Journal of Combinatorial Theory, Series A **92**, 17–28 (2000)
doi:10.1006/jcta.1999.3033, available online at http://www.idealibrary.com on IDEAL®

Finite-Ring Combinatorics and MacWilliams' Equivalence Theorem

M. Greferath and S. E. Schmidt

AT&T Labs-Research, 180 Park Avenue, Florham Park, New Jersey 07932
Email: {greferath,schmidt}@research.att.com

Communicated by the Managing Editors

Received April 26, 1999

F. J. MacWilliams proved that Hamming isometries between linear codes extend to monomial transformations. This theorem has recently been generalized by J. Wood who proved it for Frobenius rings using character theoretic methods. The present paper provides a combinatorial approach: First we extend I. Constantinescu's concept of *homogeneous* weights on arbitrary finite rings and prove MacWilliams' equivalence theorem to hold with respect to these weights for all finite Frobenius rings. As a central tool we then establish a general inversion principle for real functions on finite modules that involves Möbius inversion on partially ordered sets. An application of the latter yields the aforementioned result of Wood.

Key Words: codes over rings; MacWilliams' equivalence theorem; homogeneous weights; real functions on modules; Möbius inversion on posets; Frobenius rings.

INTRODUCTION

The classical notion of code equivalence is based on a theorem by F. J. MacWilliams [9] who proved that Hamming isometries between linear codes over finite fields can be extended to monomial transformations of the ambient vector space justified the equivalence notion for classical algebraic coding theory. This theorem justified the equivalence notion for classical algebraic coding theory and may be viewed as a coding theoretic analogue of the famous Witt-type extension theorems in geometric algebra. It has enjoyed intensive reexamination and generalization in the literature (cf. [2, 13]).

Initiated by discoveries of A. Nechaev [10] and later R. Hammons *et al.* [6] clarifying the role of $\mathbb{Z}_4$ in the interpretation of the behaviour of certain non-linear binary code classes, an increasing interest in algebraic coding theory over finite rings has led to generalizations of the above theorem for specific classes of rings. The most important of these are the papers by

0097-3165/00 $35.00

I. Constantinescu *et al.* [5] and by J. Wood [15–17], the former proving an extension theorem for *homogeneous* weight functions over integer residue rings, the latter developing extension results for the up to now most general class of rings, i.e., the class of finite Frobenius rings.

The present article was motivated by these results in two different ways. On the one hand a careful analysis shows that the results and methods in [5] are not restricted to codes over integer residue rings; on the other hand it was a challenge to use these methods in order to achieve results as general as those in [15–17].

Wood [16, Remark 2] gives an example of a non-QF-ring which does not allow a positive answer for the extension problem. We extend this and finish our considerations with an example of a QF-ring that does not admit MacWilliams' equivalence theorem.

1. HOMOGENEOUS WEIGHTS ON FINITE RINGS

I. Constantinescu [3] (cf. also [4]) establishes so-called homogeneous weights on integer residue rings. This class of weight functions is defined by two properties: on the one hand a homogeneous weight takes constant values on classes of associated elements; on the other hand the total weight of every non-zero ideal is in a constant ratio of its cardinality. Its existence is proved using the classical Möbius function on $\mathbb{Z}$. The triangle inequality is satisfied for all m which are not divisible by 6.

In this section we prove the existence of such a weight function on arbitrary finite rings. As a preparation we recall the general Möbius inversion on partially ordered sets (cf. [1, Chap. IV; 11; 12, Chap. 3.6; 14]).

For a finite poset P consider the function $\mu : P \times P \to \mathbb{R}$ implicitly defined by $\mu(x, x) = 1$ and

$$\sum_{y \leqslant t \leqslant x} \mu(t, x) = 0$$

if $y < x$, and $\mu(y, x) = 0$ if $y \nleqslant x$. It is called the *Möbius function* on P and induces for arbitrary pairs of real-valued functions f, g on P the following equivalence, referred to as *Möbius inversion*:

$$g(x) = \sum_{y \leqslant x} f(y) \quad \text{for all} \quad x \in P \Leftrightarrow f(x) = \sum_{y \leqslant x} g(y)\, \mu(y, x) \quad \text{for all} \quad x \in P.$$

Let now R be a finite ring and (unless stated otherwise) μ the Möbius function on the set $\{Rx \mid x \in R\}$ of its principal left ideals (partially ordered by set inclusion). Further let $R^{\times}$ denote the set of units of R. Then Möbius inversion yields the next statement:

LEMMA 1.1. *For each $x \in R$ the set $R^{\times}x$ is the set of all generators of Rx, and there holds*

$$|R^{\times}x| = \sum_{Ry \leqslant Rx} |Ry| \, \mu(Ry, Rx).$$

DEFINITION 1.2. A weight w on the finite ring R is called (left) *homogeneous*, if $w(0)=0$ and the following is true:

(H1) If $Rx = Ry$ then $w(x) = w(y)$ for all $x, y \in R$.

