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DUALITY AND SENSITIVITY ANALYSIS
FOR FRACTIONAL PROGRAMS

by

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and

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ABSTRACT

In this paper, we consider algorithms, duality and sensitivity analysis for optimization problems, called fractional, whose objective function is the ratio of two real valued functions. We discuss a procedure suggested by Dinkelbach for solving the problem, its relationship to certain approaches via variable transformations, and a variant of the procedure which has convenient convergence properties. The duality correspondences that are developed do not require either differentiability or the existence of optimal solution. The sensitivity analysis applies to linear fractional problems, even when they "solve" at an extreme ray, and includes a primal-dual algorithm for parametric right-hand-side analysis.

Introduction

The problem of maximizing (or minimizing) the ratio of two real valued functions $n(\cdot)$ and $d(\cdot)$ over a subset F of R^n is called a fractional program. When n(x) and d(x) are linear affine functions and the feasible region F is a polyhedral set the problem becomes the much analyzed linear fractional program. Our purpose here is to consider algorithmic approaches to the problem, to provide saddlepoint duality correspondence, and to develop sensitivity procedures for the linear fractional case. We also present a primal-dual procedure for parametric right-hand-side analysis for the linear fractional problem and introduce an algorithm for the general fractional problem that is a variant of a procedure developed previously by Dinkelbach [18].

Much previous research on the fractional program has dealt with the linear fractional case. Two approaches have been taken. The well known algorithm of Charnes and Cooper [13] transforms the problem into a linear program by modifying the feasible region. It applies when the feasible region has an optimal solution. The other approach solves a sequence of linear programs (or at least one pivot step of each linear program) over the original feasible region F by updating the linear programs objective function. Algorithms in this category are all related to ideas originating with Isbell and Marlow [26] and have been proposed by Abadie and Williams [1], Bitran and Novaes [8], Dorn [19], Gilmore and Gomory [23] and Martos [36]. These algorithms can be viewed as a specialization of either the Frank-Wolfe approach for nonlinear objective functions [21] or Martos' adjacent vertex programming methods [37]. As Yuan and Wagner have observed [45], these two approaches are equivalent when the feasible region is compact in the sense they lead to an identical sequence of pivoting operations.

The algorithms in the second category, though they exploit the underlying linearity of the linear fractional model, are specializations of Dinkelbach's [18] algorithm for the general fractional problem. In the next section we review this algorithm and mention a modification with useful a priori error bounds. We also show that this algorithm is a dual method for solving versions of the problem that have been suggested by Bradley and Frey [10] and Schaible [39] which generalize the Charnes and Cooper variable transformation. This connection between these two alternative approaches extends the observation of Yuan and Wagner concerning the linear fractional problem.

The saddlepoint duality theory that we consider in section 2 uses a Lagrangian function introduced by Gol'štein[24] and studied by Bector [6] which uses multipliers dependent upon the primal variables. The theory leads to a dual problem that is again a fractional program. To our knowledge, these results provide one of the few instances where saddlepoint duality applies to a large class of nonconvex problems, geometric programming duality [20] being a notable illustration.

In section 3, we provide sensitivity procedures for variations in the problem data of a linear fractional program which extends material developed by Aggarwal [2], [3], and [4] for compact feasible regions. The results are analogous to the usual sensitivity procedures of linear programming, but now include variations in both the numerator and denominator of the objective function as well as right-hand-sides of the constraints. The next section continues this study by introducing a primal-dual algorithm for parametric right-hand-side analysis. This algorithm suggests a branch and bound procedure for the integer programming version of the linear fractional problem which we discuss briefly.

The fractional model arises naturally for maximizing return per unit time in dynamic situations or return per unit trip in transportation settings [22]. It also arises when minimizing the ratio of return to risk in financial applications [10].

For a number of reasons, applications of the linear fractional model have been far less numerous than those of linear programming. The essential linearity of many models is certainly a contributing factor. In addition, linear fractional applications are easily disguised as linear programs when the feasible region is bounded. In this case, the fractional model can be reduced to a linear program through variable transformations. Finally, the fact that direct sensitivity analysis is not widely available for the fractional model on commercial programming systems may have some bearing on potential applications. In any event, the model has been applied to study changes in the cost coefficients of a transportation problem [14], to the cutting stock problem [23], to Markov decision processes resulting from maintenance and repair problems [17], [29] and to fire programming games [26]. It also has been used for a marine transportation problem [22], for Chebyshev maximization problems [9], and for primal-dual approaches to decomposition procedures [7], [32].

1. Transformations and Algorithms

In this section, we discuss methods for analyzing fractional programming problems, particularly an extension of the well-known Charnes and Cooper approach for transforming the problem into an alternate form and the relationship between this approach and an algorithm due to Dinkelbach. We also discuss a variant of Dinkelbach's algorithm which has convenient convergence properties.

For notation, assume that the fractional problem is written as:

$$v = \sup\{f(x) = \frac{n(x)}{d(x)} : x \in F\} \tag{P}$$

where $F = \{x \in X \subseteq R^n : g(x) \geq 0\}$ and $n(\cdot)$, $d(\cdot)$ and the component functions $g_i(\cdot)$ for $i = 1,\ldots,m$ are real valued functions defined on R^n. We assume that $d(x) > 0$ for all $x \in X$ so that the problem is well posed (if $d(\cdot) < 0$ on X write $f(x)$ as $\frac{-n(x)}{-d(x)}$). When F is a convex set, $n(x)$ is concave on X and $d(x)$ is convex on X, we say that the problem is concave-convex. In this case, $f(x)$ is a strictly quasi-concave function. Finally, when both $n(x)$ and $d(x)$ are linear-affine (linear plus a constant) and F is polyhedral, (P) is called a linear fractional program.

Transformations

We can decouple the numerator and denominator in (P) by introducing a (necessarily positive) real valued variable t to form the equivalent problem:

$$v = \sup\{n(x)t : d(x)t = 1,\ x \in F,\ t > 0\}. \tag{1.1}$$

Charnes and Cooper [14] first used this transformation for linear fractional programs. As Schaible noted [39], if $n(\bar{x}) \geq 0$ for at least one point $\bar{x}$ of F, then the transformed problem can be expressed as:

$$v = \sup\{n(x)t : d(x)t \leq 1, x \in F, t > 0\}. \tag{1.2}$$

The values v in (P) and (1.2) are the same and x^* is an optimal solution to (P) if and only if x^* and $t^* = 1/d(x^*)$ solves (1.2).

The transformed problems are especially helpful for analyzing concave-convex fractional problems. In this case, elementary results on convex analysis [38] show that the function $n(\frac{y}{t})t$ is concave and the function $d(\frac{y}{t})t$ is convex on the convex set $Y = \{(y,t) \in R^{n+1} : (y/t) \in F, t > 0\}$. Making the substitution of variables, $y = xt$, the original (nonconvex) problem is reformulated as the convex programming problem:

$$v = \sup\{n(\tfrac{y}{t})t : d(\tfrac{y}{t})t \leq 1, (y,t) \in Y\}. \tag{1.3}$$

(If d(x) is linear affine, this substitution in (1.1) also produces a convex program.) This version of the problem is useful since convex analysis can be applied to (1.3) to develop qualitative insight such as duality theory about the fractional problem (P) and concave programming algorithms can be applied directly to (1.3) in order to solve (P).

In practice, to provide closed feasible regions it is attractive to use $t \geq 0$ instead of $t > 0$ in the transformed problems as in Charnes and Cooper's original paper. For the transformed problem to remain equivalent to the original problem then requires further hypothesis such as the existence of an optimal solution x^* to (P). For details see [39].

These transformations to fractional problems have been studied by Schaible [39] and independently by Bradley and Frey [10] under slightly different hypothesis. Though Bradley and Frey formally consider fractional programs whose numerator and denominator are both positively homogeneous of degree one, their results apply, in general, by rewriting (P) in homogeneous form as

$$v = \sup\{f(y/t) = \frac{n(y/t)t}{d(y/t)t} : (y,t) \in Y,\ t = 1\}$$

To conform with the Bradley-Frey paper, we also would write the constraints $g(x) \geq 0$ in homogeneous form as $g(y/t)t \geq 0$.

Dinkelbach's Algorithm

Dinkelbach [18] suggests a "parametric" algorithm for solving the fractional problem which seems to be quite different then these parameter free transformation approaches. He introduces a real valued parameter to decouple the numerator and denominator via an auxiliary problem:

$$v(k) = \sup\{r(x,k) = n(x) - kd(x) : x \in F\}. \qquad \text{(A)}$$

Observe that $v(k) > 0$ if and only if $n(x) - kd(x) > 0$ for some point $x \in F$. Consequently, given any $k = f(x)$ with $x \in F$ or merely $k = \lim_{j\to+\infty} f(x^j)$ with $x^j \in F$, we may determine whether or not $k = v$ by solving the auxiliary optimization problem. There are two possible outcomes to (A):

(1) $v(k) \leq 0$. Then $n(x) - kd(x) \leq 0$ or $n(x)/d(x) \leq k$ for all $x \in F$ so that $v = k$.

