Geometry and structure of Lie pseudogroups from infinitesimal defining systems

Ian G. Lisle[1]
Faculty of Environmental Science
Griffith University
Nathan. Q4111. Australia.
E-mail: I.Lisle@ens.gu.edu.au

and

Gregory J. Reid[1]
Department of Mathematics
University of British Columbia
Vancouver, B.C. Canada. V6T 1Z2.
E-mail: reid@math.ubc.ca

[1]We gratefully acknowledge support from the Natural Sciences and Engineering Research Council of Canada. We especially acknowledge N. Kamran for his lectures on Cartan's theory of infinite Lie pseudogroups at the West Coast Symmetry Workshop (U.B.C., May 1992). We also thank N. Kamran and P. Winternitz for discussions and support during a visit to the Centre de Recherches Mathématiques at the Université de Montréal.

Abstract

In this paper we give a method which uses a finite number of differentiations and linear operations to determine the Cartan structure of structurally transitive Lie pseudogroups from their infinitesimal defining equations. These equations are the linearized form of the pseudogroup defining system – the system of PDEs whose solutions are the transformations belonging to the pseudogroup.

In many applications the explicit form of the transformations is not available and we only have access to their pseudogroup and infinitesimal defining systems, neither of which is initially in involutive form. The usual method for calculating Cartan structure takes as its starting point the pseudogroup defining system in involutive form. However there is currently no constructive algorithm which can always reduce nonlinear pseudogroup defining systems to involutive form. In contrast there are many algorithms which can constructively reduce linear infinitesimal defining systems to involutive form. Our method for calculating structure exploits these algorithms and the work of Singer and Sternberg on the structure of transitive Lie pseudogroups. We also give a constructive method for determining whether a Lie pseudogroup is structurally transitive from its infinitesimal defining system. Our method makes feasible the calculation of the Cartan structure of infinite Lie pseudogroups of symmetries of differential equations. Examples including the KP equation and Liouville's equation are given.

1 Introduction

To motivate our work we begin with the following example. The collection of local volume preserving diffeomorphisms on $\mathbb{R}^n$ is an example of a *pseudogroup*. This collection satisfies most but not all the axioms of a group. In particular composition is not defined for all pairs of diffeomorphisms, since the range of one diffeomorphism may not overlap the domain of a second. Members f of this collection which map x to $X = f(x)$ are naturally defined by the nonlinear analytic PDE

$$\begin{vmatrix} X^1_{x^1} & \cdots & X^1_{x^n} \\ \vdots & & \vdots \\ X^n_{x^1} & \cdots & X^n_{x^n} \end{vmatrix} = 1$$

where $X^i_{x^j} = \frac{\partial X^i}{\partial x^j}$. In general a pseudogroup whose diffeomorphisms satisfy an analytic system of PDEs, such as that above, will be called a *Lie pseudogroup* and the system of PDEs will be called the *pseudogroup defining system*. To obtain the *infinitesimal defining system* of this Lie pseudogroup we linearize the pseudogroup defining system about the identity transformation. In particular, by substituting $X^i = x^i + \varepsilon\xi^i(x) + O(\varepsilon^2)$ into the pseudogroup defining system above, we obtain its infinitesimal defining system as the single linear homogeneous partial differential equation (LHPDE)

$$\xi^1_{x^1} + \xi^2_{x^2} + \cdots + \xi^n_{x^n} = 0$$

with corresponding *infinitesimal generator*

$$\xi^1\partial_{x^1} + \xi^2\partial_{x^2} + \cdots + \xi^n\partial_{x^n} \quad .$$

When $n = 1$ the pseudogroup and infinitesimal defining systems above are easily integrated to give

$$X = x + a, \qquad \xi = b$$

respectively, where a and b are constants. Thus $n = 1$ corresponds to the finite (one) parameter Lie pseudogroup of local translations. This is a *transitive* pseudogroup since the dimension of its orbits is the same as the dimension of the space on which it acts. The case $n > 1$ however yields an *infinite Lie pseudogroup* and neither defining system can be explicitly integrated. To study such Lie pseudogroups we need methods which do not depend on knowledge of the solutions of either of the defining systems. In particular we are interested in determining structural properties of such pseudogroups—that is properties preserved under pseudogroup isomorphism (see §2).

In this paper we present and justify a constructive algorithm for determining the Cartan structure of structurally transitive Lie pseudogroups from their linear infinitesimal defining systems. By an algorithm being *constructive* we mean that it attains its goal in a finite number of steps, each of which only involves a differentiation or a linear operation. In particular such a constructive algorithm does not involve integration, which is in general a heuristic process. Part of our method is the constructive determination of geometric features of a Lie pseudogroup: the dimension of its orbits and isotropy pseudogroup.

The linear systems we deal with generally involve functions as their coefficients. Unless otherwise stated we assume that these functions belong to some computable domain (e.g. the rational functions over $\mathbb{Q}$, or some computable extension of this field). In this way we will be always able to determine whether a given coefficient is zero or nonzero and thus effectively perform Gaussian elimination on such systems. In addition we make the following

Blanket hypothesis:

The terms 'diffeomorphism', 'differential equation' and 'vector field' are henceforth understood to be real analytic without exception.[1]

We were motivated to develop the approach of this paper by our success in developing a constructive algorithm to determine the structure of *finite* parameter Lie pseudogroups [38] from their infinitesimal defining systems. This algorithm is particularly well suited to computer algebra implementation since it bypasses the heuristic step of integrating infinitesimal defining equations used by other methods. In [38] we also generalized this method to infinite Lie pseudogroups but it appears difficult to extract structural information from our generalization.

In contrast to the huge amount of research on (finite parameter) Lie groups, infinite Lie pseudogroups have received relatively little attention and rarely feature in texts (although see [33]). Part of the problem here is the difficulty of separating structural aspects of an infinite Lie pseudogroup from the manifold on which it acts. A Lie pseudogroup depending on a finite number of parameters is a realization of a bona-fide (local) Lie group. The local structure of a Lie group is determined by the structure constants of its Lie algebra; moreover the structural study of finite dimensional Lie algebras can proceed axiomatically, without reference to a manifold at all. However, it is not clear how to realize an infinite Lie pseudogroup in an analogous way. We wish to simultaneously handle both finite and infinite pseudogroups, so we will not view a finite Lie pseudogroup as a local Lie transformation group (i.e. as a realization of a local Lie group). In contrast, defining Lie pseudogroups in terms of solutions of defining equations has the virtue of being equally valid for finite and infinite Lie pseudogroups. The price paid is that this definition ties the pseudogroup to the manifold upon which it acts.

The theory of finite parameter Lie pseudogroups and algebras was initiated late last century by Sophus Lie [25] as a method for studying differential equations. The idea is analogous to Galois theory for polynomial equations where one seeks a 'permutation' or *symmetry* group on the equation's solution set as a means to deciding questions such as whether a solution in quadratures is pos-

[1] Although some progress is possible in the C^∞ case [50].

sible. The obvious way to determine the symmetries of a differential equation is to substitute a general transformation $x \mapsto f(x)$ into the equation[2] and impose the condition that the equation take the same form in the new coordinates. The result of this process often gives an intractable nonlinear overdetermined system of PDEs for the pseudogroup transformations $f(x)$. By linearization of this pseudogroup defining system the corresponding system of linear infinitesimal defining equations is obtained.

Lie's greatest contribution was the development of an infinitesimal linear theory of finite parameter Lie pseudogroups. Given the infinitesimal defining system of such a Lie pseudogroup, the vector field or infinitesimal generator corresponding to its infinitesimals, ξ^i, is of form[3]

$$\mathbf{X} = \xi^i(x)\partial_{x^i} .$$

These vector fields form a vector space equipped with a skew symmetric bracket operation $[\mathbf{X}, \mathbf{Y}] = \mathbf{XY} - \mathbf{YX}$ induced by composition at the pseudogroup level. If this *Lie algebra* is finite dimensional, with basis $\mathbf{X}_k$, then closure under the commutator bracket takes the form

$$[\mathbf{X}_i, \mathbf{X}_j] = c_{ij}^k \mathbf{X}_k, \tag{1}$$

where the constants c_{ij}^k are called the *structure constants* of the Lie algebra.

Lie's infinitesimal methods have proved to be particularly fruitful for symmetry analysis of differential equations [1, 31, 32]. In almost all cases of interest the infinitesimal defining system of Lie pseudogroups of symmetries of differential equations can be algorithmically derived [10]. Many programs are now available for deriving and attempting to integrate infinitesimal defining systems of these Lie pseudogroups of symmetries [18]. Recently methods which directly extract information from infinitesimal defining systems without integration have become available, and several computer algebra programs now implement these methods [47, 36, 49, 11]. Reid [34, 36] and Schwarz [47, 48] independently showed how to obtain the dimension of the symmetry algebra from the infinitesimal defining system and Reid [34, 36] also showed how to obtain the structure constants c_{ij}^k in the finite-parameter case. The central idea behind these methods is that the infinitesimal defining system as initially derived can, by a process of Gauss reduction and appending integrability conditions, be constructively reduced to a coordinate dependent canonical form [3, 11, 28, 36, 47, 49] for which a local existence uniqueness theorem is available [41, 21]. The coordinate independent counterpart of these approaches involve the reduction of systems of PDEs to involutive form [33, 2]. The canonical form and involutive form of a system of PDEs are related in that it has been shown that the canonical form of a system of linear PDEs can be constructively prolonged to yield an involutive system

[2] The coordinates x would in general include the dependent and independent variables of the differential equation.

[3] Unless otherwise stated we will assume the Einstein summation convention.

[28, 29, 4]. The reduction algorithms above will be of crucial use in enabling us to extract the Cartan structure of infinite Lie pseudogroups.

Lie proposed a definition of Lie pseudogroups as the local transformations constituting the general solution of an analytic system of PDEs [26]. Lie's definition is markedly different to the modern definition of a Lie group, which is given in terms of the manifold coordinatized by its parameters. The modern definition is independent of the manifold on which the pseudogroup acts. Eventually Kuranishi [23, 24] defined an abstraction of infinite Lie pseudogroups called a Lie (F)-group, which is independent of the manifold on which the pseudogroup acts and coincides with the modern definition in the finite parameter case.

Although Lie made several attempts he was unsuccessful in developing an infinitesimal structural theory of Lie pseudogroups. Cartan [6, 7, 9] and Vessiot [55] devised structure theories of Lie pseudogroups. Like Lie they defined their pseudogroups as the general solution of analytic defining equations. Cartan's structure theory is stated in terms of 1-forms ω^i invariant under the action of the pseudogroup: taking exterior derivatives, Cartan obtains

$$d\omega^k = a^k_{i\rho}\pi^\rho \wedge \omega^i - \tfrac{1}{2}c^k_{ij}\omega^i \wedge \omega^j \qquad (2)$$

where π^ρ are a certain number of additional 1-forms. The structure coefficients are c^k_{ij}, $a^k_{i\rho}$. If the pseudogroup is of finite type then the $a^k_{i\rho}$ are absent, the c^k_{ij} are constant, and the Cartan structure equations reduce to the Maurer-Cartan equations which are dual to Lie's commutation relations (1). There are several differences in the infinite case. Firstly, nontrivial $a^k_{i\rho}$ terms appear. Secondly, it is possible for the pseudogroup to have *essential scalar invariants.* For instance in the pseudogroup $X = x$, $Y = y + f(x)$, the invariant x is essential: it must appear in any realization of this pseudogroup. Thus for infinite pseudogroups there is a fundamental divide between the *transitive* pseudogroups which have no essential invariants, and the *intransitive* pseudogroups, which have essential invariants. The third difference is that in the intransitive case, the structure can vary from point to point. In particular the coefficients $a^k_{i\rho}$, c^k_{ij} can be functions of the essential invariants, whereas in the transitive (including the finite parameter) case, they are *structure constants.*

In addition Cartan gave an algorithmic procedure for computing the 1-forms ω^i and calculating his structure equations directly from the involutive form of the defining system of the pseudogroup. Although Cartan obtained many important results in this way, the method is difficult to apply to symmetry analysis of differential equations. The difficulty lies not with Cartan's structure algorithm, nor with the creation of the pseudogroup defining system, but rather with the fact that there is currently no effective algorithm for reducing nonlinear pseudogroup defining systems to involutive form. This is in strong contrast to the availability of many algorithms for reducing the linear infinitesimal defining systems to involutive form (also see [39]). Cartan's method of equivalence [7, 8, 13] can also be used for calculating the structure of Lie pseudogroups of symmetries of differential equations. This method is best suited to the symmetry analysis of classes of differential equations rather than the symmetry analysis of a particular equation. In practice to apply this method to a particular differential equation it has to be embedded in a class of differential equations with a known equivalence group [20] and the size of the calculations can become a significant barrier. In [37] we give a method which is applicable to infinitesimal defining systems and gives us Cartan's starting point in implicit form. The calculations are completed by using Cartan's algorithm *at the pseudogroup level*. The success of these calculations encouraged us to develop the fully infinitesimal method given in the current paper.

Despite Lie's failed attempts and Cartan's scepticism [9, p.1335] Kuranishi [23, 24] and Singer and Sternberg [50] eventually developed an infinitesimal interpretation of Cartan's structure theory of infinite Lie pseudogroups of transitive type (see also [14, 15, 16, 27]). We use this theory in the rigorous justification of our constructive algorithm for calculating Cartan structure coefficients $a^k_{i\rho}$, c^k_{ij} from the infinitesimal defining system in the transitive case. Our results generalize those of [38] for finite dimensional Lie algebras to the infinite case. In the case of symmetry analysis of PDEs, this will imply that *we can pass from the infinitesimal defining equations for the symmetries of the original* PDE *to the Cartan structure of its symmetry group by a process of differentiation and linear algebra only.* Thus our procedure is suitable for computer algebra implementation.

