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FACETS OF THE COMPLEMENTARITY KNAPSACK POLYTOPE

I. R. DE FARIAS, JR., E. L. JOHNSON, AND G. L. NEMHAUSER

We present a polyhedral study of the complementarity knapsack problem. Traditionally, complementarity constraints are modeled by introducing auxiliary binary variables and additional constraints, and the model is tightened by introducing strong inequalities valid for the resulting MIP. We use an alternative approach, in which we keep in the model only the continuous variables, and we tighten the model by introducing inequalities that define facets of the convex hull of the set of feasible solutions in the space of the continuous variables. To obtain the facet-defining inequalities, we extend the concepts of cover and cover inequality, commonly used in 0–1 programming, for this problem, and we show how to sequentially lift cover inequalities. We obtain tight bounds for the lifting coefficients, and we present two families of facet-defining inequalities that can be derived by lifting cover inequalities. We show that unlike 0–1 knapsack polytopes, in which different facet-defining inequalities can be derived by fixing variables at 0 or 1, and then sequentially lifting cover inequalities valid for the projected polytope, any sequentially lifted cover inequality for the complementarity knapsack polytope can be obtained by fixing variables at 0.

1. Introduction. Let $M = \{1, \ldots, m\}$, $N_i = \{1, \ldots, n_i\}$, $i \in M$, and $u_{ij} \in \Re_+ \cup \{\infty\}$, $j \in N_i$, $i \in M$. The complementarity knapsack problem (CKP) is

$$\max \quad \sum_{i\in M}\sum_{j\in N_i} c_{ij}x_{ij}$$

$$(1) \qquad \sum_{i\in M}\sum_{j\in N_i} a_{ij}x_{ij} \le b$$

$$(2) \qquad x_{ij}x_{ij'} = 0, \qquad j, j' \in N_i,\ j \neq j',\ i \in M$$

$$(3) \qquad x_{ij} \le u_{ij}, \qquad j \in N_i,\ i \in M$$

$$(4) \qquad x_{ij} \ge 0, \qquad j \in N_i,\ i \in M.$$

CKP was first studied by Ibaraki et al. (1978), who presented a branch-and-bound algorithm and two heuristics and called it *the continuous multiple-choice knapsack problem*. Ibaraki (1980) proved that CKP is NP-hard, and presented a polynomial approximation scheme for it. Beale and Tomlin (1970) studied constraint (2), and called each set $\{x_{i1}, \ldots, x_{in_i}\}$, $i \in M$, a *special ordered set of type 1*. Johnson and Padberg (1981) studied the *binary CKP*. Constraints (1)–(4) appear in the formulation of several problems, such as linear complementarity (Cottle et al. 1992), production scheduling (de Farias et al. 2000), generalized assignment (de Farias and Nemhauser 2001), capacity planning (Wolsey 1990), etc.

In this paper we study the inequalities that define facets of the convex hull of the set of feasible solutions of CKP. The motivation of our study is the use of these inequalities as cuts in a branch-and-cut scheme for the general complementarity problem, in which there is more than one knapsack constraint of the type (1).

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Traditionally, (2) is modeled by introducing binary variables and additional constraints that relate the continuous and the binary variables (Dantzig 1960). This approach has several computational disadvantages, including increasing the size of the problem and losing structure, see, e.g., (de Farias et al. 2001). Alternatively, Beale and Tomlin suggested keeping in the model only the continuous variables and enforcing (2) directly in the branch-and-bound algorithm through the use of a specialized branching scheme. We follow Beale and Tomlin's suggestion, and we conduct our polyhedral study in the space of the continuous variables. The idea of dispensing with the use of auxiliary binary variables to model combinatorial constraints on continuous variables, and enforcing the combinatorial constraints directly in the enumeration algorithm appears also, for example, in Beaumont (1990), Bienstock (1996), de Farias (1995), de Farias et al. (2000, 2001), de Farias and Nemhauser (2001, 2001a), and Ibaraki et al. (1978). This idea is particularly pervasive in constraint programming; see, for example, Hooker and Osorio (1999), Hooker et al. (2000), and van-Hentenryck (1988, 1999), and we believe that the present work provides means for building an effective approach that uses the strengths of both mathematical programming and constraint programming in the context of complementarity problems.

Let S be the set of feasible solutions of CKP. The complementarity knapsack polytope is $PS = \text{conv}(S)$. We denote by $V(PS)$ the set of vertices of PS, and by d the number of variables in the problem, i.e., $d = \sum_{i\in M} n_i$. The set $LPS = \{x \in \Re^d : x \text{ satisfies (1), (3), and (4)}\}$ is the solution set of the LP relaxation. To simplify notation, we denote by ij the ordered pair (i, j) and any set with one element by the element itself. We denote $I = \bigcup_{i\in M}(i \times N_i)$, i.e., I is the set of indices of x. For $T \subseteq I$, $M_T = \{i \in M : ij \in T \text{ for some } j \in N_i\}$.

We assume that:

Assumption 1. *$n_i \geq 2$ for some $i \in M$.*

Assumption 2. *$\sum_{i\in M} \max\{a_{i1}, \ldots, a_{in_i}\} > b$.*

Assumption 3. *$b > 0$, $c_{ij} > 0$, $a_{ij} > 0$ $\forall\, ij \in I$.*

Assumption 4. *a_{ij}, $ij \in I$, is scaled so that $a_{ij} \leq b$ and $u_{ij} = 1$.*

If Assumptions 1 and 2 do not hold, the problem is trivial. Assumption 4 can be made without loss of generality once Assumption 3 is made. If $c_{ij} \leq c_{ij'}$ for some $ij, ij' \in I$, $j \neq j'$, we can fix $x_{ij} = 0$ when $a_{ij} \geq a_{ij'}$ or $c_{ij}/a_{ij} \leq c_{ij'}/a_{ij'}$. So, we also assume that $\forall\, i \in M$ with $n_i \geq 2$:

Assumption 5. *$c_{i1} > \cdots > c_{in_i}$.*

Assumption 6. *$a_{i1} > \cdots > a_{in_i}$.*

Assumption 7. *$c_{i1}/a_{i1} < \cdots < c_{in_i}/a_{in_i}$.*

Throughout the paper we will use the following well-known result about the LP relaxation of CKP:

Proposition 1. *The point x^* is an optimal solution to the problem* $\max\{cx : x \in LPS\}$ *only if*

$$\frac{c_{rs}}{a_{rs}} > \frac{c_{uv}}{a_{uv}} \quad \text{and} \quad x^*_{uv} > 0 \Rightarrow x^*_{rs} = 1,$$

for all $rs, uv \in I$. □

The paper is organized as follows. In §2, we introduce a few simple and basic results about the inequalities that define facets of PS. In §3, we extend the concepts of cover and cover inequality, commonly used in 0–1 programming (Balas 1975, Hammer et al. 1975, Wolsey 1975) to obtain facet-defining inequalities for lower-dimensional projections of PS.

Lifting these inequalities leads to a family of valid inequalities that we call *fundamental complementarity inequalities* (FCIs). We show that by sequentially lifting FCIs we can obtain any nontrivial sequentially lifted cover inequality. We present tight bounds for the lifting coefficients of FCIs, and we derive two families of facet-defining inequalities for *PS* that can be obtained by lifting FCIs in a specific order. In §4, we show that any sequentially lifted FCI can be derived by considering projections of *PS* obtained by fixing variables at 0. In §5 we discuss directions for further research.

2. Facet-defining inequalities. In this section we introduce a few simple and basic results about the inequalities that define facets of *PS*. The following three propositions are easy to prove:

PROPOSITION 2. *PS is full-dimensional.* □

PROPOSITION 3. *If x is a vertex of LPS, then x has at most one fractional component. The vertices of PS are the vertices of LPS that satisfy* (2). □

PROPOSITION 4. *Inequality* (1) *is facet-defining for PS iff* $\sum_{i\in M-i'} a_{i1}+a_{i'n_{i'}} \geq b\ \forall\, i' \in M$. *Inequality* (4) *is facet-defining for PS* $\forall\, ij \in I$. *For* $i \in M$,

$$\sum_{j\in N_i} x_{ij} \leq 1 \tag{5}$$

is facet-defining for PS iff $a_{in_i} < b$. *Also, any facet-defining inequality for PS, with the exception of* (4), *is of the form* $\sum_{ij\in I} \alpha_{ij}x_{ij} \leq \beta$, *with* $\alpha_{ij} \geq 0$, $ij \in I$, *and* $\beta > 0$. □

Inequality (5) cuts off every vertex of *LPS* that does not satisfy constraint (2), as we show next.