(2) $v(k) > 0$. In solving the auxiliary problem we will identify a point $y \in F$ with $f(y) > k$, i.e., $n(y) - kd(y) > 0$.

Dinkelbach uses this observation to devise an algorithm for the fractional problem which can be viewed as a dual method for solving the transformed problem (1.1). To develop this interpretation of the algorithm, suppose that we apply generalized programming [16] to (1.1). That is, given m points (x^j, t^j) with $x^j \varepsilon F$, $t^j > 0$, we solve the following linear programming approximation in the variables $\Theta_1, \Theta_2, \ldots, \Theta_m$:

$$\begin{aligned} z^m = \max \ & \sum_{j=1}^{m} n(x^j)t^j\Theta_j \\ \text{subject to} \ & \sum_{j=1}^{m} d(x^j)t^j\Theta_j = 1 \\ & \sum_{j=1}^{m} \Theta_j = 1 \\ & \Theta_j \geq 0 \quad (j = 1,\ldots,m). \end{aligned}$$

Substituting $u_j = t^j\Theta_j$, the problem becomes

$$\begin{aligned} z^m = \max \ & \sum_{j=1}^{m} n(x^j)u_j \\ \text{subject to} \ & \sum_{j=1}^{m} d(x^j)u_j = 1 \\ & u_j \geq 0 \quad (j = 1,\ldots,m). \end{aligned} \tag{1.4}$$

The solution to this linear program is

$$z^m = \max\{\frac{n(x^j)}{n(x^j)} : 1 \leq j \leq m\}$$

with an optimal shadow price of z^m associated with the single equality constraint.The generalized programming, or column generation, algorithm when applied to (1.4) next solves the auxiliary problem (A) with $k = z^m$. If $v(k) \leq 0$, the algorithm terminates; otherwise, assume that x^{m+1} solves the auxiliary problem with objective value $r(x^{m+1}, k) > 0$. Then x^{m+1} and an associated variable u_{m+1} are added to the linear program (1.4) and the process is repeated. The algorithm is illustrated pictorially in Figure 1.

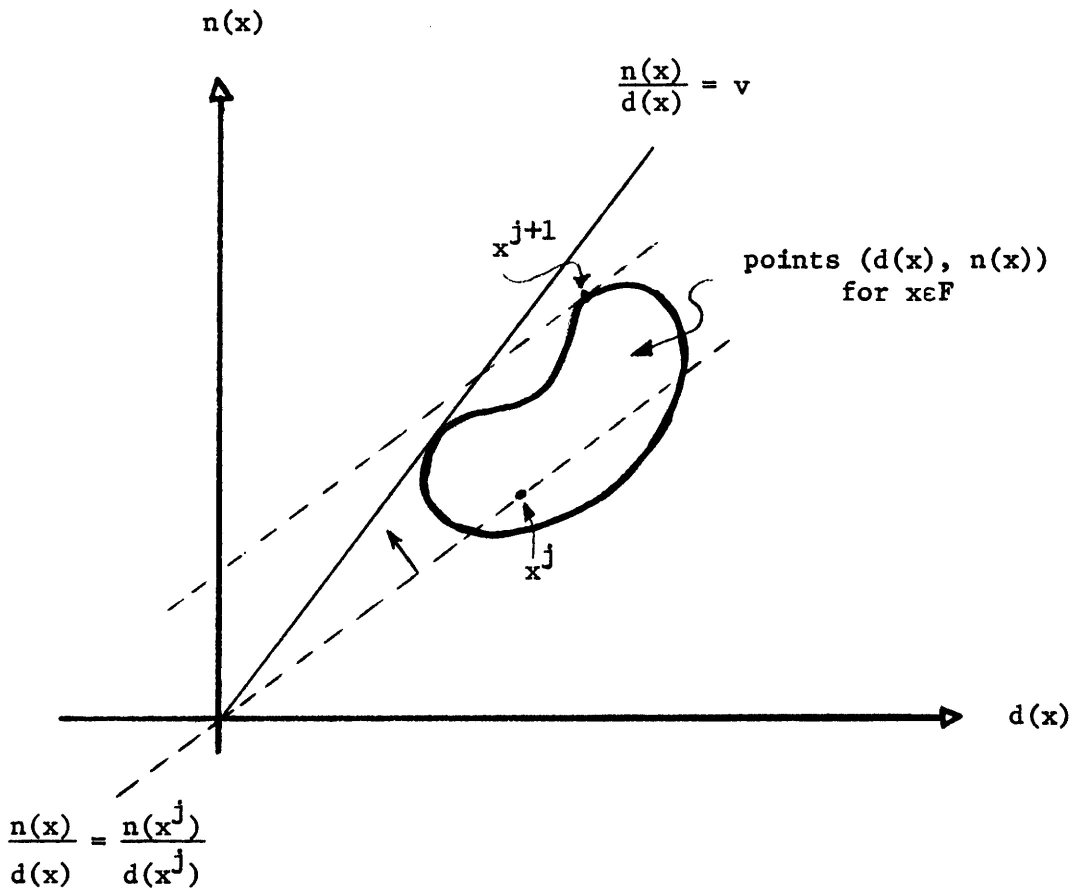

Figure 1. Generalized Programming Algorithm.

These steps are those proposed by Dinkelbach except that he uses a termination criterion of stopping when $v(k) \leq \delta$ for some predetermined tolerance $\delta > 0$. Since generalized programming is a dual approach for solving arbitrary mathematical programs (see [35] for example), the preceding discussion shows that Dinkelbach's algorithm is a dual method for solving the transformed problem (1.1).

By modifying this algorithm slightly, we can insure that the auxiliary problem is solved only finitely many times to determine an ε-optimal solution for any given $\varepsilon > 0$. We use the generalized programming algorithm with $\delta = 0$, but set $k = z^m + \varepsilon$ is solving the auxiliary problem at each step. Then

(i) $v(k) \leq 0$ implies that $n(x) - (z^m + \varepsilon)d(x) \leq 0$ or that $\frac{n(x)}{d(x)} \leq z^m + \varepsilon$ for all $x \in F$; and

(ii) $v(k) > 0$ implies that $r(x^{m+1}, k) > 0$ for some $x \in F$ or that $z^{m+1} \equiv f(x^{m+1}) = \frac{n(x^{m+1})}{d(x^{m+1})} > z^m + \varepsilon$.

Step (ii) implies that the objective value z^m of the best known feasible point increases by at least ε after each solution to the auxiliary problem; therefore assuming that $v < +\infty$, the algorithm must terminate after finitely many auxiliary problems have been solved. Moreover, at termination condition (i) implies that:

$$v = \sup\{\frac{n(x)}{d(x)} : x \in F\} \leq z^m + \varepsilon.$$

That is, the point x^m with objective value $z^m = f(x^m)$ is an ε-optimal solution.

Observe that none of these generalized programming algorithms require any convexity or concavity assumptions for these applications and that there is no "duality gap" between v and the limiting value of the z^m. For Dinkelbach's method, though, we must be able to solve the auxiliary problem to completion to obtain an optimal solution, x^{m+1}. For our modification, we must determine a point x^{m+1} such that $r(x^{m+1},k) > 0$ or be able to show that no such point exists.

Also, note that Dinkelbach obtains a δ-optimal solution to the auxiliary problem at termination, but not necessarily a δ-optimal solution to the original fractional problem because

$$v(z^m) = n(x^{m+1}) - z^m d(x^{m+1}) \leq \delta$$

implies that

$$z^m \leq z^{m+1} = \frac{n(x^{m+1})}{d(x^{m+1})} \leq z^m + \frac{\delta}{d(x^{m+1})} \leq v + \frac{\delta}{d(x^{m+1})} .$$

Thus z^{m+1} is an ε-optimal solution for $\varepsilon = \delta/d(x^{m+1})$, which is usually unknown in advance unless we have an a priori lower bound for $d(x)$ over the feasible region F. When $v(z^m) = r(x^{m+1}, z^m) \geq \delta$ in condition (ii), then

$$z^{m+1} \equiv \frac{n(x^{m+1})}{d(x^{m+1})} \geq z^m + \frac{\delta}{d(x^{m+1})} .$$

so that z^m increases by at least a fixed amount δ/σ if $\sigma = \sup\{d(x) : x \in F\} < +\infty$. (In particular, if F is compact and $d(x)$ is continuous as in Dinkelbach's paper). When $\sigma < +\infty$ this argument provides an alternate proof of convergence of his algorithm even for $\delta = 0$, which additionally shows that the auxiliary problem is solved only finitely many times if $\delta > 0$.

Other Applications

Mangasarian [34] studies application of the Frank-Wolfe algorithm [21] for solving the fractional problem when the feasible set is polyhedral. Sharma [40] presents a feasible direction approach to the fractional problem when the feasible region F is convex and the objective function is the quotient of two linear affine functions. Sinha and Wadhwa [43] consider the case where the feasible set is bounded and the objective function is the product or the quotient of two functions each homogeneous of degree one plus a constant. Wagner and Yuan [45] establish the equivalence between Charnes and Cooper [14] and Martos algorithms [36] for linear fractional programs (in the sense that these algorithms visit identical sequences of basis in solving the problem). Jagannathan [28] presents several results for the auxiliary problem (A) when F is bounded.

