In $H_2 - e$ the root s of the special blocks is a cutpoint since any path from a vertex in a special block to a vertex outside the special blocks passes through s. Thus, if $F = \emptyset$, our claim is proved. Henceforth, assume $F \neq \emptyset$.

If no edge of F is incident with a vertex in a core of a *special block*, then s is a cutpoint in $(H_2 - e) + F$ and hence e is essential in $H_2 + F$. Thus, suppose that some edge $e_F \in F$ is incident on a vertex in a special block (see figure 9). By the definition of a special block no external edge is incident with the core of a special block. It follows that e_F is not external, i.e., it links a vertex in a special block with a vertex in some other external block of H_1 having root s. Since each of these two blocks is contained in a single internal block of H_2, the fundamental cycle of e_F in T_H contains edges of at most two internal blocks of H_2. Thus, because e_F is essential in $H_2 + F$ (by the assumption of the lemma), it contains edges of exactly two internal blocks of H_2, say B_1 and B_2. Both of these blocks are internal in H_2. Hence, one of them, say B_1, is rooted at s and the other block is rooted at a proper ancestor of s in T_H. This is illustrated in figure 9.

Without loss of generality all the special blocks are contained in B_1 (the case where they are contained in B_2 is treated similarly). Thus, edge e_F connects a vertex in the core of a special block with a vertex in B_2 (see figure 9). Let V_0 be the set of vertices other than s that belong to B_1 but do not belong to a special block or that belong to a block rooted at a proper descendant of s in T_H. We shall show that s is a cutpoint in $(H_2 - e) + F$ by establishing that no nontree

edge in $(H_2 - e) + F$ connects a vertex in V_0 with a vertex outside V_0 other than s. Suppose for a contradiction that there exists a nontree edge e' with this property. No edge in $H_2 - e$ has this property. Thus, e' belongs to F. Note that if e' has an endpoint in the core of a special block, then e' is not an external edge and the other endpoint of e' is contained in an external block rooted at s (the root of the special blocks). By the definition of V_0 this external block is contained in B_1. This contradicts the assumption that e' is essential in $H_2 + F$. Hence, e' does not have an endpoint in the core of a special block . It follows that the fundamental cycle of e' contains edges from both B_1 and B_2. But then e_F is redundant in $H_2 + F$, contradicting the minimality of F.

We now prove the second part of lemma 11. Suppose that a critical endpoint $v \in V_e$ of an edge $e \in A_2^{(1)}$ is not incident with an edge in F. Consider the spanning subgraph of Q induced by the edges in $A_2^{(2)}$. Let U be the unique connected component in this subgraph that contains v. Note that no vertex in U is incident with an external edge in $A_2^{(1)} - e$. By assumption no edge of F is incident with v. It follows that no external edge in $(H_2 - e) + F$ is incident with a vertex in U. Thus we can use the argument for part 1 of this proof to show that the root of the blocks whose cores are represented by vertices in U is a cutpoint in $(H_2 - e) + F$. Hence e is essential in $H_2 + F$. □

If T is reduced to a single node, every block of H_1 is an external block having as root node the root of T_H. Thus $V_e = \emptyset$, $A_2^{(1)} = \emptyset$ and $H_2 = H_1 + A_2^{(2)}$ is biconnected because $H_1 + C$ is biconnected. In this case, by lemma 11 (with $F = \emptyset$), all edges in A_2 are essential in H_2. Thus, we may immediately proceed to algorithm 10 which removes redundant edges in A_1. Henceforth, we shall assume that T is not reduced to a single node. Thus, there is a unique block, say B_0, that has the root of T_H as its root node. We denote the root of B_0 by r_0.

We shall now describe stage 3 of our minimal augmentation procedure. In this stage we compute a minimal augmentation for H_2, i.e., a minimal set $A_3 \subseteq C$ such that $H_2 + A_3$ is biconnected. For the following discussion we fix a leaf chain L of T. Number the vertices on the stem of L in T_H that are roots of blocks of H_2 consecutively as $r_1 \ldots r_k$, starting at the root closest to r_0. By lemma 8 and lemma 10 no two distinct blocks in H_2 have the same root. Denote the block of H_2 having root r_j by B_j.

Define the *range* of an edge $e = (u, v)$ of C whose lca lies on L as the integer pair (i, j), $i \leq j$, such that u is a vertex in B_i but is different from r_{i+1} and v is a vertex in the core of B_j. Note that the fundamental cycle of an edge with range (i, j) intersects exactly the blocks $B_i \ldots B_j$ in at least one edge. As an immediate consequence of this observation we get the following result.

Lemma 12 *Let e be an edge of C whose lca lies on leaf chain L and whose range is (i, j) $(i \leq j)$. Then the blocks $B_i \ldots B_j$ of H_2 are all contained in a single block of $H_2 + e$ while the remaining*

blocks of H_2 are also blocks of $H_2 + e$. □

We reduce the problem of computing a minimal augmentation for H_2 to the problem of covering stem edges considered in the previous section (algorithm 4).

Algorithm 8 Augmenting H_2 into a biconnected graph.
Input Graph H_2, edge set $C - A_1 - A_2$.
Output Minimal subset A_3 of $C - A_1 - A_2$ such that $H_2 + A_3$ is biconnected.

In parallel for each leaf chain L, do :

(1) Let P_L be the path $B_0 \dots B_k$, i.e., the vertices of P_L are the blocks $B_0 \dots B_k$ and there is an edge from B_{i-1} to B_i for $i = 1, \dots, k$.