PROPOSITION 5. *Let $\tilde{x}$ be a vertex of LPS that does not satisfy* (2). *Then there are inequalities among* (5) *that cut off $\tilde{x}$.*

PROOF. Suppose that $\tilde{x}_{ij} > 0$ and $\tilde{x}_{ij'} > 0$ for some ij, $ij' \in I$, $j \neq j'$. From Proposition 3, at least one of $\tilde{x}_{ij}$ or $\tilde{x}_{ij'}$ must be equal to 1. Thus, $\tilde{x}$ is cut off by (5). □

EXAMPLE 1. Let $m=5$, $n_1=n_2=n_3=n_5=2$, $n_4=3$, and (1) be given by

$$(6x_{11}+x_{12})+(2x_{21}+x_{22})+(4x_{31}+3x_{32})+(8x_{41}+6x_{42}+x_{43})+(9x_{51}+4x_{52}) \leq 13. \quad \square$$

The point $\tilde{x}$, given by $\tilde{x}_{11}=\tilde{x}_{12}=\tilde{x}_{42}=1$ and $\tilde{x}_{21}=\tilde{x}_{22}=\tilde{x}_{31}=\tilde{x}_{32}=\tilde{x}_{41}=\tilde{x}_{43}=\tilde{x}_{51}=\tilde{x}_{52}=0$, is a vertex of *LPS* that does not belong to *PS*, and is cut off by $x_{11}+x_{12} \leq 1$. □

Inequalities (1), (4), and (5) are called the trivial facet-defining inequalities of *PS*. In the remainder of the paper we will discuss some nontrivial facet-defining inequalities for *PS*.

Given a facet-defining inequality $\sum_{ij\in I} \alpha_{ij}x_{ij} \leq \beta$, if $\sum_{ij\in I} \alpha_{ij}x_{ij} = \beta \Rightarrow x_{i'j'}=0$ for some $i'j' \in I$, the inequality is $x_{i'j'} \geq 0$. Likewise, if $\sum_{ij\in I} \alpha_{ij}x_{ij} = \beta \Rightarrow \sum_{j\in N_{i'}} x_{i'j} = 1$, for some $i' \in M$, the inequality is $\sum_{j\in N_{i'}} x_{i'j} \leq 1$. We then have

PROPOSITION 6. *Let* $\sum_{ij\in I} \alpha_{ij}x_{ij} \leq \beta$ *be a nontrivial facet-defining inequality, and let* $\{x^{(1)},\ldots,x^{(d)}\}$ *be a set of d affinely independent points of S that satisfy the inequality at equality. Then, for each* $ij \in I$ $\exists r \in \{1,\ldots,d\}$, *such that* $x^{(r)}_{ij} > 0$. *Also,* $\forall\, i \in M$ $\exists s \in \{1,\ldots,d\}$, *such that* $\sum_{j\in N_i} x^{(s)}_{ij} < 1$. □

We now establish a relation among the coefficients α_{ij}, $ij \in I$, of the nontrivial facet-defining inequalities for *PS*.

PROPOSITION 7. *Let* $\sum_{ij\in I} \alpha_{ij}x_{ij} \leq \beta$ *be a nontrivial facet-defining inequality for PS. For any* $i \in M$ *either* $\alpha_{ij}=0\ \forall\, j \in N_i$ *or* $\alpha_{ij} > 0\ \forall\, j \in N_i$. *Also,* $\alpha_{i1} \geq \cdots \geq \alpha_{in_i}$.

PROOF. Suppose that $\alpha_{i'j'} = 0$ for some $i'j' \in I$. Since the inequality is nontrivial, by Proposition 6, S has a point $\tilde{x}$ with $\tilde{x}_{i'j'} > 0$ that satisfies the inequality at equality, i.e., $\sum_{i\in M-i'} \sum_{j\in N_i} \alpha_{ij}\tilde{x}_{ij} = \beta$. However, for any $j'' \in N_{i'} - j'$, $\hat{x}$ given by

$$\hat{x}_{ij} = \begin{cases} 0 & \text{if } ij = i'j', \\ \min\left\{1, \dfrac{a_{i'j'}\tilde{x}_{i'j'}}{a_{i'j''}}\right\} & \text{if } ij = i'j'', \\ \tilde{x}_{ij} & \text{otherwise,} \end{cases}$$

belongs to S. This implies that $\alpha_{i'j''} = 0$.

Now, if $n_{i'} \geq j'' > j'$, x' given by

$$x'_{ij} = \begin{cases} 0 & \text{if } ij = i'j', \\ \tilde{x}_{i'j'} & \text{if } ij = i'j'', \\ \tilde{x}_{ij} & \text{otherwise,} \end{cases}$$

belongs to S. This implies that $\alpha_{i'j'} \geq \alpha_{i'j''}$. □

3. Facet-defining inequalities derived from fundamental complementarity inequalities. In this section we extend the concepts of cover and cover inequality, commonly used in 0–1 programming, to complementarity programming. Unlike 0–1 programming, our cover inequalities are valid for *LPS*, and cannot be used as cuts. However, by lifting them with respect to a single variable, it is possible to derive a family of cuts, which we call fundamental complementarity inequalities (FCIs), and by lifting FCIs we can derive nontrivial facet-defining inequalities for *PS*. Moreover, we show that any nontrivial sequentially lifted cover inequality is a sequentially lifted FCI. We give tight bounds for the coefficients of sequentially lifted FCIs, and we present two families of facet-defining inequalities for *PS* that can be obtained by sequentially lifting FCIs in a certain order.

DEFINITION 1. Let $C = \{i_1 j_1, \ldots, i_k j_k\} \subset I$, where $i_1, \ldots, i_k$ are all distinct. The set C is called a cover if $\sum_{ij\in C} a_{ij} > b$. Given a cover C, the inequality

$$\sum_{ij\in C} a_{ij}x_{ij} \leq b \tag{6}$$

is called a cover inequality. □

It is easy to see that

PROPOSITION 8. *Inequality* (6) *defines a facet of* $PS \cap \{x \in \Re^d : x_{ij} = 0\ \forall\, ij \in I - C\}$. □

The sequential lifting procedure consists of applying the following lemma one variable at a time; see de Farias (1995) and Wolsey (1976) for a proof of a more general result.

LEMMA 1. *Let* $\tilde{x} \in S$, $L \subset I$, *and*

$$\sum_{ij\in L} \alpha_{ij}x_{ij} \leq \beta \tag{7}$$

be a facet-defining inequality for $PS \cap \{x \in \Re^d : x_{ij} = \tilde{x}_{ij}\ \forall\, ij \in I - L\}$. *Let* $rs \in I - L$,

$$\alpha_{rs}^{\max} = \min\left\{\frac{\beta - \sum_{ij\in L} \alpha_{ij}x_{ij}}{x_{rs} - \tilde{x}_{rs}} : x \in V(PS), x_{ij} = \tilde{x}_{ij}\ \forall\, ij \in I - (L \cup rs) \textit{ and } x_{rs} > \tilde{x}_{rs}\right\}, \tag{8}$$

and

$$\alpha_{rs}^{\min} = \max\left\{ \frac{\beta - \sum_{ij\in L} \alpha_{ij} x_{ij}}{x_{rs} - \tilde{x}_{rs}} : x \in V(PS),\ x_{ij} = \tilde{x}_{ij}\ \forall\, ij \in I - (L \cup rs) \text{ and } x_{rs} < \tilde{x}_{rs} \right\}. \tag{9}$$

(When $\{x \in V(PS) : x_{ij} = \tilde{x}_{ij}\ \forall\, ij \in I - (L\cup rs) \text{ and } x_{rs} > \tilde{x}_{rs}\} = \varnothing$, $\alpha_{rs}^{\max} = \infty$. *Likewise, when* $\{x \in V(PS) : x_{ij} = \tilde{x}_{ij}\ \forall\, ij \in I - (L\cup rs) \text{ and } x_{rs} < \tilde{x}_{rs}\} = \varnothing$, $\alpha_{rs}^{\min} = -\infty$.*) Then,*

$$\sum_{ij\in L} \alpha_{ij} x_{ij} + \alpha_{rs} x_{rs} \le \beta + \alpha_{rs}\tilde{x}_{rs} \tag{10}$$

is a valid inequality for PS iff

$$\alpha_{rs}^{\min} \le \alpha_{rs} \le \alpha_{rs}^{\max}. \tag{11}$$

If, in addition to (11), $\alpha_{rs} \in \{\alpha_{rs}^{\min}, \alpha_{rs}^{\max}\} - \{-\infty, \infty\}$, *then* (10) *defines a facet of* $PS \cap \{x \in \Re^d : x_{ij} = \tilde{x}_{ij}\ \forall\, ij \in I - (L \cup rs)\}$. □

Note that when $\alpha_{rs}^{\min} > \alpha_{rs}^{\max}$, it is not possible to lift (7) with respect to x_{rs}. Also, when $\{x \in V(PS) : x_{ij} = \tilde{x}_{ij}\ \forall\, ij \in I - (L\cup rs) \text{ and } x_{rs} > \tilde{x}_{rs}\} \ne \varnothing$, the minimization problem in (8) has an optimal solution, since $V(PS)$ has a finite number of elements. Likewise, when $\{x \in V(PS) : x_{ij} = \tilde{x}_{ij}\ \forall\, ij \in I - (L\cup rs) \text{ and } x_{rs} < \tilde{x}_{rs}\} \ne \varnothing$, the maximization problem in (9) has an optimal solution.

