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# Convergence, Oscillations, and Chaos in a Discrete Model of Combat*

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**Abstract.** A piecewise smooth mapping of the three-dimensional Euclidean space is derived from a discrete-time model of combat. The mathematical analysis of this mapping focuses on the effects of discontinuity caused by the defender's withdrawal strategy—a prime component of the original model. Both the asymptotics and the transient behavior are discussed, and all the behavior types noted in the title are established as possible outcomes.

**Key words.** transient behavior, asymptotic behavior, limit cycles, chaos, attrition rates, withdrawal rate

**AMS subject classifications.** 39A11, 37E99, 37N99

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## 1. Introduction.

Deterministic combat models often serve as foundations upon which more complex war games (often with stochastic add-ons) may be based. In this paper, we study the ground version of a combat model proposed by Epstein in [2]. The salient feature of this model is a withdrawal mechanism that, in addition to conventional warfare, enables one to model specialized types of combat like guerrilla warfare (where the battle front is frequently in motion). Although Epstein gave a full derivation of his model, plus historical background and some numerical simulations, in [2, 3], he did not offer a mathematical analysis. In particular, it is by no means clear from the presentations given in [2] or [3] that the model is capable of exhibiting all the behavior types mentioned in the title of this paper.

The ground version of Epstein's model (i.e., without the air support component) may be formulated as a three-dimensional nonlinear system of difference equations that can be represented by a piecewise smooth map. We wish mainly to examine the consequences of a jump discontinuity in the state space—a phenomenon caused by the defender's aforementioned withdrawal strategy.

The methodology for analyzing the discontinuous mapping is based on a somewhat intuitive approach. When a trajectory jumps back and forth between regions with different dynamical regimes, we say that it is *mode-switching*. Taking advantage of a damping effect, we show that in spite of mode-switching (seen as damped oscillations in the time series), most trajectories converge to a fixed point of the system (Theorem 1). However, the asymptotic behavior is less predictable in a two-dimensional (invariant) subspace where damping is not present. Some aspects of the asymptotic behavior

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in this subspace are considered (Theorems 3 and 4) and, in particular, it is shown that chaotic behavior is a consequence of the discontinuity of the map (Theorem 5 and subsequent remarks).

In studying the transient behavior, we use difference inequalities and linear bounds to examine the influence of both the initial balance of forces and the attrition rate thresholds (some of the system's basic parameters) on the mode-switching frequency and duration (Theorem 2). The implications of these results for the maximum displacement of the front line are also considered.

**2. The Ground Combat Model.** The classical Lanchester theory of combat [5] appears recurrently as a core component in combat simulations (see, e.g., [4] and similar references). This theory is based on special types of factorable systems of two differential equations (see [7] for a general discussion of the mathematical properties of such systems). Finding Lanchester's differential equations overly simplistic, however, Epstein proposed a model in [2] in which the combatants are more adaptive in using feedback from action on the front. This model was used in [3] to simulate battles between the United States and the Soviet Union in the Middle East, the likelihood of which was perceived to be rather high in the 1980s after the Soviet invasion of Afghanistan.

Our focus is on the ground component of Epstein's model (and its main dynamical engine), which involves the following nonnegative quantities:

$A_n$ is the attacker's combat power (or lethality) on day $n$ of combat;[1]
$D_n$ is the defender's combat power (or lethality) on day $n$;
$\alpha_n$ is the attrition rate of the attacker on day $n$;
$\delta_n$ is the attrition rate of the defender on day $n$;
$a \in (0, 1)$ is the attacker's prescribed attrition rate threshold;[2]
$d \in (0, 1)$ is the defender's prescribed attrition rate threshold;
$W_n$ is the defender's rate of withdrawal on day $n$;
$W_{\max}$ is the defender's prescribed maximum rate of withdrawal;
$\gamma_n$ is the attacker's "prosecution rate" of combat on day $n$.

In Epstein's words, $\gamma_n$ "is the rate of ground attrition that the attacker himself is prepared to suffer in order to press the combat at his chosen pace" [2, p. 16]. This is distinguished from the actual rate of attrition $\alpha_n$, which is directly affected not only by $\gamma_n$ but also by the defender's tenacity (modeled in terms of the withdrawal rate in (3) below). For fuller descriptions of the above quantities as well as a semantic description of the withdrawal feedback mechanism that is mathematically captured in the following equations, see [2].

The following relationships exist by definition between the various quantities defined above:

$$W_n \leq W_{\max} \quad \text{for all } n, \quad W_0 = 0,\ W_{\max} > 0,$$

$$A_{n+1} = A_n - \alpha_n A_n = (1 - \alpha_n)A_n,\ A_0 > 0, \tag{1}$$

$$D_{n+1} = D_n - \delta_n D_n = (1 - \delta_n)D_n,\ D_0 > 0. \tag{2}$$

[1] The quantities $A_n$ and $D_n$ are measured in terms of standard military scores based primarily on weaponry; for instance, a standard U.S. armored division has a score of 47,490. See [6].

[2] In simulations offered in [2, 3] for conventional warfare, the values of $a, d$ are often taken in a middle subinterval of (0,0.1). However, as noted in [3, p. 125], $a, d$ may range over the entire unit interval if one considers extreme cases; e.g., $d$ is nearly 0 in the case of guerrilla combat, but nearly 1 for trench warfare.

Following Epstein, we further postulate the relationships

$$\alpha_n = \gamma_n(1 - W_n/W_{\max}), \quad \alpha_0, \gamma_0 \in (0, a], \tag{3}$$

$$\gamma_{n+1} = \gamma_n + \frac{1}{a}(a - \gamma_n)(a - \alpha_n), \tag{4}$$

$$W_{n+1} = \begin{cases} W_n + [(W_{\max} - W_n)/(1-d)](\delta_n - d), & \delta_n \geq d, \\ 0, & \delta_n < d. \end{cases} \tag{5}$$

Equation (5), which is of particular interest to us here, incorporates the defender's decision process: they will withdraw by the indicated amount only when their attrition rate $\delta_n$ reaches the threshold value $d$. Since the value of $W_n$ may not be zero when $\delta_n = d$, (5) clearly injects a discontinuity into the system.