2. Duality

Ordinary Lagrangian saddlepoint duality theory is inadequate for dealing with fractional programs since it leads to duality gaps even for linear fractional programs. Table 2.1 at the end of this section includes an example (number 1) of such duality gaps.

Gol'štein [24], however, introduced a "fractional Lagrangian"

$$L(x,u) = \frac{n(x)}{d(x)} + u\frac{g(x)}{d(x)}$$

and showed the saddlepoint duality correspondence

$$w \equiv \inf_{u\geq 0} \sup_{x\varepsilon X} L(x,u) = \sup_{x\varepsilon X} \inf_{u\geq 0} L(x,u) \tag{2.1}$$

under hypothesis to be discussed below. The inf sup problem is called the <u>fractional dual problem</u> and the sup inf problem is the original problem in the sense of the following lemma.

<u>Lemma 2.1</u>: $v = \sup_{x\varepsilon X} \inf_{u\geq 0} L(x,u)$. Moreover, for any $x \varepsilon F$

$$\inf_{u\geq 0} L(x,u) = f(x).$$

Additional results concerning fractional duality theory have been developed by Bector [6], Jagannathan [28] and Schaible [39] for concave-convex programs and by Sharma and Swarup [41], Craven and Mond [15], Kornbluth and Salkin [30] and Kydland [31] for linear fractional programs.

Bector applies the Kuhn-Tucker conditions to the following equivalent formulation of the problem

$$v = \sup\{f(x) = \frac{n(x)}{d(x)} : x \in X \text{ and } \frac{g(x)}{d(x)} \geq 0\}. \qquad (2.2)$$

Assuming that F is compact and that n(x), d(x) and each $g_i(x)$ are differentiable, he exhibits the duality correspondence (2.1) for concave-convex problem in differentiable form. He also discusses several converse duality properties and relates duality to the auxiliary problem (A).

Since the fractional Lagrangian agrees with the usual Lagrangian applied to (2.2), many results from Lagrangian duality are valid for fractional duality. In particular, we note

Lemma 2.2 (Weak duality and complementary slackness): Assume that $\bar{x} \in F$ and $u \in R^m$, $u \geq 0$. Then $f(\bar{x}) \leq \sup_{x \in X} L(x,u)$. Consequently, $v \leq w$. Moreover, if $\bar{x}$ solves the primal problem (P), if $\bar{u}$ solves the fractional dual problem, and if $v = w$, then $\bar{u}g(\bar{x}) = 0$. Thus $v = \sup_{x \in X} L(x,\bar{u}) = L(\bar{x},\bar{u})$.

Schaible [39] obtains related results by applying ordinary Lagrangian duality to the transformed problem (1.3). Jagannathan [28] also develops similar differentiable duality properties.

In this section, we complement these results by exhibiting two theorems for providing fractional duality in (2.1) without enforcing compactness or differentialibity assumptions. We show that in many instances (P) inherits duality from the auxiliary problem (A) when k = v:

$$\sup\{r(x,v) = n(x) - vd(x) : x \in X \text{ and } g(x) \geq 0\}. \qquad (A1)$$

That is, generally, when ordinary duality holds for (A1), fractional duality holds for (P). Our first result assumes that there is a dual variable $\bar{u}$ solving the Lagrangian dual of (A1).

Theorem 2.1: Suppose that $F \neq \emptyset$, $v < +\infty$ and that

$$\sup_{x \in F}\{n(x) - vd(x)\} = \min_{u \geq 0} \sup_{x \in X}\{n(x) - vd(x) + ug(x)\}$$

then

$$v = \min_{u \geq 0} \sup_{x \in X}\{f(x) + u\frac{g(x)}{d(x)}\} = w.$$

Proof: From our previous observations in section 1 about the auxiliary problem

$$\sup_{x \in F}\{n(x) - vd(x)\} \leq 0$$

Thus by hypothesis there is a $\bar{u} \geq 0$ satisfying

$$\sup_{x \in X}\{n(x) - vd(x) + \bar{u}g(x)\} \leq 0 \qquad \text{so that}$$

$$n(x) - vd(x) + \bar{u}g(x) \leq 0 \qquad \text{for all } x \in X$$

since $d(x) > 0$ for $x \in X$, dividing by $d(x)$ preserves the inequality

$$\frac{n(x)}{d(x)} + \bar{u}\frac{g(x)}{d(x)} \leq v \qquad \text{for all } x \in X.$$

Thus

$$w = \inf_{u \geq 0} \sup_{x \in X}\{L(x,u)\} \leq \sup_{x \in X}\{L(x,\bar{u})\} \leq v.$$

But, weak duality states that $w \geq v$ and consequently $w = v$. //

The existence of a dual variable $\bar{u}$ solving the Lagrangian dual to the auxiliary problem is guaranteed, for example, if the auxiliary problem is concave and satisfies the Slater condition [34], or is a linear program, or more generally a concave quadratic programming program.

Fractional duality also is valid when Lagrangian duality holds for the auxiliary problem, but without optimizing dual variables, as long as d(x) is bounded away from zero on X.

Theorem 2.2: Suppose that $F \neq \emptyset$, that $v < +\infty$ and that

$$\sup_{x\in F}\{n(x) - vd(x)\} = \inf_{u\geq 0}\ \sup_{x\in X}\{n(x) - vd(x) + ug(x)\}$$

If there is a $\delta > 0$ such that $d(x) \geq \delta$ for all $x \in X$, then

$$v = \inf_{u\geq 0}\ \sup_{x\in X}\left\{\frac{n(x)}{d(x)} + u\frac{g(x)}{d(x)}\right\} = w.$$

Proof: As in Theorem 1, $\sup_{x\in F}\{n(x) - vd(x)\} \leq 0$ and thus by hypothesis given $\varepsilon > 0$ there is a $u^{\varepsilon} \geq 0$ satisfying

$$\sup_{x\in X}\{n(x) - vd(x) + u^{\varepsilon}g(x)\} \leq \varepsilon$$

since $d(x) \geq \delta > 0$ for all $x \in X$ this implies that

$$\frac{n(x)}{d(x)} + u^{\varepsilon}\frac{g(x)}{d(x)} \leq v + \frac{\varepsilon}{d(x)} \leq v + \frac{\varepsilon}{\delta} \qquad \text{for all } x \in X$$

But $\varepsilon > 0$ is arbitrary and consequently

$$w = \inf_{u\geq 0}\ \sup_{x\in X}\left\{\frac{n(x)}{d(x)} + u\frac{g(x)}{d(x)}\right\} \leq v$$

Coupled with weak duality $w \geq v$, this provides the desired result. //

Note that if $v = +\infty$, then weak duality states that $w = +\infty$ so that fractional duality is valid in this case as well.

Gol'štein gives a combined version of Theorems 2.1 and 2.2. He assumes both the Slater condition and that d(x) is bounded away from zero on X and is able to give infinite dimensional versions of the basic duality correspondence (2.1). His results do not, however, apply directly to linear fractional programs such as example 2 of Table 2.1 since the Slater condition does not hold. This example also illustrates that the saddle-point theory of Theorems (2.1) and (2.2) is not a byproduct of the differentiable approaches discussed above since there is no point x^* solving the original fractional problem.

Finally, we should note that the results in Theorem 2.1 and 2.2 can be obtained by applying duality to the transformed problem (1.3) instead of the auxiliary problem (A1). It is easy to show, for example, that if the Slater condition applies to (A1), then it applies to (1.3). Also, the dual multiplier π associated with the constraint $d(y/t)t \leq 1$ in the dual problem to (1.2) must equal v. Thus the dual problems to (1.2) and (A1) are the same and the given proofs apply to (1.2).

We conclude this section with additional examples. Example 3 shows that some requirement such as the hypothesis in Theorem 2.1 is needed to assure that fractional duality $v = w$ holds. On the other hand, example 4 shows that fractional duality may hold when $d(x)$ is not bounded from below over X. The final example illustrates a situation when ordinary Lagrangian duality applies and fractional Lagrangian duality does not. This example does not violate any of the above results since $[n(x) - vd(x)]$ is not concave.

Observe that ordinary Lagrangian duality does hold for the auxiliary problem of example 3 and that there is no optimizing dual variables to the Lagrangian dual to the auxiliary problem of example 4. Also, for the last example the usual Lagrangian gives $\inf_{u \geq 0} \sup_{x \in X}[f(x) + ug(x)] = -\frac{2}{5}$ by taking $u = .12$.