The remainder of the paper is organized as follows.

In §2 we present background material about Lie pseudogroups, canonical and involutive forms for systems of LHPDEs. In particular in §2.1 we define Lie pseudogroups and isomorphism of Lie pseudogroups, and give illustrative examples of such pseudogroups. In §2.2 we describe canonical form algorithms for systems of LHPDEs, and give a discussion of Lie algebra systems in §2.3.

In §3 methods are developed for extracting geometric properties of a Lie pseudogroup from its infinitesimal defining system. In particular we show how to find the *pseudogroup distribution* in §3.1. That is, we construct vector fields $\mathbf{Y}_i$ such that every $\mathbf{X}$ in the Lie algebra of the Lie pseudogroup is a (nonconstant coefficient) linear combination of $\mathbf{Y}_i$. This gives a differential characterization of the orbits and scalar invariants of the pseudogroup. Section 3.2 is devoted to the *isotropy subgroup* of a point. We describe a sequence of isotropy (stabilizer) subgroups at a point, characterize them in terms of the infinitesimal defining system, and explicitly construct the *linear isotropy algebra.*

In §4, we use the methods from §3 to show how to obtain Cartan structure of a transitive infinite Lie pseudogroup from its infinitesimal defining equations. We begin with a restricted case, then show how to reduce the general case to the restricted one. In §4.1 we show how to calculate structure of transitive infinite Lie pseudogroups with *no scalar invariants* and a defining system of *first order*. The structure coefficients $a^k_{i\rho}$ and c^k_{ij} in the Cartan structure relations (2) are found from the infinitesimal defining system, without knowledge of either ω^i or π^ρ in (2). In §4.2 we treat the case of a first order defining system where scalar invariants are present, and provide a transitivity test, based on detection of essential invariants. The test uses the infinitesimal defining system and *does not require knowledge of the invariants.* If the test of §4.2 shows that a pseudogroup of first order type is transitive, we can restrict the pseudogroup action to an orbit without losing any of its structure. In §4.3 we show how to find Cartan structure of the restricted pseudogroup from the original infinitesimal defining system, *without explicitly knowing either the vector fields or the orbits of the pseudogroup.* Finally, if the pseudogroup has defining equations of order higher than one, then in §4.4 we give a prescription for adjoining additional variables so that the pseudogroup acting on the enlarged space has first order defining system, in which case the methods of §4.1, §4.2, §4.3 can be applied.

In the Appendix we give the results of applying a preliminary computer algebra implementation of our methods to calculating the structure of the infinite Lie symmetry pseudogroups of several physically important PDEs including: Liouville's equation, the KP equation, and the steady state boundary layer equations.

2 Lie pseudogroups and canonical form algorithms for PDE systems

2.1 Lie pseudogroups

The definition of a pseudogroup which we use [23, 24, 50] is:

Definition 1 (Pseudogroup) Let M be a (real) manifold, and $\mathcal{G}$ be a collection of diffeomorphisms of open subsets of M into M. Then $\mathcal{G}$ is a *pseudogroup* if

i. $\mathcal{G}$ is closed under restriction: if $\tau: U \to M$ is in $\mathcal{G}$, then so is $\tau|_V$ for any open $V \subseteq U$.

ii. If $U \subseteq M$ is an open set with $U = \bigcup_s U_s$, and $\tau: U \to M$ is a diffeomorphism with $\tau|_{U_s} \in \mathcal{G}$, then $\tau \in \mathcal{G}$.

iii. $\mathcal{G}$ is closed under composition: if $\tau : U \to M$, and $\sigma : V \to M$ are any two members of $\mathcal{G}$, then $\sigma \circ \tau \in \mathcal{G}$ also, wherever this composition is defined.

iv. $\mathcal{G}$ contains the identity diffeomorphism of M.

v. $\mathcal{G}$ is closed under inverse: if $\tau : U \to M$ is in $\mathcal{G}$, then τ^{-1} (the domain of which is $\tau(U)$) is also in $\mathcal{G}$.

Properties (iii), (iv), (v) provide the group-like character of the pseudogroup, while (i), (ii) ensure that neighbourhoods can be shrunk and enlarged in the obvious ways.

Definition 2 (Lie pseudogroup) A *Lie pseudogroup* is a pseudogroup whose diffeomorphisms are local analytic solutions of an analytic system of defining partial differential equations.

Example 3 Take $\mathcal{G}$ as the collection of local diffeomorphisms f of $\mathbb{R}$

$$f(x) = X = \frac{ax+b}{cx+d}, \qquad ad - bc = 1. \tag{3}$$

The pseudogroup defining system, obtained by eliminating a, b, c, d, is

$$X_{xxx} - \frac{3}{2}\frac{X_{xx}^2}{X_x} = 0 \tag{4}$$

where $X_x = \frac{\partial X}{\partial x}$ etc. Each transformation (3) is defined only on an open subset of $\mathbb{R}$, and composition is only locally defined. Since $\mathcal{G}$ is specified by an analytic differential equation, it is a Lie pseudogroup. The infinitesimal defining equation corresponding to (4), obtained by setting $X = x + \varepsilon\xi + O(\varepsilon^2)$, is

$$\xi_{xxx} = 0. \tag{5}$$

A Lie pseudogroup is of *finite type* if the solution set of its group defining equations depends on a finite number of parameters, and infinite otherwise. Example 3 is a 3-parameter finite Lie pseudogroup. Infinite Lie pseudogroups have no finite parametrization, and involve arbitrary *functions* in their transformations.

Example 4

a. The pseudogroup of local analytic diffeomorphisms of $\mathbb{R}$. The defining system in this case is null. The pseudogroup is of Lie type.

b. The pseudogroup $\mathcal{G}$ of local diffeomorphisms τ of a manifold M which fix a particular point $P \in M$. The pseudogroup $\mathcal{G}$ is not of Lie type, because the condition $\tau(P) = P$ is not expressible as a differential equation.

c. Let $\mathcal{G}_1$ be the Lie pseudogroup of Example 4a. Let $\mathcal{G}_2$ be the pseudogroup of local diffeomorphisms τ of $\mathbb{R}^2$ of the form $X = f(x)$, $Y = f(y)$, where $f \in \mathcal{G}_1$. It is not possible to specify this pseudogroup by differential equations, and hence it is not of Lie type.

Examples 4b, 4c were given both by Lie and Cartan. They demonstrate that many infinite pseudogroups are not of Lie type, even those which arise as sub-pseudogroups of Lie pseudogroups. Example 4c shows that taking copies of a Lie pseudogroup may not yield a Lie pseudogroup.

The finite Lie pseudogroups are realizations of an underlying Lie group, and the usual development of local transformation groups follows this approach [1, 31, 32]. Finding such a definition of a Lie pseudogroup, freed from the manifold on which the pseudogroup acts, is much more difficult in the infinite case. Without such a definition, it is necessary to *define* isomorphism of Lie pseudogroups in terms of their transformations of the manifold. We use the definition due to Cartan [6] and Vessiot [55].

Definition 5 Let $\mathcal{G}$, $\hat{\mathcal{G}}$ be Lie pseudogroups on manifolds M, $\hat{M}$ respectively. We say that $\hat{\mathcal{G}}$ is an *isomorphic prolongation* of $\mathcal{G}$ if the following conditions hold:

a. $\hat{M}$ is fibred over M, $\pi: \hat{M} \to M$.

b. For each $\hat{\tau} \in \hat{\mathcal{G}}$ there is a $\tau \in \mathcal{G}$ such that the following diagram commutes:

$$\begin{array}{ccc} \hat{M} & \xrightarrow{\hat{\tau}} & \hat{M} \\ \pi \downarrow & & \downarrow \pi \\ M & \xrightarrow{\tau} & M \end{array}$$

c. For all $\tau \in \mathcal{G}$, $\tau \circ \pi = \pi$ implies that $\hat{\tau} = \mathrm{id}_{\hat{M}}$.

In other words, $\hat{\mathcal{G}}$ is an isomorphic prolongation of $\mathcal{G}$ if to every transformation $X = \tau(x)$ in $\mathcal{G}$, there is exactly one associated transformation $X = \tau(x)$, $Y = \sigma(x, y)$ in $\hat{\mathcal{G}}$.

Definition 6 Two pseudogroups $\mathcal{G}_1$, $\mathcal{G}_2$ on a manifold M are *similar* if there is a diffeomorphism $\phi: M \to M$ such that $\mathcal{G}_2 = \phi^{-1} \circ \mathcal{G}_1 \circ \phi$.

In other words, $\mathcal{G}_1$, $\mathcal{G}_2$ are related by a change of variables.

Definition 7 (Isomorphism) Two Lie pseudogroups $\mathcal{G}$, $\mathcal{H}$ are *isomorphic* if they admit isomorphic prolongations $\hat{\mathcal{G}}$, $\hat{\mathcal{H}}$ which are similar.

This definition of isomorphism appears to be at variance with the usual definition of isomorphism. However Kuranishi [24, p.91] resolved this discrepancy when he introduced the Lie (F)-group associated with a pseudogroup, and showed that two transitive Lie pseudogroups are isomorphic according to Cartan's definition (above), if their corresponding Lie (F)-groups are isomorphic in the usual sense.

A *structural* property of a Lie pseudogroup is one which is preserved by isomorphism in the sense of Definition 7. Hence we can calculate the structure of a given Lie pseudogroup by equivalently calculating the structure of any of its isomorphic prolongations. This property is exploited in §4.4.

2.2 Ordered canonical form for systems of linear homogeneous PDEs

A cornerstone of this paper will be our ability to constructively reduce systems of analytic linear homogeneous PDEs to a canonical form subject to certain admissible orderings of derivatives in the system. In particular we will apply these algorithms to infinitesimal defining systems. Several algorithms for this reduction are available and the main purpose of this section will be to review these well-known methods. We will centre our discussion around the classical Riquier-Janet algorithm and a canonical form which can be obtained from the complete orthonomic passive form given by this algorithm.

From the canonical form of a PDE system, a set of analytic initial data can be constructed which determines unique local analytic solutions of the system.

Although the ordered canonical form obtained for a system is generally lost if a change of coordinates is applied to the system, geometric information can be be obtained from it. In particular the results of [28, 29] and [4] show how such a form can be prolonged to a system which is involutive by the definition of Pommaret [33]. The property of involutivity is a geometric one which is not disturbed by changes of coordinates and crucial use of this transformation to involutive form is made in §4.4 of our paper.

We begin with a discussion of the Riquier-Janet Method.

2.2.1 The Riquier-Janet reduction algorithm

Suppose we have an analytic system of LHPDEs on an n-dimensional manifold M. That is the coefficients of the derivatives (which we take to include order zero derivatives—i.e. the dependent variables themselves) are analytic functions of the independent variables x for all $x \in M$ except at certain points of M which we call *singular* points. An essential ingredient of all the Riquier-Janet algorithms are certain rankings of partial derivatives. In particular:

Definition 8 (Orderly ranking of derivatives.) Let $\mathrm{ord}(v)$ denote the order of a derivative v in a system of PDEs[4]. An orderly ranking of derivatives $\succ$ which respects total order of derivative is a ranking of derivatives that, in addition to the usual properties of rankings (transitivity, comparability and trichotomy), satisfies

$$\mathrm{ord}(v) \succ \mathrm{ord}(w) \Rightarrow v \succ w \tag{6}$$

and

$$v \succ w \Rightarrow \partial_{x^j} v \succ \partial_{x^j} w, \tag{7}$$

for $j = 1, ..., n$.

[4]If v is a trivial derivative, i.e. a dependent variable, then $\mathrm{ord}(v)= 0$.

Property (6) means that it first respects total order of derivative and property (7) means that the ranking is preserved under differentiation.

Riquier [41] gave a large class of such rankings which are specified by matrices with nonnegative integer entries. See [5, 43, 56], for recent work on extending this class.

An example of an orderly ranking for a system of PDEs in two independent variables $(x, y) \in M = \mathbb{R}^2$ and a single dependent variable ξ is:

$$\xi \prec \xi_x \prec \xi_y \prec \xi_{xx} \prec \xi_{xy} \prec \xi_{yy} \prec \xi_{xxx} \cdots \tag{8}$$

Note that given an orderly ranking then an equation in a system of LHPDEs analytic at $p \in M$ will have a unique derivative amongst the derivatives appearing in the LHPDE at p which is highest in the ranking. We call this derivative the *leading derivative* of the PDE. The PDE can then be expressed in solved form with respect to its highest derivative in the obvious way. For example the LHPDE

$$x\partial_x \xi - y\partial_y \xi - \xi = 0, \tag{9}$$

has leading derivative $\partial_y \xi$ with respect to the ranking above, and when expressed in solved form with respect to that ranking it becomes

$$\partial_y \xi = \frac{x}{y}\partial_x \xi - \frac{1}{y}\xi. \tag{10}$$

By convention we will place the leading derivative of the solved form on the LHS of the equation. An important property of the canonical form which we employ is that the system is in the solved (orthonomic) form defined by Riquier [41] (also see [42, p.153]).

Definition 9 (Orthonomic form) An analytic system of LHPDEs on a manifold M is said to be in orthonomic form with respect to an orderly ranking if

a. Each equation of the system is in solved form for its leading derivative and has the same leading derivative throughout M.

b. No two equations have the same leading derivative.

c. No derivative of any leading derivative of the system appears in the RHS of the system.

An orthonomic system is *complete* [41, 21] if the monomials associated with the systems leading derivatives form a complete set [42, p.153,154]. A complete orthonomic system is *passive* if the set of integrability conditions for the system as defined by Janet are satisfied [21],[51, p.305].