Next, we define the following "exchange ratio" of attacker units lost per defender units lost in day $n$:

$$\rho_n = \frac{\alpha_n A_n}{\delta_n D_n}. \tag{6}$$

As in [2], we take $\rho_n$ to be constant, say, $\rho_n = \rho$. Let us now define the following three nonnegative *pure rates* variables:

$$x_n = \gamma_n, \quad y_n = \frac{W_n}{W_{\max}}, \quad z_n = \frac{\delta_n}{\alpha_n}.$$

From (1) and (2) we obtain

$$\frac{A_{n+1}}{D_{n+1}} = \frac{(1 - \alpha_n)A_n}{(1 - \delta_n)D_n},$$

which is transformed, using (6) with constant $\rho$, into

$$\frac{\delta_{n+1}}{\alpha_{n+1}} = \frac{(1 - \alpha_n)}{(1 - \delta_n)} \frac{\delta_n}{\alpha_n}. \tag{7}$$

Given that $\delta_n = \alpha_n z_n$, (7) may be written as

$$z_{n+1} = \frac{1 - \alpha_n}{1 - \alpha_n z_n} z_n.$$

Transforming the rest of the preceding equations into $x, y, z$, we obtain the dynamical system

$$\begin{aligned} x_{n+1} &= x_n + \frac{1}{a}(a - x_n)[a - x_n(1 - y_n)], \\ y_{n+1} &= \begin{cases} y_n + \frac{1-y_n}{1-d}[z_n x_n(1 - y_n) - d], & z_n x_n(1 - y_n) \geq d, \\ 0, & z_n x_n(1 - y_n) < d, \end{cases} \\ z_{n+1} &= \frac{1 - x_n(1 - y_n)}{1 - z_n x_n(1 - y_n)} z_n, \end{aligned} \tag{8}$$

with initial values given as follows:

$$x_0 \in (0, a], \quad y_0 = 0, \quad z_0 = \frac{A_0}{\rho D_0}. \tag{9}$$

The dynamical system consisting of (8) and (9) constitutes Epstein's model without air support. Its dynamics are discussed in the next two sections. Clearly, if $x_n$, $y_n$, and $z_n$ are known, the remaining variables $\alpha_n$, $\delta_n$, etc., are easily determined from the remaining (passive) equations.

**3. General Asymptotics.** Our first result shows that all trajectories starting within the open set $B_0$ in Theorem 1 below (which includes the semantically inspired initial values in (9) for $x_0$ and $y_0$) converge to an equilibrium of the system. As we will see later, the key condition here is that $z_0 < 1$ (i.e., the initial balance of forces satisfy $A_0 < \rho D_0$).[3]

THEOREM 1. (a) *If $F_E(x_n, y_n, z_n)$ is the mapping defined by the right-hand side of* (8)*, and if $B_0 = (0, a] \times [0, 1) \times [0, 1)$, then $F_E(B_0) \subset B_0$.*

(b) *The point $(a, 0, 0) \in B_0$ is a fixed point of* (8) *and a stable attractor of all trajectories with initial point $(x_0, y_0, z_0)$ in $B_0$, with $x_n$ increasing to $a$, $z_n$ decreasing to 0, and $y_n = 0$ for all sufficiently large values of $n$.*

*Proof.* (a) To show that $F_E(B_0) \subset B_0$, let $(x, y, z) \in B_0$. Subtracting the first component of $F_E$, namely,

$$F_{E,1}(x, y, z) = x + (1/a)(a - x)[a - x(1 - y)],$$

from $a$ and combining terms, we obtain

$$\begin{aligned} a - F_{E,1}(x, y, z) &= (a - x)[1 - (1/a)[a - x(1 - y)]] \\ &= (a - x)(x/a)(1 - y), \end{aligned}$$

which shows that for $y \in [0, 1)$ and any value of $z$,

$$F_{E,1}(x, y, z) \in (x, a] \tag{10}$$

whenever $x \in (0, a]$. Next, we consider the second component

$$F_{E,2}(x, y, z) = \begin{cases} y + [(1 - y)/(1 - d)][zx(1 - y) - d], & zx(1 - y) \geq d, \\ 0, & zx(1 - y) < d. \end{cases}$$

For nontriviality, suppose that the values of $x, y, z$ are such that $zx(1 - y) \geq d$. Then

$$\begin{aligned} 1 - F_{E,2}(x, y, z) &= (1 - y)\left[1 - \frac{zx(1 - y) - d}{1 - d}\right] \\ &= (1 - y)\left[\frac{1 - zx(1 - y)}{1 - d}\right], \end{aligned}$$

which readily implies that

$$F_{E,2}(x, y, z) \in [y, 1) \tag{11}$$

for points in $B_0$. Finally, $z < 1$ implies

$$F_{E,3}(x, y, z) = \left[\frac{1 - x(1 - y)}{1 - zx(1 - y)}\right] z \in [0, z), \tag{12}$$

and the invariance of $B_0$ is established.

[3]The simulations in [2, 3] use $\rho = 1.5$, so $z_0 < 1$ if $A_0/D_0 < 1.5$. This is conservative by conventional and historical standards, which indicate the range $3 \leq \rho \leq 5$; see [3, pp. 47–75]. Therefore, the condition $z_0 < 1$ is likely to be satisfied in typical conflicts where the attacker's forces $A_0$ will not outnumber the defender's by a 5:1 ratio or worse.

(b) It is not difficult to see that $F_E(a,0,0) = (a,0,0)$, so $(a,0,0)$ is a fixed point of $F_E$ in $B_0$. Let $(x_0, y_0, z_0) \in B_0$. By (10), for every $n \geq 1$,

$$x_n < x_{n+1} < a,$$

and the sequence $\{x_n\}$ is increasing and bounded by $a$. If $\bar{x}$ denotes the supremum of $x_n$, then we have for all $y_n$,

$$x_{n+1} - x_n \geq \frac{1}{a}(a - x_n)^2 \geq \frac{1}{a}(a - \bar{x})^2,$$

which shows that $\bar{x} = a$. Similarly, using (12) we see that the sequence $\{z_n\}$ is decreasing and

$$z_n - z_{n+1} = \frac{z_n x_n (1 - y_n)(1 - z_n)}{1 - z_n x_n (1 - y_n)}.$$

If $\bar{z}$ is the infimum of $z_n$ and $\bar{z} > 0$, then there are two possible cases:
(i) there is an $n_0$ such that for all $n \geq n_0$, $x_n(1 - y_n) \geq d/\bar{z}$, or
(ii) for all $k \geq 1$, there is an $n \geq k$ such that $x_n(1 - y_n) < d/\bar{z}$.
In case (i), we see that

$$z_n - z_{n+1} \geq \frac{d(1 - z_0)}{1 - d}$$

for all $n \geq n_0$, which is not possible if $z_n$ converges. In case (ii), it is possible to choose an $n$ so large that $z_n x_n (1 - y_n) < d$. Then $y_{n+1} = 0$ and

$$z_{n+1} - z_{n+2} \geq \frac{\bar{z} x_0 (1 - z_0)}{1 - \bar{z} x_0}$$

for infinitely many $n$. Again this cannot happen if $z_n$ converges. Thus $\bar{z} = 0$. Note that $x_n$ and $z_n$ converge as they do regardless of what $y_n$ does. Finally, since for all $n \geq 1$,

$$z_n x_n (1 - y_n) \leq z_n x_n$$

and there is a $k$ such that

$$z_n x_n \leq z_k a < d, \quad n \geq k,$$

it follows that $y_n = 0$ for all $n > k$ and the proof is complete (stability of $(a,0,0)$ is evident from the monotonic nature of $\{x_n\}$, $\{z_n\}$, and the fact that $\{y_n\}$ is eventually 0). ∎

*Remark* 1. The act of withdrawal results in reduced attrition rates for both the defender and the attacker. For the attacker, this is most easily seen by setting $\gamma_n = a$ in (3), since by Theorem 1, $a$ is the limit of $x_n = \gamma_n$. For the defender, we recall from section 2 that $\delta_n = z_n \alpha_n \leq \alpha_n$. These observations hold for $z_0 = 1$ also, due to the invariance of the plane $z = 1$ (see section 5 below).