In the case of cover inequalities, all variables are initially fixed at 0. As a consequence, $\alpha_{rs}^{\min} = -\infty$ and $-\infty < \alpha_{rs}^{\max} < \infty$, and therefore it is always possible to lift cover inequalities sequentially in any order. In principle, variables could be fixed for subsequent lifting at any value between 0 and 1. However, as we show in §4, there is no loss of generality in defining cover inequalities for projections of *PS* obtained by fixing variables exclusively at 0. Thus, for the remainder of this section, variables will be fixed for subsequent lifting at 0 only, and the lifting coefficients will be given by (8) with $\tilde{x} = 0$.

Since cover inequalities are valid for *LPS*, they cannot be used as cuts. However, by lifting cover inequalities with respect to a single variable, it is possible to derive a family of inequalities that are valid for *PS* but not for *LPS*, as we show next.

PROPOSITION 9. *Let C be a cover, and suppose that*

$$\sum_{ij\in C - i'j'} a_{ij} + a_{i'j''} < b \tag{12}$$

for some $i'j' \in C$ *and* $j'' \in N_{i'} - j'$. *Then the inequality,*

$$\sum_{ij\in C} a_{ij} x_{ij} + \left(b - \sum_{ij\in C - i'j'} a_{ij}\right) x_{i'j''} \le b, \tag{13}$$

defines a facet of $PS \cap \{x \in \Re^d : x_{ij} = 0\ \forall\, ij \in I - (C \cup i'j'')\}$.

PROOF. Inequality (13) is clearly valid when $x_{i'j''} = 0$. If $x_{i'j''} > 0$,

$$\begin{aligned}\sum_{ij\in C} a_{ij} x_{ij} + \left(b - \sum_{ij\in C - i'j'} a_{ij}\right) x_{i'j''} &= \sum_{ij\in C-i'j'} a_{ij} x_{ij} + \left(b - \sum_{ij\in C - i'j'} a_{ij}\right) x_{i'j''} \\ &\le \sum_{ij\in C-i'j'} a_{ij} + \left(b - \sum_{ij\in C - i'j'} a_{ij}\right) = b.\end{aligned}$$

Therefore (13) is valid.

Because $\sum_{ij\in C} a_{ij} > b$, $PS \cap \{x \in \Re^d : x_{ij} = 0\ \forall ij \in I - C\}$ has $|C|$ affinely independent points that satisfy (13) at equality. Additionally, the point $\hat{x}$ given by

$$\hat{x}_{ij} = \begin{cases} 1 & \text{if } ij \in C - i'j' \text{ or } ij = i'j'', \\ 0 & \text{otherwise,} \end{cases}$$

belongs to $PS \cap \{x \in \Re^d : x_{ij} = 0\ \forall ij \in I - (C \cup i'j'')\}$ and satisfies (13) at equality. Therefore (13) is facet defining. □

We call (13) a fundamental complementarity inqualit (FCI). Proposition 9 shows that lifting a cover inequality with respect to one variable leads to an FCI, provided that (12) is satisfied. Note that when (12) is not satisfied then the lifting simply yields another cover inequality. So by continuing the lifting we either get an FCI or the original inequality (1).

In Proposition 5, we showed that (5) cuts off all vertices of LPS that do not satisfy (2). We now show that (13) cuts off all vertices of $LPS \cap \{x \in \Re^d : x \text{ satisfies } (5)\}$ that do not satisfy (2).

PROPOSITION 10. *Let $\tilde{x}$ be a vertex of $LPS \cap \{x \in \Re^d : x$ satisfies $(5)\}$ that does not satisfy* (2). *Then there is an FCI that is violated by $\tilde{x}$.*

PROOF. Suppose that $\tilde{x}_{i'j'}$ and $\tilde{x}_{i'j''}$ are positive for some $i'j', i'j'' \in I$, $j' < j''$. Because $\tilde{x}$ is a vertex of $LPS \cap \{x \in \Re^d : x$ satisfies $(5)\}$, for each positive component of $\tilde{x}$ there must be an inequality among (1) and (5) satisfied at equality and such that $\tilde{x}$ is the unique solution of the corresponding system of equations. Because $x_{i'j'}$ and $x_{i'j''}$ appear only in (1) and in $\sum_{j\in N_{i'}} x_{i'j} \le 1$, we have that

$$\sum_{ij\in I} a_{ij}\tilde{x}_{ij} = b$$

and

$$\sum_{j\in N_{i'}} \tilde{x}_{i'j} = 1.$$

Also, $\tilde{x}$ cannot have any fractional components other than $\tilde{x}_{i'j'}$ and $\tilde{x}_{i'j''}$. Let $C = \{ij \in I : \tilde{x}_{ij} > 0\} - i'j''$. Then,

$$\sum_{ij\in C} a_{ij}\tilde{x}_{ij} + a_{i'j''}\tilde{x}_{i'j''} = \sum_{ij\in C-i'j'} a_{ij} + a_{i'j'}\tilde{x}_{i'j'} + a_{i'j''}\tilde{x}_{i'j''} = b. \tag{14}$$

Because $(\tilde{x}_{i'j'}, \tilde{x}_{i'j''})$ is a solution of the system of equations

$$\begin{cases} a_{i'j'}x_{i'j'} + a_{i'j''}x_{i'j''} = b - \sum_{ij\in C-i'j'} a_{ij} \\ x_{i'j'} + x_{i'j''} = 1, \end{cases}$$

and $a_{i'j'} > a_{i'j''}$, then $\sum_{ij\in C} a_{ij} > b$. Also, note that $i_1j_1 \ne i_2j_2 \Rightarrow i_1 \ne i_2\ \forall i_1j_1, i_2j_2 \in C$. So C is a cover. On the other hand, $\sum_{ij\in C-i'j'} a_{ij} + a_{i'j''} < b$. Thus, (13) is valid and it cuts off $\tilde{x}$. □

EXAMPLE 2. With the data of Example 1, $\tilde{x}$ with $\tilde{x}_{11} = \frac{1}{5}$, $\tilde{x}_{12} = \frac{4}{5}$, $\tilde{x}_{21} = \tilde{x}_{32} = \tilde{x}_{42} = 1$, and $\tilde{x}_{22} = \tilde{x}_{31} = \tilde{x}_{41} = \tilde{x}_{43} = \tilde{x}_{51} = \tilde{x}_{52} = 0$ is a vertex of $\{x \in \Re^{11} : (6x_{11} + x_{12}) + (2x_{21} + x_{22}) + (4x_{31} + 3x_{32}) + (8x_{41} + 6x_{42} + x_{43}) + (9x_{51} + 4x_{52}) \le 13,\ \sum_{j\in N_i} x_{ij} \le 1,\ i \in M$, and $x_{ij} \ge 0,\ ij \in I\}$. This point is cut off by the FCI

$$(6x_{11} + 2x_{12}) + 2x_{21} + 3x_{32} + 6x_{42} \le 13, \tag{15}$$

with $C = \{11, 21, 32, 42\}$, $i'j' = 11$, and $j'' = 2$. □

As a consequence of Proposition 10, we have the following.

COROLLARY 1. *PS is given by* (1), (4), *and* (5) *iff* $\sum_{ij\in C-i'j'} a_{ij} + a_{i'j''} \geq b$ *for every cover* C, $i'j' \in C$, *and* $j'' \in N_{i'} - j'$.

PROOF. If $\sum_{ij\in C-i'j'} a_{ij} + a_{i'j''} \geq b$ for every cover C, $i'j' \in C$, and $j'' \in N_{i'} - j'$, no FCI can be defined, and by Proposition 10, every vertex of $LPS \cap \{x \in \Re^d : x$ satisfies (5)$\}$ satisfies (2).