Further Riquier gave a process which after a finite number of steps would obtain a system of PDEs in either incompatible form (i.e. a form containing an equation only involving the system's independent variables) or in complete orthonomic passive form (see [41, 21] and [51, p.308]). When specialized to the case of analytic systems of LHPDEs his result is:

Proposition 10 (Reduction to complete orthonomic passive form) *Any analytic system of* LHPDEs *on a manifold M can be constructively reduced by the reduction algorithm of Riquier-Janet to complete orthonomic passive form with respect to a given orderly ranking.*

We make a few comments about the effectiveness of this algorithm. Each step of the Riquier-Janet algorithm involves either solving a PDE for its leading derivative or differentiating a PDE. Hence for nonlinear systems of PDEs, the method is not constructive. Also a PDE involving only the system's independent variables may be found. However for systems of LHPDEs, only new LHPDEs can result during the application of his method, so neither of these difficulties arise.

As LHPDEs are solved for their leading derivatives during the reduction process, division by the coefficient functions of the leading derivatives occurs. The output complete orthonomic form has the same local analytic solutions as the original system away from a set of points on the manifold where these (pivot) functions vanish. We will call this set the *pivot locus* of the system. For example the pivot locus of the equation (10) is $y = 0$.

The only remaining barrier to the Riquier-Janet algorithm being fully effective for systems of LHPDEs is determining whether the coefficient of a leading derivative is zero or nonzero. This difficulty does not occur if the coefficients come from a computable domain (e.g. rational functions on M over $\mathbb{Q}$, or some finite extension of this field). Unless otherwise stated we will assume that the coefficients come from such a computable domain.

2.2.2 The Riquier-Janet initial data algorithm and existence-uniqueness theorem

Suppose that a system of LHPDEs is in the complete orthonomic passive form with respect to an orderly ranking of derivatives. One of the basic principles underlying the Riquier-Janet initial data algorithm is that it partitions the set of all derivatives of the dependent variables of such a system into two sets: those that are not derivatives of the leading derivatives (the set $\mathcal{P}$ of parametric derivatives), and the set of remaining derivatives (the set of principal or non-parametric derivatives $\overline{\mathcal{P}}$). Janet [21] and [35] give finite methods for describing this generally infinite set of initial data. Specification of the values of the parametric derivatives at $x_0 \in M$ determines values of all the principal derivatives, by using derivatives of the equations of the system as substitution rules, and thus determines a unique formal power series solution of the system. Consequently the space of formal power series solutions at $x = x_0$ depends on $\#(\mathcal{P})$ parameters. Convergence of the formal power series solution in the analytic case is established through the use of majorants [41, 21, 51, 42].

Of particular interest to us will be the set of k-th order parametric and principal derivatives of a system which we will denote by $\mathcal{P}_k$ and $\overline{\mathcal{P}}_k$ respectively. Then the set of initial conditions corresponding to the initial data for the k-th order parametric derivatives is obtained by assigning constant values to these derivatives at x_0.

We specialize Riquier's existence-uniqueness theorem [41, 21, 51, 52, 42] to the case of analytic LHPDEs.

Theorem 11 *Consider a system of* LHPDE*s on a manifold M, which is both analytic and in complete orthonomic passive form with respect to an orderly ranking of derivatives. Then*

1. *From the leading derivatives of the system the Riquier-Janet initial data algorithm for a given point $x_0 \in M$ determines a set of analytic initial data at x_0.*
2. *There is a unique analytic solution of the system satisfying the initial data given at the point x_0 which is valid in some neighbourhood of x_0.*

2.2.3 Ordered Canonical Form

The complete orthonomic passive form of a system of analytic LHPDEs will in general contain PDEs whose leading derivatives are derivatives of other leading derivatives in the system. These other PDEs are redundant in that their removal from the system alters neither the solution of the system nor the sets of parametric and principal derivatives. In particular the assumption that they are not redundant would lead to a contradiction of the Riquier-Janet existence uniqueness theorem. We call the system obtained by deleting all such equations from the system the *ordered canonical form* of the system. It is canonical in the following sense. Suppose another system of LHPDEs with the same solutions, posed in the same coordinates, is reduced to complete orthonomic passive form with respect to the same orderly ranking, and then to ordered canonical form. Then (away from points where the pivots vanish) the two canonical forms will agree.

Several computer algebra implementations of the Riquier-Janet method are available (see especially Schwarz [46], who later automated the Riquier-Janet initial data algorithm [47, 48], and Topunov [53]). Schwarz [47, 48] has implemented the deletion process described above and hence can obtain the ordered canonical of systems of LHPDEs. The standard form algorithm of Reid [35] produces the ordered canonical form and initial data set of a system of LHPDEs directly without first creating the complete orthonomic passive form of a system. In addition, differential Gröbner basis algorithms [30, 28, 3, 11] when applied to systems of LHPDEs also produce the ordered canonical form of the system.

As a consequence we have the following weakened specialization of Riquier's results:

Theorem 12 *Any analytic system of* LHPDE*s on a manifold M can be constructively reduced to ordered canonical form with respect to an orderly ranking of derivatives. Further, given a positive integer k, then for each nonsingular $x_0 \in M$ it is possible to algorithmically determine the set of parametric derivatives of order i, $1 \leq i \leq k$. Finally, if these parametric derivatives are ascribed values at x_0 then there exists a local solution of the system satisfying the given initial value problem.*

We will make repeated use of this result.

The effectiveness of the algorithms of this paper will depend ultimately on the analysis of first order systems for their initial data. In particular we will only require the 0-th order parametrics $\mathcal{P}_0$ and the first order parametric derivatives $\mathcal{P}_1$, and these will be algorithmically available to us.

2.2.4 Prolongation of ordered canonical form to involutive form

We define the *ordinary prolongation to order q* of a system of PDEs of order k as the equivalent system obtained by taking all possible total derivatives of all equations in the system so that the resulting system is of order q [33].

The ordered canonical form obtained for a system is generally lost if a change of coordinates is applied to the system. The coordinate independent counterpart of the canonical form above is the involutive form of a system. An involutive system contains all its integrability conditions and has an involutive symbol. In particular the property of involutivity is not destroyed by coordinate changes [33, 2]. We have the following theorem [28, 29, 4] linking the two:

Theorem 13 (Prolongation to involutive form) *Suppose S is a k-th order analytic system of* LHPDE*s on a manifold M which is in ordered canonical form with respect to an orderly ranking of derivatives. Then an integer $q \geq k$ can be constructively determined such that the ordinary prolongation to order q of the system is involutive in the sense of Pommaret away from points where the system is singular.*

Consequently one way of obtaining the involutive form of a system of PDEs is to first obtain it in ordered canonical form and then prolong it to involutive form. Alternatively Pommaret's involution form algorithm can be used (see [44] for a computer implementation of this algorithm). For an algorithm which checks involutivity of an exterior differential system in the sense of Cartan, see [17]. We further note that Theorem 12 is also valid upon replacement of ordered canonical form with respect to an orderly ranking by 'involutive form' (see [33]).

2.3 Lie algebra system

Let $\mathcal{G}$ be a Lie pseudogroup on a manifold M with coordinates x. Its pseudogroup defining system has independent variables x and dependent variables X.

Now consider a 1-parameter local group ϕ_ε of local analytic diffeomorphisms of M, generated by a vector field $\mathbf{X} = \xi^i(x)\partial_{x^i}$. If ϕ_ε is contained in $\mathcal{G}$, then the components $\xi^i(x)$ satisfy a linear *infinitesimal defining system,* which can formally be obtained by inserting $X = x + \varepsilon\xi + \cdots$ into the pseudogroup defining system and retaining terms of first order in ε. For instance, the volume preserving condition in the introduction is replaced by the linear condition $\sum_i \xi^i_{x^i} = 0$. This defining system satisfies many but not all of the properties for $\xi^i\partial_{x^i}$ to be a Lie algebra of vector fields.

For a finite parameter Lie pseudogroup on a manifold M, linearization about the identity does indeed give rise to a Lie algebra of vector fields on M. This is not so in the infinite case: analytic solutions will in general only be defined in a neighbourhood $U \subseteq M$, and if their domains of definition do not overlap, the commutator $[\mathbf{X}, \mathbf{Y}]$ of two vector fields $\mathbf{X} = \xi^i\partial_{x^i}$, $\mathbf{Y} = \eta^i\partial_{x^i}$ is not defined. However if we fix a point x_0, and consider the subset of solutions of the infinitesimal defining system defined and analytic at x_0, then the commutator on this subset is defined. These vector fields therefore do form a Lie algebra L_{x_0} attached to the point x_0.

However L_{x_0} is not necessarily a Lie algebra of local vector fields on any fixed neighbourhood U of x_0. For example, $\mathbf{X}_n = 1/\sqrt{1 - n^2x^2}\,\partial_x$, $n = 0, 1, 2, \ldots$, are local vector fields analytic at $x_0 = 0$, but there is no neighbourhood of $x = 0$ on which all $\mathbf{X}_n$ are defined. This situation occurs whenever the pseudogroup is infinite. To deal with this technical difficulty we use analytic vector field *germs.* A vector field germ at x_0 is an equivalence class of vector fields, two vector fields being equivalent if they agree on some neighbourhood of x_0. In local coordinates about x_0, a vector field can be written $\mathbf{X} = \xi^i(x)\partial_{x^i}$. Since we assume analyticity, the germ of $\mathbf{X}$ at x_0 can be identified with the power series for $\xi^i(x)$ about $x = x_0$. We denote such a vector field germ by $\{\mathbf{X}, x_0\}$. In fact the only distinction between an analytic vector field and its germ is that the vector field has a domain of definition. The germ suppresses this domain information, retaining only the basepoint x_0. However, the distinction between germs and vector fields is a small one in the analytic case, and we shall often speak of 'vector fields based at x_0' instead of germs.

Definition 14 (Lie algebra system) A *Lie algebra system* $\mathcal{L}$ is a collection of analytic vector field germs $\{\mathbf{X}, x_0\}$ on a manifold M with $\mathbf{X} = \xi^i(x)\partial_{x^i}$, such that

i. Each $x_0 \in M$ is a *nonsingular point* of a linear homogeneous system of analytic partial differential equations S.

ii. $\xi^i(x)\partial_{x^i}$ is a germ of a *local solution* of S in some neighbourhood of x_0.

iii. If $\{\mathbf{X}, x_0\}$ and $\{\mathbf{Y}, x_0\}$ are in $\mathcal{L}$, then their Lie bracket is also in $\mathcal{L}$.

The system S of PDEs is called the *infinitesimal defining system* of $\mathcal{L}$. Note that a Lie algebra system is specified by the defining equations S along with a commutator bracket defined on its solutions.

The Lie algebra system $\mathcal{L}$ consists of the totality of analytic vector field germs of local solutions of the infinitesimal defining system. These vector field germs constitute a *sheaf*, with the Lie algebra L_{x_0} of vector field germs based at x_0 being the *stalk over* x_0. Most of our constructions are concerned with the algebras L_{x_0}. The point is that in the infinite case we are dealing not with *one* Lie algebra of vector fields, but with a collection of Lie algebras L_{x_0}, one at each basepoint x_0. The approach of Singer and Sternberg [50] makes heavy use of sheaf theory for developing their infinitesimal interpretation of Cartan structure. We use the term 'Lie algebra system' to emphasize that we take the infinitesimal defining system as our starting point. Although we make no use of sheaf theory, our 'Lie algebra system' is indeed a 'Lie algebra sheaf' in the sense of Singer and Sternberg [50]: it is shown in [50] that their axiomatically defined Lie algebra sheaves could indeed be identified with the solutions of a system of partial differential equations.

We also note that the computational methods of [35, 36, 38] for finding dimension, structure and Taylor series for Lie symmetry algebras fit this framework. Reid [35, 36] chooses an 'initial data point' x_0, and gives an algorithm for constructing Taylor series solutions to any order about x_0. Thus at each point x_0, Reid is constructing (a finite order approximation to) L_{x_0}. In [38], the structure constants of L_{x_0} are constructed when L_{x_0} is of finite dimension. In this case the vector fields generated by the germs in L_{x_0} share a nontrivial domain of definition and the 'Lie algebra system' gives rise to a Lie algebra of vector fields.

The set of values $\mathbf{X}|_{x_0}$ taken by vector fields in L_{x_0} span a subspace $\Sigma(L_{x_0})$ of tangent space $T_{x_0}M$ to M at x_0. We shall see in §3.1 that condition (i) ensures that $\Sigma(L_{x_0})$ has the same dimension at each x_0. The dimension of the subspace of $T_{x_0}M$ spanned by the vector fields in L_{x_0} is the dimension of the pseudogroup's orbit through x_0; thus all the orbits have the same dimension. We shall always understand that any singular points have been removed from the domain of the defining PDE system R. For example, consider the Lie algebra system consisting of vector fields $\mathbf{X} = \xi(x,y)\partial_x + \eta(x,y)\partial_y$ on $\mathbb{R}^2$ which are local analytic solutions of the infinitesimal defining system, with dependent variables $(\xi, \eta) \in \mathbb{R}^2$ and independent variables $(x,y) \in \mathbb{R}^2$,

$$y\xi_x - \eta = 0, \quad \xi_y = 0, \qquad y\eta_y - \eta = 0. \tag{11}$$

The manifold for this Lie algebra system is $\mathbb{R}^2$ minus the set $\{(x,y) \in \mathbb{R}^2 : y = 0\}$ of singular points of the infinitesimal defining system.

The set $\{\Sigma(L_{x_0}) \mid x_0 \in M\}$ of subspaces of $T_{x_0}M$ of the same dimension constitute a *distribution* $\Sigma(\mathcal{L})$ on M, which we shall call the pseudogroup distribution. A special case is:

Definition 15 (Transitivity) A Lie algebra system is *transitive* if $\Sigma(L_{x_0}) = T_{x_0}M$ for all $x_0 \in M$.