(2) If several edges in $C - A_1 - A_2$ (whose lca's lie on leaf chain L) have the same range, eliminate all but one edge having this range (for any range (i, j)). Denote the resulting subset of $C - A_1 - A_2$ by C_0. By mapping each edge in C_0 of range (i, j) to the edge (B_i, B_j), we obtain a set C_L of edges connecting vertices of P_L. Compute a minimal subset $A_L \subseteq C_L$ whose edges cover the edges on P_L using the same method as in algorithm 4. Let $A_3^L \subseteq C_0$ be the collection of those edges in C_0 that are mapped to an edge in A_L.

Let $A_3 = \bigcup_L A_3^L$.

Let H_3 be the graph $H_2 + A_3$.

Lemma 13 *The set A_3 is a minimal augmentation for H_2, i.e., H_3 is biconnected and all edges of A_3 are essential in H_3.*

Proof. With lemma 12 it follows that H_3 is biconnected if and only if each edge of P_L is covered by an edge of A_L (i.e., it lies on the fundamental cycle in P_L of an edge of A_L) for any L. Since $H_2 + C$ is biconnected, the edges in C_L cover the edges in P_L (for each L). By lemma 6 algorithm 4 produces a minimal subset A_L of C_L covering the edges of P_L. We conclude that $\bigcup_L A^L$ is a minimal set of edges covering all the P_L's and hence A_3 is a minimal augmentation for H_2. □

As in the last section we are now faced with the problem that edges chosen at earlier stages may have become redundant because of edges added during later stages. The procedure for pruning excess edges is more complicated here than in the last section because we have added edges in three different stages.

In stage 4 of our minimal augmentation procedure we eliminate redundant edges in A_2 by resorting to a strategy similar to that used in the last section for removing edges of A_1 (algorithm 5). By

lemma 11 only the edges in $A_2^{(1)}$ may possibly be removed. The idea is to remove from H_3 several edges in $A_2^{(1)}$ at once and check if the blocks of H_2 that contain them now contain a cutpoint. We shall put back only those edges for which the corresponding block does indeed contain a cutpoint. This strategy will work provided that the fundamental cycles in T_H of the edges are vertex disjoint. To guarantee this, we shall select at each step a subset of edges that do not lie on adjacent blocks of H_2 (i.e., blocks that intersect the same stem and share a common vertex).

The following result (reminiscent of observations 1 and 2) tells us that we only need to look at at most 4 edges of $A_2^{(1)}$ in each block of H_2.

Lemma 14 *At most 4 vertices of any block of H_2 are incident with an edge of A_3^L.*

Proof. Fix a vertex B_j on P_L. By the definition of C_L (see algorithm 8, step (2)), an edge of A_3^L can only be incident in H_2 with a vertex in block B_j of H_2 if the corresponding edge of C_L has its upper endpoint (i.e., the one closer to B_0 on P_L) at B_{j+1} or its lower endpoint at B_{j-1} or is incident with B_j. By observation 2 at most 2 edges of A_L are incident with B_j. By observation 1 at most one edge of A_L has its upper endpoint at B_{j+1} and at most one edge of A_L has its lower endpoint at B_{j-1}. We conclude that at most 4 edges of A_3^L are incident with vertices in B_j. Since the two endpoints of an edge of A_3 lie in different blocks of H_2, the claim follows. □

Algorithm 9 Removing redundant edges from A_2.
Input Biconnected graph H_3, edge set A_2.
Output Biconnected spanning subgraph H_3' of H_3 such that every edge of A_2 contained in H_3' is essential in H_3'.

(0) Let $H^* = H_3$.

Process the blocks of H_2 in two phases: in phase 1 we perform steps (1) and (2) in parallel for each block B of H_2 that is at an even distance from B_0 on some P_L; in phase 2 we perform these steps in parallel for each block B that is at an odd distance from B_0 on some P_L (for all leaf chains L). (This ensures that in each phase we consider a collection of edges of $A_2^{(1)}$ whose fundamental cycles are vertex-disjoint.)

(1) Determine the subset V_B of vertices in the block B of H_2 that are incident with an edge of A_3^L. (By lemma 14 we have $|V_B| \leq 4$.)

(2) Process the vertices v of V_B sequentially as follows:

(2.1) If v is a node in the core of an external block of H_1 incident with a unique edge of $A_2^{(1)}$, remove that edge from the graph H^*.

(2.2) Determine if H^* has a cutpoint that belongs to B. If this is the case, put the edge back into the graph H^*.

(3) Let $H_3' = H^*$.

Denote by A_2' the subset of edges in A_2 that are contained in H_3'.

Lemma 15 *H_3' is a spanning biconnected subgraph of H and each edge of A_2' is essential in H_3'.*

Proof. We first observe a more general fact. Let H' be a subgraph of H containing the tree T_H. Let e be a nontree edge in H' and let w be a vertex in H' not contained on the fundamental cycle of e in T_H. We claim that w is a cutpoint in H' if and only if w is a cutpoint in $H' - e$. The *only if* part is clear. The *if* part of the statement follows from the observation that any path in H' avoiding w yields such a path in $H' - e$: simply replace each occurrence of e on this path by the path in T_H connecting the endpoints of e.