Suppose now that $\sum_{ij\in C-i'j'} a_{ij} + a_{i'j''} < b$ for some cover C, $i'j' \in C$, and $j'' \in N_{i'} - j'$. Since C is a cover and $a_{i'j'} > a_{i'j''}$, $\hat{x}$ given by

$$\hat{x}_{ij} = \begin{cases} 1 & \text{if } ij \in C - i'j', \\ \dfrac{b - a_{i'j''} - \sum_{ij\in C-i'j'} a_{ij}}{a_{i'j'} - a_{i'j''}} & \text{if } ij = i'j', \\ \dfrac{\sum_{ij\in C} a_{ij} - b}{a_{i'j'} - a_{i'j''}} & \text{if } ij = i'j'', \\ 0 & \text{otherwise,} \end{cases}$$

is a vertex of $LPS \cap \{x \in \Re^d : x$ satisfies (5)$\}$ that does not satisfy (2). □

By lifting FCIs, we obtain facet-defining inequalities for PS. Moreover, we can derive the complete theory of sequentially lifted cover inequalities from FCIs, since the following proposition shows that we can derive any nontrivial sequentially lifted cover inequality by sequentially lifting FCIs.

PROPOSITION 11. *Any nontrivial sequentially lifted cover inequality is a sequentially lifted FCI.*

PROOF. Suppose that after some iterations of the lifting procedure applied to a cover inequality, the current inequality is

$$\sum_{ij\in T} a_{ij} x_{ij} \leq b \tag{16}$$

(all lifting coefficients so far are a_{ij}, and x_{ij} is presently fixed at 0 $\forall\, ij \in I - T$). Let $rs \in I - T$. We lift (16) next with respect to x_{rs}. Assume that the lifting coefficient is $\alpha_{rs} \neq a_{rs}$ (if the lifting coefficient at every iteration is equal to the corresponding knapsack coefficient, the final lifted cover inequality is (1)). Let $j_i \in N_i$ be such that $a_{ij_i} = \max\{a_{ij} : ij \in T\}$ $\forall\, i \in M_T$. If $\sum_{i\in M_T - r} a_{ij_i} + a_{rs} \geq b$, $\sum_{ij\in T} a_{ij} x_{ij} + a_{rs} x_{rs} \leq b$ is facet defining for $PS \cap \{x \in \Re^d : x_{ij} = 0\ \forall\, ij \in I - (T \cup rs)\}$, and $\alpha_{rs} = a_{rs}$. Thus,

$$\sum_{i\in M_T - r} a_{ij_i} + a_{rs} < b.$$

By using an argument similar to the one in the proof of Proposition 9, it follows that $\alpha_{rs} = b - \sum_{i\in M_{T-r}} a_{ij_i}$. Now, let $C = \{ij_i : i \in M_T\}$. The set C is clearly a cover. Now, note that $r \in M_T$ (otherwise $\sum_{i\in M_T - r} a_{ij_i} + a_{rs} > b$). Therefore,

$$\sum_{ij\in C} a_{ij} x_{ij} + \Big(b - \sum_{i\in M_C - r} a_{ij_i}\Big) x_{rs} \leq b \tag{17}$$

is an FCI. By again using an argument similar to the one in the proof of Proposition 9, and the fact that $a_{rj_r} > b - \sum_{i\in M_C - r} a_{ij_i}$, it can be shown that the lifting coefficient of x_{ij} $\forall\, ij \in T - (C \cup rs)$ when lifting (17), is a_{ij}. Thus,

$$\sum_{ij\in T} a_{ij} x_{ij} + \Big(b - \sum_{i\in M_T - r} a_{ij_i}\Big) x_{rs} \leq b$$

can be derived by sequentially lifting the FCI (17). □

As a result of Proposition 11, from now on, we will focus on the lifting of FCIs. We now give tight bounds for the coefficients of the facet-defining inequalities obtained by sequentially lifting FCIs.

PROPOSITION 12. *Let C be a cover that satisfies* (12). *Let $\sum_{ij\in I} \alpha_{ij}x_{ij} \le b$ be a facet-defining inequality for PS obtained by lifting* (13). *Then,*

(1) $\alpha_{ij} = 0\ \forall\, i \in M - M_C,\ j \in N_i$,

(2) *If* $rt \in C$, $s \in N_r$, *and* $s > t$, $a_{rs} \le \alpha_{rs} \le \max\{a_{rs}, b - \sum_{ij\in C-rt} a_{ij}\}$,

(3) *If* $rt \in C - i'j'$, $s \in N_r$, *and* $s < t$,

$$a_{rt} \le \alpha_{rs} \le a_{rt} \max\left\{1, \frac{a_{rs}}{b - \sum_{ij\in C-\{i'j',rt\}} a_{ij} - a_{i'j''}}\right\},$$

(4) *If* $s \in N_{i'}$ *and* $s < j'$, $\alpha_{i's} \le a_{i's}$.

PROOF. Let $p \in M - M_C$ and $q \in N_p$. Since (12) holds, $\hat{x}$ given by

$$\hat{x}_{ij} = \begin{cases} 1 & \text{if } ij \in C - i'j' \text{ or } ij = i'j'', \\ \min\left\{1, \dfrac{b - \sum_{ij\in C-i'j'} a_{ij} - a_{i'j''}}{a_{pq}}\right\} & \text{if } i = p \text{ and } j = q, \\ 0 & \text{otherwise}, \end{cases}$$

belongs to S. Since

$$\sum_{ij\in C} a_{ij}\hat{x}_{ij} + \left(b - \sum_{ij\in C-i'j'} a_{ij}\right)\hat{x}_{i'j''} = b$$

and $\hat{x}_{pq} > 0$, $\alpha_{pq} = 0$. This proves (1).

If $s > t$, $a_{rs} < a_{rt}$. If $\alpha_{rs} < a_{rs}$, then

$$\frac{\alpha_{rs}}{a_{rs}} < \frac{a_{rt}}{a_{rt}} = \frac{\alpha_{rt}}{a_{rt}}. \tag{18}$$

Now, $a_{rs} < a_{rt}$ and (18) imply that

$$z = \max\left\{\sum_{ij\in I} \alpha_{ij}x_{ij} : x \in PS\right\}$$

has an optimal solution $\tilde{x}$ with $\tilde{x}_{rs} = 0$, which means that if we increase the value of α_{rs}, and α_{rs} remains not greater than a_{rs}, z will not increase. In other words,

$$\sum_{ij\in I-rs} \alpha_{ij}x_{ij} + (\alpha_{rs} + \epsilon)x_{rs} \le b$$

is valid for PS for $\epsilon > 0$ sufficiently small, which contradicts the assumption that $\sum_{ij\in I} \alpha_{ij}x_{ij} \le b$ is facet defining. This proves that $\alpha_{rs} \ge a_{rs}$.

If $a_{rs} > b - \sum_{ij\in C-rt} a_{ij}$, PS has a point $\tilde{x}$ with $\tilde{x}_{rs} > 0$ and

$$\sum_{ij\in C-rt} a_{ij}\tilde{x}_{ij} + a_{rs}\tilde{x}_{rs} = b.$$

If $\alpha_{rs} > a_{rs}$, then

$$\sum_{ij\in C-rt} a_{ij}\tilde{x}_{ij} + \alpha_{rs}\tilde{x}_{rs} > b.$$

Thus, $a_{rs} > b - \sum_{ij\in C-rt} a_{ij} \Rightarrow \alpha_{rs} \le a_{rs}$.

On the other hand, if $b-\sum_{ij\in C-rt} a_{ij} \geq a_{rs}$, x' given by

$$x'_{ij} = \begin{cases} 1 & \text{if } ij \in C-rt \text{ or } ij = rs, \\ 0 & \text{otherwise,} \end{cases}$$

belongs to S. If $\alpha_{rs} > b-\sum_{ij\in C-rt} a_{ij}$,

$$\sum_{ij\in C-rt} a_{ij}x'_{ij} + \alpha_{rs}x'_{rs} > b.$$

Thus, $b-\sum_{ij\in C-rt} a_{ij} \geq a_{rs} \Rightarrow \alpha_{rs} \leq b-\sum_{ij\in C-rt} a_{ij}$. This proves (2). The proofs of (3) and (4) are similar to the proof of (2). □

EXAMPLE 3. Using the data of Example 1, we start with the FCI (15). Let $\alpha_{22}, \alpha_{31}, \alpha_{41}, \alpha_{43}, \alpha_{51}$, and α_{52} be the lifting coefficients of $x_{22}, x_{31}, x_{41}, x_{43}, x_{51}$, and x_{52}, respectively. From (1) of Proposition 12, $\alpha_{51} = \alpha_{52} = 0$. From (2), $1 \leq \alpha_{22} \leq \max\{1, -2\} = 1$, and $1 \leq \alpha_{43} \leq 2$. From (3), $3 \leq \alpha_{31} \leq 3\max\{1, \frac{4}{4}\} = 3$, and $6 \leq \alpha_{41} \leq 6\max\{1, \frac{8}{7}\} = \frac{48}{7}$.