Example	X	n(x)	d(x)	g(x)	v	$r = \sup_{x\epsilon F}[n(x) - vd(x)]$	$w = \inf_{u\geq 0} \sup_{x\epsilon X} \left[\frac{n(x) + ug(x)}{d(x)}\right]$
1	$\{x\epsilon R\colon x \geq 0\}$	1	$x + 1$	$x - 1$	$\frac{1}{2}$	0	1/2
2	$\{(x_1,x_2)\epsilon R^2\colon x_1 \geq 0,\ x_2 \geq 0\}$	$-x_1 + 2x_2 - 2$	x_2	$x_1 - x_2 - 1$ $-x_1 + x_2 + 1$	1	-3	1
3	$\{(x_1,x_2)\epsilon R^2\colon x_1 \geq 0,\ x_2 \leq \sqrt{x_1}\}$	x_2	$\sqrt{x_1}$ if $x_1 > 0$ 1 if $x_1 = 0$	$-x_1$	0	0	1
4	$\{(x_1,x_2)\epsilon R^2\colon x_1 \geq 0,\ x_2 \leq x_1^{3/4}\}$	x_2	$\sqrt{x_1}$ if $x_1 > 0$ 1 if $x_1 = 0$	$-x_1$	0	0	0
5	$\{x\epsilon R\colon x \geq 1\}$	$-x$	$1 + x^2$	$2 - x$	$-\frac{2}{5}$	0	0

Table 2.1

3. Sensitivity Analysis for the Linear Fractional Program

The linear fractional model assumes that $f(x)$ is the ratio of two affine functions $n(x) = c_o + cx$ and $d(x) = d_o + dx$, that $g(x) = Ax - b$ and that $X = \{x \in R^n : x \geq 0\}$. (We assume that $d_o + dx > 0$ for $x \in F$ and may include the redundant condition $d_o + dx > 0$ in X if we like, i.e., replace X with $X \cap \{x \in R^n : d_o + dx > 0\}$, to conform with the basic assumptions that we previously have imposed on X.) In this case, the auxiliary problem becomes

$$v(k) = \sup_{x \in F}\{r(x,k) = (c_o - kd_o) + (c - kd)x\} \tag{A2}$$

where

$$F = \{x \in R^n : x \geq 0 \text{ and } Ax = b\}.$$

Since (A2) is a linear program much of the usual linear programming sensitivity analysis can be extended to this more general setting. Aggarwal [2], [3] and [4], Swarup [44], Chadha [11], Chadha and Shivpuri [12] and Gupta [25] have developed much of this analysis when the feasible region F is compact. In this section we summarize some of their results and indicate extensions that apply when compactness assumptions are relaxed.

As shown by Abadie and Williams [1] and others, the linear fractional problem "solves" either at an extreme point or extreme ray of the feasible region. The sensitivity analysis will indicate when the solution remains optimal. First let us set some notation. Suppose that $A = [B,N]$ where B is a feasible basis and that x, c and d are partitioned conformally as

(x^B, x^N), (c^B, c^N) and (d^B, d^N). Then $x^B = \bar{b} - \bar{N}x^N$ where $\bar{b} = B^{-1}b$, $\bar{N} = B^{-1}N$ and $f(x)$ can be written in terms of the nonbasic variables x^N as

$$f(x) = \frac{c_o + c^B(\bar{b} - \bar{N}x^N) + c^N x^N}{d_o + d^B(\bar{b} - \bar{N}x^N) + d^N x^N} = \frac{\bar{c}_o + \bar{c}^N x^N}{\bar{d}_o + \bar{d}^N x^N} \tag{3.1}$$

where

$$\bar{c}_o = c_o + c^B\bar{b} \qquad \bar{d}_o = d_o + d^B\bar{b} \qquad \bar{c}^N = c^N - c^B\bar{N} \qquad \bar{d}^N = d^N - d^B\bar{N}.$$

Using (3.1) we see that if $k = f(\bar{x})$ for any given extreme point $\bar{x} \varepsilon F$, then the objective function $n(x) - kd(x)$ for (A2) is expressed in terms of $\bar{x}^N$ as

$$\bar{c}_o - f(\bar{x})\bar{d}_o + \bar{t}^N x^N$$

where $\bar{t}^N = \bar{c}^N - f(\bar{x})\bar{d}^N$. But since $\bar{x}^N = 0$ for the extreme point, the objective value $f(\bar{x}) = {}^{\bar{c}_o}/_{\bar{d}_o}$ and the auxiliary objective function becomes simply

$$\bar{t}^N x^N \tag{3.2}$$

The simplex multipliers corresponding to the objective function in the auxiliary problem (A2) and the basis B are $\pi = (c^B - kd^B)B^{-1} = \pi^n - k\pi^d$ where $\pi^n = c^B B^{-1}$ and $\pi^d = d^B B^{-1}$. Note that this data for $\bar{c}_o$, $\bar{c}^N$ and π^n is precisely that carried by the simplex method with objective function $c_o + cx$ and similarly $\bar{d}_o$, $\bar{d}^N$, and π^d is the data carried with objective function $d_o + dx$.

Suppose first that the fractional problem solves at an extreme point $\bar{x} = (\bar{x}^B, \bar{x}^N)$ of the feasible region and that B is the corresponding basis of A. By the observations made in section 1 concerning the auxiliary problem $v = f(x^o)$ as long as $v(f(x^o)) = 0$ in (A2). But since the auxiliary objective function in (3.2) is $\bar{t}^N x^N$ the extreme point $\bar{x}$ remains optimal as long as the coefficients of the nonbasic variables x^N are nonpositive, that is $\bar{t}^N + \Delta\bar{t}^N \leq 0$ where

$$\Delta\bar{t}^N = \Delta\bar{c}^N - f(x^o)\Delta\bar{d}^N - [\Delta f(x^o)]\bar{d}^N - \Delta f(x^o)\Delta\bar{d}^N. \tag{3.3}$$

Table 1 summarizes results obtained by Aggarwal for altering the initial data c_o, d_o, c^B, d^B, c^N, d^N and b by translations parametrized by the variable δ. We include this summary for contrast with sensitivity analysis given below when the problem solves at an extreme ray. Observe that except for case 6, the interval of variation of δ, for which the basis remains optimal, are specified by linear inequalities in δ. These results are derived by substitution in (3.3) and the usual arguments of linear programming sensitivity analysis.

Next suppose that the fractional problem "solves" at an extreme ray r, i.e., for any $x \in F$, $f(x + \lambda r) = \frac{cx + \lambda cr}{dx + \lambda dr}$ approaches $\frac{cr}{dr} = v$ as $\lambda \to +\infty$. The analysis changes somewhat in this case. If $v = \bar{f} = +\infty$ then $cr > 0$, $dr = 0$ and the solution remains $+\infty$ as long as these conditions are maintained. When $v = \bar{f} < +\infty$ the final auxiliary problem is

$$\bar{v} = v(\bar{f}) = \sup_{x\varepsilon F}\{(c_o - \bar{f}d_o) + (c - \bar{f}d)x\} \tag{A2}$$

Let $\bar{x}$ be the optimal extreme point solution to this problem and let B be the corresponding basis. B will be optimal whenever it is feasible and

$$\bar{t}^N = \bar{c}^N - \bar{f}\bar{d}^N = (c^N - \bar{f}d^N) - \pi A^N \leq 0. \tag{3.4}$$

The optimal value to (A2) determined by this solution will be nonpositive if

$$\bar{v} = v(\bar{f}) = (c_o - \bar{f}d_o) + (c - \bar{f}d)\bar{x} = (c_o - \bar{f}d_o) + \pi b \leq 0. \tag{3.5}$$

Consequently, the ray r remains optimal whenever (3.4) and (3.5) are satisfied after the data change. Note that the optimal basis B to (A2) can change, yet maintaining $\bar{v} \leq 0$ and r optimal. Considering only (3.4) and (3.5) provides conservative bounds for $\bar{f}$ to remain optimal. These can be read directly from the final linear programming tableau to (A2). When the optimal basis B changes, the sensitivity analysis can be continued by pivoting to determine the new optimal basis.

The resulting ranges on δ that satisfy these conditions are summarized in Table 2. Except for cases (11) and (13), the ranges are obtained by substituting the parameter change δ and its effect upon $\bar{f}$ and π into (3.4) and (3.5). We distinguish two possibilities for cases (11) and (13). When $r_j = 0$ in either case, the updated denominator $\hat{d}r = dr + \delta r_j = 0$ for any δ and the problem remains unbounded. If $r_j > 0$, however, then $\hat{d}r \neq 0$ for any $\delta \neq 0$ so that the extreme ray does not continue to give $\bar{f} = +\infty$. If $\delta \leq 0$, then $\hat{d}(x + \lambda r) = \hat{d}x + \lambda\hat{d}r$ becomes negative for any x as $\lambda \to +\infty$ since $\hat{d}r < 0$. Thus as long as there is a point $x^o \in F$ with $\hat{d}(x^o) > 0$, the problem will be ill posed because $\hat{d}(x)$ will be zero for some $x \in F$. Since $\hat{d}x = dx + \delta x_j \leq 0$ for all $x \in F$ is equivalent to

$$\delta \leq \delta_2 \equiv \sup\{-\frac{dx}{x_j} : x \in F\}$$

the problem is ill posed for $0 \geq \delta \geq \delta_2$. For $\delta < \delta_2$, $d(x) < 0$ for all $x \in F$ and we may analyze the problem as indicated above.