With the aid of difference inequalities and linear bounds, the next result furnishes the decreasing sequence $\{z_n\}$ with a positive lower bound that may be used in estimating the maximum movement of the front line (see the remark following Theorem 2).

LEMMA 1. *Let* $0 < x_0 \leq a$ *and* $0 < z_0 < 1$. *Then*

$$z_n \geq \frac{1}{x_0/a + (1/z_0 - x_0/a)(1 - a)^{-n}}, \quad n \geq 1.$$

*Proof.* If we substitute $u_n = 1 - y_n$, then the second half of (8) takes the form

$$u_{n+1} = \begin{cases} u_n(1 - x_n z_n u_n)/(1-d), & x_n z_n u_n \ge d, \\ 1, & x_n z_n u_n < d, \end{cases} \tag{13}$$

$$z_{n+1} = \frac{(1 - x_n u_n) z_n}{1 - x_n z_n u_n},$$

where $u_0 = 1$. Since by Theorem 1(a), $0 < u_n \le 1$ and

$$\frac{\partial}{\partial u_n} \frac{1 - x_n u_n}{1 - x_n u_n z_n} = \frac{-x_n(1 - z_n)}{(1 - x_n u_n z_n)^2} < 0,$$

we may conclude that for all $n \ge 1$,

$$z_{n+1} \ge \frac{(1 - x_n) z_n}{1 - x_n z_n}. \tag{14}$$

The unique solution $\{s_n\}$ of the first-order equation

$$s_{n+1} = \frac{(1 - x_n) s_n}{1 - x_n s_n} \tag{15}$$

with $s_0 = z_0$ (treating the $x_n$ as variable coefficients) is found to be

$$s_n = \frac{p_n}{1/z_0 - x_0 - \sum_{k=1}^{n-1} x_k p_k}, \qquad p_k = \prod_{j=0}^{k-1} (1 - x_j), \quad k = 1, 2, 3, \ldots. \tag{16}$$

To see this, substitute $s_n = 1/t_n$ in (15) and transform it into the linear difference equation

$$t_{n+1} = \frac{1}{1 - x_n}(t_n - x_n), \quad t_0 = \frac{1}{z_0}.$$

This equation readily yields the solution

$$t_n = \prod_{k=0}^{n-1} (1 - x_k)^{-1} \left( \frac{1}{z_0} - x_0 - \sum_{k=1}^{n-1} x_k \prod_{j=0}^{k-1} (1 - x_j) \right),$$

which may be transformed into (16).

Next, note that $s_1 = z_1$, and by (14), $s_2 \le z_2$. If $s_n \le z_n$ for $n \ge 2$, then

$$s_{n+1} \le \frac{(1 - x_n) z_n}{1 - x_n z_n}$$

because for all $r \in (0, 1)$,

$$\frac{\partial}{\partial r} \frac{(1 - x_n) r}{1 - x_n r} = \frac{1 - x_n}{(1 - x_n r)^2} > 0.$$

Therefore, we conclude by (14) and induction that $z_n \ge s_n$ for all $n \ge 1$. To complete the proof, notice that since by Theorem 1, $x_0 \le x_n \le a$ for all $n$, we have

$$p_n = \prod_{k=0}^{n-1} (1 - x_k) \ge (1 - a)^n,$$

$$x_0 + \sum_{k=1}^{n-1} x_k p_k \ge \sum_{k=0}^{n-1} x_k (1 - a)^k \ge \frac{x_0}{a} [1 - (1 - a)^n].$$

Using these inequalities in (16), we obtain

$$z_n \geq s_n \geq \frac{(1-a)^n}{1/z_0 - x_0/a + (x_0/a)(1-a)^n},$$

where the last ratio is equivalent to that in the statement of the lemma. ∎

In particular, from (6) and the preceding result, the following lower bound on the ratio $A_n/D_n$ is obtained:

$$\frac{A_n}{D_n} \geq \frac{\rho}{\gamma_0/a + (\rho D_0/A_0 - \gamma_0/a)(1-a)^{-n}}.$$

In the special case where no withdrawal occurs, a solution for (8) can be obtained as follows.

COROLLARY 1. *Let $z_0 \in (0,1)$, i.e., $0 < A_0 < \rho D_0$. If $z_0 a < d$ (in particular, if $a \leq d$), then $y_n = 0$ for all $n$, and*

$$z_n = \frac{z_0 p_n}{1 - x_0 z_0 - z_0 \sum_{k=1}^{n-1} x_k p_k}, \qquad p_k = \prod_{j=0}^{k-1}(1-x_j), \quad k = 1,2,3,\ldots,$$

*where the values of $x_n$ are obtained from the first-order recursion*

$$x_{n+1} = x_n + \frac{1}{a}(a - x_n)^2. \tag{17}$$

*Proof.* By Theorem 1, $\{z_n\}$ is decreasing to zero and $x_n$ is increasing to $a$, so for $n \geq 1$ we have

$$z_n x_n \leq z_0 a < d,$$

which implies that $y_n = 0$ for all $n$. This also reduces the first equation of (8) to the form in (17). Now applying the argument in Lemma 1 with $s_n = z_n$ (since now $u_n = 1$ for all $n$) completes the proof. ∎

*Remark* 2. It may be noted that $a$ is the unique semistable fixed point of equation (17) and that the same equation transforms into the more familiar logistical equation $r_{n+1} = r_n(1 - r_n)$ under the substitution $x_n = a(1 - r_n)$.

**4. The Transient Behavior.** Theorem 1 establishes the asymptotic behavior of (8) over $B_0$, but does not say much about the *transient* behavior of the trajectory when $y_n$ assumes nonzero values. This latter consideration is of interest with regard to the semantic interpretation of the model, since it determines whether, and by how much, the defender withdraws (i.e., the extent of the front's movement; see the remark following the next theorem).