Lifting the inequality with respect to x_{41} first, $\alpha_{41} = 48/7$. If we now lift with respect to x_{43}, $\alpha_{43} = 2$. Therefore, the following inequality is valid and facet defining,

$$(6x_{11}+2x_{12})+(2x_{21}+x_{22})+(3x_{31}+3x_{32})+\left(\tfrac{48}{7}x_{41}+6x_{42}+2x_{43}\right) \leq 13. \quad \square$$

In principle, the value of x_{rs} in an optimal solution of the lifting problem (8) can be any number in the interval (0,1]. In some cases, however, it is possible to fix the value of x_{rs} at 1 before solving (8), as shown in the next proposition.

PROPOSITION 13. *Let C be a cover and suppose that*

$$\sum_{ij\in L} \alpha_{ij}x_{ij} \leq b \tag{19}$$

is a facet-defining inequality for $PS \cap \{x \in \Re^d : x_{ij} = 0\ \forall\, ij \in I-L\}$ obtained by lifting (13). *Let $rs' \in C$, $rs \in I-L$, assume that* (19) *is lifted next with respect to x_{rs}, and let α_{rs} be its lifting coefficient. If $s > s'$,*

$$\alpha_{rs} = b - \max\left\{\sum_{ij\in L} \alpha_{ij}x_{ij} : x \in V(PS) \text{ and } x_{rs} = 1\right\}. \tag{20}$$

PROOF. Consider the optimization problem,

$$\max\left\{\sum_{ij\in L} \alpha_{ij}x_{ij} + \alpha_{rs}x_{rs} : x \in V(PS) \text{ and } x_{rs} > 0\right\}. \tag{21}$$

Clearly the optimal value of (21) is b. Note that (20) holds iff (21) has an optimal solution with $x_{rs} = 1$. Let x^* be an optimal solution of (21). From Proposition 3, x^* has at most one fractional component. Suppose that $x^*_{rs} \in (0, 1)$. Let $P = \{ij \in L : x^*_{ij} > 0\}$. From (2) of Proposition 12, $\alpha_{rs}/a_{rs} \geq 1$. Because x^*_{rs} is the only fractional variable, by Proposition 1, there cannot be $ij \in P$ with $\alpha_{ij}/a_{ij} < 1$. If $\alpha_{ij}/a_{ij} = 1\ \forall\, ij \in P-rs$, we can obtain an optimal solution $\tilde{x}$ for (21) with $\tilde{x}_{rs} = 1$ by introducing x_{rs} into the knapsack first.

So suppose that $P^{(>)} = \{ij \in P : \alpha_{ij}/a_{ij} > \alpha_{rs}/a_{rs}\} \neq \varnothing$. Because $\alpha_{uv} > 0\ \forall\, uv \in P^{(>)}$, it follows from (1) of Proposition 12 that $M_{P^{(>)}} \subseteq M_C$. For $u \in M_{P^{(>)}}$, let $j_u \in N_u$ be such that $uj_u \in C$. Because $\alpha_{uv} > a_{uv}\ \forall\, uv \in P^{(>)}$, it follows from Proposition 12 that

$$\sum_{ij\in C-uj_u} a_{ij} + a_{uv} < b, \tag{22}$$

and therefore,

(23) $$a_{uj_u} > a_{uv} \qquad \forall\, uv \in P^{(>)}.$$

Now, let $pq \in P^{(>)}$. We have

$$\begin{aligned}\sum_{uv\in P^{(>)}} a_{uv} + a_{rs} < \sum_{uv\in P^{(>)}} a_{uv} + a_{rs'} &= \sum_{uv\in P^{(>)}-pq} a_{uv} + a_{pq} + a_{rs'} \\ &\le \sum_{ij\in C-\{pj_p, rs'\}} a_{ij} + a_{pq} + a_{rs'} = \sum_{ij\in C-pj_p} a_{ij} + a_{pq} < b,\end{aligned}$$

where the second inequality follows from (23), and the last inequality follows from (22). Because $\sum_{uv\in P^{(>)}} a_{uv} + a_{rs} < b$, (21) has an optimal solution $\hat{x}$ in which $\hat{x}_{rs} = 1$. □

When $s < s'$, Proposition 13 does not necessarily hold, as we show next.

EXAMPLE 4. With the data of Example 1, consider the cover $C = \{21, 42, 51\}$ and the FCI

(24) $$2x_{21} + 6x_{42} + (9x_{51} + 5x_{52}) \le 13.$$

We lift (24) with respect to x_{41}. Note that

$$13 - \max\{2x_{21} + 6x_{42} + (9x_{51} + 5x_{52}) : x \in S \text{ and } x_{41} = 1\} = 7.$$

However,

$$\min\left\{\frac{13 - 2x_{21} + 6x_{42} + (9x_{51} + 5x_{52})}{x_{41}} : x \in S,\, x_{41} = \frac{7}{8},\, x_{22} = x_{31} = x_{32} = x_{43} = 0\right\} = \frac{48}{7},$$

which is the lifting coefficient of x_{41}. □

We now present, in Theorems 1 and 2, two families of facet-defining inequalities for PS that can be derived by lifting FCIs. The elements of the cover in the first family have the highest values of a_{ij} among the indices in their special ordered sets. The elements of the cover in the second family, with the exception of one, have the lowest values of a_{ij} among the indices in their special ordered sets.

THEOREM 1. *Let C be a cover, and suppose that $j = 1\ \forall\, ij \in C$. Assume that C satisfies* (12). *Then*

(25) $$\sum_{i\in M_C} a_{i1}x_{i1} + \sum_{i\in M_C}\sum_{j\in N_i-1} \max\left\{a_{ij}, b - \sum_{k\in M_C-i} a_{k1}\right\} x_{ij} \le b$$

is valid and facet-defining.

PROOF. Let $\tilde{x} \in S$. If $\tilde{x}_{ij} = 0\ \forall\, j \in N_i - 1$ and $i \in M_C$ with $\max\{a_{ij}, b - \sum_{k\in M_C-i} a_{k1}\} = b - \sum_{k\in M_C-i} a_{k1}$, $\tilde{x}$ clearly satisfies (25). So suppose that $\tilde{x}_{rs} > 0$ for some $s \in N_r - 1$, $r \in M_C$, and $\max\{a_{rs}, b - \sum_{k\in M_C-r} a_{k1}\} = b - \sum_{k\in M_C-r} a_{k1}$. Then,

$$\begin{aligned}&\sum_{i\in M_C} a_{i1}\tilde{x}_{i1} + \sum_{i\in M_C}\sum_{j\in N_i-1} \max\left\{a_{ij}, b - \sum_{k\in M_C-i} a_{k1}\right\}\tilde{x}_{ij} \\ &\quad= \sum_{i\in M_C-r} a_{i1}\tilde{x}_{i1} + \sum_{i\in M_C-r}\sum_{j\in N_i-1} \max\left\{a_{ij}, b - \sum_{k\in M_C-i} a_{k1}\right\}\tilde{x}_{ij} + \left(b - \sum_{k\in M_C-r} a_{k1}\right)\tilde{x}_{rs} \\ &\quad\le \sum_{i\in M_C-r} a_{i1} + b - \sum_{k\in M_C-r} a_{k1} = b,\end{aligned}$$

where the first equality holds because $\tilde{x}_{rs} > 0 \Rightarrow \tilde{x}_{rt} = 0 \ \forall\, t \in N_r - s$, and the inequality follows from $a_{i1} > \max\{a_{ij}, b - \sum_{k\in M_C - i} a_{k1}\} \ \forall\, j \in N_i - 1, i \in M_C$. This proves that (25) is valid.