If $\delta > 0$, then the auxiliary problem evaluated at

$k = \lim_{\lambda\to+\infty} \dfrac{n(x + \lambda r)}{\hat{d}(x + \lambda r)} = \dfrac{cr}{\delta r_j}$ becomes

$$\max\{[c - \frac{cr}{\delta r_j}(d + \delta e_j)]x : x \in F\}$$

where e_j denotes the j^{th} unit vector and the optimality conditions reduce to case (ii).

In both Tables 1 and 2, changes that involve a variation in $d(x) = d_o + dx$ include the condition $d(x) > 0$ for all $x \in F$ so that the problem remains well posed. For changing d_o to $d_o + \delta$, this condition requires

$$\delta + d_o + dx > 0 \qquad \text{for all } x \in F$$

or equivalently

$$\delta > \delta_o \equiv -\inf\{d_o + dx : x \in F\} \tag{3.6}$$

For changing d_j to $d_j + \delta$, the condition requires

$$d_o + dx + \delta x_j > 0 \qquad \text{for all } x \in F$$

or that

$$\delta > \delta_1 \equiv \sup\{\frac{-d(x)}{x_m} : x \in F\} \tag{3.7}$$

When the right-hand-side of the p^{th} constraint is being varied, the appropriate range on δ can be determined by solving the parametric linear program

$$\inf\{d(x) : x \geq 0 \text{ and } Ax = b + \delta e_p\}$$

The behavior of the problem as δ approaches its lower bound in expressions (3.6) or (3.7) is specified in the following two lemmas. As above let $-\delta_o = \inf\{d(x) : x \in F\}$; also, let $D = \{x \in F : d(x) = -\delta_o\}$.

<u>Lemma 3.1</u>: Assume that $n(x) > 0$ for at least one $x \in D$. Then the following two conditions are equivalent:

(i) $\bar{x}$ solves the linear fractional program

$$\max\{\frac{n(x)}{d(x) + \delta} : x \in F\} \tag{3.8}$$

for all $\delta_o < \delta \leq \bar{\delta}$ for some $\bar{\delta} > \delta_o$.

(ii) $\bar{x} \in D$ and $\bar{x}$ solves the linear program

$$\max\{n(x) : x \in D\} \tag{3.9}$$

<u>Proof</u>: <u>(i) => (ii)</u>. Let $\delta_o < \delta \leq \bar{\delta}$. By hypothesis $\frac{n(\bar{x})}{d(\bar{x}) + \delta} \geq \frac{n(x)}{d(x) + \delta}$ for all $x \in F$. Since $d(x) + \delta > 0$ on F and $n(x) > 0$ for some $x \in F$, then necessarily $n(\bar{x}) > 0$. If $\bar{x} \notin D$, then $d(\bar{x}) = -\delta_o + \varepsilon$ for some $\varepsilon > 0$ and

$$\frac{n(\bar{x})}{d(x) + \delta} = \frac{n(\bar{x})}{\delta - \delta_o + \varepsilon} < \frac{n(\bar{x})}{\varepsilon} \tag{3.10}$$

But for any $x^* \in D$ with $n(x^*) > 0$, $d(x^*) = -\delta_o$ and

$$\frac{n(x^*)}{d(x^*) + \delta} = \frac{n(x^*)}{\delta - \delta_o} \tag{3.11}$$

approaches $+\infty$ as $\delta \to \delta_o$. This conclusion with (3.10) implies that $\bar{x} \in D$.

Next note from (3.11), that $\frac{n(\bar{x})}{d(\bar{x}) + \delta} \geq \frac{n(x^*)}{d(x^*) + \delta}$ for any $x^* \in D$ if and only if $n(\bar{x}) \geq n(x^*)$. Thus for $n(\bar{x})$ to solve the linear fractional program at value δ, requires then $\bar{x}$ solves the linear program (3.9).

<u>(ii) => (i)</u>. Recall that for any value of $\delta > \delta_o$, the linear fractional program (3.8) solves at either an extreme point or extreme ray of the feasible region. Let x^j for $j = 1,\ldots,J$ denote the extreme points of F not lying in D, and let r^i for $i=1,\ldots,I$ denote the extreme rays of F. For each x^j, since $d(x^j) = -\delta_o + \gamma_j$ with $\gamma_j > 0$

$$\frac{n(x^j)}{d(x^j) + \delta} < \frac{n(x^j)}{\gamma_j} \quad \text{or} \quad \frac{n(x^j)}{d(x^j) + \delta} \leq 0.$$

depending upon whether $n(x^j) > 0$ or $n(x^j) \leq 0$. Let

$$\varepsilon_1 = \max\{\frac{n(x^j)}{\gamma_j} : 1 \leq j \leq J\}$$

$$\varepsilon_2 = \max\{\frac{cr^i}{dr^i} : 1 \leq i \leq I\}.$$

Then for any $\bar{x} \in D$ with $n(\bar{x}) > 0$ and any $\delta > \delta_o$

$$\frac{n(\bar{x})}{d(\bar{x}) + \delta} = \frac{n(\bar{x})}{\delta - \delta_o}$$

is larger than the objective function evaluated at any x^j or r^i as long as

$$\frac{n(\bar{x})}{\delta - \delta_o} \geq \varepsilon = \max(\varepsilon_1, \varepsilon_2, 1)$$

or $0 < \delta - \delta_o \leq \frac{n(\bar{x})}{\varepsilon}$. Moreover, if $\bar{x}$ solves the linear program (3.9), then (3.11) implies that $\bar{x}$ solves the linear fractional program (3.8) over $x \in D$. Thus for $\delta_o < \delta \leq \bar{\delta} \equiv \delta_o + \frac{n(\bar{x})}{\varepsilon}$, $\frac{n(\bar{x})}{d(\bar{x}) + \delta}$ is as large as the objective function evaluated at any extreme point or extreme ray of F, and consequently is optimal in (3.8). //

Next consider changing d_j to d_j to $d_j + \delta$ and let δ_1 be the minimum δ such that $d(x) + \delta x_j \geq 0$ for all $x \in F$, i.e., $\delta_1 = \sup\{-\frac{d(x)}{x_j} : x \in F\}$. Let $D' = \{x \in F : \frac{d(x)}{x_j} = -\delta_1\}$.

<u>Lemma 3.2</u>: Assume that $n(x) > 0$ for at least one $x \in D'$. Then the following are equivalent:

(i) $\bar{x}$ solves the linear fractional program

$$\max\{\frac{n(x)}{d(x) + \delta x_j} : x \in F\} \tag{3.12}$$

for all $\delta_1 < \delta \leq \delta'$ for some $\delta' > \delta_1$.

(ii) $\bar{x} \in D'$ and $\bar{x}$ solves the linear fractional program

$$\max\{\frac{n(x)}{x_j} : x \in D'\}. \tag{3.13}$$

The proof of this lemma is similar to that of lemma 3.2 and is omitted.

Note that these lemmas provide simple methods for computing the optimal solution to the fractional program as δ approaches the appropriate lower bound. We first solve for a point contained in D (or D') and then append the linear constraint $d(x) = -\delta_o$ (or $d(x) = -\delta_1 x_j$) to F to define D (or D'). The point $\bar{x}$ then can be found by solving the linear program (3.9) in lemma 3.1 or the linear fractional program (3.13) in lemma 3.2. An alternative procedure is to use a parametric linear fractional program similar, but more elementary, than the procedure introduced in the next section to determine the optimal solution to the linear fractional program as δ is decreased.

DATA CHANGE	$\Delta \bar{c}_k^N$	$\Delta \bar{d}_k^N$	$\Delta f(\bar{x})$	Necessary and sufficients conditions for the optimal basis to remain optimal. $\bar{x}$ is the optimal extreme point.
1) $\Delta c_o = \delta$	0	0	$\frac{\delta}{d(\bar{x})}$	$\bar{t}^N - \frac{\delta}{d(\bar{x})} \bar{d}^N \leq 0$
2) $\Delta d_o = \delta$	0	0	$\frac{-f(\bar{x})\delta}{d(\bar{x}) + \delta}$	$\bar{t}^N + \frac{\delta f(\bar{x})\bar{d}^N}{d(\bar{x}) + \delta} \leq 0$ $\delta > \delta_o$
3) $\Delta c_j = \delta$ ($\bar{x}_j$ nonbasic)	δ for $k = j$ 0 for $k \neq j$	0	0	$\bar{t}_j^N + \delta \leq 0$
4) $\Delta d_j = \delta$ ($\bar{x}_j$ nonbasic)	0	δ for $k = j$ 0 for $k \neq j$	0	$\bar{t}_j^N - \delta f(\bar{x}) \leq 0$ $\delta > \delta_1$
5) $\Delta c_j = \delta$ ($\bar{x}_j$ i^{th} basic variable)	$-\delta B_{i}^{-1} . A^N$	0	$\frac{\delta \bar{x}_j}{d(\bar{x})}$	$\bar{t}^N - \delta B_{i:}^{-1} A^N - \frac{\delta \bar{x}_j}{d(\bar{x})} \bar{d}^N \leq 0$
6) $\Delta d_j = \delta$ ($\bar{x}_j$ i^{th} basic variable)	0	$-\delta B_i^{-1} . A^N$	$-\frac{f(\bar{x})\delta \bar{x}_j}{d(\bar{x}) + \delta \bar{x}_j}$	$\bar{t}^N + f(\bar{x})\delta B_i^{-1} . A^N + \frac{f(\bar{x})\delta \bar{x}_j}{d(\bar{x}) + \delta \bar{x}_j}[\bar{d}^N - \delta B_i^{-1} . A^N] \leq 0$ $\delta > \delta_1$
7) $\Delta b_i = \delta$	0	0	$\frac{\pi_i \delta}{d(\bar{x}) + \pi_i^d \delta}$	$\bar{t}^N - \frac{\pi_i \delta}{d(\bar{x}) + \pi_i^d} \bar{d}^N \leq 0$ $\bar{x}^B + \delta B_{.i}^{-1} \geq 0$ and $d_o + dx > 0$ for all $x \in F(\delta)$