A transitive Lie algebra system $\mathcal{L}$ generates a transitive pseudogroup $\mathcal{G}$ in a natural way [50]. Transitivity of the pseudogroup implies that the point x_0 can be mapped to every other point x in some neighbourhood U of x_0. Hence the Lie algebra L_{x_0} can also be transported to any other point $x \in U$. All the algebras L_{x_0} are therefore isomorphic, and we need only consider L_{x_0} at *one* arbitrary point. We shall use this fact to normalize the structure coefficients in our calculation of Cartan structure.

3 Geometry of pseudogroup

3.1 Pseudogroup distribution and invariants

The distribution $\Sigma(\mathcal{L})$ defined by a Lie algebra system is completely integrable by virtue of closure of the Lie algebras L_{x_0} under commutator bracket. The leaves of the foliation induced by $\Sigma(\mathcal{L})$ are orbits of the pseudogroup.

Proposition 16 *Suppose an analytic infinitesimal defining system on a manifold M is in canonical form with respect to an orderly ranking. Then the pseudogroup distribution $\Sigma(\mathcal{L})$ can be constructively obtained from this canonical form.*

Proof: Since we are assuming an orderly ranking the 0-th order equations in the canonical form have the form

$$\xi^r = \sum_{\alpha:\xi^\alpha \in \mathcal{P}_0} e^r_\alpha(x)\xi^\alpha, \qquad \text{for } r \text{ such that } \xi^r \in \overline{\mathcal{P}}_0, \tag{12}$$

where $\mathcal{P}_0$, $\overline{\mathcal{P}}_0$ denote the sets of 0-th order parametric and principal derivatives respectively. An operator $\mathbf{X} = \xi^i(x)\partial_{x^i}$ solving the defining system can therefore be written

$$\begin{aligned} \mathbf{X} &= \sum_{\alpha:\xi^\alpha \in \mathcal{P}_0} \xi^\alpha \partial_{x^\alpha} + \sum_{r:\xi^r \in \overline{\mathcal{P}}_0} \xi^r \partial_{x^r} \\ &= \sum_{\alpha:\xi^\alpha \in \mathcal{P}_0} \xi^\alpha \mathbf{Y}_\alpha, \end{aligned}$$

where, from (12),

$$\mathbf{Y}_\alpha = \partial_{x^\alpha} + \sum_{r:\xi^r \in \overline{\mathcal{P}}_0} e^r_\alpha(x)\partial_{x^r}, \qquad \text{for } \alpha \text{ such that } \xi^r \in \mathcal{P}_0. \tag{13}$$

At each point x we have $\text{span}\{\mathbf{X}\} \subseteq \text{span}\{\mathbf{Y}_\alpha\}$. Moreover, by the Riquier existence Theorem 12, there exists a solution vector field $\mathbf{X}$, for any assignment

of values to the 0-th order parametric derivatives. Hence $\text{span}\{\mathbf{X}\} = \text{span}\{\mathbf{Y}_\alpha\}$.
■

As a consequence, the dimension of the group orbits is equal to the number $\#(\mathcal{P}_0)$ of parametric derivatives of order 0. The number of functionally independent scalar invariants is $n - \#(\mathcal{P}_0)$. In fact we can directly characterize invariants I. For a scalar function I to be an invariant of the pseudogroup, it is necessary and sufficient that $\mathbf{X}I = 0$ identically for all $\mathbf{X} \in \mathcal{L}$. Proposition 16 then implies that it is necessary and sufficient that

$$\mathbf{Y}_\alpha I = 0, \qquad \alpha : \xi^\alpha \in \mathcal{P}_0,$$

in order for I to be an invariant. As a special case, a Lie algebra system is transitive (no scalar invariants) if and only if its defining system contains no zeroth order equations. Note that explicit knowledge of the operators $\mathbf{X}$ is not needed to derive the above equations.

As remarked in §2.3, Definition 14 ensures that the subspaces $\Sigma(L_{x_0})$ are all of the same dimension. This is because i of the definition demands that each $x_0 \in M$ be a nonsingular point of the defining system. Proposition 16 then gives constancy of the orbit dimension. If we allowed x_0 to be a singular point of the defining system, we would not be able to apply the Riquier theorem, so Proposition 16 would not be available to give the orbit dimension through x_0.

3.2 Isotropy subgroup

Let $\mathcal{G}$ be a Lie pseudogroup acting on a manifold M and let $x \in M$. The *isotropy* or stabilizer sub-pseudogroup[5] at x is the set $\{\tau \in \mathcal{G} \mid \tau(x) = x\}$ of transformations in $\mathcal{G}$ leaving x fixed. Any vector field $\mathbf{X}$ generating a 1-parameter local subpseudogroup of the isotropy subgroup must satisfy $\mathbf{X}|_{x_0} = 0$ and we define the *isotropy algebra* $L^0_{x_0}$ at x_0 by

$$L^0_{x_0} = \{\mathbf{X} \in L_{x_0} \mid \mathbf{X}|_{x_0} = 0\}.$$

Our blanket analyticity hypothesis implies that $L^0_{x_0}$ consists of vector fields in L_{x_0} with no 0-th order terms in their Taylor expansions about x_0.

Following [50], we analogously define higher order isotropy algebras $L^k_{x_0}$, consisting of vector fields in L_{x_0} whose Taylor coefficients vanish to order k. For $k \geq 1$, $L^k_{x_0}$ is an ideal in $L^{k-1}_{x_0}$, so that we have the chain of subalgebras

$$L_{x_0} \supset L^0_{x_0} \rhd L^1_{x_0} \rhd \cdots \rhd L^k_{x_0} \rhd \cdots$$

We shall only require the algebras L_{x_0}, $L^0_{x_0}$, $L^1_{x_0}$ of this sequence. In the finite dimensional case, there is some maximal order q isotropy subalgebra such that $L^k_{x_0}$ vanishes for $k > q$. In the infinite case, all $L^k_{x_0}$ are infinite dimensional. Note that each algebra $L^k_{x_0}$ is a coordinate free object.

[5] Note that it is not in general a *Lie* pseudogroup however.

We now characterize the algebras $L^k_{x_0}$ in terms of initial data for the canonical form of their infinitesimal defining equations. Since $L^k_{x_0}$ has Taylor coefficients vanishing to order k, all parametric derivatives of order less than or equal to k vanish at x_0. Since the ranking respects total order, all principal derivatives up to order k also vanish. Thus $\mathbf{X} \in L^k_{x_0}$ if and only if the initial data up to order k vanishes. We give a simple example.

Example 17 Consider the finite dimensional Lie algebra with infinitesimal defining system

$$\xi_{xxx} = 0. \tag{14}$$

The parametric derivatives are ξ, ξ_x, ξ_{xx} and the Lie algebra L_{x_0} has basis

$$\mathbf{X}_1 = \partial_x, \qquad \mathbf{X}_2 = (x - x_0)\partial_x, \qquad \mathbf{X}_3 = (x - x_0)^2\partial_x.$$

The isotropy algebra $L^0_{x_0}$ has basis $\{\mathbf{X}_2, \mathbf{X}_3\}$; and $L^1_{x_0}$ has basis $\{\mathbf{X}_3\}$. Note that $L^0_{x_0}$ is the general solution of (14) with initial condition $\xi(x_0) = 0$. Similarly $L^1_{x_0}$ is the general solution of (14) with initial conditions $\xi(x_0) = 0$, $\xi_x(x_0) = 0$.

3.2.1 Linear isotropy algebra

Any transformation τ in the isotropy subgroup at x_0 induces a linear map $\tau_{x_0*}: T_{x_0}M \to T_{x_0}M$ on the tangent space at x_0. In coordinates τ_{x_0*} is the Jacobian matrix of τ at x_0. The collection of all such maps is a matrix Lie group $G_{x_0} \subseteq GL_n$, the *linear isotropy group* G_{x_0} of $\mathcal{G}$ at x_0. The matrix Lie algebra associated with G_{x_0} is the *linear isotropy algebra,* denoted by g_{x_0}. An element of g_{x_0} is a linear map $T_{x_0}M \to T_{x_0}M$. In the case where $\mathcal{G}$ is transitive, neither G_{x_0} nor g_{x_0} can vary from point to point, and we can omit the subscript x_0.

An alternative construction of g_{x_0} which better suits our purposes is as follows [50]. It was noted above that $L^1_{x_0} \triangleleft L^0_{x_0}$.

Definition 18 The *linear isotropy algebra* g at a point x_0 is the quotient algebra $L^0_{x_0}/L^1_{x_0}$, regarded as a matrix Lie algebra operating on $T_{x_0}M$.

In this definition each member of $L^0_{x_0}/L^1_{x_0}$ is an equivalence class of vector fields in $L^0_{x_0}$, two vector fields $\mathbf{Z}$, $\mathbf{W}$ being identified if $\mathbf{W} - \mathbf{Z} \in L^1_{x_0}$. Let

$$\begin{aligned} \mathbf{Z} &= a^i_j(x^j - x^j_0)\partial_{x^i} + O(|x - x_0|^2) \\ \mathbf{W} &= b^i_j(x^j - x^j_0)\partial_{x^i} + O(|x - x_0|^2) \end{aligned} \tag{15}$$

be two vector fields in $L^0_{x_0}$. In the remainder terms, $|x - x_0|^2 = \sum (x^i - x^i_0)^2$, and the notation $O(|x - x_0|^2)$ indicates that each component of the vector field is $O(|x - x_0|^2)$. Then $\mathbf{Z} - \mathbf{W} \in L^1_{x_0}$ if and only if their difference vanishes to order 1, that is, if and only if $a^i_j = b^i_j$ for $i, j = 1, \ldots, n$. An element of $g = L^0_{x_0}/L^1_{x_0}$ is therefore identified with the matrix of leading order Taylor coefficients of any of its class representatives. Let $A \in g_{x_0}$ and let $\mathbf{Z}$ be a class representative of A. We regard A as a linear map $T_{x_0}M \to T_{x_0}M$ defined by

$$AX = [\mathbf{X}, \mathbf{Z}]|_{x_0} \tag{16}$$

where $X \in T_{x_0}M$, and $\mathbf{X}$ is any vector field satisfying $\mathbf{X}|_{x_0} = X$. This definition does not depend on the choice of class representatives $\mathbf{Z}$, $\mathbf{X}$. Take $\mathbf{e}_i = \partial_{x^i}|_{x_0}$ as a basis of $T_{x_0}M$, and let $X = \beta^i \mathbf{e}_i \in T_{x_0}M$ be a tangent vector. Taking

$$\mathbf{X} = \beta^i \partial_{x^i} + O(|x - x_0|)$$

as a vector field solving the defining system with $\mathbf{X}|_{x_0} = X$, and $\mathbf{Z} \in L^0$ given by (15) as a class representative of A, gives

$$[\mathbf{X}, \mathbf{Z}]|_{x_0} = \beta^i a_i^k \mathbf{e}_k,$$

so a_i^k operates on components β^i in the obvious way. Although we will always work in coordinates, definition (16), being coordinate free, will adapt easily to our requirements in §4.3.

The vector field bracket in $L^0_{x_0}$ induces a matrix commutator bracket on $L^0_{x_0}/L^1_{x_0}$. Computing $[\mathbf{Z}, \mathbf{W}] = c^i_j(x^j - x^j_0)\partial_{x^i} + O(|x - x_0|^2)$, we find

$$c^i_j = a^i_k b^k_j - b^i_k a^k_j$$

or equivalently $C = AB - BA$.

4 Cartan structure

The name 'structure' equations for (2) is justified by a number of results due to Cartan. In [6, part I] he shows that two transitive Lie pseudogroups with the same structure equations are similar. Then in [6, part II] he shows that it is possible to decide if one Lie pseudogroup is the prolongation of another on the basis of knowledge only of $a^k_{i\rho}$ and c^k_{ij}. The long paper [7] is devoted to extracting information about sub-pseudogroups of infinite Lie pseudogroups. Cartan uses his method of equivalence to show how to decide the question whether two transitive Lie pseudogroups are isomorphic, based on knowledge only of their structure equations [8, 13]. Hence we need only provide an algorithm for finding the Cartan structure equations since Cartan has provided a method for resolving the isomorphism question based on the knowledge of their structure equations.

The connection between Cartan's structure constants c^k_{ij} and the vector field viewpoint in the transitive case was given by Kuranishi [24], and Singer and Sternberg [50, §2.1]. Our goal is to show that their methods can be realized directly from knowledge of the infinitesimal defining system in the transitive pseudogroup case, and that the process is constructive, involving only linear algebra and differentiation.