Since $\sum_{i\in M_C} a_{i1} > b$, S has $|C|$ linearly independent points with $x_{ij} = 0 \ \forall\, ij \in I - C$ that satisfy (25) at equality. Now, let $rs \in I - C$ be such that $r \in M - M_C$. Since $\sum_{i\in M_C} a_{i1} + a_{rs} > b$, S has a point with $x_{rs} > 0$ which satisfies (25) at equality. Finally, let $uv \in I - C$ be such that $u \in M_C$. Since $\sum_{i\in M_C - u} a_{i1} + \max\{a_{uv}, b - \sum_{k\in M_C - u} a_{k1}\} \geq b$, S has a point with $x_{uv} > 0$ which satisfies (25) at equality. This proves that (25) is facet-defining. □

EXAMPLE 5. With the data of Example 1, let $C = \{41, 51\}$. Then, $M_C = \{4, 5\}$, $\max\{a_{42}, b - \sum_{k\in M_C - 4} a_{k1}\} = 6$, $\max\{a_{43}, b - \sum_{k\in M_C - 4} a_{k1}\} = 4$, and $\max\{a_{52}, b - \sum_{k\in M_C - 5} a_{k1}\} = 5$. So,

$$(8x_{41} + 6x_{42} + 4x_{43}) + (9x_{51} + 5x_{52}) \leq 13$$

is valid and facet defining. □

THEOREM 2. *Let C be a cover that satisfies* (12) *with $j = n_i \ \forall\, i \in M_C - i'$, $j' < n_{i'}$, and $j'' = n_{i'}$, i.e. $a_{i'j'} + \sum_{i\in M_C - i'} a_{in_i} > b$ and $\sum_{i\in M_C} a_{in_i} < b$. Then,*

$$\sum_{j\in N_{i'}} \max\left\{a_{i'j}, b - \sum_{k\in M_C - i'} a_{kn_k}\right\} x_{i'j} + \sum_{i\in M_C - i'} a_{in_i} x_{in_i} + \sum_{i\in M_C - i'} \sum_{j\in N_i - n_i} a_{in_i} \max\left\{1, \frac{a_{ij}}{b - \sum_{k\in M_C - i} a_{kn_k}}\right\} x_{ij} \leq b \tag{26}$$

is valid and facet defining.

PROOF. We prove the proposition by lifting the FCI

$$a_{i'j'} x_{i'j'} + \left(b - \sum_{k\in M_C - i'} a_{kn_k}\right) x_{i'n_{i'}} + \sum_{i\in M_C - i'} a_{in_i} x_{in_i} \leq b. \tag{27}$$

From (1) of Proposition 12, $\alpha_{ij} = 0 \ \forall\, i \in M - M_C$. Now we lift (27) with respect to $x_{i'j}$, $j \in N_{i'} - \{j', n_{i'}\}$. By using an argument similar to the one in the proof of Proposition 9, it can be shown that the lifting coefficient is given by

$$\alpha_{i'j} = \max\left\{a_{i'j}, b - \sum_{k\in M_C - i'} a_{kn_k}\right\}. \tag{28}$$

Thus,

$$\sum_{j\in N_{i'}} \max\left\{a_{i'j}, b - \sum_{r\in M_C - i'} a_{rn_r}\right\} x_{i'j} + \sum_{i\in M_C - i'} a_{in_i} x_{in_i} \leq b \tag{29}$$

is valid and facet defining for $PS \cap \{x \in \Re^d : x_{ij} = 0 \ \forall\, j \in N_i - n_i, i \in M_{C-k'}\}$.

Next, we lift (29) with respect to x_{ij}, $j \in N_i - n_i$, $i \in M_C - i'$, with ij satisfying

$$\sum_{k\in M_C - i} a_{kn_k} + a_{ij} > b, \tag{30}$$

and we show that the lifting coefficient is

$$\alpha_{ij} = \frac{a_{in_i} a_{ij}}{b - \sum_{k\in M_{C-i}} a_{kn_k}}. \tag{31}$$

The lifting order is the following. Let $r \in M_C - i'$ be such that $a_{rn_r} = \min\{a_{sn_s} : s \neq i'\}$ (in case of a tie, break it arbitrarily). We then pick, in any order, all the variables $x_{r1}, \ldots, x_{rn_r - 1}$ for which (30) holds. Then, we pick, in any order, the variables $x_{t1}, \ldots, x_{tn_t - 1}$, where $t \in M_C - \{i', r\}$ is such that $a_{tn_t} = \min\{a_{sn_s} : s \neq i', r\}$, for which (30) holds, and so on. Let

$$T = C \cup i'n_{i'} \cup \{ij : (27) \text{ has been lifted with respect to } x_{ij}\}.$$

Suppose that the lifting coefficient of x_{ij} is given by (31) $\forall\, ij \in T$ such that $j \in N_i - n_i$, $i \in M_C - i'$, and ij satisfies (30). Let uv be such that $v \in N_u - n_u$, $u \in M_C - i'$, uv satisfies (30), $uv \notin T$, and x_{uv} is the next variable $\sum_{ij \in T} \alpha_{ij} x_{ij} \leq b$ is lifted with respect to. The lifting coefficient of x_{uv} is given by

$$\alpha_{uv} = \frac{a_{un_u} a_{uv}}{b - \sum_{k \in M_C - u} a_{kn_k}} \tag{32}$$

if and only if

$$\max\left\{\sum_{ij \in T} \alpha_{ij} x_{ij} + \frac{a_{un_u} a_{uv}}{b - \sum_{k \in M_C - u} a_{kn_k}} x_{uv} : x \in S \text{ and } x_{uv} > 0\right\} = b. \tag{33}$$

We now prove that (32) holds by proving (33).

Consider the continuous knapsack problems, (L_t), $t \in N_{i'}$,

$$\max\left\{\sum_{ij \in T} \alpha_{ij} x_{ij} + \frac{a_{un_u} a_{uv}}{b - \sum_{i \in M_C - u} a_{in_i}} x_{uv} : \sum_{ij \in T} a_{ij} x_{ij} \leq b, 0 \leq x_{ij} \leq 1, ij \in T,\right.$$
$$\left. x_{un_u} = 0, \text{ and } x_{i'j} = 0, j \in N_{i'} - t\right\}.$$

Note that $\alpha_{i't}/a_{i't} \geq 1$, $\alpha_{in_i}/a_{in_i} = 1\ \forall\, i \in M_C - i'$,

$$\frac{\alpha_{ij}}{a_{ij}} = \frac{a_{in_i}}{b - \sum_{r \in M_C - i} a_{rn_r}} < 1$$

for all $ij \in T$ with $i \neq i'$ and $j \neq n_i$, and

$$\frac{1}{a_{uv}} \frac{a_{un_u} a_{uv}}{b - \sum_{r \in M_C - u} a_{rn_r}} < 1. \tag{34}$$

Note also that because of the lifting order, $a_{un_u} \geq a_{in_i}$ $\forall\, i \in M_T - i'$, and therefore

$$\frac{1}{a_{uv}} \frac{a_{un_u} a_{uv}}{b - \sum_{r \in M_C - u} a_{rn_r}} \geq \frac{\alpha_{ij}}{a_{ij}}$$

for all $ij \in T$ with $i \neq i'$ and $j \neq n_i$.

We can obtain an optimal solution for (L_t) by selecting $x_{i't}$ to enter the knapsack first, x_{in_i}, $i \in M_C - \{i', u\}$, in any order, until they are all in the knapsack, or until there is no more room in the knapsack, x_{uv}, in case there is room in the knapsack, and finally, if there is still room in the knapsack, x_{ij}, $ij \in T$, with $i \neq i'$ and $j \neq n_i$, in nonincreasing order of α_{ij}/a_{ij}, until the knapsack is full or all of them are included.

If $a_{i't} + \sum_{k \in M_C - i'} a_{kn_k} \geq b$, by (28), $\alpha_{i't} = a_{i't}$, and because of (34), the optimal value of (L_t) is no greater than b. If $a_{i't} + \sum_{k \in M_C - i'} a_{kn_k} < b$, (L_t) has a basic optimal solution $x^{(t)}$ with $x^{(t)}_{uv} > 0$. Also, by (28), $\alpha_{i't} = b - \sum_{k \in M_C - i'} a_{kn_k}$. Because we are considering the case where $\sum_{k\in M_C-u} a_{kn_k} + a_{uv} > b$, $\sum_{k\in M_C-\{i',u\}} a_{kn_k} + a_{i't} + a_{uv} > b$, and $x_{ij}^{(t)} = 0 \; \forall\, ij \in T$ with $i \neq i'$ and $j \neq n_i$. Also, $x_{in_i}^{(t)} = x_{i't}^{(t)} = 1 \; \forall\, i \in M_C - \{i', u\}$, and

$$x_{uv}^{(t)} = \frac{b - \sum_{k\in M_C-\{i',u\}} a_{kn_k} - a_{i't}}{a_{uv}}.$$

The optimal value of (L_t) in this case is

$$b - \frac{a_{i't} - a_{i'n_{i'}}}{b - \sum_{k\in M_{C-u}} a_{kn_k}} a_{un_u} \leq b,$$

and it is equal to b if and only if $t = n_{i'}$. So,

$$\max\left\{\sum_{ij\in T} \alpha_{ij} x_{ij} + \frac{a_{un_u} a_{uv}}{b - \sum_{k\in M_C-u} a_{kn_k}} x_{uv} : x \in V(PS) \text{ and } x_{uv} > 0\right\} = b.$$

This shows that the lifting coefficient of x_{uv} is given by (32).