We put this observation to use as follows: fix an execution of steps (2.1) and (2.2). Let H' and H'' be the graph H^* before and after this execution. Let H' be biconnected. Suppose that after the execution of step (2.1) a vertex of B becomes a cutpoint of H^*. By the previous observation we must have removed an edge e in step (2.1) whose fundamental cycle contains w. Since all edges removed in step (2.1) belong to nonadjacent blocks of H_2, their fundamental cycles in T_H are vertex-disjoint. Hence, in step (2.1) we removed a *unique* edge e whose fundamental cycle includes w. Note that e is put back in step (2.2). By the above observation w is a cutpoint of H'' if and only if w is a cutpoint in H'. Since H' is biconnected, w is not a cutpoint of H''. Because this holds for any vertex w, the graph H'' is biconnected. Applying the above observation one more time we see that the graph $H' - e$ has w as a cutpoint. Thus, e is essential in H'. Since H'' is a biconnected spanning subgraph of H', e is also essential in H''. By lemma 11 each edge of A_2 that is redundant in H_3 will be considered in some execution of step (2.1). The claim follows. □

Above we have shown how to minimally augment graph H_1 into the biconnected graph H_3' using algorithms 7, 8 and 9. The minimal augmentation is based on the fact that graph H_1 has all of its cutpoints on stems of leaf chains in T_H. Any spanning subgraph of H containing T_H and having this property can be minimally augmented using algorithms 7, 8, and 9.

In the fifth and final stage of the minimal augmentation procedure we remove a subset of edges of A_1 that are redundant in H_3'. This stage is done roughly as follows: remove from H_3' a maximal set of edges in A_1 such that the resulting subgraph has all of its cutpoints on stems. Minimally augment this subgraph into a biconnected graph using algorithms 7, 8, and 9. Repeat these two steps one more time. We now give a more detailed description of this stage and establish its correctness.

Our goal is to remove from H_3' a subset of edges in A_1 such that the resulting graph is biconnected and all edges of A_1 in this graph are essential. Let V_l, as defined in algorithm 6, denote the set of leaves in the block condensation of $H_0 = T_H + IN + R$ that represent the cores of blocks whose root does not lie on a stem in T_H. Recall that the set A_1 forms a minimal subset in C with the property that each vertex in V_l is incident with at least one edge of A_1. Thus, each edge in A_1 has at least one endpoint in V_l that is not incident with another edge of A_1. We call such an endpoint *special*.

By lemma 9 an edge $e \in A_1$ can only be redundant in H_3' if each of its special endpoints (it has at least one) is incident with an edge of $A_2' \cup A_3$. Let $B_0 \subseteq A_1$ contain all those edges in A_1 whose special endpoints are incident with an edge of $A_2' \cup A_3$. All edges of A_1 that will be removed from H_3' in stage 5 will belong to B_0.

We start by removing the edges in B_0 from H_3'. Let $H^{(1)}$ denote the resulting graph (i.e., $H^{(1)} = H_3' - B_0$). Note that a vertex v in V_l may not be incident with an edge of C in $H^{(1)}$. In this case we say that vertex $v \in V_l$ is *exposed* in $H^{(1)}$. We take care of this problem by having each exposed vertex in V_l select a single edge of B_0 incident with it. Denote the set of selected edges by B_1 (a subset of B_0) and let $H^{(2)}$ denote the graph $H_3' - B_0 + B_1$. Although no vertex of V_l is exposed in $H^{(2)}$ (i.e., every vertex in V_l is incident with an edge of C in $H^{(2)}$), the graph $H^{(2)}$ may not be biconnected. Since each vertex of V_l is incident with an edge of C in $H^{(2)}$, we may use algorithms 7, 8 and 9 to compute a minimal augmentation B_2 for $H^{(2)}$. Let $H^{(3)}$ denote the graph $H^{(2)} + B_2 (= H_3' - B_0 + B_1 + B_2)$. By lemmas 13 and 15 the graph $H^{(3)}$ is indeed biconnected and all edges of B_2 are essential in $H^{(3)}$.

We are not done yet since some edges of $B_1 \subseteq A_1$ may be redundant in $H^{(3)}$. Note that an edge $e \in B_1$ can only be redundant in $H^{(3)}$ if both of its endpoints in V_l are incident with some edge in $A_2' \cup A_3 \cup B_2$. Let B_3 be the subset of those edges in B_1 that have this property. Remove the edges in B_3 from $H^{(3)}$ and call the resulting graph $H^{(4)}$ $(= H_3' - B_0 + B_1 + B_2 - B_3)$. Note that no vertex of V_l is exposed in $H^{(4)}$. Thus, we can compute a minimal augmentation $B_4 \subseteq B_3$ for $H^{(4)}$ using algorithms 7, 8 and 9. Let H_3'' denote the resulting graph, i.e., $H_3'' = H^{(4)} + B_4$ $(= H_3' - B_0 + B_1 + B_2 - B_3 + B_4)$. Again by lemmas 13 and 15 the graph H_3'' is indeed biconnected and the edges of B_4 are essential in H_3''. We have to show that all edges of A_1 that belong to H_3'' are essential in H_3''. We observed earlier that the edges of B_2 are essential in $H^{(3)} = H_3' - B_0 + B_1 + B_2$. Since H_3'' is a biconnected spanning subgraph of $H^{(3)}$, the edges of B_2 are also essential in H_3''. Finally, we note that the edges in $B_1 - B_3$ all have an endpoint in V_l that is not incident with an edge of $A_2' \cup A_3 \cup B_2 \cup B_4$ and not incident with another edge of B_1. Therefore, they are essential in H_3''. We conclude that all edges of A_1 are essential in H_3''.

The following is a more compact description of the algorithm that we have just outlined.

Algorithm 10 Pruning A_1.
Input Graph H_3', edge set A_1.
Output Biconnected spanning subgraph H_3'' of H_3' such that each edge of A_1 contained in H_3'' is essential.