4.1 Structure: first order systems with no invariants

Suppose that the Lie algebra system $\mathcal{L}$ of a Lie pseudogroup is transitive, so that its defining system contains no zeroth order equations. Then the vector field germs $\mathbf{X} \in L_{x_0}$ at each basepoint x_0 span tangent space $T_{x_0}M$. In a fixed coordinate system x, we take $\mathbf{e}_i = \partial_{x^i}|_{x_0}$ as a basis for $T_{x_0}M$. Because $\mathcal{L}$ is transitive, there exists a vector field $\mathbf{X}_i \in L_{x_0}$ such that $\mathbf{X}_i|_{x_0} = \mathbf{e}_i$. In fact there is some freedom in our choice of $\mathbf{X}_i$, since any vector field in the isotropy algebra $L^0_{x_0}$ can be added to $\mathbf{X}_i$ without changing the value $\mathbf{X}_i|_{x_0}$. Select one such representative for each i. Take the n vector fields $\mathbf{X}_i$ as a basis for a space $K \subseteq L$. Note that K is a complement of $L^0_{x_0}$, i.e. $L = K \oplus L^0$ as a vector space. Now compute the Lie bracket $[\ ,\ ]: K \wedge K \to L$. Resolve $[\mathbf{X}_i, \mathbf{X}_j]$ with respect to the direct sum $K \oplus L^0$, and resolve the component in K with respect to the basis $\{\mathbf{e}_i\}$ to get

$$[\mathbf{X}_i, \mathbf{X}_j] = c^k_{ij}\mathbf{X}_k + \mathbf{Z}_{ij}$$

where $\mathbf{Z}_{ij} \in L^0$. Evaluating at x_0 yields

$$[\mathbf{X}_i, \mathbf{X}_j]\Big|_{x_0} = c^k_{ij}\mathbf{e}_k.$$

On taking linear combinations of the basis $\mathbf{e}_i$ we have determined a map

$$\underline{c}: T_{x_0}M \wedge T_{x_0}M \to T_{x_0}M$$

which is the map described in [50, §2.1]. In their 'Lexicon to Cartan' §2.14, Singer and Sternberg [50] show that c^k_{ij} *so constructed can be identified with those of Cartan,* when the algebra is transitive and of 'first order type' (first order defining system).

Before showing how to calculate c^k_{ij} from the infinitesimal defining system we first clarify the above process. The crucial point is that although we require *existence* of $\mathbf{X}_i$, the explicit form of $\mathbf{X}_i$ is not required. In fact, only the zeroth order terms in the Taylor expansion of the commutator $[\mathbf{X}_i, \mathbf{X}_j]$ are needed: these depend only on the Taylor coefficients of $\mathbf{X}_i$, $\mathbf{X}_j$ of orders 0 and 1. Hence we do not require the $\mathbf{X}_i$ explicitly, but only their zeroth and first order Taylor terms.

We consider the case where the canonical form of an infinitesimal system is of first order, with respect to an orderly ranking, and there are no invariants Consequently the canonical form contains no 0-th order equations and every 0-th order derivative $\xi^1, \ldots, \xi^n$ is parametric. To make the connection with Cartan structure we introduce new variables ϕ^μ defined by

$$\frac{\partial \xi^k}{\partial x^i} = \phi^\mu, \qquad \mu = 1, \ldots, \#(\mathcal{P}_1), \tag{17}$$

where $\frac{\partial \xi^k}{\partial x^i} \in \mathcal{P}_1$ (i.e. the $\frac{\partial \xi^k}{\partial x^i}$ are first order parametric derivatives). We then eliminate the first order parametrics from the canonical form of the infinitesimal defining system by using (17), and append the equations (17) to the simplified canonical form to obtain the infinitesimal defining system as

$$\frac{\partial \xi^k}{\partial x^i} = \sum_{j=1}^{n} b_{ij}^k(x)\xi^j + \sum_{\rho=1}^{\#(\mathcal{P}_1)} A_{i\rho}^k(x)\phi^\rho, \qquad \text{for } i,k = 1,\ldots,n. \tag{18}$$

Note that cases $\frac{\partial \xi^k}{\partial x^i} \in \mathcal{P}_1$, and $\frac{\partial \xi^k}{\partial x^i} \in \overline{\mathcal{P}}_1$, are covered. In particular when $\frac{\partial \xi^k}{\partial x^i} \in \mathcal{P}_1$, then $b_{ij}^k = 0$, for $j = 1,\ldots,n$ and $A_{i\mu}^k = \delta_\mu^\rho$ so that (18) yields (17). The main result of this section is the following:

Proposition 19 *Let x_0 be a nonsingular point for the infinitesimal defining system (18) in ordered canonical form with respect to an orderly ranking of derivatives. Let $a_{i\rho}^k = A_{i\rho}^k(x_0)$, and*

$$c_{ij}^k = b_{ij}^k(x_0) - b_{ji}^k(x_0). \tag{19}$$

Then $a_{i\rho}^k$, c_{ij}^k can be identified with those in the Cartan structure equations (2).

Proof: Fix the basepoint x_0. First we calculate the linear isotropy algebra $L_{x_0}^0$ of the Lie algebra L of vector field germs at x_0 generating local solutions of (18). Let $\mathbf{X} = \xi^i(x)\partial_{x^i} \in L_{x_0}^0$, with $\xi^i(x_0)$ vanishing. Then

$$\xi^k(x) = \alpha_j^k(x^j - x_0^j) + O(|x - x_0|^2). \tag{20}$$

Substitution of $\xi^i(x)$ into the infinitesimal defining system (18) and evaluation at x_0 gives

$$\alpha_i^k = A_{i\rho}^k(x_0)\phi^\rho(x_0). \tag{21}$$

Moreover, for any choice of $\phi^\rho(x_0)$, the Riquier existence theorem guarantees that a solution of the form (20) exists, with α_i^k given by (21). In §3.2.1 we identified the linear isotropy algebra $g_{x_0} = L_{x_0}^0/L_{x_0}^1$ with the matrices of leading coefficients of vector fields in $L_{x_0}^0$. Thus α_i^k of (21) is in g_{x_0} if and only if it is in the span of the matrices $A_{i\rho}^k(x_0)$, $\rho = 1,\ldots,\#(\mathcal{P}_1)$. Both Cartan [9] and Singer and Sternberg [50] show that $a_{i\rho}^k$ in the structure equations form a basis for the linear isotropy algebra, and we have shown that the a_{ij}^k can be identified with those in Cartan's structure equations.

To determine the relationship between our c_{ij}^k and those of Cartan we note that the construction above asserts that c_{ij}^k are the leading (0-th) order Taylor terms for $[\mathbf{X}_i, \mathbf{X}_j]$ and perform the expansions explicitly. Let $\mathbf{X}_i = \xi_i^k\partial_{x^k}$ be a solution of the initial value problem (18) with initial data given by

$$\begin{aligned} \xi_i^k(x_0) &= \delta_i^k, \\ \phi^\rho(x_0) &= 0, \qquad \rho = 1,\ldots,\#(\mathcal{P}_1). \end{aligned}$$

The Riquier theorem guarantees existence of such a solution. Following the Taylor expansion algorithm in [35],

$$\mathbf{X}_i = \xi_i^k \partial_{x^k} = \partial_{x^i} + b_{ij}^k(x_0)(x^j - x_0^j)\partial_{x^k} + O(|x - x_0|^2)$$

Computing commutators,

$$[\mathbf{X}_i, \mathbf{X}_j] = \Big(b_{ji}^k(x_0) - b_{ij}^k(x_0)\Big)\partial_{x^k} + O(|x - x_0|^2).$$

and evaluating at x_0,

$$[\mathbf{X}_i, \mathbf{X}_j]\Big|_{x_0} = c_{ij}^k \mathbf{e}_k$$

with c_{ij}^k given by (19).

Note that, since the Lie algebra system $\mathcal{L}$ is assumed transitive, the choice of basepoint x_0 is arbitrary and all the Lie algebras L_{x_0} are identical. ∎

We remark that construction of the structure constants from the defining equations requires only expressions for the first order derivatives $\partial_{x^i}\xi^j$. The c_{ij}^k and $a_{i\rho}^k$ are constructed from the coefficients of the 0-th order and 1-st order parametric derivatives respectively, *evaluated at the basepoint* x_0.

Finally, as remarked in the proof, transitivity of the pseudogroup ensures that the Lie algebras L_{x_0} attached to each point x_0 are identical. Hence we may choose x_0 to be *any* convenient point. In fact the structure constants c_{ij}^k and $a_{i\rho}^k$ computed by the above method will differ according to the choice of x_0. This does not contradict the assertion that L_{x_0} all have the same structure: it merely means that our method chooses a different basis for L_{x_0} at different x_0. In practice, however, a wise choice of x_0 can force many of the c_{ij}^k to vanish.

Example 20 Consider the Lie algebra system of vector fields $\mathbf{X} = \xi\partial_x + \eta\partial_y$ on $M = \mathbb{R}^2$ with coordinates (x, y), and first order infinitesimal defining system in canonical form

$$\begin{array}{ll} \xi_x = \frac{1}{y}\eta & \\ \xi_y = 0 & \eta_y = \frac{1}{y}\eta \end{array} \tag{22}$$

were the singular locus $y = 0$ has been removed from the domain. The parametric derivatives of order 0 are ξ, η; and η_x is the only parametric first order derivative. We construct c_{ij}^k and $a_{i\rho}^k$ directly by the above method. The infinitesimal Jacobian matrix evaluated at an initial data point $(x_0, y_0) \in \mathbb{R}^2$ is

$$J = \begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix}\Bigg|_{(x_0,y_0)} = \begin{pmatrix} \frac{1}{y}\eta & 0 \\ \eta_x & \frac{1}{y}\eta \end{pmatrix}\Bigg|_{(x_0,y_0)} \tag{23}$$

after simplification modulo the defining system (23). The values of the first order partials of ξ and η at (x_0, y_0) are also given by (22)

The linear isotropy algebra g at (x_0, y_0) is found by evaluating the infinitesimal Jacobian (23) subject to vanishing 0-th order initial data $\xi(x_0, y_0) =$

0, $\eta(x_0, y_0) = 0$. Then assigning the value a to the first order initial data $\eta_x(x_0, y_0) = a$ gives J as

$$\begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} \tag{24}$$

which is the linear isotropy algebra at (x_0, y_0). We calculate c^k_{ij}:

$$\begin{aligned} c^1_{12} &= (\text{coeff. of } \eta \text{ in } J_{11}) - (\text{coeff. of } \xi \text{ in } J_{12}) = 1/y_0 \\ c^2_{12} &= (\text{coeff. of } \eta \text{ in } J_{12}) - (\text{coeff. of } \xi \text{ in } J_{22}) = 0 \end{aligned}$$

On formally replacing a by a 1-form π^1 our construction has yielded Cartan structure equations

$$\begin{aligned} d\omega^1 &= -\tfrac{1}{y_0}\omega^1 \wedge \omega^2 \\ d\omega^2 &= \pi^1 \wedge \omega^1 \end{aligned} \tag{25}$$

Any value $y_0 \neq 0$ is suitable as an initial data point, so we choose $y_0 = 1$.

It is also instructive to directly calculate the Taylor series expansion of solutions of (22) to first order and evaluate the commutator at x_0.

4.1.1 Residual freedom in the c^k_{ij} and absorption of torsion

In deriving the formula (19) for c^k_{ij}, we chose the first order initial data to be 0 when we constructed the Taylor expansion of the vector fields $\mathbf{X}_i$. This choice is coordinate dependent, and any values of first order initial data could be imposed, leading to differing values of c^k_{ij}. We show that this freedom in c^k_{ij} is parametrized by $a^k_{i\rho}$, and is related to the 'absorption of torsion' step in the Cartan method.

In the Cartan structure equations (2), the 1-forms π^ρ are determined modulo ω^i, that is, it is permissible to replace π^ρ by $\bar{\pi}^\rho + v^\rho_i \omega^i$, with v^ρ_i arbitrary. Making this replacement transforms the structure equations (2) into

$$d\omega^i = a^k_{i\rho}\bar{\pi}^\rho \wedge \omega^i - \tfrac{1}{2}\bar{c}^k_{ij}\omega^i \wedge \omega^j$$

where

$$\bar{c}^k_{ij} = c^k_{ij} + a^k_{i\rho}v^\rho_j - a^k_{j\rho}v^\rho_i. \tag{26}$$

Proposition 21 *Let c^k_{ij} be structure constants for vector fields $\mathbf{X}_i$ computed according to (19) with first order initial data $\phi^\rho(x_0) = 0$. Let $\bar{c}^k_{ij}$ be the structure constants for $\bar{\mathbf{X}}_j$ with first order initial data $\phi^\rho(x_0) = v^\rho_j$. Then $\bar{c}^k_{ij}$ and c^k_{ij} are related by (26).*

Proof: The representatives $\mathbf{X}_i = \partial_{x^i} + O(|x - x_0|)$ used in computing c_{ij}^k are indeterminate up to a choice of vector fields in the isotropy algebra $\mathbf{X}_i \mapsto \bar{\mathbf{X}}_i = \mathbf{X}_i + \mathbf{Z}_i$, with $\mathbf{Z}_i \in L_{x_0}^0$. Since c_{ij}^k depend only on $\mathbf{X}_i$ up to first order, we require only the leading (first order) terms of $\mathbf{Z}_i$. However, as remarked in §3.2.1, the first order terms in an isotropy vector field specify a matrix in the linear isotropy algebra. So expanding $\mathbf{Z}_i$ to first order gives

$$\mathbf{Z}_i = v_i^\rho a_{j\,\rho}^k (x^j - x_0^j)\partial_{x^k} + O(|x - x_0|^2)$$

where v_i^ρ is the value of $\phi^\rho(x_0)$ chosen when constructing $\mathbf{X}_i$. Thus

$$\bar{\mathbf{X}}_i = \mathbf{X}_i + \mathbf{Z}_i = \mathbf{X}_i + v_i^\rho a_{j\,\rho}^k (x^j - x_0^j)\partial_{x^k} + O(|x - x_0|^2)$$

Computing commutators yields

$$\begin{aligned} [\bar{\mathbf{X}}_i, \bar{\mathbf{X}}_j] &= [\mathbf{X}_i, \mathbf{X}_j] + \left(v_j^\rho a_{i\rho}^k - v_i^\rho a_{j\rho}^k\right)\partial_{x^k} + O(|x - x_0|) \\ &= \bar{c}_{ij}^k \bar{\mathbf{X}}_k + O(|x - x_0|), \end{aligned}$$

with $\bar{c}_{ij}^k$ given by (26). ■

By starting with $\mathbf{e}_i$ at the basepoint and building vector fields $\mathbf{X}_i$ with $\mathbf{X}_i|_{x_0} = \mathbf{e}_i$, we have constructed a frame on a neighbourhood $U \subseteq M$. In the case of a finite parameter Lie pseudogroup, the frame is determined by the choice of basis $\mathbf{e}_i$ at the base point. In the infinite case, however, the frame $\mathbf{X}_1, \ldots, \mathbf{X}_n$ has the freedom to vary by addition of vector fields in $L_{x_0}^0$. This residual freedom in the frame is parametrized by the linear isotropy algebra $a_{i\rho}^k$, and manifests itself in the structure through (26).