Finally, we lift with respect to x_{ij}, $j \in N_i - n_i$, $i \in M_C - i'$, satisfying

$$\sum_{k\in M_C-i} a_{kn_k} + a_{ij} \leq b.$$

From (3) of Proposition 12, it follows that the lifting coefficient of x_{ij} is given by $\alpha_{ij} = a_{in_i}$. □

EXAMPLE 6. Using the data of Example 1, consider the FCI,

$$x_{22} + 3x_{32} + x_{43} + (9x_{51} + 8x_{52}) \leq 13.$$

Let $\alpha_{11}, \alpha_{12}, \alpha_{31}, \alpha_{41}$, and α_{42} be the lifting coefficients of $x_{11}, x_{12}, x_{31}, x_{41}$, and x_{42}, respectively. Since $1 \in M - M_C$, $\alpha_{11} = \alpha_{12} = 0$. Also,

$$\alpha_{21} = a_{22} \max\left\{1, \frac{a_{21}}{b - \sum_{ij\in C-\{51,22\}} a_{ij} - a_{52}}\right\} = 1,$$

$$\alpha_{31} = a_{32} \max\left\{1, \frac{a_{31}}{b - \sum_{ij\in C-\{32,51\}} a_{ij} - a_{52}}\right\} = 3,$$

$$\alpha_{42} = a_{43} \max\left\{1, \frac{a_{42}}{b - \sum_{ij\in C-\{43,51\}} a_{ij} - a_{52}}\right\} = \frac{6}{5},$$

and

$$\alpha_{41} = a_{43} \max\left\{1, \frac{a_{41}}{b - \sum_{ij\in C-\{43,51\}} a_{ij} - a_{52}}\right\} = \frac{8}{5}.$$

Therefore,

$$(x_{21} + x_{22}) + (3x_{31} + 3x_{32}) + \left(\frac{8}{5}x_{41} + \frac{6}{5}x_{42} + x_{43}\right) + (9x_{51} + 8x_{52}) \leq 13$$

is valid and facet defining. □

4. Variable values for polytope projection. The inequalities studied in §3 were derived by first fixing some of the variables at 0, and then sequentially lifting the cover inequality defined by the free variables. In principle, however, variables could be fixed for subsequent lifting at any value between 0 and 1. The main result of this section is that there is no loss of generality in fixing variables for subsequent lifting exclusively at 0.

Formally, consider the following more general definition of cover and cover inequality that will be used throughout this section.

DEFINITION 2. Let $\tilde{x} \in S$. Let $C = \{i_1 j_1, \ldots, i_k j_k\} \subset I$, where $i_1, \ldots, i_k$ are all distinct, and

$$\tilde{x}_{ij} = 0 \ \forall\, ij \in I - C \text{ with } i \in M_C. \tag{35}$$

Let $F_0 = \{ij \in I - C : \tilde{x}_{ij} = 0\}$, $F_1 = \{ij \in I - C : \tilde{x}_{ij} = 1\}$, and $F_2 = \{ij \in I - C : \tilde{x}_{ij} \in (0, 1)\}$. We say that C is a cover for $PS \cap \{x \in \Re^d : x_{ij} = \tilde{x}_{ij} \ \forall\, ij \in F_0 \cup F_1 \cup F_2\}$ iff

$$\sum_{ij \in C} a_{ij} > b - \sum_{ij \in F_1 \cup F_2} a_{ij} \tilde{x}_{ij}.$$

The inequality

$$\sum_{ij \in C} a_{ij} x_{ij} \le b - \sum_{ij \in F_1 \cup F_2} a_{ij} \tilde{x}_{ij}, \tag{36}$$

is called a cover inequality for $PS \cap \{x \in \Re^d : x_{ij} = \tilde{x}_{ij} \ \forall\, ij \in F_0 \cup F_1 \cup F_2\}$. □

Note that the variables indexed by F_0, F_1, and F_2 are fixed at 0, 1, and fractional values, respectively. The reason for Condition (35) is that when variable x_{ij} is fixed at a positive value, all other variables $x_{ij'}$, $j' \in N_i - j$, are automatically fixed at 0. The main result of this section is shown below.

THEOREM 3. *Let*

$$\sum_{ij \in I} \alpha_{ij} x_{ij} \le \beta \tag{37}$$

be a nontrivial facet-defining inequality for PS obtained by sequentially lifting (36). *Then, it is possible to obtain* (37) *by sequentially lifting the cover inequality*

$$\sum_{ij \in C \cup F_1 \cup F_2} a_{ij} x_{ij} \le b, \tag{38}$$

which is valid for $PS \cap \{x \in \Re^d : x_{ij} = 0 \ \forall\, ij \in F_0\}$. □

Theorem 3 is the result of the following conditions:

(1) The variables are continuous.

(2) At most one variable in each set $\{x_{i1}, \ldots, x_{in_i}\}$, $i \in M$, can be positive.

It is well known that Theorem 3 may not hold when there are binary variables. Likewise, as we show next, Theorem 3 may not hold when condition (2) of Theorem 3 does not hold.

EXAMPLE 7. Consider the set,

$$S = \{x \in [0, 1]^3 : 5x_1 + 4x_2 + 2x_3 \le 7 \text{ and at most two variables can be positive}\}.$$

If we fix $x_2 = 1$ and $x_3 = 0$, we obtain

$$5x_1 \le 3, \tag{39}$$

which defines a facet of conv(S) $\cap \{x \in \Re^3 : x_2 = 1 \text{ and } x_3 = 0\}$. We first lift (39) with respect to x_3, and we obtain

$$5x_1 + 3x_3 \le 3. \tag{40}$$

(Note that because $x_2 = 1$, at most one of x_1 or x_3 can be positive.) Finally, we lift (40) with respect to x_2. The lifting coefficient of x_2, α_2, is given by

(41) $$5x_1 + \alpha_2 x_2 + 3x_3 \leq 3 + \alpha_2.$$

It can be shown that $\alpha_2 = 5$, and therefore that

(42) $$5x_1 + 5x_2 + 3x_3 \leq 8$$

defines a facet of conv(S). (Note that when $x_2 = 0$, the left-hand side of (41) is at most 8, and therefore $\alpha_2 \geq 5$.) Clearly, (42) cannot be derived by lifting cover inequalities which define facets of projections of conv(S) obtained by fixing variables exclusively at 0. □

In the remainder of the section, we will prove Theorem 3.

We show next that any nontrivial sequentially lifted cover inequality is a sequentially lifted FCI.

PROPOSITION 14. *Let C be a cover for $PS \cap \{x \in \Re^d : x_{ij} = \tilde{x}_{ij}\ \forall ij \in F_0 \cup F_1 \cup F_2\}$. Let $T \subseteq F_0$. Suppose that*

(43) $$\sum_{ij \in C} a_{ij} x_{ij} + \sum_{ij \in T} a_{ij} x_{ij} \leq b - \sum_{ij \in F_1 \cup F_2} a_{ij} \tilde{x}_{ij}$$

is a facet-defining inequality for $PS \cap \{x \in \Re^d : x_{ij} = \tilde{x}_{ij}\ \forall ij \in (F_0 - T) \cup F_1 \cup F_2\}$. Let $rs \in F_1 \cup F_2$, and lift (43) next with respect to x_{rs}. Then, the lifting coefficient is a_{rs}.

PROOF. Clearly,

(44) $$\sum_{ij \in C} a_{ij} x_{ij} + \sum_{ij \in T} a_{ij} x_{ij} + a_{rs} x_{rs} \leq b - \sum_{ij \in F_1 \cup F_2} a_{ij} \tilde{x}_{ij} + a_{rs} \tilde{x}_{rs}$$

is valid for $PS \cap \{x \in \Re^d : x_{ij} = \tilde{x}_{ij}\ \forall ij \in (F_0 - T) \cup (F_1 - rs) \cup (F_2 - rs)\}$.