(1) Let B_0 be the set of edges of A_1 whose special endpoints in V_l are incident with edges of $A_2' \cup A_3$. Let $H^{(1)} = H_3' - B_0$.

(2) Each exposed vertex of V_l selects a single edge of B_0 incident with it. Let $B_1 \subseteq B_0$ be the set of edges selected at this step and let $H^{(2)} = H^{(1)} + B_1$ $(= H_3' - B_0 + B_1)$.

(3) Compute a minimal augmentation $B_2 \subseteq B_0$ for $H^{(2)}$ using algorithms 7, 8 and 9. Let $H^{(3)} = H^{(2)} + B_2$ $(= H_3' - B_0 + B_1 + B_2)$.

(4) Let B_3 be the set of edges in B_1 whose endpoints in V_l are incident with edges of $A_2' \cup A_3 \cup B_2$. Let $H^{(4)} = H^{(3)} - B_3$ $(= H_3' - B_0 + B_1 + B_2 - B_3)$.

(5) Compute a minimal augmentation B_4 for $H^{(4)}$ using algorithms 7, 8 and 9.

(6) Let $H_3'' = H^{(4)} + B_4$ $(= H_3' - B_0 + B_1 + B_2 - B_3 + B_4)$.

Let A_1' be the set of edges in A_1 that are contained in H_3''. From the discussion preceding algorithm 10 we get the following result.

Lemma 16 *H_3'' is biconnected and every edge in A_1' is essential in H_3''.*

Corollary 4 *The set $A_1' \cup A_2' \cup A_3$ forms a minimal augmentation for $H_0 = T_H + IN + R$.*

Proof. By lemma 16 the graph H_3'' is biconnected and the edges of A_1' are essential in H_3''. By lemma 13 the edges of A_3 are essential in H_3. Since H_3'' is a biconnected spanning subgraph of H_3, the edges of A_3 are also essential in H_3''. Finally, by lemma 15 the edges of A_2' are essential in H_3'. Since H_3'' is a biconnected spanning subgraph of H_3', the edges of A_2' are also essential in H_3''. □

An analysis similar to the one done in section 3.3 shows that the algorithm for computing a minimal biconnected spanning subgraph runs within the same resource bounds as the algorithm for finding a minimal 2-edge-connected spanning subgraph.

5 Sequential Algorithms

In this section we give linear time algorithms for minimally augmenting a spanning tree into a 2-edge-connected graph or into a biconnected graph. Both for 2-edge-connectivity and biconnectivity

steps (0), (1) and (1.1) of algorithm 1 can be implemented to execute in linear time by making use of linear time procedures for finding an ear decomposition and for vertex triconnectivity given in [18] (see section 3.3). Thus, if we adhere to the high-level structure of algorithm 1, we obtain algorithms for computing a minimal 2-edge-connected or biconnected spanning subgraph that run in time $O(m + n \log n)$. Recently ([9]), linear time algorithms have been developed for these problems. These algorithms use the linear time augmentation procedures described in this section as subroutines. Similar linear time augmentation procedures (as well as linear time algorithms for finding minimal spanning subgraphs) have been found independently by [8].

5.1 A Linear Time Algorithm for Computing a Minimal Augmentation for 2-Edge-Connectivity

We shall first describe how to minimally augment a spanning tree with respect to 2-edge-connectivity. Assume we are given a 2-edge-connected graph H on p vertices and q edges and a spanning tree T_H in H rooted at an arbitrary vertex. We describe how to compute a minimal augmentation for T_H in H, i.e., a minimal set of nontree edges of H whose addition to T_H yields a 2-edge-connected graph. The sequential algorithm for finding a minimal augmentation for T_H is simpler than the parallel algorithm because it processes in one step a single lca instead of a collection of leaf chains.

In a preprocessing phase we number the vertices of T_H in preorder from 1 to p. Henceforth, we shall identify a vertex with its preorder number. We compute the lca's of nontree edges in H in linear time using the algorithm of [10]. We sort the nontree edges in H by their lca (in increasing order), and we compute for each edge $e = (x, y)$ of T_H the quantity $low(e)$ (or $low(x, y)$) defined as follows: $low(e)$ is the smallest lca of a nontree edge in H that covers e. Let (u, v) be an edge in T_H from a vertex u to its child v. The function low satisfies the recurrence:

$$low(u, v) = min(\{low(v, w) : w \text{ child of } v\} \cup \{lca(e) : e \text{ is a nontree edge incident with } v\}).$$

Thus, we may compute the low-value for each edge of T_H in a postorder traversal of T_H. Altogether, the preprocessing requires time $O(p + q)$ (where $p = n(H)$ and $q = m(H)$).

The following algorithm computes a minimal augmentation for T_H. Its structure is similar to that of algorithm 2: the main difference is that we process a single lca in each stage rather than a collection of lca's. As in algorithm 2 the set IN denotes the set of nontree edges that have been committed to the augmentation. The tree T_c represents the condensation of $T_H + IN$ (defined in section 3.2), i.e., T_c is obtained by collapsing the 2-edge-connected components in $T_H + IN$.

Algorithm 11 Minimally augmenting T_H into a 2-edge-connected graph.
Input 2-edge-connected graph H, spanning tree T_H in H.

Output Minimal subset IN of nontree edges in H s.t. $T_H + IN$ is 2-edge-connected.

(1) Initialize: $IN := \emptyset$ and $T_c := T_H$.