THEOREM 2. *Let $z_0 \in (0,1)$, i.e., $A_0 < \rho D_0$.*

(a) *If $x_0 z_0 \geq d$, then the least positive integer $N$ such that $y_n = 0$ for all $n \geq N$ satisfies*

$$N \geq N^* \doteq \frac{\ln[(x_0/d - x_0/a)/(1/z_0 - x_0/a)]}{\ln[1/(1-a)]} - 1. \tag{18}$$

*In particular, for fixed values of $a$ and $d$, $N$ can be arbitrarily large for points $(x_0, z_0)$ sufficiently close to $(a, 1)$.*

(b) *Suppose that* $m \geq 0$ *satisfies* $y_m = 0$ *and* $x_m z_m \geq d$*, and let* $n_m > m$ *be the least integer such that*

$$x_{n_m} z_{n_m}(1 - y_{n_m}) < d. \tag{19}$$

*Then* $n_m - m$ *satisfies the inequality*

$$\left[\left(\frac{1-a}{1-d}\right)^{n_m - m} - \frac{d}{a}\right](1-a)^{n_m - m} < 1 - \frac{d}{a}. \tag{20}$$

*Proof.* (a) Note that the number $N$ is the least positive integer such that $x_n z_n u_n < d$ for all $n \geq N$ ($N$ is finite by Theorem 1). Let $N_1$ be the least positive integer such that $x_n z_n < d$ for all $n \geq N_1$ and observe that $N_1 = N+1$, since $u_{N+1} = 1$. Therefore, by Lemma 1,

$$\frac{x_0}{a} + \left(\frac{1}{z_0} - \frac{x_0}{a}\right)\left(\frac{1}{1-a}\right)^{N+1} \geq \frac{1}{z_{N+1}} > \frac{x_{N+1}}{d} \geq \frac{x_0}{d},$$

which may be solved for $N$ to yield (18).

(b) From (13) we have for $m \leq n < n_m$ that

$$u_{n+1} = \frac{1 - x_n z_n u_n}{1-d} u_n \geq \frac{1-a}{1-d} u_n$$

so that for such values of $n$,

$$u_n \geq \left(\frac{1-a}{1-d}\right)^{n-m} \tag{21}$$

since $u_m = 1 - y_m = 1$. Inequality (21) with $n = n_m$ and (19) imply that

$$x_{n_m} z_{n_m} \left(\frac{1-a}{1-d}\right)^{n_m - m} \leq x_{n_m} z_{n_m} u_{n_m} < d.$$

Now, this inequality and Lemma 1 with $z_m, x_m$ instead of $z_0, x_0$ imply that

$$\frac{x_{n_m}}{d}\left(\frac{1-a}{1-d}\right)^{n_m - m} < \frac{x_m}{a} + \left(\frac{1}{z_m} - \frac{x_m}{a}\right)\left(\frac{1}{1-a}\right)^{n_m - m}.$$

Since $x_{n_m} \leq x_m$, multiplying both sides of the above inequality by $d/x_m$ yields

$$\left(\frac{1-a}{1-d}\right)^{n_m - m} < \frac{d}{a} + \left(\frac{d}{x_m z_m} - \frac{d}{a}\right)\left(\frac{1}{1-a}\right)^{n_m - m},$$

which, upon noting that $d/x_m z_m < 1$ and rearranging terms, gives (20). □

*Remark* 3. Since the quantities $A_0$ and $D_0$ are given as integers in the model, the value of $z_0$ does not get arbitrarily close to 1, so $N$ is finite in physical situations. If $N^*$ is the quantity on the right-hand side of (18) and given that

$$L_n = W_{\max} \sum_{k=1}^{n} y_k$$

is the amount by which the front line moves in $n$ days as a result of the defender's withdrawal, then Theorem 2 gives the following lower bound for the *maximum front line displacement* $L_N$:

$$L_N > L_{N^*} = W_{\max} \sum_{k=1}^{N^*} y_k.$$

Both estimates $N^*$ and $L_{N^*}$ are maximal, hence closest to $N$ and $L_N$, respectively, if $x_0 = a$. This is feasible semantically, since the attacker may choose at the outset to prosecute the combat at the maximum tolerable rate of attrition in order to force maximum withdrawal by the defender. With $x_0 = a$, one may also gain some sense of the extent of territorial gain by the attacker as a function of tolerances $a, d$ through the function $N^*$ and the next corollary, which in particular furnishes lower bounds for each summand $y_k$ when $x_0 = a$.

COROLLARY 2. *Suppose that* $z_0 \in (0,1)$ *and* $x_0 = a$. *If* $az_0 \geq d$, *then for each* $m$ *in Theorem* 2(b), *and* $1 \leq k \leq n_m - m$,

$$y_{m+k} \geq \frac{1}{1-d}\left[\frac{az_0}{z_0 + (1-z_0)(1-a)^{-m}} - d\right]. \tag{22}$$

*Further, if* $1/2 \leq d < a$ *or if* $d < 1/2$ *and* $a \geq 1-d$, *then* $n_m = m+1$. *Hence, until* $N$ *in Theorem* 2(a) *is exceeded,* $\{y_n\}$ *oscillates in the following way for* $n \geq 0$:

$$y_{2n} = 0, \quad y_{2n+1} \geq \frac{1}{1-d}\left[\frac{az_0}{z_0 + (1-z_0)(1-a)^{-2n-1}} - d\right]. \tag{23}$$

*Proof.* First, let us observe that $y_n > 0$ if and only if $m < n \leq n_m$ for some $m$ as defined in Theorem 2. For each $m$ we have from (13)

$$u_{m+k} \leq u_{m+1} = \frac{1 - az_m}{1-d}, \quad k = 1, \ldots, n_m - m.$$

Hence, using Lemma 1 to substitute for $z_m$,

$$y_{m+k} \geq \frac{az_m - d}{1-d} \geq \frac{1}{1-d}\left[\frac{a}{1 + (z_0^{-1} - 1)(1-a)^{-m}} - d\right], \tag{24}$$

which establishes (22). Next, it is evident from (20) that a necessary condition for $n_m > m+1$ is

$$\left(\frac{1-a}{1-d}\right) > \frac{d}{a}. \tag{25}$$

Inequality (25) is equivalent to the quadratic inequality

$$a^2 - a + d(1-d) < 0,$$

which gives the range $d < a < 1-d$, since for $d < 1/2$ we have

$$\frac{1}{2}\left(1 - \sqrt{1 - 4d(1-d)}\right) = d,$$
$$\frac{1}{2}\left(1 + \sqrt{1 - 4d(1-d)}\right) = 1 - d.$$

Therefore, if $1/2 \leq d < a$ or if $d < 1/2$ and $a \geq 1-d$, it follows that $n_m = m+1$. In particular, (21) is implied by (24), since $k$ can only be equal to 1. ∎