Now, because (43) defines a facet of $PS \cap \{x \in \Re^d : x_{ij} = \tilde{x}_{ij}\ \forall ij \in (F_0 - T) \cup F_1 \cup F_2\}$, (44) is satisfied at equality by $|C \cup T|$ linearly independent points of $PS \cap \{x \in \Re^d : x_{ij} = \tilde{x}_{ij}\ \forall ij \in (F_0 - T) \cup (F_1 - rs) \cup (F_2 - rs)\}$, with $x_{rs} = \tilde{x}_{rs}$. Since $r \notin M_C$ and

$$\sum_{ij \in C} a_{ij} + a_{rs} > b - \sum_{ij \in F_1 \cup F_2} a_{ij} \tilde{x}_{ij} + a_{rs} \tilde{x}_{rs},$$

$PS \cap \{x \in \Re^d : x_{ij} = \tilde{x}_{ij}\ \forall ij \in (F_0 - T) \cup (F_1 - rs) \cup (F_2 - rs)\}$ has a point which satisfies (43) at equality with $x_{rs} \neq \tilde{x}_{rs}$.

This proves that (43) defines a facet of $PS \cap \{x \in \Re^d : x_{ij} = \tilde{x}_{ij}\ \forall ij \in (F_0 - T) \cup (F_1 - rs) \cup (F_2 - rs)\}$, and therefore the lifting coefficient of x_{rs} is a_{rs}. □

Let $rs \in F_1 \cup F_2$, as in Proposition 14. Note that $C \cup rs$ is a cover for $PS \cap \{x \in \Re^d : x_{ij} = \tilde{x}_{ij}\ \forall ij \in F_0 \cup (F_1 - rs) \cup (F_2 - rs)\}$, and that (44) can be derived by lifting the cover inequality,

$$\sum_{ij \in C \cup rs} a_{ij} x_{ij} \leq b - \sum_{ij \in F_1 \cup F_2} a_{ij} \tilde{x}_{ij} + a_{rs} \tilde{x}_{rs},$$

with respect to x_{ij}, $ij \in T$.

Note also that, unless the lifting coefficient of x_{ij} is greater than a_{ij} for some $ij \in F_0$, we will obtain (1) at the end of the sequential lifting procedure. It is easy to see that Proposition 9 holds for $PS \cap \{x \in \Re^d : x_{ij} = \tilde{x}_{ij}\ \forall ij \in F_0 \cup F_1 \cup F_2\}$, C, and $b - \sum_{ij \in F_1 \cup F_2} a_{ij}$, instead of b. Therefore, as in §3, we only need to consider lifting FCIs, i.e.,

(45) $$\sum_{ij \in C} a_{ij} x_{ij} + \left(b - \sum_{ij \in F_1 \cup F_2} a_{ij} - \sum_{ij \in C - i'j'} a_{ij}\right) x_{i'j''} \leq b - \sum_{ij \in F_1 \cup F_2} a_{ij},$$

where $i'j' \in C$, $j'' \in N_{i'} - j'$, and $\sum_{ij \in C - i'j'} a_{ij} + a_{i'j''} < b - \sum_{ij \in F_1 \cup F_2} a_{ij}$.

However, as the next proposition shows, even when we lift FCIs, the lifting coefficient of x_{rs}, $rs \in F_1$, is a_{rs}. This means that we may as well start with the cover $C \cup F_1$. Since the proof of the proposition is similar to the proof of Proposition 14, it is omitted.

PROPOSITION 15. *Let $T \subseteq F_0 - i'j''$. Suppose that*

$$\sum_{ij\in C} a_{ij}x_{ij} + \left(b - \sum_{ij\in F_1\cup F_2} a_{ij} - \sum_{ij\in C-i'j'} a_{ij}\right)x_{i'j''} + \sum_{ij\in T}\alpha_{ij}x_{ij} \le b - \sum_{ij\in F_1\cup F_2} a_{ij} \tag{46}$$

defines a facet of $PS \cap \{x \in \Re^d : x_{ij} = \tilde{x}_{ij}\ \forall\, ij \in (F_0 - (T \cup i'j'')) \cup F_1 \cup F_2\}$. Let $rs \in F_1$. Lift (46) *next with respect to x_{rs}. Then, the lifting coefficient is a_{rs}.* $\square$

We now show that it is not possible to lift an FCI with respect to the variables x_{rs}, $rs \in F_2$, or, as in Lemma 1, $\alpha_{rs}^{\min} > \alpha_{rs}^{\max}$.

PROPOSITION 16. *Let $T \subseteq F_0 - i'j''$, and $rs \in F_2$. It is not possible to lift* (46) *with respect to x_{rs}.*

PROOF. Let α_{rs} be the lifting coefficient of x_{rs}. Then,

$$\sum_{ij\in C} a_{ij}x_{ij} + \left(b - \sum_{ij\in F_1\cup F_2} a_{ij} - \sum_{ij\in C-i'j'} a_{ij}\right)x_{i'j''} + \sum_{ij\in T}\alpha_{ij}x_{ij} + \alpha_{rs}x_{rs} \le b - \sum_{ij\in F_1\cup F_2} a_{ij} + \alpha_{rs}\tilde{x}_{rs}$$

for all $x \in PS \cap \{x \in \Re^d : x_{ij} = \tilde{x}_{ij}\ \forall\, ij \in (F_0 - (T \cup i'j'')) \cup F_1 \cup (F_2 - rs)\}$.

Let $\hat{x}$ be given by

$$\hat{x}_{ij} = \begin{cases} 1 & \text{if } ij \in C - i'j' \text{ or } ij = i'j'', \\ \min\left\{1, \dfrac{b - \sum_{ij\in F_1\cup F_2} a_{ij} + a_{rs}\tilde{x}_{rs} - \sum_{ij\in C-i'j'} a_{ij} - a_{i'j''}}{a_{rs}}\right\} & \text{if } ij = rs, \\ 0 & \text{otherwise.} \end{cases}$$

Then, $\hat{x} \in PS \cap \{x \in \Re^d : x_{ij} = \tilde{x}_{ij}\ \forall\, ij \in (F_0 - (T \cup i'j'')) \cup F_1 \cup (F_2 - rs)\}$. Because $\hat{x}_{rs} > \tilde{x}_{rs}$, $\alpha_{rs} = 0$.

On the other hand, x^*, given by

$$x^*_{ij} = \begin{cases} 1 & \text{if } ij \in C - i'j', \\ \min\left\{1, \dfrac{b - \sum_{ij\in F_1\cup F_2} a_{ij} + a_{rs}\tilde{x}_{rs} - \sum_{ij\in C-i'j'} a_{ij}}{a_{i'j'}}\right\} & \text{if } ij = i'j', \\ 0 & \text{otherwise,} \end{cases}$$

belongs to $PS \cap \{x \in \Re^d : x_{ij} = \tilde{x}_{ij}\ \forall\, ij \in (F_0 - (T \cup i'j'')) \cup F_1 \cup (F_2 - rs)\}$, and therefore, $\alpha_{rs} > 0$. Thus, (46) cannot be lifted with respect to x_{rs}. $\square$

The proof of Theorem 3 now follows easily from Propositions 14–16.

PROOF OF THEOREM 3. As a consequence of Proposition 16, (36) must be lifted with respect to the variables x_{ij}, $ij \in F_2$, before it is lifted with respect to $x_{i'j''}$. But then, from Proposition 14, the lifting coefficient is a_{ij}. Because of that and of Proposition 15, we may as well start with the cover $C \cup F_1 \cup F_2$, and with all other variables fixed exclusively at 0. $\square$

5. Extensions and further research. We are applying the results of this paper to construct a branch-and-cut algorithm for quadratic programming over a box and 0–1 unconstrained quadratic programming. Many important applications, such as portfolio optimization and location selection, can be formulated as an LP, or a convex quadratic program, with the additional constraint that at most k out of the n variables can be positive in a feasible solution; see Bienstock (1996), de Farias (2001), and Perold (1984). We are currently investigating how FCIs can be used to derive strong cuts for these problems (see de Farias and Nemhauser 2001a).

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I. R. de Farias, Jr.: CORE, 34 Voie du Roman Pays, 1348 Louvain-la-Neuve, Belgium; e-mail: defarias@core.ucl.ac.be

E. L. Johnson: School of Industrial and Systems Engineering, Georgia Institute of Technology, Atlanta, Georgia 30332-0205; e-mail: ellis.johnson@isye.gatech.edu

G. L. Nemhauser: School of Industrial and Systems Engineering, Georgia Institute of Technology, Atlanta, Georgia 30332-0205; e-mail: george.nemhauser@isye.gatech.edu