(2) For $u := n$ downto 1 do:

(2.1) Let C be the set of nontree edges in H with lca u. If $C \neq \emptyset$, perform the following two steps:

(2.2) Let V_l be the set of leaves of T_c such that the edge e from the leaf to its parent in T_c satisfies $low(e) = u$. Determine a minimal set $A \subseteq C$ such that each vertex in V_l is incident with an edge in A.

(2.3) Let $IN := IN \cup A$ and let T_c be the condensation of $T_H + IN$.

Lemma 17 *Upon termination of algorithm 11, IN is a minimal augmentation for T_H in H.*

Proof. First, we show that $T_H + IN$ is 2-edge-connected (upon termination of algorithm 11). We establish this by proving that every edge in T_H is covered by an edge of IN. Consider the iteration of algorithm 11 when some lca u is processed. Assume inductively that all edges of T_H with low-value $> u$ are covered by edges in IN chosen at previous iterations. Fix an edge e from a node v to its child w in T_c. If $low(e) = u$, then every leaf of T_c that is a descendant of w is incident with an edge of C and the edge e' to its parent in T_c satisfies $low(e') = u$. Hence, all leaves of T_c that are descendants of w belong to V_l and are incident with an edge in the set A computed at this iteration. We conclude that all edges of T_c with low-value u will be covered at this iteration. Therefore, every edge of T_H is covered by an edge of IN upon termination of algorithm 11.

To see why IN is minimal, consider the same iteration of algorithm 11. Let e be an edge in A at this iteration. Since the set A constructed in step (2.2) is minimal with respect to the property that each node in V_l is incident with an edge of A, at least one endpoint of e in V_l, say y, is incident with no other edge of A. Moreover, the edge e' from y to its parent in T_c is covered neither by an edge of IN whose lca is $> u$ (by the definition of T_c) nor by an edge with lca $< u$ (since $low(e') = u$). Hence, e' is covered only by e. It follows that all the edges in A are essential in the final graph $T_H + IN$. □

We show how to implement algorithm 11 so that it runs in time $O(p + q)$ (where $p = n(H)$ and $q = m(H)$).

The tree T_c is the condensation of $T_H + IN$. Hence, each vertex in T_c represents the set of vertices of a 2-edge-connected component of $T_H + IN$. We shall describe a method for maintaining T_c that runs in time $O(p + q)$. Our technique is similar to methods discovered independently by [23] and [12]. One result in [23] shows that the 2-edge-connected components of an initially connected graph

on n vertices can be maintained under m edge insertions in time $O(m\alpha(m,n)+n)$. This bound is improved to $O(m+n)$ by [12]. Although the bound of [12] matches our bound, we choose to present our method because it was discovered independently of [12]. Moreover, the presentation of our technique will be helpful in understanding the more complicated solution for the biconnected case.

The tree T_c is represented by a partition of the nodes of T_H into disjoint nonempty sets. We refer to a set in this collection as a *set of* T_H. Each set of T_H has a *name* which is a vertex in this set. Initially, each vertex v of T_H is in a singleton set $\{v\}$ whose name is v. At any point each set of T_H will represent the set of vertices of a 2-edge-connected component of $T_H + IN$. Hence, the vertices in this set induce a subtree in T_H. The name of the set is the root of this subtree. The following two operations access the sets of T_H:

(i) *find(x):* returns the name of the set of T_H containing node x of T_H;

(ii) *unite(x,y):* merges the sets of T_H containing x and y into a new set whose name is the name of the old set containing x.

We incorporate the edges of A into T_c (step (2.3) of algorithm 11) one at a time. Let $e=(v,w)$ be an edge in A (at the current iteration of algorithm 11) and let $u = lca(v,w)$. The addition of e to IN in step (2.3) creates a 2-edge-connected component in $T_H + IN$ whose vertex set is the union of the vertex sets of the components in $T_H + IN - e$ that contain a vertex on the fundamental cycle of e in T_H. This amounts to creating a new set of T_H that is the union of all old sets containing a node on the fundamental cycle of e in T_H. The following sequence of union and find operations creates such a set ($p(z)$ denotes the parent of a node z in T_H, $e=(v,w)$ and $u = lca(v,w)$):

$x := find(v)$;
while $x \neq u$ do begin $unite(p(x),x)$; $x := find(p(x))$ end;
$y := find(w)$;
while $y \neq u$ do begin $unite(p(y),y)$; $y := find(p(y))$ end;

All unions are of the form $unite(p(z),z)$ for some vertex z in T_H. Hence, we may use the algorithm of Gabow and Tarjan ([5]) to perform at most q union and find operations in time $O(p+q)$ (with T_H being the *union tree*). This is also the total time spent in step (2.3) over all iterations of the for-loop.

In step (2.2) we need to determine whether an endpoint in T_c of an edge in C is a leaf in T_c. To this end we maintain for each set of T_H the number of children of the corresponding vertex in T_c.

For a set named v we denote this quantity by $d(v)$. For some z in T_H let $d = d(find(z))$ and $d' = d(find(p(z))$. After executing $unite(p(z)), z)$ we update the d-values by setting $d(find(z)) = d + d' - 1$. Thus, the total time to maintain the quantities $d(v)$ is $O(p + q)$. We can determine in constant time whether an endpoint v in T_c of an edge of C is a leaf of T_c by checking whether $d(v) = 0$. We can compute the set A in step (2.2) by eliminating the edges in C one by one, making sure not to remove all the edges incident with a vertex of V_l. By maintaining for each vertex of V_l a counter recording the number of edges of C incident with it that have not been eliminated yet, the set A can be computed in time $O(|C|)$.

In summary we see that algorithm 11 computes a minimal augmentation for T_H in H in time $O(p + q)$.