**5. The Invariant Plane $z = 1$.** The case $z_0 = 1$ in (8), i.e., $A_0 = \rho D_0$, is particularly interesting from a mathematical point of view. The trajectory in this case exhibits a more complex asymptotic behavior than the case $z_0 < 1$, because now there are no damping effects contributed by the $z$ variable. With $z_0 = 1$, system (8) reduces to a system of two equations, namely,

$$x_{n+1} = x_n + \frac{1}{a}(a - x_n)[a - x_n(1 - y_n)],$$

$$y_{n+1} = \begin{cases} y_n + \frac{1-y_n}{1-d}[x_n(1 - y_n) - d], & x_n(1 - y_n) \geq d, \\ 0, & x_n(1 - y_n) < d, \end{cases} \tag{26}$$

and the state space is two-dimensional. The mapping $F(x, y)$ defined by the right-hand side of system (26) is discontinuous along the curve

$$y = \eta(x) = 1 - \frac{d}{x}.$$

It is helpful to note that when $d < a$, then $\eta$ contains, in particular, the point $(a, 1 - d/a)$, which is a stable node of the associated smooth system

$$u_{n+1} = u_n + \frac{1}{a}(a - u_n)[a - u_n(1 - v_n)], \tag{27}$$

$$v_{n+1} = v_n + \frac{1 - v_n}{1 - d}[u_n(1 - v_n) - d],$$

whose linearization has positive eigenvalues $d/a$ and $(1 - 2d)/(1 - d)$, both less than unity.

LEMMA 2. *Let $F_s$ be the smooth mapping defined by the right-hand side of system (27). If*

$$d \leq a \leq 1 - d, \tag{28}$$

*then $F_s(S) \subset S$, where*

$$S = \left\{(x, y) : d \leq x \leq a,\ 0 \leq y \leq 1 - \frac{d}{x}\right\}.$$

*Proof.* Let $F_{s,1}$ and $F_{s,2}$ be the first and second components of $F_s$, respectively. If $(x, y) \in S$, then

$$a - F_{s,1}(x, y) = \frac{x}{a}(a - x)(1 - y) \leq a - x,$$

which implies that

$$d \leq x \leq F_{s,1}(x, y) \leq a. \tag{29}$$

It remains to show that

$$F_{s,2}(x, y) \leq 1 - \frac{d}{F_{s,1}(x, y)}. \tag{30}$$

The last inequality is equivalent to

$$d(1 - d) \leq F_{s,1}(x, y)(1 - y)[1 - x(1 - y)],$$

which by (29) is true if, in particular,

$$d(1-d) \le x(1-y)[1-x(1-y)]. \tag{31}$$

Define $u = x(1-y)$ and note that for $(x, y) \in S$, we have $d \le u \le a$. Writing (31) as

$$u^2 - u + d(1-d) \le 0 \tag{32}$$

and using arguments similar to those given in the proof of Corollary 2, we conclude that (32) holds if $d \le u \le 1-d$. This is clearly true by (28), so (30) is established. □

LEMMA 3. *Consider the cubic polynomial*

$$C(t) = -(1-t)(t^2 - at + a^2) + ad(1-d).$$

*If $a > 1/2$, then $C$ is strictly increasing with a unique real zero $\xi \in (d, a)$ if $a > d \ge 1/2$, whereas if $d < 1/2$ and $a > 1-d$, then $\xi \in (1-d, a)$.*

*Proof.* The derivative

$$C'(t) = 3t^2 - 2(1+a)t + a(1+a)$$

has no real zeros if $a > 1/2$, so it is always positive. It follows that $C$ is strictly increasing with a unique real zero $\xi$, say. Further,

$$C(a) = a[d(1-d) - a(1-a)],$$

which is positive according to the arguments in the proof of Theorem 2 under the conditions of this lemma. Finally, since $C$ is symmetric with respect to $d$ and $1-d$ and

$$C(d) = -(1-d)(a-d)^2 < 0,$$

the proof is complete. □

THEOREM 3. *Define $B_1 = (0, a] \times [0, 1)$, and let $\eta(x) = 1 - d/x$.*

(a) *If $a < d$, then for every $(x_0, y_0) \in B_1$, it follows that $y_n = 0$ for all $n \ge 1$ and $\{x_n\}$ increases monotonically to $a$.*

(b) *Assume that* (28) *holds. Then for every $(x_0, y_0) \in B_1$, each of the sequences $\{x_n\}$ and $\{y_n\}$ is increasing for $n > 1$ and*

$$\lim_{n\to\infty}(x_n, y_n) = \left(a, 1 - \frac{d}{a}\right).$$

(c) *Let $d < 1/2$ and $a > 1-d$, and let $\xi \in (1-d, a)$ be the unique zero of $C(t)$ in* Lemma 3. *If*

$$x_0 \in (\xi, a], \qquad y_0 = 0, \tag{33}$$

*then $\{x_n\}$ increases monotonically to $a$, but for all $n$,*

$$y_{2n} = 0, \qquad y_{2n+1} = \frac{x_{2n} - d}{1-d} > \eta(x_{2n+1}). \tag{34}$$

*In particular, the trajectory $\{(x_n, y_n)\}$ converges to the 2-cycle*

$$\Gamma = \{(a, 0), (a, y_\infty)\},$$

*where $y_\infty = (a-d)/(1-d) > \eta(a)$.*

(d) *Let $a > d \geq 1/2$, and let $\xi \in (d, a)$ be the unique zero of $C(t)$ in Lemma* 3. *Then the same behavior as in part* (c) *is obtained.*

*Proof.* (a) Since $x_0(1 - y_0) \leq a < d$, it follows that $y_1 = 0$. Also, for all $n \geq 0$,

$$a - x_{n+1} = \frac{x_n}{a}(a - x_n)(1 - y_n) \leq a - x_n \tag{35}$$

if $x_n \leq a$, so the sequence $\{x_n\}$ is increasing towards $a$. In particular, $x_n < d$ so the proof is completed by induction.

(b) If $(x_0, y_0)$ is not in the set $S$ of Lemma 2, then as in part (a), $y_1 = 0$ and $x_1 > x_0$. Clearly, $y_n = 0$ until $x_k > d$ for some positive integer $k$. Thus $(x_k, y_k) \in S$, which is invariant under the discontinuous map $F$ since

$$F_s|_S = F|_S.$$

Thus by Lemma 2, the trajectory remains inside $S$. By (35) $x_n$ is increasing, and by (26) so is $y_n$. Since $S$ is bounded and $F = F_s$ is continuous on $S$, it follows that $(x_n, y_n)$ approaches the fixed point $(a, 1 - d/a)$ of $F_s$.