TABLE 1

TABLE 2

DATA CHANGE	Δc_0	Δd_0	Δc_ℓ	Δd_ℓ	$\Delta\bar{f}$	$\Delta\pi$	Conditions for the optimal ray r to remain optimal. The optimal value is $\bar{f}\cdot B^{-1}b = \bar{x}^B$, $\bar{x}$ is the optimal extreme point solving (A1) with $k = \bar{f}$.	
							$\bar{f} = +\infty$	$\bar{f} < +\infty$ $\bar{v} = (c_0-\bar{f}d_0) + \pi b$, $\bar{t}^N = c^N - \bar{f}d^N - \pi A^N$
8 $\Delta c_0 = \delta$	δ	0	0	0	0	0	The problem is unbounded for any δ	$\bar{v} + \delta \le 0$
9 $\Delta d_0 = \delta$	0	δ	0	0	0	0	$\delta > -d(\bar{x})$	$\bar{v} - \bar{f}\delta \le 0$ $\delta > \delta_0$
10 $\Delta c_j = \delta$ nonbasic	0	0	δ $\ell = j$ 0 $\ell \ne j$	0	$\frac{\delta r_j}{dr}$	$-\frac{\delta r_j}{dr}\pi^d$	$\hat{c}r = cr + \delta r_j > 0$	$\bar{t}_i^N + \delta\delta_{ij} - \frac{\delta r_j}{dr}\bar{d}_i^N \le 0$ for all i $\bar{v} - \frac{\delta r_j}{dr}(d_0+\pi^d b) \le 0$
11 $\Delta d_j = \delta$ nonbasic	0	0	0	δ $\ell = j$ 0 $\ell \ne j$	$-\frac{cr\delta r_j}{dr(dr+\delta r_j)}$	$+\frac{cr\delta r_j}{dr(dr+\delta r_j)}\pi^d$	$\hat{d}r = dr + \delta r_j = \delta r_j = 0$ (The problem remains unbounded if $r_j = 0$)	$\bar{t}_i^N + \frac{cr\delta r_j}{dr(dr+\delta r_j)}(\bar{d}_i^N + \delta\delta_{ij}) - \delta_{ij}\bar{f}\delta \le 0$ for all i $\bar{v} + \frac{cr\delta r_j}{dr(dr+\delta r_j)}(d_0+\pi^d b) \le 0$ $\delta > \delta_1$
12 $\Delta c_j = \delta$ p^{th} basic variable	0	0	δ $\ell = j$ 0 $\ell \ne j$	0	$\frac{\delta r_j}{dr}$	$\delta B_{p.}^{-1} - \frac{\delta r_j}{dr}\pi^d$	$\hat{c}r = cr = \delta r_j > 0$	$\bar{t}_i^N - \frac{\delta r_j}{dr}\bar{d}_i^N - \delta B_{p.}^{-1}A_i^N \le 0$ for all i $\bar{v} - \frac{\delta r_j}{dr}(d_0+\pi^d b) + \delta B_{p.}^{-1}b \le 0$
13 $\Delta d_j = \delta$ p^{th} basic variable	0	0	0	δ $\ell = j$ $\ell \ne j$	$-\frac{cr\delta r_j}{dr(dr+\delta r_j)}$	$-\bar{f}\delta B_{p.}^{-1}$ $+\frac{cr\delta r_j}{dr(dr+\delta r_j)}(\pi^d+\delta B_{p.}^{-1})$	$\delta r_j = 0$	$\bar{t}_i^N + \frac{cr\delta r_j}{dr(dr+\delta r_j)}[\bar{d}_i^N - \delta B_{p.}^{-1}A_i^N] + \bar{f}\delta B_{p.}^{-1}A_i^N \le 0$ $\bar{v} + \frac{cr\delta r_j}{dr(dr+\delta r_j)}[d_0 + (\pi^d+\delta B_{p.}^{-1})b] - \bar{f}\delta B_{p.}^{-1}b \le 0$ $\delta > \delta_1$
14 $\Delta b_p = \delta$	0	0	0	0	0	0	The problem is unbounded for any δ such that the feasible region is not empty.	$\bar{v} + \pi_i\delta \le 0$ $\bar{x}^B + \delta B_{.i}^{-1} \ge 0$ $d_0 + dx > 0$

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

$B_{p.}^{-1}$ and $B_{.i}^{-1}$ are the p^{th} row and i^{th} column of B^{-1}.

4. A Primal-Dual Parametric Algorithm

Having solved a linear fractional problem, we can investigate parametric right hand-side variations by a primal-dual algorithm. The linear fractional algorithm terminates by solving the linear program ($v < +\infty$)

$$\max\{(c_o - kd_o) + (c - kd)x : x \geq 0 \quad \text{and} \quad Ax \leq b\} \qquad \text{(A2)}$$

with $k = v$, the optimal value to the problem.

Suppose that y is a fixed vector and that the right-hand-side of the structural constraints is parameterized by a scalar Θ as $b + \Theta y$. For each fixed value of Θ, we can find the optimal solution first by applying the dual simplex algorithm to (A2) with b replaced by $b + \Theta y$, obtaining a feasible solution x^o, and then by using x^o as a starting point for an application of the linear fractional algorithm. This approach ignores the dependency of k on x when solving for x^o by the dual simplex algorithm.

In contrast, we can extend ideas of the self-parametric algorithm of linear programming [16] to develop a primal-dual algorithm which uses both primal and dual simplex pivots to maintain an optimal basis as the parameter Θ is varied. Suppose that $\bar{x}$ is an optimal extreme point to the fractional problem at $\Theta = 0$ and that B is the corresponding basis of $[A,I]$. This basis remains optimal as long as it is both

$$\text{primal feasible, i.e., } \bar{b} + \Theta\bar{y} \geq 0 \qquad (\bar{b} = B^{-1}b, \bar{y} = B^{-1}y) \quad (4.1)$$

and dual feasible, i.e., $\bar{t}^N(\Theta) = \bar{c}^N - f(x(\Theta))\bar{d}^N \leq 0$

where from (3.1),

$$f(x(\Theta)) = \frac{c_o + c^B\bar{b} + \Theta c^B\bar{y}}{d_o + d^B\bar{b} + \Theta d^B\bar{y}}$$

For the previous analysis to be valid, we require, in addition, that $d_o + dx$ remains positive on the feasible region. This may require solving $\min\{[d_o + dx] : Ax \leq b + \Theta y,\ x \geq 0\}$. Below we always assume that this condition is met.

Using the notation and techniques of the previous section, we easily see that the reduced cost as a function of Θ is given by

$$\bar{t}^N(\Theta) = \bar{t}^N - \frac{\Theta\pi y}{d(\bar{x}) + \Theta\pi^d y}\bar{d}^N \ , \qquad \bar{t}^N = \bar{c}^N - f(x(0))\bar{d}^N \tag{4.2}$$

and that the dual feasible condition $\bar{t}^N(\Theta) \leq 0$ becomes

$$\Theta[\pi^d y\bar{t}^N - \pi y\bar{d}^N] \leq -\bar{t}^N d(\bar{x}). \tag{4.3}$$

As long as conditions (4.1) and (4.3) hold, the basic feasible solution given by (4.1) is optimal for the parameterized problem. To study the solution behavior for $\Theta \geq 0$, Θ is increased until at $\Theta = \Theta_o$ equality occurs in one component of either (4.1) or (4.3) and any further increase in Θ causes one of these inequalities to be violated. Thus, either some basic variable reaches zero or some nonbasic objective coefficient reaches zero. We make the nondegeneracy assumption that exactly one primal or dual condition in (4.1) or (4.3) is constraining at $\Theta = \Theta_o$ and distinguish two cases:

Case 1: The i^{th} basic variable reaches zero, i.e., $\bar{b}_i + \Theta_o\bar{y}_i = 0$ and $\bar{y}_i < 0$. The new value for $\bar{t}^N(\Theta)$ is computed by (4.2) and then a usual dual simplex pivot is made in row i (if every constraint coefficient in row i is nonnegative, linear programming duality theory [16] shows that the problem is infeasible for $\Theta > \Theta_o$ and the procedure terminates). Since the i^{th} basic variable is zero at Θ_o, this pivot simply re-expresses the extreme point given by (4.1) at $\Theta = \Theta_o$ in terms of a new basis. Consequently, $k = f(x(\Theta_o))$ does not vary during the pivot.