5.2 A Linear Time Algorithm for Computing a Minimal Augmentation for Biconnectivity

We now describe a linear time sequential algorithm for minimally augmenting a spanning tree into a biconnected graph. Let H be a biconnected graph on p vertices and q edges and let T_H be a spanning tree in H.

In a preprocessing phase we compute for each vertex in T_H the preorder number (henceforth identifying a vertex with its preorder number), we compute the lca's of nontree edges in H (using the algorithm of [10]), we sort the nontree edges by their lca, and compute for each lca u a sorted list $L(u)$ of endpoints in T_H of nontree edges with lca u (by traversing T_H in preorder). Furthermore, we compute during a postorder traversal of T_H for each vertex v in T_H the smallest lca of a nontree edge incident with a descendant of v (we consider a vertex to be a descendant of itself). We denote this quantity by $low(v)$. The preprocessing phase requires time $O(p + q)$.

The following algorithm computes a minimal augmentation for T_H with respect to biconnectivity. As usual the set IN contains those edges that have already been chosen for the augmentation. The tree T_c represents the block condensation of $T_H + IN$ (defined in section 4.2). For a node v in T_c define $low(v)$ to be the minimum low-value of any node in T_H that is collapsed into v. Steps (2.2), (2.3) and (2.4) in the following algorithm are similar to algorithms 6, 7 and 10, respectively.

Algorithm 12 Minimally augmenting T_H into a biconnected graph.
Input Biconnected graph H, spanning tree T_H.
Output Minimal subset IN of nontree edges in H such that $T_H + IN$ is biconnected.

(1) Initialize: $IN := \emptyset$, $T_c := T_H$.

(2) For $u := n$ downto 1 do:

(2.1) Let C be the set of nontree edges whose lca is u.
If $C \neq \emptyset$, perform steps (2.2)-(2.5) below:

(2.2) Let V_l be the set of leaves v of T_c such that $low(v) = u$ and the block of $T_H + IN$ whose core is represented by v has a root different from u in T_H. Compute a minimal set $A_1 \subseteq C$ such that each vertex in V_l is incident with an edge in A_1.

(2.3) Define auxiliary graph G_u as follows: the vertices of G_u are the children $u_1, \ldots, u_l$ of u in T_H. To each edge e of C connecting a descendant of u_i to a descendant of u_j in T_H corresponds an edge (u_i, u_j) in G_u. Mark the vertices in G_u whose low-value is less than u. For $B \subseteq C$ let $G_u(B)$ be the subgraph of G_u induced by those edges in G_u that correspond to edges in B. We say that a set $B \subseteq C$ is *good* for G_u if $u = 1$ and $G_u(B)$ is connected or $u > 1$ and in $G_u(B)$ there is a path from any vertex to some marked vertex. Compute a minimal subset $A_2 \subseteq C$ such that $A_1 \cup A_2$ is good for G_u.

(2.4) Eliminate some edges from A_1 as follows:

(2.4.1) Let B_0 be the subset of edges in A_1 whose endpoints in V_l are incident with other edges in $A_1 \cup A_2$. Let $A_1^{(1)} = A_1 - B_0$.

(2.4.2) Each vertex in V_l not incident with an edge in $A_1^{(1)} \cup A_2$ selects a single edge of B_0 incident with it. Let $B_1 \subseteq B_0$ be the set of edges selected at this step and let $A_1^{(2)} = A_1^{(1)} \cup B_1$ $(= (A_1 - B_0) \cup B_1)$.

(2.4.3) Determine a minimal subset $B_2 \subseteq B_0$ such that $A_1^{(2)} \cup B_2 \cup A_2$ is good for G_u. Let $A_1^{(3)} = A_1^{(2)} \cup B_2$ $(= (A_1 - B_0) \cup B_1 \cup B_2)$.

(2.4.4) Let B_3 be the set of edges in B_1 whose endpoints in V_l are incident with edges in $B_2 \cup A_2$. Let $A_1^{(4)} = A_1^{(3)} - B_3$ $(= ((A_1 - B_0) \cup B_1 \cup B_2) - B_3)$.

(2.4.5) Determine a minimal subset $B_4 \subseteq B_3$ such that $A_1^{(4)} \cup B_4 \cup A_2$ is good for G_u.
Let $A = A_1^{(4)} \cup B_4 \cup A_2$ $(= (((A_1 - B_0) \cup B_1 \cup B_2) - B_3) \cup B_4 \cup A_2)$.

(2.5) Let $IN := IN \cup A$. Let T_c be the block condensation of $T_H + IN$. Update *low* (see below).

Lemma 18 *Upon termination of algorithm 12, IN is a minimal augmentation for T_H.*

Proof. Fix an iteration of step (2). Let IN_u denote the set of edges in IN with lca $> u$ and let R_u be the set of nontree edges with lca $< u$. Let H_0 be the graph $T_H + IN_u + R_u$. As in the proof of lemma 2 it suffices to establish that A (as defined in step (2.4.4)) is a minimal subset of C such that $H_0 + A$ is biconnected.

Assume inductively that $H_0 + C$ is biconnected. First, we prove that for any $B \subseteq C$ the graph $H_0 + B$ is biconnected if and only if each vertex in V_l is incident with an edge in B *and* B is good

for G_u. For the only-if direction we prove the contrapositive. If a vertex of V_l is not incident with an edge of B, then this vertex represents the core of a block in $H_0 + B$ whose root is a proper descendant of u, implying that $H_0 + B$ is not biconnected. Also if B is not good for G_u then, in the graph $G_u(B)$, there exist at least two connected components at least one of which does not contain a marked vertex. The union of the cores corresponding to vertices in a component that does not contain a marked vertex forms the core of a block of $H_0 + B$ that does not contain all the vertices of T_H, implying again that H_0 is not biconnected.