4.2 Transitivity test for essential invariants

If a defining system is of first order, but contains some 0-th order equations, the method described above does not apply, because the vector fields in L_{x_0} no longer span tangent space $T_{x_0}M$ at a base point x_0. In other words the Lie algebra system is no longer transitive. Following Cartan, we distinguish two cases: (i) the pseudogroup $\mathcal{G}$ generated by $\mathcal{L}$ is isomorphic to a transitive pseudogroup (ii) no such isomorphism exists. Although Cartan calls these two cases 'transitive' and 'intransitive' respectively, we refer to them as *structurally transitive* and *structurally intransitive,* so as to avoid confusion with the usual geometric meaning of transitivity. Thus the pseudogroup $X = x$, $Y = f(y)$ is intransitive, but is structurally transitive, since it is an isomorphic prolongation of the transitive pseudogroup $Y = f(y)$.

Structural intransitivity is related to the presence of *essential invariants,* so we provide a test that gives a characterization of essential invariants, and in particular counts them.

First note that the construction of the linear isotropy algebra g described in §3.2.1 is general, and does not rely on transitivity of the pseudogroup, or even on the defining system being first order. A matrix in the Lie algebra g_{x_0} attached to the point x_0 is regarded as a linear map $T_{x_0}M \to T_{x_0}M$, so that it operates on tangent vectors.

Cartan in [9, p.18] defined the 'groupe plan de stabilité' as the matrix group G' acting on cotangent space $T_{x_0}M$ induced by the isotropy group at x_0. Thus Cartan's linear isotropy group (algebra) is the contragredient representation of the linear isotropy group as we defined it. The quantities $a^k_{i\rho}$ occurring in the structure equations span the Lie algebra g' of G'. Cartan showed that $a^k_{i\rho}$ could be used to characterize essential invariants. His construction is as follows. First define the Pfaff system $a^k_{i\rho}\omega^i$, which Cartan calls the *systatic system.* This system is completely integrable. An integral element ('systatic element') of the systatic system is the subspace of $T_{x_0}M$ annihilated by $a^k_{i\rho}\omega^i$. If a function $I: M \to M$ is both an integral of the systatic system and an invariant of the group, Cartan shows that I is an *essential invariant,* and must appear in any group isomorphic to $\mathcal{G}$.

We now interpret the above in terms of the tangent space $T_{x_0}M$. Let $\mathbf{e}_i$ be the basis of $T_{x_0}M$ dual to $\omega^i|_{x_0}$. Then $\beta^i\mathbf{e}_i$ is a 1-dimensional integral element of the systatic system if

$$a^k_{i\rho}\beta^i = 0, \qquad k = 1, \ldots, \rho, \quad \rho = 1, \ldots, \#(\mathcal{P}_1).$$

Note that Cartan's $a^k_{i\rho}$ are resolved with respect to the basis of $T^*_{x_0}M$ provided by ω^i. Our $a^k_{i\rho}$ are resolved with respect to the basis ∂_{x^i} of $T_{x_0}M$. Since the integral elements of the systatic system are geometrically defined quantities, independent of the basis in which they are expressed, we can make the following definition, which is equivalent to Cartan's.

Definition 22 The *systatic distribution* $\Gamma(\mathcal{L})$ of a Lie algebra system $\mathcal{L}$ with linear isotropy algebra g is the collection of subspaces $\Gamma_{x_0} \subseteq T_{x_0}M$ defined by

$$\Gamma_{x_0} = \Big\{ Z \in T_{x_0}M \mid AZ = 0 \;\; \forall A \in g_{x_0} \Big\}.$$

Thus a tangent vector lies in Γ_{x_0} if and only if it is in the common nullspace of the matrices A_ρ, $\rho = 1, \ldots, \#(\mathcal{P}_1)$, spanning the linear isotropy algebra. Explicitly, if

$$\mathbf{Z} = \alpha^i(x)\partial_{x^i}$$

is a vector field whose values at each point $x \in M$ lie in Γ_x, then

$$a^k_{i\rho}(x)\alpha^i(x) = 0, \qquad i = 1, \ldots, n, \quad \rho = 1, \ldots, \#(\mathcal{P}_1).$$

Note that this characterization of $\Gamma(\mathcal{L})$ not only does not require knowledge of invariant 1-forms ω^i, but also uses only the $a^k_{i\rho}$, which are available from the infinitesimal defining system R.

Definition 23 Let $\mathcal{L}$ be a Lie algebra system with defining equations of first order. A function $I: M \to \mathbb{R}$ is an *essential invariant* of $\mathcal{L}$ if it is (i) an integral of the group distribution $\Sigma(\mathcal{L})$ and (ii) an integral of the systatic distribution $\Gamma(\mathcal{L})$.

A Lie pseudogroup is isomorphic to a transitive Lie pseudogroup if and only if it has no essential invariants. Construction of essential invariants from the distributions $\Sigma(\mathcal{L})$, $\Gamma(\mathcal{L})$ is an integration process. However, we can count the number of common integrals of $\Sigma(\mathcal{L})$, $\Gamma(\mathcal{L})$ by finding the dimension of the subspace spanned by the completion of $\Sigma(\mathcal{L})$, $\Gamma(\mathcal{L})$. The condition that there be no essential invariants gives

Proposition 24 *A Lie algebra system with first order defining system in canonical form with respect to an orderly ranking of derivatives is structurally transitive if and only if the completion of the differential system* $\{\Sigma(\mathcal{L}), \Gamma(\mathcal{L})\}$ *spans tangent space* T_xM *at each point* $x \in M$.

Example 25 Consider the Lie algebra system $\mathcal{L}$ of vector fields $\mathbf{X} = \xi\partial_x + \eta\partial_y + \zeta\partial_z$ on $M = \mathbb{R}^3$ with coordinates (x, y, z) and first order defining system in canonical form

$$\begin{array}{ll} \xi = 0 & \zeta_y = -\frac{x}{y}\zeta_x \\ \eta = 0 & \zeta_z = 0 \end{array} \tag{27}$$

As usual the singular points $y = 0$ are removed from the domain. The group distribution $\Sigma(\mathcal{L})$ is 1-dimensional, and is spanned by the vector field ∂_z. There are therefore 2 scalar invariants. The linear isotropy algebra at the point (x, y, z) is 2-dimensional, consisting of matrices

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a & -ax/y & b \end{pmatrix}$$

Let $\mathbf{Z} = \alpha^1\partial_x + \alpha^2\partial_y + \alpha^3\partial_z$. Then $\mathbf{Z}$ lies in $\Gamma(\mathcal{L})$ at each point if

$$\alpha^1 - \frac{x}{y}\alpha^2 = 0, \qquad \alpha^3 = 0.$$

Hence the systatic distribution $\Gamma(\mathcal{L})$ is 1-dimensional, and is spanned by the vector field $x\partial_x + y\partial_y$. Thus essential invariants $I(x, y, z)$ are the solutions of

$$\partial_z I = 0, \qquad (x\partial_x + y\partial_y)I = 0 \tag{28}$$

This system is already complete, and since a two dimensional subspace of T_xM is spanned at each point, we conclude that there is one essential invariant, and the pseudogroup is structurally intransitive. ■

We note that the pseudogroup described by (27) is

$$X = x, \qquad Y = y, \qquad Z = z + f(\tfrac{x}{y}),$$

with two functionally independent invariants x, y (integrals of $\Sigma(\mathcal{L}) : \partial_z I = 0$). Clearly $\frac{x}{y}$ is essential: note that it is an integral of (28). The infinitesimal characterization in Example 25 is therefore correct. Note also that it is reasonable to demand that the basis of invariants of $\mathcal{L}$ should contain as a subset a basis of the essential invariants. Thus $\{x/y, y\}$ is a more natural basis of the scalar invariants in the example.

4.3 Suppression of inessential invariants

Although we can detect structural intransitivity, we shall not pursue methods for getting the Cartan structure in this case. Part of the difficulty in the intransitive case is that $a^k_{i\rho}$ and c^k_{ij} can be functions of the essential invariants, which is local rather than infinitesimal information. Hence our goal of structure by infinitesimal methods would appear to be difficult to attain.

Suppose the pseudogroup $\mathcal{G}$ is structurally transitive and has $n-r$ invariants I^ν, $\nu = 1, \ldots, n-r$. Then the I^ν are incidental from a structural viewpoint, and we shall give a method for suppressing them.

In particular, choose a base point x_0 and consider the orbit N passing through x_0. That is, N is the leaf of the foliation induced by the group distribution $\Sigma(\mathcal{L})$ through x_0. If the invariants $I^\nu(x)$ are known in some neighbourhood of x_0, then N is specified locally by equations $I^\nu(x) = I^\nu(x_0)$, $\nu = 1, \ldots, n-r$ and N is r-dimensional.

The pseudogroup $\mathcal{G}$ restricts naturally to a transitive pseudogroup on an orbit N which we denote $\mathcal{G}|_N$. Similarly the Lie algebra system $\mathcal{L}$ restricts to a transitive Lie algebra system on N which is denoted by $\mathcal{L}|_N$. Moreover, since $\mathcal{L}$ is structurally transitive, no structural change is made when I^ν are discarded in this way. There is no theoretical difficulty in performing such restriction, but as always we are concerned with the question of construction from the infinitesimal defining system. In particular we shall show that structure constants $a^k_{i\rho}$ and c^k_{ij} are available for $\mathcal{L}|_N$ without explicitly knowing $\mathcal{L}$ or N.

Let $\mathcal{L}$ act on an n-dimensional manifold M. Let the infinitesimal defining system contain $n-r$ independent relations of order 0. Then according to §3.1, the integral submanifolds of the group distribution $\Sigma(\mathcal{L})$ are of dimension r. Suppose the 0-th order equations are resolved into principal and parametric derivatives as in (12), and that we have constructed vector fields $\mathbf{Y}_i$, $i = 1, \ldots, r$ (13) spanning the distribution $\Sigma(\mathcal{L})$. The construction in §3.1 gives $\mathbf{Y}_\alpha$ in Gauss reduced form, and because $\Sigma(\mathcal{L})$ is completely integrable this implies that $\{\mathbf{Y}_\alpha\}$ given by (13) mutually commute. Each local vector field $\mathbf{X} \in \mathcal{L}$ is tangent to the leaves of the foliation induced by $\Sigma(\mathcal{L})$, and so $\mathbf{X}$ restricts naturally to the integral submanifolds of $\Sigma(\mathcal{L})$. Let N be the integral submanifold passing through the point $x_0 \in M$; the vector field $\mathbf{X}$ restricted to N is denoted by $\mathbf{X}|_N$. The vector fields $\mathbf{Y}_\alpha$ given by (13) also restrict to N. Moreover, because they commute, the $\mathbf{Y}_\alpha|_N$ define a coordinate system on N. We make this coordinate system more explicit. Let us arrange that the parametric derivatives of order 0 are $\xi^1, \ldots, \xi^r$. Also, let $I^1(x), \ldots, I^{n-r}(x)$ be functionally independent scalar invariants, so that $\mathbf{Y}_\alpha I^\nu = 0$ for $\nu = 1, \ldots, n-r$. With respect to the coordinate system X given by

$$\begin{aligned} X^i &= x^i, & i &= 1, \ldots r \\ X^{r+\nu} &= I^\nu(x), & \nu &= 1, \ldots n-r, \end{aligned} \tag{29}$$

the operators $\mathbf{Y}_i$ are just ∂_{X^i}, for $i = 1, \ldots, r$. Restricting to N by fixing $I^\nu(x) = I^\nu(x_0) = \text{const}$, we see that $X^i|_N$, $i = 1, \ldots, r$ are coordinates on N. In other words, the x^i corresponding to the parametric derivatives of order 0 can be used to coordinatize N in a neighbourhood of x_0.

Because the restricted Lie algebra system $\mathcal{L}|_N$ is transitive, its defining system must contain no 0-th order equations, and the method of §4.1 applies. If we write

$$\mathbf{X} = \xi^i(x)\partial_{x^i} = \Xi^i(X)\partial_{X^i}$$

then we have

$$\Xi^i(X) = \xi^i(x), \qquad i = 1, \ldots, r$$

and $\Xi^i(X) = 0$ for $i = r+1, \ldots, n$. We shall apply the method of (4.1) in the new coordinates, then return to the original x coordinates where explicit calculation is possible.

There is no necessity to construct the defining system for $\Xi|_N$ as a function of $(X^1, \ldots, X^r)$ explicitly. The method of §4.1 requires only expressions for $\frac{\partial \Xi^i|_N}{\partial X^j}$, for $i, j = 1, \ldots, n-r$. Hence we evaluate

$$\begin{pmatrix} \partial_{X^1}\Xi^1 & \cdots & \partial_{X^n}\Xi^1 \\ \vdots & & \vdots \\ \partial_{X^1}\Xi^r & \cdots & \partial_{X^n}\Xi^r \end{pmatrix}$$

which in our original coordinate system is

$$\begin{pmatrix} \mathbf{Y}_1\xi^1 & \cdots & \mathbf{Y}_r\xi^1 \\ \vdots & & \vdots \\ \mathbf{Y}_1\xi^r & \cdots & \mathbf{Y}_r\xi^r \end{pmatrix}.$$

The original infinitesimal defining system for $\xi^i(x)$ furnishes relations

$$\mathbf{Y}_i\xi^k = A^k_{i\rho}(x)\phi^\rho(x) + b^k_{ij}\xi^j(x)$$

where $A^k_{i\rho}(x)$ and $b^k_{ij}(x)$ are explicitly known. Proposition 19 then shows that Cartan structure constants of the restricted pseudogroup are given by $a^k_{i\rho} = A^k_{i\rho}(x_0)$, and $c^k_{ij} = b^k_{ij} - b^k_{ji}$.