Case 2(a): The j^{th} reduced cost $\bar{t}_j^N(\Theta)$ reaches zero. A primal simplex pivot is made in column j (the unbounded situation is considered below). Since $\bar{t}_j^N(\Theta) = 0$ and $\bar{t}_i^N(\Theta) < 0$ for $i \neq j$ at $\Theta = \Theta_o$, after this pivot the resulting basis will be optimal.

By our nondegeneracy assumption, the new basis B_o determined after the pivot in either case will be optimal for Θ in some interval $[\theta_o, \Theta_1]$, $\Theta_1 > \Theta_o$. Expressing $b + \Theta y$ for $\Theta \geq \Theta_o$ as $(b + \Theta_o y) + (\Theta - \Theta_o)y$, the parametric analysis given above can be repeated with $(b + \Theta_o y)$ replacing b. The procedure will continue in this fashion by increasing Θ and successively reapplying cases 1 and 2. Since both conditions (4.1) and (4.3) are linear in Θ, a given basis will be optimal (i.e., both primal and dual feasible) for some interval of Θ. Evoking nondegeneracy, this implies that no basis will repeat as Θ increases and establishes finite convergence of the method.

To complete the analysis, let us see how to proceed in case 2 if:

Case 2(b): Every constraint coefficient in column j is nonpositive, so that no primal pivot can be made. Then an extreme ray r is identified which for $\Theta = \Theta_o$ satisfies

$$f(x(\Theta_o) + \lambda r) \rightarrow \frac{cr}{dr} = f(x(\Theta_o)) \quad \text{as } \lambda \text{ approaches } +\infty.$$

Thus at $\Theta = \Theta_o$ the extreme ray r becomes optimal and $k = \frac{cr}{dr}$. Since k does not depend upon Θ, (A2) now is solved by the dual simplex algorithm as a linear program with a parametric right-hand-side. A lemma below will show that the ray r remains optimal as Θ is increased as long as the problem remains feasible.

Having completed the analysis if the problem solves at an extreme point at $\Theta = 0$, let us now suppose that the extreme ray r is optimal at $\Theta = 0$. The dual simplex procedure of Case 2(b) is applied until at $\Theta = \Theta_o$, the objective value reaches zero and further increase of Θ causes it to become positive. In this case the optimal basis at $\Theta = \Theta_o$ corresponds to an extreme point satisfying $f(x(\Theta)) > \frac{cr}{dr}$ for $\Theta = \Theta_o^+$ and we revert to cases 1 and 2 above with this extreme point to continue the analysis for $\Theta > \Theta_o$.

A final point to note here is that the optimal value of the linear fractional problem, or more generally a concave-convex fractional problem, as a function of Θ is quasi-concave. Formally,

Lemma 4.1: Let y be a fixed vector in R^m and let $v(\Theta) = \sup\{f(x) = \frac{n(x)}{d(x)} : g(x) \geq \Theta y\}$ be the optimal value function for a concave-convex fractional program with g(x) concave that is parameterized by the scalar Θ. Then $v(\Theta)$ is a quasi-concave function on the set $\Omega = \{\Theta \varepsilon R : g(x) \geq \Theta y \text{ for some } x \varepsilon R^m \text{ and } v(\Theta) < +\infty\}$.

Proof: We must show that $\{\Theta \varepsilon \Omega : v(\Theta) \geq k\}$ is convex for any real number k. Let $\Theta^1, \Theta^2 \varepsilon \Omega$ with $v(\Theta^1) \geq k$, $v(\Theta^2) \geq k$ and let $\lambda \varepsilon [0,1]$. For any $\varepsilon > 0$, there are $x^1, x^2 \varepsilon R^n$ with $g(x^j) \geq \Theta^j y$ and $f(x^j) \geq k - \varepsilon$ for $j = 1, 2$. By concavity of $g(\cdot)$, $g(\hat{x}) \geq \hat{\Theta} y$ where $\hat{x} = \lambda x^1 + (1 - \lambda)x^2$ and $\hat{\Theta} = \lambda\Theta^1 + (1 - \lambda)\Theta^2$. Also,

$$v(\hat{\Theta}) \geq f(\hat{x}) = \frac{n(\hat{x})}{d(\hat{x})} \geq \frac{\lambda n(x^1) + (1-\lambda)n(x^2)}{\lambda d(x^1) + (1-\lambda)d(x^2)} =$$

$$= \frac{\lambda f(x^1)d(x^1) + (1-\lambda)f(x^2)d(x^2)}{\lambda d(x^1) + (1-\lambda)d(x^2)}$$

by concavity of n(x) and convexity of $d(\cdot)$. Thus $v(\hat{\Theta}) \geq \min[f(x^1), f(x^2)] \geq k - \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $v(\hat{\Theta}) \geq k$ and $v(\Theta)$ is quasi-concave on Ω. //

<u>Corollary 4.1</u>: If the supremum is attained for every $\Theta\varepsilon\Omega$ in the previous lemma, then $v(\Theta)$ is strictly quasi-concave on Ω.

<u>Proof</u>: Let $f(x^1) = v(\Theta^1)$, $f(x^2) = v(\Theta^2)$ and suppose $v(\Theta^1) > v(\Theta^2)$ in the proof of the lemma. Then for $\lambda\varepsilon(0,1)$

$$v(\hat{\Theta}) \geq \frac{\lambda v(\Theta^1)d(x^1) + (1-\lambda)v(\Theta^2)d(x^2)}{\lambda d(x^1) + (1-\lambda)d(x^2)} > v(\Theta^2) \qquad //$$

<u>Remarks</u>: (1) This lemma is valid for multiparameter variations. That is, the same proof applies if Θ is a vector, and y is given matrix so that $\theta y = \Theta_1 y^1 + \Theta_2 y^2 + \ldots + \Theta_k y^k$ for vectors $y^1,\ldots,y^k$. In fact, the result also is valid in infinite dimensional spaces.

(2) For linear fractional programs, the lemma also shows that the extreme ray maximizing $\frac{cr}{dr}$ is optimal for all Θ in some intervals $(-\infty, \Theta_o]$, $[\Theta_1, -\infty)$ (possibly $\Theta_o = -\infty$ and/or $\Theta_1 = +\infty$) as long as the problem remains feasible. This is a direct result of $v(\Theta) \geq \frac{cr}{dr}$ and quasi-concavity of $v(\Theta)$.

<u>Corollary 4.2</u>: Suppose that $y \geq 0$ for a linear fractional program. Then $v(\Theta)$ is nondecreasing in its argument Θ and if an extreme ray r is optimal for some Θ there is a $\Theta_o \geq -\infty$ such that r is optimal for $\Theta\varepsilon(-\infty, \Theta_o)$.

Proof: For $\Theta_1 > \Theta_2$, the feasible region for $\Theta = \Theta_1$ contains the feasible region for $\Theta = \Theta_2$, so that $v(\Theta_1) \geq v(\Theta_2)$. The last conclusion is a consequence of the lemma. //

Of course if $y \leq 0$ a similar result can be stated. These results include, as a special case, variations in only one component of the right-hand-side.

Example: The following example illustrates many of the above results. For simplicity, we present only the problem geometry, omitting the detailed pivoting calculations.

$$\max\{f(x) = \frac{n(x)}{d(x)} = \frac{-x_1 + 5}{x_2}\}$$

$$\text{subject to} \quad -x_1 + x_2 \geq 0 - 2\Theta$$

$$x_1 + x_2 \geq 4$$

$$x_1 \geq 1 + 2\Theta$$

$$x_1 \leq 6$$

$$x_1 \geq 0, \; x_2 \geq 0$$

The feasible region is indicated for Θ = 0, 1, 2 and 2.5 by respectively the bold faced region, solid region, dashed region and hatched region on Figure 2. As shown in [8], the region where $f(x) = k$ is given by a hyperplane [in this case a line in (x_1, x_2) space] and as k varies this hyperplane rotates about the point where $n(x) = d(x) = 0$.

For $\Theta = 0$, the solution is at point (a) and the basis is determined by equality of the first two constraints. As Θ increases (or for $\Theta \leq 0$) this basis remains optimal until at point (b) increasing Θ above 1 causes it to be infeasible. A dual simplex pivot is made to replace the third constraint's surplus variable in the basis with the surplus variable of the second constraint.

The new basis remains optimal until at point (c) with $\Theta = 2$ the extreme ray $r = (0, 1)$ becomes optimal. This ray is optimal for $2 \leq \Theta \leq 2.5$. For $\Theta > 2.5$, the problem is infeasible.

By using the tight constraints for $\Theta \leq 2$ to solve for x_1 and x_2 in terms of Θ and noting that $\frac{cr}{dr} = 0$ for $r = (0, 1)$, we plot the optimal objective value $v(\Theta)$ in Figure 3. It is quasi-concave.