(c) Suppose that (33) holds, and note that

$$y_1 = \frac{x_0 - d}{1 - d}, \qquad x_1 = x_0 + \frac{1}{a}(a - x_0)^2. \tag{36}$$

Now $x_1(1 - y_1) < d$ if and only if

$$x_1 \frac{1 - x_0}{1 - d} < d.$$

Inserting the value of $x_1$ from (36) and rearranging, the preceding inequality is seen to be equivalent to

$$C(x_0) > 0,$$

which is true by Lemma 3 when $x_0 > \xi$. It then follows that $y_1 > \eta(x_1)$, and thus $y_2 = 0$. Repeating the preceding argument but replacing $y_0$ with $y_2$, and continuing the process, it follows inductively that (34) is true. The assertion about $\Gamma$ now follows easily since $x_n \to a$, and thus by (34), $y_{2n-1}$ converges to $(a - d)/(1 - d) = y_\infty$; this is greater than $\eta(a)$, since $a > 1 - d$.

(d) This is done in essentially the same way as (c). □

It may be noted that the behavior in (c) and (d) occurs *because* of the discontinuity; the limit cycles mentioned there do not exist in the continuous system, which, as noted earlier, has a stable node at $(a, \eta(a))$. Next, we focus on some other consequences of this discontinuity in the case $x_0 > a$, including the occurrence of imbedded periodic and aperiodic behavior.

LEMMA 4. *Consider the quintic polynomial*

$$Q(t) = a + t(1 - t)(t - a)\frac{t^2 - at + a^2}{a^3(1 - d)}.$$

(a) *If $d \leq 1 - (1 - a)/4$, then $Q$ has a unique zero $\zeta \in (1, 1 + a)$; in fact, there exists $\varepsilon > 0$ such that $Q$ is strictly decreasing on the interval $(1 - \varepsilon, \infty)$, and $Q$ maps the interval $[0, a]$ homeomorphically onto $[1, \zeta]$.*

(b) *Assume that $d < a$. Then all fixed points of $Q$ that exceed $d$ are in the interval $[a, 1)$. If $a \geq 1/2$ then $a$ is the only fixed point of $Q$ that is larger than $d$. On the other hand, if $a < 1/2$ and*

$$d \geq 1 - \frac{1}{4a}, \tag{37}$$

*then $Q$ has a fixed point in $(a, \beta^-)$ and another in $(\beta^+, 1)$, where*

$$\beta^{\pm} = \frac{1 \pm \sqrt{1 - 4a(1-d)}}{2}.$$

*Proof.* (a) The existence of $\zeta$ is established by examining the values of $Q$ at the points 1 and $1 + a$. As for uniqueness, since the term $t^2 - at + a^2$ is always positive, it is clear that

$$Q(t) > a \quad \text{if} \quad a < t < 1 \text{ or } t < 0.$$

Further, for $0 < t < a$, we have $t(a - t) \leq a^2/4$ and $t^2 - at + a^2 \leq a^2$ so that

$$\begin{aligned} Q(t) &> a - (1-a)t(a-t)\frac{t^2 - at + a^2}{a^3(1-d)} \\ &\geq a - (1-a)\frac{a^2}{4}\left[\frac{a^2}{a^3(1-d)}\right] \\ &\geq a - \frac{a(1-a)}{4(1-d)} \\ &= \left[1 - \frac{1-a}{4(1-d)}\right] a, \end{aligned}$$

and the last expression is nonnegative if $d \leq 1 - (1 - a)/4$. Finally,

$$\begin{aligned} Q'(t) = {} & [(1-t)(t-a) - t(t-a) + t(1-t)]\frac{t^2 - at + a^2}{a^3(1-d)} \\ & + t(1-t)(t-a)\frac{2t - a}{a^3(1-d)}, \end{aligned}$$

which shows that the derivative $Q'(t) < 0$ if $t > 1$, and

$$Q'(1) = -\frac{(1-a)(1-a+a^2)}{a^3(1-d)} < 0$$

so that for some $\varepsilon > 0$, $Q$ is decreasing on $(1 - \varepsilon, \infty)$. In particular, $\zeta$ is unique. The last assertion is now clear since $Q(1) = a$.

(b) Since a fixed point of $Q$ is a solution of the equation $Q(t) = t$, $a$ is obviously a fixed point of $Q$ and fixed points of $Q$ other than $a$ are zeros of the quartic polynomial

$$P(t) = t(1-t)(t^2 - at + a^2) - a^3(1-d).$$

Since $P(t) < -a^3(1 - d)$ if $t > 1$, we next consider the interval $(d, a)$. On this interval,

$$P(t) < a^3[(1-t) - (1-d)] = -a^3(t-d) < 0.$$

Hence, fixed points of $Q$ other than $a$, if any, must occur in $(a, 1)$.

Let $a \geq 1/2$; for $t \in (a, 1)$, we find that $t(1-t) \leq a(1-a)$, which yields

$$P(t) \leq a^3[(1-a) - (1-d)] = -a^3(a-d) < 0,$$

as desired. Now, consider the opposite case and assume that $a < 1/2$. Then for $a < t < 1$,

$$P(t) > a^2[t(1-t) - a(1-d)].$$

If (37) holds, then the quadratic expression inside the brackets has zeros at $\beta^{\pm}$ and is positive on the interval $(\beta^-, \beta^+)$. Thus $P$ is also positive on $(\beta^-, \beta^+)$, an interval that is contained in $(a, 1)$ if (37) holds with $a < 1/2$. However, $P(a)$ and $P(1)$ are both negative, so there must be zeros of $P$ in $(a, \beta^-)$ and $(\beta^+, 1)$. □

THEOREM 4. *Assume that one of the following conditions holds:*

(i) $a > d \geq 1/2$*;*
(ii) $d < 1/2$ *and* $a > 1 - d$*;*
(iii) $d < a < 1 - d$*.*

*Then the following statements are true:*

(a) *Every trajectory with* $x_0 \in (a, 1)$ *and* $y_0 = 0$ *converges to the cycle* $\Gamma$ *of Theorem* 3 *from the right, with* $\{y_n\}$ *having the same behavior as in Theorem* 3(c)*, but now* $\{x_n\}$ *converges nonmonotonically to* $a$ *from the right with* $x_{2n+1} > x_{2n} > x_{2n+2}$ *for every* $n$*.*

(b) *Suppose that either condition* (i) *or condition* (ii) *holds, and let* $\zeta$ *and* $\xi$ *be the zeros of* $Q$ *and* $C$*, respectively, as in Lemmas* 4 *and* 3*; also, let* $\xi' \in (1, \zeta)$ *be the unique point such that* $Q(\xi') = \xi$*. If* $x_0 \in [1, \xi')$ *and* $y_0 = 0$*, then* $x_2 \in (\xi, a]$ *and* $y_2 = 0$ *so that the behavior in Theorem* 3(c) *or* (d) *results.*

(c) *Suppose that condition* (iii) *holds, and let* $x_0 \in [1, \zeta)$ *and* $y_0 = 0$*. Then* $x_2 \in (0, a]$ *and* $y_2 = 0$*, so that the behavior in Theorem* 3(b) *results.*

(d) *If for some* $k \geq 0$*,* $x_k > \zeta$ *and* $y_k = 0$*, then* $x_{k+2} < 0$*.*