Example 26 Consider the Lie algebra system of vector fields

$$\mathbf{X} = \xi\partial_x + \eta\partial_y + \zeta\partial_z$$

satisfying the defining system

$$\begin{array}{lll} \xi_x = \frac{1}{y}\eta & & \\ \xi_y = 0 & \eta_y = \frac{1}{y}\eta & \zeta = -\xi \\ \xi_z = 0 & \eta_z = 0 & \end{array} \tag{30}$$

with 0-th order parametric derivatives ξ, η. We find

$$\mathbf{X} = \xi(\partial_x - \partial_z) + \eta\partial_y$$

so that the group distribution is spanned by the commuting operators $\mathbf{Y}_1 = \partial_x - \partial_z$, $\mathbf{Y}_2 = \partial_y$. There is one scalar invariant. The systatic distribution is two dimensional, and spanned by $\mathbf{Z}_1 = \partial_y$, $\mathbf{Z}_2 = \partial_z$. Hence $\mathbf{Y}_1$, $\mathbf{Y}_2$, $\mathbf{Z}_1$, $\mathbf{Z}_2$ span the tangent space at each point, the pseudogroup is structurally transitive, and restriction to an orbit is justified.

We compute

$$J = \begin{pmatrix} \mathbf{Y}_1\xi & \mathbf{Y}_2\xi \\ \mathbf{Y}_1\eta & \mathbf{Y}_2\eta \end{pmatrix} = \begin{pmatrix} \frac{1}{y}\eta & 0 \\ \eta_x & \frac{1}{y}\eta \end{pmatrix},$$

after evaluating modulo the infinitesimal defining system (30). The same calculation as in Example 20 then yields the structure equations (25). ∎

The pseudogroup associated with this example has one scalar invariant $I = xz$: the above calculation effectively restricts to the orbit $I = x_0z_0$ passing through the basepoint (x_0, y_0, z_0) *without knowing* I.

Note that the above computations can be carried out for a structurally intransitive pseudogroup to obtain the structure of $\mathcal{G}|_N$. The difference is that some of the pseudogroup structure is lost in this transition. For example, the pseudogroup $X = x$, $Y = y + f(x)$ is structurally intransitive: restricting to an orbit $x = x_0$ leaves only a 1-parameter translation group.

4.4 Reduction of higher order system to the first order case

We have already shown how to determine the Cartan structure of transitive Lie pseudogroups with first order infinitesimal defining equations in canonical form. The purpose of this section is to exhibit an algorithmic prolongation procedure for reducing the higher order case to first order. Our construction is guided by Cartan's method for determination of the structure of Lie pseudogroups, which proceeds from the pseudogroup defining system in first order involutive form (i.e. as an involutive system of 1-forms).

Consider an infinitesimal defining system for the infinitesimals $\zeta(z)$ with corresponding vector field $\zeta^1\partial_{z^1}+\cdots+\zeta^p\partial_{z^p}$ which we suppose to be involutive at order q. For the remainder of this section, i, j, k, l will be indices ranging between 1 and p. At the infinitesimal level we mimic Cartan by first reducing the infinitesimal defining system for $\zeta^i(z)$ to first order involutive form. Cartan works with the pseudogroup defining system whose solutions $Z = \tau(z)$ are the group transformations. In his process he prolongs the pseudogroup action on z to a pseudogroup action on (z, Z) with trivial action on Z. That is, a transformation $z \mapsto \tau(z)$ prolongs to $(z, Z) \mapsto (\tau(z), Z)$. At the infinitesimal level we prolong the vector field $\zeta^i\partial_{z^i}$ to a vector field on (z, Z) with trivial action on Z, that is to $\zeta^i\partial_{z^i} + \psi^i \cdot \partial_{Z^i}$, with $\psi^i = 0$. Cartan then prolongs the pseudogroup transformations to the derivatives of Z^i up to order $q-1$; similarly, we prolong the vector field to

$$\mathbf{Z}^{(q-1)} = \zeta^i\partial_{z^i} + \psi^i\partial_{Z^i} + \psi^i_J\partial_{Z^i_J} \tag{31}$$

where $\psi^i = 0$, the ψ^i_J are given by the standard extension formula [31, p.113], and there is summation on the repeated index i and the repeated multi-index J, $1 \le \#(J) \le q-1$. Then $\mathbf{Z}^{(q-1)}$ is a vector field on the $(q-1)$-th order jet bundle with coordinates z, Z, Z^l_J. Here the derivatives of the Z's are denoted by $Z^l_J = \partial^n Z^l/\partial z^{j_1}\partial z^{j_2}\cdots\partial z^{j_n}$ where $J = (j_1, ..., j_n)$, $1 \le j_v \le p$, is a symmetric multi-index of order $n = \#(J)$. We will show that:

Theorem 27 *Let $\mathcal{L}$ be a Lie algebra system of vector fields $\mathbf{Z} = \zeta^i\partial_{z^i}$ whose infinitesimal defining system is involutive at order q. Then the prolongation $\mathbf{Z}^{(q-1)}$ of $\mathbf{Z}$ to (z^i, Z^i, Z^i_J)-space is a Lie algebra system with an infinitesimal defining system for ζ^i, ψ^i, ψ^i_J which is involutive at order 1. Furthermore this system can be constructively determined.*

Note that the independent variables in the infinitesimal defining system of the original Lie algebra are z^i. For the prolonged Lie algebra the independent variables in the first order defining system are (z^i, Z^i, Z^i_J), and the corresponding dependent variables ζ^i, ψ^i, ψ^i_J, $1 \le \#(J) \le q-1$. The remainder of this section will be devoted to proving this result.

First we have

Lemma 28 *Any analytic infinitesimal defining system can be constructively transformed to an equivalent first order involutive system.*

Proof: According to Theorem 12 a defining system can be reduced to ordered canonical form, and by Theorem 13 this system can then be constructively prolonged to a q-th order involutive system. Applying the method of Pommaret [33, p.109, p.161] for constructively reducing a q-th order system to an equivalent first order system with equivalent symbol then establishes the result. ■

In Pommaret's transformation to first order, the derivatives ζ^l_L, $1 \le \#(L) \le q-1$ are relabelled as new dependent variables, and the given system is expressed as a first order system with respect to these variables. In addition certain first order differential relations between the ζ^l_L are appended to this first order system (see [33, p.109] for an exact description).

The ψ^l_J are determined in terms of the ζ^i and ζ^l_L by the standard extension formula [31, p.113] which in our case is recursively defined by:

$$\begin{aligned} \psi^l &= 0 \\ \psi^l_{J,i} &= D_i \psi^l_J - \zeta^k_i Z^l_{J,k}, \qquad 0 \leq \#(J) \leq q-1, \end{aligned}$$

and D_i is the total derivative operator with respect to z^i

$$D_i = \partial_{z^i} + \sum_{\#(J)\geq 0} \zeta^l_{J,i} \partial_{\zeta^l_J}. \tag{32}$$

For the vector field (31) the required extensions are

$$\begin{aligned} \psi^l &= 0 \\ \psi^l_i &= D_i \psi^l & -\zeta^k_i Z^l_k, \\ \psi^l_{j,i} &= D_i \psi^l_j & -\zeta^k_i Z^l_{j,k}, \\ &\vdots \\ \psi^l_{J,i} &= D_i \psi^l_J & -\zeta^k_i Z^l_{J,k}, \end{aligned}$$

where J is a symmetric multi-index with $\#(J) = q-2$. Evaluating the total derivative using (32) the above system becomes

$$\begin{aligned} \psi^l_i &= -\zeta^k_i Z^l_k, \\ \psi^l_{j,i} &= -\zeta^k_{j,i} Z^l_k &+& R^l_{j,i}, \\ &\vdots \\ \psi^l_{J,i} &= -\zeta^k_{J,i} Z^l_k &+& R^l_{J,i}, \end{aligned} \tag{33}$$

where each of the remainders $R^l_{K,i}$ depends on $\zeta^k_{L,i}$ only for $\#(L) < \#(K)$.

Lemma 29 *In a neighbourhood of $Z^l_k = \delta^l_k$, $Z^l_L = 0$, $2 \leq \#(L) \leq q-1$ the relations (33) define an invertible linear map from ζ^l_L to ψ^l_L.*

Proof: First note that the system (33), including the remainder terms, is indeed linear in ζ^l_L. Secondly the highest order terms ζ^l_L (i.e. those with maximum $\#(L)$) occur in the explicitly displayed terms in (33), so that the equations have a block triangular structure. When $Z^l_k = \delta^l_k$ the diagonal blocks are identity matrices, relations (33) reduce to

$$\begin{aligned} \psi^l_i &= -\zeta^l_i, \\ \psi^l_{j,i} &= -\zeta^l_{j,i} &+& R^l_{j,i}, \\ &\vdots \\ \psi^l_{J,i} &= -\zeta^l_{J,i} &+& R^l_{J,i}, \end{aligned}$$

which are clearly invertible. Since the coefficients of ζ^l_L in (33) are analytic, invertibility holds in some neighbourhood of $Z^l_k = \delta^l_k$. ■

The main Theorem 27 now follows easily:

Proof: Let S denote the first order involutive system with independent variables z^i obtained by the transformation of Pommaret from the q-th order involutive infinitesimal defining system. Let T denote S augmented with the equations

$$\partial_{Z^l_J}\zeta^i = 0, \qquad \partial_{Z^l_J}\zeta^i_L = 0, \qquad 1 \le \#(L) \le q-1, \quad 0 \le \#(J) \le q-1, \quad (34)$$

Thus the system T has independent variables z^i, Z^i_J, and dependent variables ζ^i_J, with $0 \le \#(J) \le q-1$. Since the system S has no Z^i_J, the integrability conditions between the new equations and S are trivial, and the system T is also first order involutive.

The map

$$(z^i, Z^i, Z^i_J, \zeta^i, \zeta^i_J) \mapsto (z^i, Z^i, Z^i_J, \zeta^i, \psi^i_J), \qquad 1 \le \#(J) \le q-1$$

induced by (33), is an analytic invertible change of coordinates on the space of independent and dependent variables of the system T, by virtue of Lemma 29. Both involutivity and the order of a system are geometric properties which are preserved under invertible changes of coordinates. Consequently the system obtained by making the change of coordinates above and adjoining the conditions $\psi^l = 0$ is first order and involutive, and Theorem 27 is proved. ■

Note that the variables Z^i play a trivial role in the defining system for the prolonged vector field $\mathbf{Z}^{(q-1)}$. Their infinitesimals ψ^i vanish, and it is readily confirmed that all ψ^i_J are independent of Z^i. We retain them only for formal convenience.

A Appendix: Determination of structure of Lie pseudogroups of PDEs

When the results of this article are applied to the determination of Lie pseudogroups of symmetries of differential equations we have the following result:

Theorem 30 *Given an analytic infinitesimal defining system of the Lie pseudogroup of symmetries of a system of PDEs then the algorithms of this paper can constructively determine*

(a) whether the Lie pseudogroup is structurally transitive,

(b) the Cartan structure of the Lie pseudogroup, if it is structurally transitive.

We have programmed preliminary computer algebra versions of many of the algorithms in this paper.

We now present a sequence of examples of differential equations with infinite Lie symmetry pseudogroups. In each case the infinitesimal defining system for the point symmetry vector fields was derived by the usual method [1, 31, 32]. In particular we used the Maple program [19] to obtain the infinitesimal defining systems. These systems were then automatically brought to canonical form using the program [40]. None of the defining systems contained 0-th order equations so the pseudogroups are all transitive. An involutivity check showed that in each case the defining system was involutive at second order and they were automatically reduced to a first order involutive system. Application of our programs to these first order systems automatically yielded their Cartan structure.

Example 31 (Liouville's equation) Liouville's equation

$$u_{xy} = e^u$$

admits an infinite Lie pseudogroup of symmetries with two arbitrary functions of 1 variable. If we seek symmetry vector fields of the form

$$\xi\partial_x + \tau\partial_y + \eta\partial_u$$

then we obtain the ordered canonical form of the infinitesimal defining system

$$\begin{aligned}
\tau_{yy} &= -\eta_y \\
\eta_{xy} &= 0 \\
\xi_x &= -\tau_y - \eta \\
\xi_y &= 0 \\
\xi_u &= 0 \\
\tau_x &= 0 \\
\tau_u &= 0 \\
\eta_u &= 0
\end{aligned}$$

After reducing to first order as described in §4.4, the method of §4.3 yields Cartan structure equations

$$\begin{aligned}
d\omega^1 &= -\omega^1 \wedge \omega^6 \\
d\omega^2 &= -\omega^2 \wedge \omega^3 + \omega^2 \wedge \omega^6 \\
d\omega^3 &= -\omega^1 \wedge \omega^4 - \omega^2 \wedge \omega^5 \\
d\omega^4 &= \pi^1 \wedge \omega^1 + \omega^4 \wedge \omega^6 \\
d\omega^5 &= \pi^2 \wedge \omega^2 - \omega^3 \wedge \omega^5 - \omega^5 \wedge \omega^6 \\
d\omega^6 &= -\omega^1 \wedge \omega^4
\end{aligned}$$

In [37] we also derive the above structure equations using a different method.