One immediate application of this primal-dual algorithm is for branch and bound when integrality conditions are imposed upon the variables of a linear fractional model. If, for example, x_j is restricted to be an integer and solving problem P without the integrality conditions gives x_j basic at value 3.5, two new problems P_1 and P_2 are constructed by adding, respectively, the constraints $x_j \leq 3$ and $x_j \geq 4$. Taking P_1 for example, we can suppose that the problem was initially formulated with the $(m + 1)^{\underline{st}}$ constraint $x_j + s_{m+1} = 3 + \Theta$. Eliminating the basic variable from this constraint gives an updated equation with right-hand-side equal to $(-.5 + \Theta)$. For $\Theta \geq .5$ the optimal basis to P together with s_{m+1} forms an optimal basis to the parameterized problem; the parametric algorithm can be applied to decrease Θ to 0 and solve the modified problem. A similar procedure is applied to P_2 and other modified problems generated by the branching rules. In every other way, the branch and bound procedure is the same as that for (mixed) integer linear programs and all the usual fathoming tests can be applied.

For another approach for integer fractional programs, see Anzai [5].

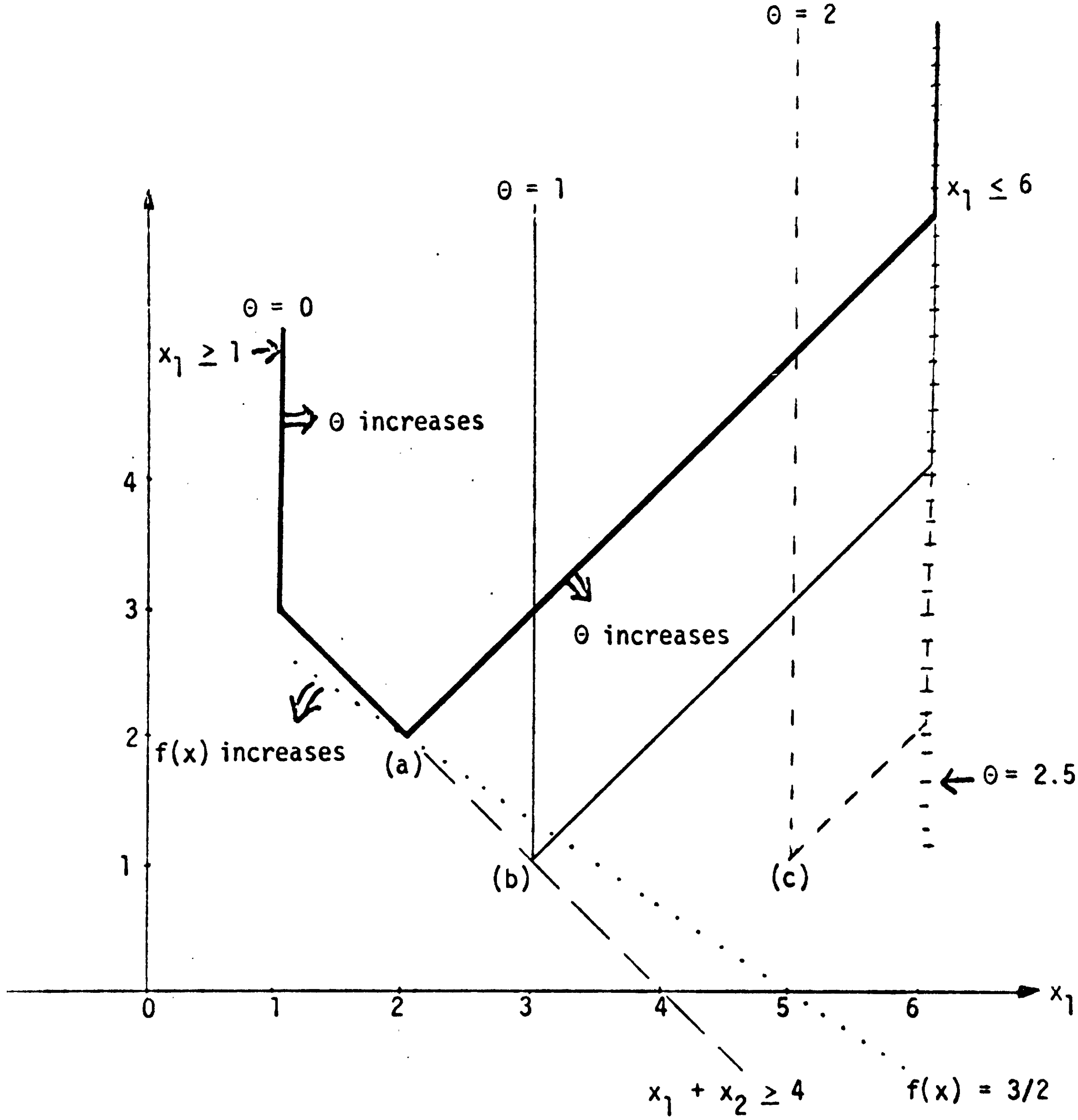
Θ = 2
Θ = 1
$x_1 \leq 6$
Θ = 0
$x_1 \geq 1$
Θ increases
Θ increases
f(x) increases
(a)
(b)
(c)
Θ = 2.5
0
1
2
3
4
5
6
x_1
1
2
3
4
$x_1 + x_2 \geq 4$
f(x) = 3/2

Figure 2

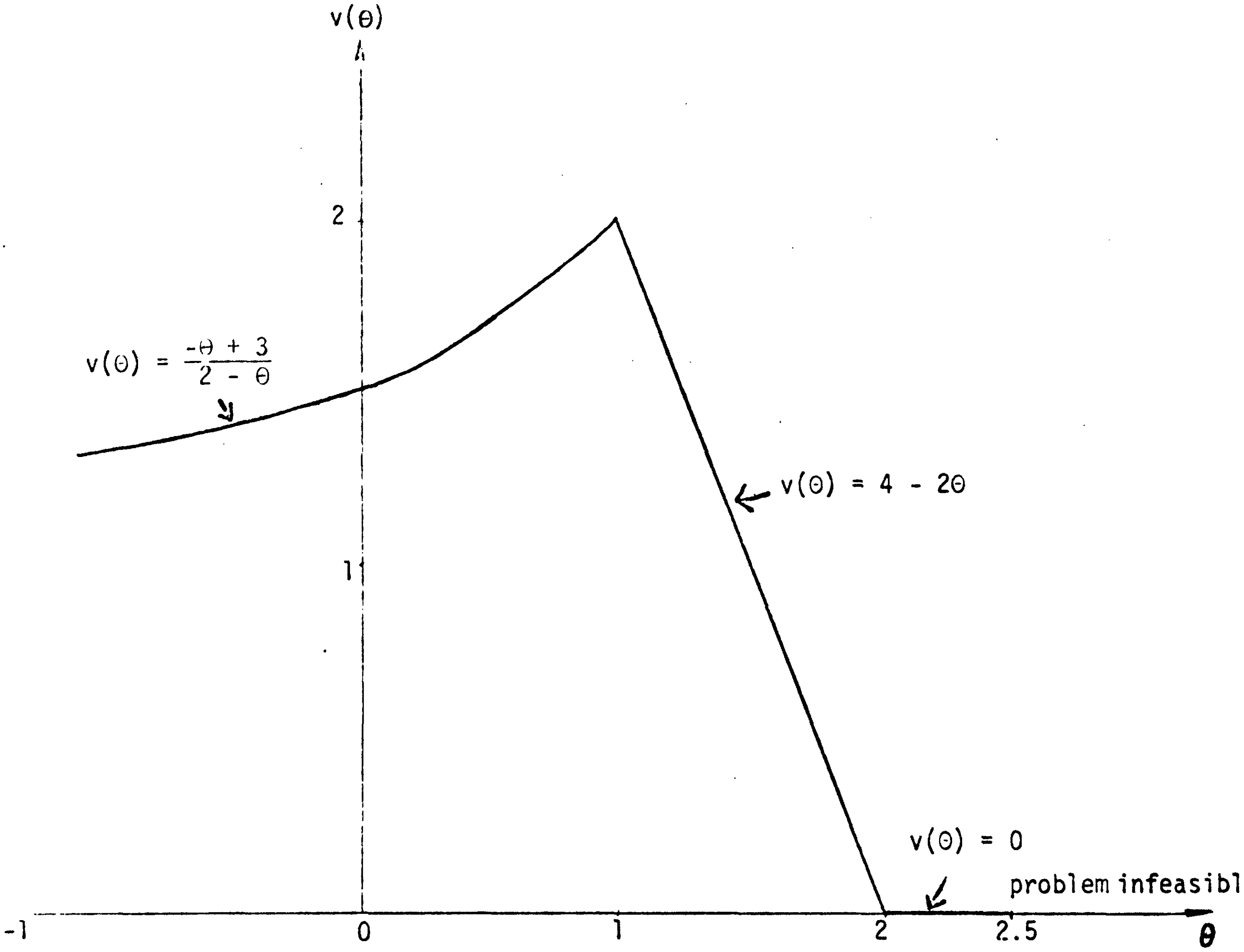
v(θ)
2
1
$v(\Theta) = \frac{-\Theta + 3}{2 - \Theta}$
$v(\Theta) = 4 - 2\Theta$
$v(\Theta) = 0$
problem infeasible
-1
0
1
2
2.5
θ

Figure 3

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