*Proof.* (a) Straightforward computation shows that if $y_k = 0$ and $x_k > d$ for any $k \geq 0$, then

$$y_{k+1} = \frac{x_k - d}{1 - d}, \quad x_{k+1} = x_k + \frac{(x_k - a)^2}{a}, \quad x_{k+2} = Q(x_k).$$

If any of the three conditions (i)–(iii) holds, then by Lemma 4, $Q$ has no fixed points (except $a$). It follows that if $x_0 \in (a, 1)$ and $y_0 = 0$, then $x_1 > x_0$, and $x_2 = Q(x_0)$ is in the interval $(a, x_0)$, since

$$Q'(a) = \frac{1-a}{1-d} < 1$$

and the graph of $Q$ must lie below the identity line on the interval $(a, 1)$. In addition, if $\xi$ is as defined in Theorem 3, then $\xi < a$ so $y_1 > \eta(x_1)$ and $y_2 = 0$. Now we apply induction as in Theorem 3(c) to complete the proof of part (a).

(b) and (c) These parts follow immediately from Lemma 4(a) since $Q$ maps $[1, \zeta]$ homeomorphically onto $[0, a]$.

(d) This is clear from the definition of $\zeta$ and the strictly decreasing nature of $Q$ on the interval $(1, \infty)$. □

For the sake of the next result, Figures 1 and 2 show the zeros of $C$ and $P$ plotted as surfaces over the $a, d$ parameter space. Here we have taken advantage of the symmetry of $C$ in $d$ to show two cut-away views of these "zero surfaces" for greater clarity.

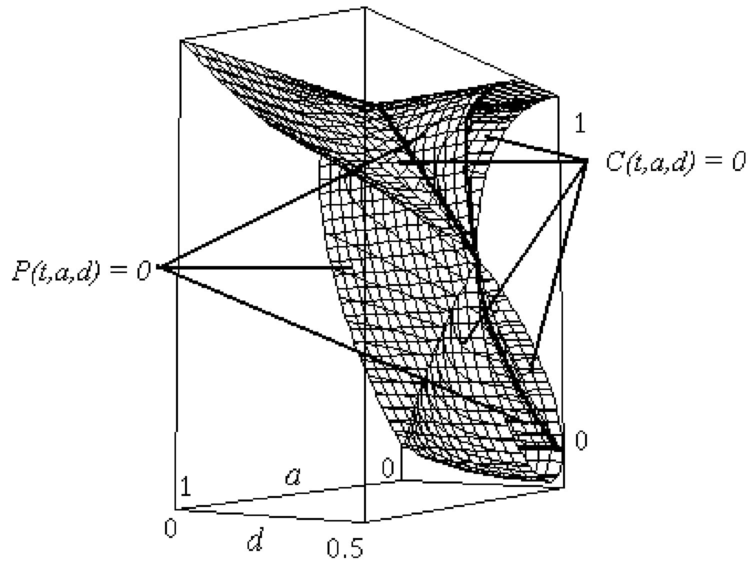


**Fig. 1** *Zeros of $C$ and $P$ as functions of $(a, d)$, first half.*

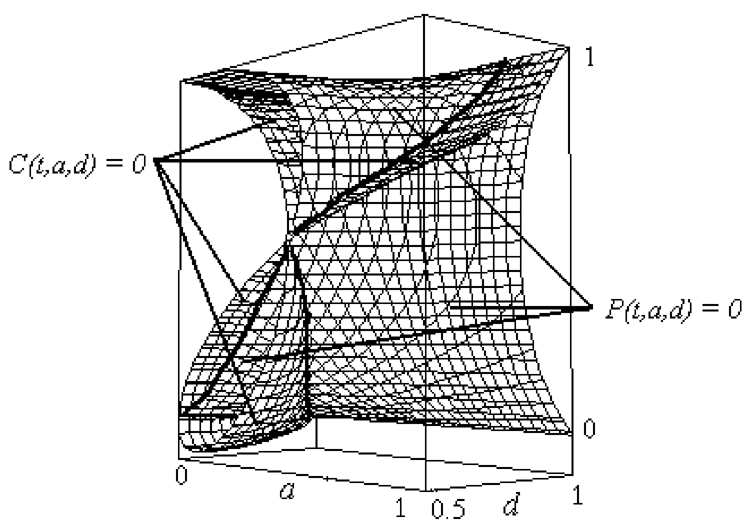


**Fig. 2** *Zeros of $Q$ and $C$ as functions of $(a, d)$, second half.*

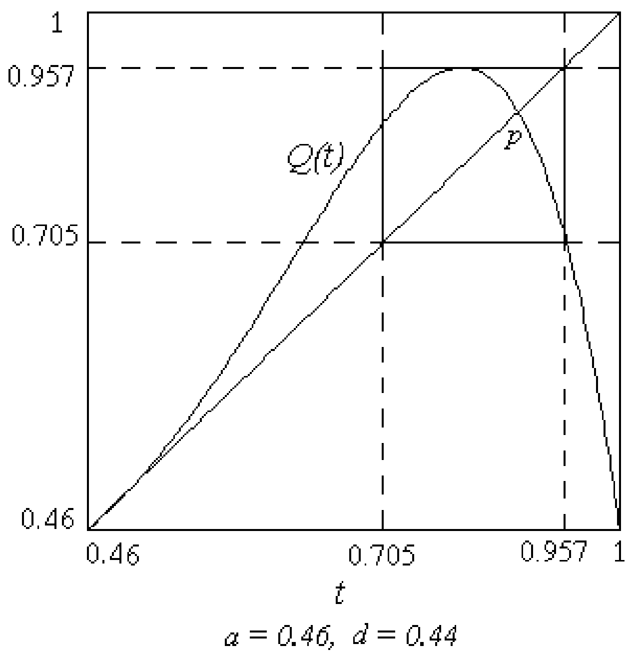


**Fig. 3** *Graph of $Q$ together with its invariant interval shown as the box containing the fixed point $p = 0.9$ (approximate).*

These figures show, in particular, that the quartic $P$ has at most two zeros for each given pair of values for $a$ and $d$, while the cubic can have as many as three, the maximum possible. We also see the relationships between the two sets of zeros for different values of $a$ and $d$ in terms of the way the two surfaces cross each other.