Example 32 (KP equation) The Kadomtsev-Petviashvili equation

$$u_{yy} + \left(u_t + u_{xxx} + 2uu_x\right)_x = 0$$

has an infinite Lie pseudogroup of symmetries depending on 3 arbitrary functions of 1 variable. Its structure equations are

$$\begin{aligned}
d\omega^1 &= \omega^1 \wedge \omega^8 + 2\,\omega^3 \wedge \omega^9 \\
d\omega^2 &= -\omega^1 \wedge \omega^9 + \frac{\omega^2 \wedge \omega^8}{2} + 2\,\omega^3 \wedge \omega^4 \\
d\omega^3 &= \frac{3\,\omega^3 \wedge \omega^8}{2} \\
d\omega^4 &= -\omega^1 \wedge \omega^5 - \omega^2 \wedge \omega^6 - \omega^3 \wedge \omega^7 - \omega^4 \wedge \omega^8 \\
d\omega^5 &= \pi^1 \wedge \omega^1 + \pi^2 \wedge \omega^3 - 2\,\omega^5 \wedge \omega^8 + \omega^6 \wedge \omega^9 \\
d\omega^6 &= -\pi^1 \wedge \omega^3 - \frac{3\,\omega^6 \wedge \omega^8}{2} \\
d\omega^7 &= \pi^2 \wedge \omega^1 - \pi^1 \wedge \omega^2 + \pi^3 \wedge \omega^3 - 2\,\omega^4 \wedge \omega^6 - 2\,\omega^5 \wedge \omega^9 - \frac{5\,\omega^7 \wedge \omega^8}{2} \\
d\omega^8 &= -4\,\omega^3 \wedge \omega^6 \\
d\omega^9 &= -2\,\omega^1 \wedge \omega^6 + 2\,\omega^3 \wedge \omega^5 + \frac{\omega^8 \wedge \omega^9}{2}
\end{aligned}$$

In [12] the explicit form of the infinitesimal generators of symmetries of the KP equation is given. The generators depend on arbitrary functions and these are used to parametrize the commutation relations of the algebra. Laurent expansion of the generators is used to show that the symmetry algebra has a Kac-Moody-Virasoro structure. It would be interesting to see whether this structure could be directly determined from the Cartan structure given above.

Example 33 (Steady boundary layer equations) The steady state boundary layer equations [32]

$$\begin{aligned}
uu_x + vu_y + p_x &= u_{yy} \\
p_y &= 0 \\
u_x + v_y &= 0
\end{aligned}$$

have an infinite Lie pseudogroup of symmetries depending on one arbitrary function of one variable. Its Cartan structure equations are

$$\begin{aligned}
d\omega^1 &= \omega^1 \wedge \omega^3 - 2\,\omega^1 \wedge \omega^7 \\
d\omega^2 &= \omega^1 \wedge \omega^4 - \omega^2 \wedge \omega^7 \\
d\omega^3 &= 0 \\
d\omega^4 &= -\omega^1 \wedge \omega^4 - \omega^1 \wedge \omega^6 + \omega^3 \wedge \omega^4 + \omega^4 \wedge \omega^7
\end{aligned}$$

$$
\begin{aligned}
d\omega^5 &= -2\,\omega^3 \wedge \omega^5 \\
d\omega^6 &= \pi^1 \wedge \omega^1 + \omega^3 \wedge \omega^4 + 2\,\omega^3 \wedge \omega^6 + 2\,\omega^4 \wedge \omega^7 + 3\,\omega^6 \wedge \omega^7 \\
d\omega^7 &= 0
\end{aligned}
$$

It is interesting to compare the above results with those in [38] where the commutation relations of its infinite dimensional Lie algebra of symmetries are parametrized using arbitrary functions.

References

[1] Bluman, G.W. and S. Kumei. 1989. *Symmetries and Differential Equations.* (Springer Verlag, New York).

[2] Bryant, R.L., S.S. Chern, R.B. Gardner, H.L. Goldschmidt and P.A. Griffiths. 1991 *Exterior Differential Systems*, Mathematical Sciences Research Institute Publications **18** (Springer Verlag, New York).

[3] Carrà-Ferro, G. 1987. Gröbner Bases and Differential Algebra. *Lecture Notes in Comp. Sci.* **356** 128–140.

[4] Carrà-Ferro, G. and S.V. Duzhin. 1993. Differential-algebraic and differential-geometric approach to the study of involutive symbols. In *Modern Group Analysis: Advanced Analytical and Computational Methods in Mathematical Physics.* Edited by N.H. Ibragimov, M. Torrisi and A. Valenti. 93–99. (Kluwer, Dordrecht).

[5] Carrà-Ferro, G. and W. Y. Sit. 1993. On Term-Orderings and Rankings, *Lecture Notes in Pure and Applied Math.*, Computational Algebra, Dekker, **151**.

[6] Cartan, É.J. 1904. Sur la structure des groupes infinis de transformations. *Oeuvres Complètes* Part II, Vol. 2, 571–714. (Gauthier-Villars, Paris).

[7] Cartan, É.J. 1908. Les sous-groupes des groupes continus de transformations. *Oeuvres Complètes* Part II, Vol. 2, 719–856. (Gauthier-Villars, Paris).

[8] Cartan, É.J. 1937. Les problèmes d'équivalence. *Oeuvres Complètes* Part II, Vol. 2, 1311–1334. (Gauthier-Villars, Paris).

[9] Cartan, É.J. 1937. La structure des groupes infinis. *Oeuvres Complètes* Part II, Vol. 2, 1335–1384. (Gauthier-Villars, Paris).

[10] Clarkson, P.A. and E.L. Mansfield. 1993 *Algorithms for Nonclassical Symmetries.* Preprint, Department of Mathematics. (University of Exeter, Exeter, United Kingdom).

[11] Czichowski, G. and M. Thiede. 1992 *Gröbner Bases, Standard Forms of Differential Equations and Symmetry Computation.* Seminar Sophus Lie Darmstadt-Erlangen-Greifswald-Leipzig.

[12] David, D., N. Kamran, D. Levi, and P. Winternitz. 1986. Symmetry reduction for the Kadomtsev-Petrviashvili equation using a loop algebra. *J. Math. Phys.* **27** 1225–1337.

[13] Gardner, R.B. 1989. *The Method of Equivalence and Its Applications.* SIAM, Philadelphia.

[14] Guillemin, V. 1966. A Jordan-Hölder decomposition for certain classes of infinite-dimensional Lie algebras. *J. Diff. Geom.* **2** 313–345.

[15] Goldschmidt, H. 1972–76. Sur la structure des équations de Lie, I, II, III. *J. Diff. Geom.* **6** 357–373, **7** 67–95, **11** 167–223.

[16] Goldschmidt, H. and D.C. Spencer. 1976–78. On the nonlinear cohomology of Lie equations I, II. *Acta Math.* **136** 103–239, III, IV. *J. Diff. Geom.* **13** 409–526.

[17] Hartley, D. and R. W. Tucker. 1991 A Constructive Implementation of the Cartan-Kähler Theory of Exterior Differential Systems, *J. Symb. Comp.* **12** 655-667.

[18] Hereman, W. 1994. Review of symbolic software for the computation of Lie symmetries of differential equations. *Euromath Bull.*, **1** 45–79.

[19] Hickman, M. 1993. *The Use of Maple in the Search for Symmetries*, Research Report no. 77, Department of Mathematics (University of Canterbury, Christchurch, New Zealand).

[20] Hsu, L. and N. Kamran. 1989. Classification of second-order ordinary differential equations admitting Lie groups of fibre-preserving symmetries. *Proc. Lond. Math. Soc.* **58** 387–416.

[21] Janet, M. 1920. Sur les systèmes d'équations aux dérivées partielles. *J. de Math* **3** 65–151.

[22] Kamran, N. 1989. Contributions to the study of the equivalence problem of Élie Cartan and its applications to partial and ordinary differential equations. Preprint, Institute for Advanced Study. (Princeton Univ., Princeton).

[23] Kuranishi, M. 1959. On the local theory of continuous infinite pseudogroups. I. *Nagoya Math. J.* **15** 225–260.

[24] Kuranishi, M. 1961. On the local theory of continuous infinite pseudogroups. II. *Nagoya Math. J.* **19** 55–91.

[25] Lie, S. 1881. Über die Integration durch bestimmte Integrale von einer Klasse linearer partieller Differentialgleichungen. *Arch. for Math.* **6** 328–368; also *Gesammelte Abhandlungen,* Vol. III, B.G. Teubner, Leipzig, 1922, 492–523.

[26] Lie, S. 1891. Foundations of the theory of infinite transformation groups. (English translation in Hermann, R. 1980. *Cartanian geometry, nonlinear waves and solitons, Part B.* (MathSci Press, Brookline Mass).

[27] Malgrange, B. 1972. Équations de Lie, I, II. *J. Diff. Geom.* **6** 503–522), **7** 117–141.

[28] Mansfield, E. 1991. *Differential Gröbner Bases*, Ph.D Thesis, University of Sydney.

[29] Mansfield, E. 1994. *A simple criterion for involutivity*. Univ. of Exeter Preprint M94/16.

[30] Mansfield, E.L. and E.D. Fackerell. 1993 *Differential Gröbner Bases.* Preprint 92-108 School of Mathematics, Physics, Computer Science, and Electronics (Macquarie University, Sydney, Australia, 1992).

[31] Olver, P.J. 1986. Application of Lie groups to differential equations. (Springer Verlag, New York).

[32] Ovsiannikov, L.V. 1982. Group analysis of differential equations. (Academic Press, New York).

[33] Pommaret, J. F. 1978. *Systems of Partial Differential Equations and Lie Pseudogroups* (Gordon and Breach science publishers, Inc.)

[34] Reid, G.J. 1990. *Algorithmic Determination of Lie Symmetry Algebras of Differential Equations*, In V. Hussin (ed.), *Lie Theory, Differential Equations and Representation Theory* Proc. Annual Seminar of the Canadian Math. Soc. (Les Publications de Centre de Recherches Mathématiques, Montréal, Canada).

[35] Reid, G.J. 1991. Algorithms for reducing a system of PDEs to standard form, determining the dimension of its solution space and calculating its Taylor series solution. *Euro. J. Appl. Maths.* **2** 293–318.

[36] Reid, G.J. 1991. Finding abstract Lie symmetry algebras of differential equations without integrating determining equations. *Euro. J. Appl. Maths.* **2** 319–340.

[37] Reid, G.J., I.G. Lisle and A. Boulton. Characterising Lie equations by their infinitesimal symmetries. Preprint.

[38] Reid, G.J., I.G. Lisle, A. Boulton and A. D. Wittkopf. 1992. Algorithmic determination of commutation relations for Lie symmetry algebras of PDEs, in: *Proc. ISSAC '92*, Berkeley, California, Ed.: P.S. Wang (ACM Press, New York, 1992) 63-68.

[39] Reid, G.J., D.T. Weih and A.D. Wittkopf. 1993. A Point symmetry group of a differential equation which cannot be found using infinitesimal methods. In *Modern Group Analysis: Advanced Analytical and Computational Methods in Mathematical Physics.* Edited by N.H. Ibragimov, M. Torrisi and A. Valenti. 93–99. Kluwer, Dordrecht.

[40] Reid, G.J. and A. D. Wittkopf. 1993. The long guide to the Standard Form package. Programs and documentation available by anonymous ftp to `math.ubc.ca` in directory `/pub/reid/standardform.`

[41] Riquier, C. 1910. *Les Systèmes d'Équations aux Dérivées Partielles.* (Gauthier-Villars, Paris).

[42] Ritt, J.F. 1950. *Differential Algebra*, Amer. Math. Soc. Colloq. Publns. **13** (A.M.S., New York).

[43] Rust, C. 1993. *On The Classification of Rankings of Partial Derivatives*, Preprint.

[44] Schü, J., W.M. Seiler, and J. Calmet. 1992. Algorithmic Methods for Lie Pseudogroups. In *Proc. Modern Group Analysis.* Acireale.

[45] Schwarz, F. 1984. The Riquier-Janet theory and its application to nonlinear evolution equations. *Physica* **D11** 243–251.

[46] Schwarz, F. 1985. Automatically determining symmetries of differential equations. *Computing.* **34** 91–106.

[47] Schwarz, F. 1992. An Algorithm for Determining the Size of Symmetry Groups. *Computing.* **49** 95–115.

[48] Schwarz, F. 1992. Reduction and completion algorithms for Partial Differential Equations. In *Proc. ISSAC '92.* 49–56. (ACM Press, Berkeley).

[49] Sherring, J. and G. Prince. 1992. *DIMSYM - Symmetry Determination and Linear Differential Equations Package.* Preprint, Department of Mathematics (LaTrobe University, Bundoora, Australia).

[50] Singer, I.M. and S. Sternberg. 1965. The infinite groups of Lie and Cartan. I. The transitive groups. *J. d'Analyse Math.* **15:** 1–115.

[51] Thomas, J.M. 1929. Riquier's existence theorems. *Annals of Maths.* **30** 285–310.

[52] Thomas, J. M. 1931. Riquier's Existence Theorems. *Annals of Math.* **35** 306–311.

[53] Topunov, L. 1989. Reducing systems of linear differential equations to a passive form. *Acta Appl. Math.* **16** 191–206.

[54] Tresse, A. 1894. Sur les invariants différentiels des groupes continus de transformations. *Acta Mathematica* **18** 1–88. (English translation I. Lisle 1989, available from the author).

[55] Vessiot, E. 1904. Sur l'intégration des systèmes différentiels. *Acta Mathematica* **28** 307–349.

[56] Weispfenning, V. 1993. Differential-Term Orders. In *Proc. ISSAC 1993.* ACM Press.