Now we prove the if-part of the above claim. We consider the two cases $u = 1$ and $u > 1$. If $u = 1$ we argue as follows. Since $H_0 + C$ is biconnected, the tree T_c has no leaf with a low-value > 1 because the root of the corresponding block in H_0 would be a cutpoint in $H_0 + C$. Since all the vertices in V_l are incident with an edge in B, it follows that there can be no cutpoint in $H_0 + B$ other than u (the root of T_H). Since B is good for G_u, $G_u(B)$ is connected and any two vertices in $H_0 + B$ other than the root of T_H are connected by a path in $H_0 + B$ not containing the root of T_H. Hence, the root of T_H is not a cutpoint of $H_0 + B$ and $H_0 + B$ is indeed biconnected. Now consider the case $u > 1$. As before we observe that no leaf in T_c has a low-value $> u$. Since any vertex in V_l is incident with an edge in B, there can be no cutpoint in $H_0 + B$ other than u. Since B is good for G_u, there is a path between any two vertices in $H_0 + B$ other than u that avoids vertex u. Hence, u is not a cutpoint in $H_0 + B$. We conclude that $H_0 + B$ is biconnected.

We now check that A (as defined in step (2.4.5)) is a minimal subset of C such that each vertex in V_l is incident with an edge in A *and* A is good for G_u. (The following argument is very similar to that preceding algorithm 10.) Let us denote these two properties by (P1) and (P2), respectively. By step (2.4.5) of algorithm 12 the set A satisfies property (P2). We need to verify that each vertex in V_l is incident with an edge in A (i.e., A satisfies property (P1)). By step (2.4.2) each vertex of V_l is incident with an edge of $A_1^{(2)} \cup A_2$ and hence with an edge in $A_1^{(3)} \cup A_2$. Let e be an edge in B_3. By the definition of B_1 any edge in B_1 has exactly one endpoint that is incident with an edge of A_2. Since $B_3 \subseteq B_1$, this also holds for edge e. Again by the definition of B_1 the endpoint of an edge in B_1 that is not incident with an edge of A_2 is incident only with this edge of B_1. It then follows from the definition of B_3 that both endpoints of edge $e \in B_3$ are incident with edges in $A_2 \cup B_2$. Since $(A_2 \cup B_2) \cap B_3 = \emptyset$, it follows that each vertex of V_l is incident with an edge in A.

Let us say that an edge $e \in A$ is *essential for* (P1) if $A - \{e\}$ does not satisfy property (P1) and edge $e \in A$ is *essential for* (P2) if $A - \{e\}$ does not satisfy property (P2). We have to show that each edge A is essential for (P1) or (P2). (Note that an edge may be essential for both properties.) By step (2.4.5) of algorithm 12 each edge in B_4 is essential for (P2). Furthermore, each edge in $B_1 - B_3$ has one endpoint in V_l that is not incident with another edge in $A_1^{(3)} \cup A_2$. Since $A \subseteq A_1^{(3)} \cup A_2$, this implies that the edges in $B_1 - B_3$ are essential for (P1). By step (2.4.3) B_2 is a minimal subset of

B_0 such that $A_1^{(2)} \cup B_2 \cup A_2$ is good for G_u. Since $A \subseteq A_1^{(2)} \cup B_2 \cup A_2$, the edges in B_2 are essential for (P2). Finally we observe that each edge in $A_1 - B_0$ has at least one endpoint in V_l that is not incident with another edge in $A_1 \cup A_2$; it is therefore essential for (P1). □

We show how to implement the various steps so that the running time of algorithm 12 is $O(p+q)$ (where $p = n(H)$ and $q = m(H)$).

We first describe how to maintain the block condensation T_c of $T_H + IN$ at each step. Some methods are known (e.g., [23] and [12]) for maintaining the biconnected components of a graph under edge insertions. These methods however are superlinear in the number of edge insertions. We shall explain how T_c can be maintained in linear time.

We represent T_c by a partition of the nodes of T_H; as before, we refer to a set in this partition as a *set of* T_H. Let us first assume that each set of T_H corresponds to a node of T_c, the tree T_c being the block condensation of $T_H + IN$. Hence, a set of T_H is either the singleton set consisting of the root of T_H or it contains the nodes in the core of a block of $T_H + IN$. Below we shall see that this requirement needs to be relaxed.

In step (2.5) we compute T_c by incorporating the edges in A one by one. Fix an edge $e = (v, w)$ of A with $lca(e) = u$. Adding e to IN in step (2.5) results in a block in $T_H + IN + e$ whose root is u and whose core is the union of the cores in $T_H + IN$ containing a vertex other than u on the fundamental cycle of e in T_H. Unfortunately, the unions of the corresponding sets of T_H are not all of the form $unite(p(z), z)$ where $p(z)$ denotes the parent of a node z in T_H (unless $v = u$ or $w = u$). For instance if u has two children v and w that are both in a set by themselves then the addition of edge $e = (v, w)$ in step (2.5) amounts to taking the union of sets $\{v\}$ and $\{w\}$; this union is clearly not of the form $unite(p(z), z)$. Thus, the algorithm of Gabow and Tarjan ([5]) does not apply directly.