THEOREM 5. *Assume that the polynomial $Q$ has two fixed points in the interval $(a, 1)$, e.g., if $a < 1/2$ and inequality (37) in Lemma 4 holds—see Figure 3. Assume further that $p$ is the larger fixed point, and $p > \xi$. If the interval $(\xi, 1)$ contains a subinterval $I$ with $p \in Q(I) \subset I$, and if $x_0 \in I$, $y_0 = 0$, then for $n \geq 1$,*

$$x_{2n} = Q(x_{2(n-1)}), \quad y_{2n} = 0, \tag{38}$$

*and*

$$x_{2n-1} = x_{2(n-1)} + \frac{(x_{2(n-1)} - a)^2}{a}, \quad y_{2n-1} = \frac{x_{2(n-1)} - d}{1 - d}. \tag{39}$$

*Proof.* Since $p$ is the larger of the two fixed points of $Q$, it must be true that $Q(t) > t$ for $t < p$ and $Q(t) < t$ for $t > p$. With $x_0 \in I$, $y_0 = 0$, we obtain $x_1$, $y_1$ as given by (39) with $n = 1$; furthermore, since $Q$ maps $I$ into itself and $x_0 > \xi$, it follows that $x_2 = Q(x_0) \in I$ and $y_2 = 0$. Hence, by induction, the even indexed terms of $x_n$ and $y_n$ satisfy (38), while the odd indexed terms satisfy (39). □

COROLLARY 3. *Under the hypotheses of Theorem 5, if the derivative $|Q'(p)| < 1$, then all trajectories of (26) with $x_0 \in I$ and $y_0 = 0$ converge to the 2-cycle*

$$\Psi = \left\{ (p, 0), \left( p + \frac{(p-a)^2}{a}, \frac{p-d}{1-d} \right) \right\}.$$

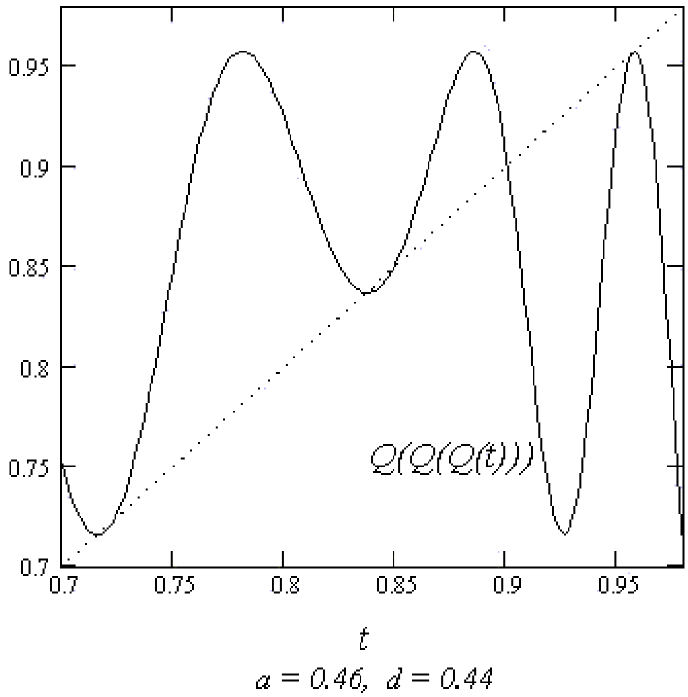


**Fig. 4** *Plot of a portion of the third iterate of* $Q(t)$ *showing a* 3*-cycle in the interval* $(0.705, 0.957)$.

*Further, if* $Q'(p) \leq -1$*, and the first-order initial value problem*

$$u_n = Q(u_{n-1}),\ u_0 \in I, \quad n = 1, 2, 3, \ldots,$$

*has an asymptotically stable* $k$*-periodic solution* $\{c_1, \ldots, c_k\}$*, then all trajectories of* (26) *with* $x_0 \in I$ *and* $y_0 = 0$ *converge to a* $2k$*-cycle whose even indexed terms are* $(c_i, 0)$*,* $i = 1, \ldots, k$.

It is evident that under the hypotheses of Theorem 5, if $Q'(p) < -1$, then any stable aperiodic or chaotic solution of the first-order equation in Corollary 3 will also give rise to a similar solution for (26) (although odd periods cannot occur for $F$, they do occur for $F^2$). Note that *this situation occurs because of the discontinuity in* (26).

To see that these cases are in fact possible, consider a special case: $a = 0.46$, $d = 0.44$. The computer-generated Figure 3 shows the mapping $Q$ with the invariant interval to the right of $\xi$ highlighted in the square.

Figure 4 displays the third iterate $Q^3$ showing the existence of a period 3 solution for the first-order difference equation in Corollary 3 in the interval highlighted in Figure 3. The existence of the period 3 solution indicates that for a certain range of values of $a$ and $d$, chaotic solutions (in the one-dimensional sense) are possible; see, e.g., [1].

**6. Concluding Remarks.** Let us recall that the model discussed in this paper is not a complete model of combat but a possible core component of one. A realistic combat model (used in computer-simulated war games, but perhaps of little analytical interest) would need to take into account not only the role of air support, as Epstein did numerically, but also deterministic and stochastic factors related to weather conditions, terrain composition, the opponent's psychological profile, logistical issues, etc. It may be possible to give an analytical treatment for certain extended versions of the model in this paper with some of these factors added on, or for a modified version using more generic functions than the linear and quadratic ones used by Epstein; but these alternatives will need to be the subject of another paper.

As is the case with virtually all social science models, the above observations have certain implications for the semantic interpretation of the results of our partial model. For instance, the plane $z = 1$ is not stable according to Theorem 1, so the *ground version* of Epstein's model is sensitive to perturbations in the $z$-coordinate. (However, Epstein's numerical simulations, which include his nonautonomous air support equations, do not seem to exhibit this sensitivity.) Unlike a complete model, the existence of complex behavior and chaos in the partial ground version (as indicated by Theorems 3–5) is significant not only mathematically, but also because such behavior can play a major role in any extension of the model discussed here. In particular, the existence of complexity in the core model suggests that unexpected turns on the battlefield may be attributable as much to basic deterministic elements discussed here as to such things as stochastic disturbances or nonquantifiable aspects of human behavior (e.g., leadership, ingenuity, and error).

We close with a remark about the methodology. Models containing thresholds and trigger mechanisms (like the withdrawal strategy discussed here) are seen to partition the state space into disjoint regions that are governed by different smooth maps; such pairs of maps and regions may be called *modes*. We did not analyze these maps collectively as a single piecewise smooth mapping (which might be "made continuous" by using a pointwise close smooth map that patches things using sigmoid functions). Rather, we gained an understanding of the system as a whole by tracking various representative trajectories and examining the effect of each mode's dynamics on a trajectory as the latter crosses the threshold and undergoes mode-switching. Using this approach, we found that the Epstein system exhibits a different dynamic from what its individual parts would produce separately. For example, the oscillations of Theorem 2 and the limit cycles of Theorems 3 and 4 and Corollary 3 are caused by mode-switching trajectories rather than by any single-map dynamics.

Thresholds and mode-switching are a common feature of social science modeling. For additional models and a further discussion of technical and methodological issues surrounding classes of mode-switching dynamical systems such as (8) and (26), the reader is referred to [8].

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