We overcome this problem by deferring some of the union operations to a later point. Inductively, we assume that the collection of sets of T_H has the following two properties: (1) the vertices in a set of T_H induce a subtree of T_H; we take the root of that subtree as the *name* of the set. (2) The vertex set of a core of a block in $T_H + IN$ is equal to the union of sets of T_H whose names are children of the same node of T_H. We implement this by assigning to each set a *block number*: two sets have the same block number iff they are contained in the core of a single block of $T_H + IN$. We choose for the block number the smallest name (w.r.t. preorder numbering) of a set of T_H that is contained in the core of a given block of $T_H + IN$. We maintain for each block number a list of sets that have this block number (in order to perform the union operations efficiently).

Let $e = (v, w)$ be an edge of A with $lca(e) = u$. After adding edge e to IN in step (2.5), we update the sets of T_H as follows (assume v and w are both different from u): we create one new set by

taking the union of all sets containing a vertex other than u on the path from v to u in T_H and then adding those sets having the same block number as one of these sets; we create another set by proceeding similarly for the path from w to u. If we do this for every edge of A, we get a collection of sets whose names are children of u. We determine the block number of each such set using an auxiliary graph G_A. The vertices of G_A are the children $u_1, \ldots, u_l$ of u in T_H. There is an edge in G_A from u_i to u_j if an edge of A connects a descendant of u_i with a descendant of u_j. Compute the connected components of G_A. All sets whose names are part of the same component receive the same block number, namely the smallest name in the component. We associate with each block number i the low-value $low(i)$ of the core of the corresponding block of T_H (i.e., the minimum low-value of any vertex of T_H in the core). After we have computed T_c in step (2.5) we update the quantity $low(i)$ as follows: let C_i be a component of G_A that gets assigned block number i. The quantity $low(i)$ is set to the minimum low-value of any child of u in C_i. All of these steps can be performed in time $O(|C| + l)$ where l is the number of children of u in T_H. (To construct the adjacency list for G_A, see the implementation of step (2.3) below.)

To see that the sets of T_H satisfy the two properties stated above, note that as soon as a set x becomes part of a larger set y, all sets with the same block number as x are also merged into y. Furthermore, each new set at the end of the current execution of step (2.5) has a child of u as its name. It follows that the sets of T_H do indeed satisfy the two properties mentioned above. We conclude that the total number of union and find operations is $O(p+q)$. Since all unions are of the form $unite(p(z), z)$, we can use the algorithm of Gabow and Tarjan ([5]). This gives us an overall time bound of $O(p+q)$ for all executions of step (2.5).

We now describe the implementation of steps (2.2), (2.3), and (2.4) of algorithm 12. In step (2.2) we need to be able to check quickly whether an edge of C is incident with a leaf in V_l. For this we maintain, for each block number i, the number of children of the corresponding node in T_c. Let $d(i)$ denote the number of children of the node in T_c corresponding to block number i. We update the d-values as follows: recall that we incorporate an edge $e = (v, w)$ with lca u into T_c by taking the union of sets along the paths from v to u and from w to u. After we execute $unite(z, p(z))$ where z is a vertex on one of these two paths with block number j and its parent $p(z)$ in T_H has some block number $k < j$, we set $d(k) := d(j) + d(k) - 1$. After incorporating all the edges in A we have a collection of sets of T_H whose names are children of u in T_H. Let C_i be the connected component of G_A (see above) whose minimum name is i. As described above all sets whose name is in C_i receive block number i. We set $d(i)$ to the sum of the d-values of the names contained in component C_i. With the low-values and d-values, we can compute the set A_1 in step (2.2) in time $O(|C|)$.

To construct the adjacency list for G_u in step (2.3), we need to determine for each endpoint other than u of an edge of C the unique child of u of which it is a descendant in T_H. Using the sorted

sequence $L(u)$ (see preprocessing stage) of endpoints of edges of C in T_H and the preorder numbers of the children of u in T_H, this can be done in time $O(|C| + l)$ where l is the number of children of u in T_H. To compute a minimal set A_2 such that $A_1 \cup A_2$ is good for G_u, we first collapse the vertex set of each connected component in $G_u(A_1)$, then collapse all vertices in the resulting graph corresponding to a component of $G_u(A_1)$ that contains a marked vertex into a single new vertex. In the resulting graph we compute a spanning tree and let A_2 be the set of those edges in C that correspond to an edge of this spanning tree. Thus, we spend $O(p+q)$ time on all executions of step (2.3). Step (2.4) requires time proportional to the size of G_u, i.e., $O(|C| + l)$. Using the techniques we described above we can implement one execution of steps (2.4.1)-(2.4.4) in time $O(|C| + l)$.

Altogether, we see that algorithm 12 runs in time $O(p + q)$ as claimed.

6 Concluding Remarks

In this paper we have presented efficient parallel and sequential algorithms for the problems of finding a minimal 2-edge-connected spanning subgraph of a 2-edge-connected graph and finding a minimal biconnected spanning subgraph of a biconnected graph.

The algorithms for both problems have a similar high-level structure: repeatedly compute a spanning tree of the input graph with the smallest possible number of redundant edges and minimally augment this tree. This strategy is useful for finding minimal subgraphs of a graph with respect to other properties. In particular this approach gives similar algorithms for the problem of finding a minimal k-connected subgraph of a graph (for any k), assuming we have a method for augmenting a spanning tree with respect to these properties.

In [9] we describe refinements for algorithm 1 that yield linear time sequential algorithms for the above problems. The algorithms for both problems use the linear time augmentation procedures described in section 5 as subroutines. These results reduce the parallel work required for these problems by a factor of $\Theta(\log n)$.

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