(H2) There exists a real number $c \geqslant 0$ such that

$$\sum_{y \in Rx} w(y) = c \, |Rx| \qquad \text{for all} \quad x \in \mathrm{R} \setminus \{0\}.$$

Right homogeneous weights are defined accordingly, and since we are dealing with left homogeneous weights in the sequel we will refer to these simply as homogeneous weights. As a first result we obtain an existence and characterization theorem as a generalization of [3, 4].

THEOREM 1.3. *A weight w on the finite ring R is homogeneous if and only if the following holds*:

(H) *There exists a real number $c \geqslant 0$ such that $w(x) = c(1 - \mu(0, Rx)/|R^{\times}x|)$ for all $x \in R$.*

Proof. For a given weight w let us always assume (H1) from Definition 1.2 because this condition results from (H) by Lemma 1.1. If now (H2) or (H) holds with respect to a positive real number c then the expression $f(Rx) := (c - w(x)) \, |R^{\times}x|$ is well-defined for all $x \in R$, and it follows $f(0) = c$ as well as

$$\sum_{Ry \subseteq Rx} f(Ry) = \sum_{y \in Rx} (c - w(y))$$

for all $x \in R$. Now (H2) is equivalent to $\sum_{Ry \subseteq Rx} f(Ry) = 0$ for all $x \in R \setminus \{0\}$ which by Möbius inversion is seen to be equivalent to $f(Rx) = c\mu(0, Rx)$ for all $x \in R$. The latter is finally equivalent to (H). ∎

Remark 1.4. As a consequence of Theorem 1.3 and the fact that $Rx \cong Rxu$ for all $x \in R$ and $u \in R^{\times}$ the condition (H1) is left-right symmetric for a homogeneous weight. We will refer to this in the inductive proof of Theorem 2.5.

Essential for our later results is the question, when a homogeneous weight satisfies a stronger version of condition (H2):

LEMMA 1.5. *For a finite ring R the following are equivalent*:

(a) $\mathrm{soc}({}_R R)$ *is left principal.*

(b) *For all nonzero* $I \leqslant {}_R R$ *there holds* $\sum_{Rx \leqslant I} \mu(0, Rx) = 0$.

Proof. First assume $\mathrm{soc}({}_R R)$ to be left principal and let $I \leqslant {}_R R$ be a nonzero Ideal. Then $I \cap \mathrm{soc}({}_R R)$ is a nonzero principal left ideal and therefore we obtain the equation

$$\sum_{Rx \leqslant I} \mu(0, Rx) = \sum_{\substack{Rx \leqslant I \\ Rx \leqslant \mathrm{soc}({}_R R)}} \mu(0, Rx) + \sum_{\substack{Rx \leqslant I \\ Rx \nleqslant \mathrm{soc}({}_R R)}} \mu(0, Rx)$$

$$= \sum_{\substack{Rx \leqslant I \\ Rx \nleqslant \mathrm{soc}({}_R R)}} \mu(0, Rx).$$

Our claim follows if we show that $\mu(0, Rx) = 0$ for all $x \notin \mathrm{soc}({}_R R)$. Assume there exists $Rx \leqslant I$ such that $Rx \nleqslant \mathrm{soc}({}_R R)$ and $\mu(0, Rx) \neq 0$, and let Rx be minimal with respect to these properties. Then we obtain

$$0 = \sum_{Ry \leqslant Rx} \mu(0, Ry) = \sum_{\substack{Ry \leqslant Rx \\ Ry \leqslant \mathrm{soc}({}_R R)}} \mu(0, Ry)$$

$$+ \sum_{\substack{Ry < Rx \\ Ry \nleqslant \mathrm{soc}({}_R R)}} \mu(0, Ry) + \mu(0, Rx) = \mu(0, Rx),$$

a contradiction. After all we therefore have $\sum_{Rx \leqslant I} \mu(0, Rx) = 0$.

Conversely, let $\sum_{Rx \leqslant I} \mu(0, Rx) = 0$ for all nonzero $I \leqslant {}_R R$. We assume $\mathrm{soc}({}_R R)$ is not left principal and consider a non-principal left ideal I contained in $\mathrm{soc}({}_R R)$ which we let be minimal with respect to this property. Denoting the Möbius function of the lattice of all left ideals of R by μ_L we obtain

$$0 = \sum_{J \leqslant I} \mu_L(0, J) = \sum_{J < I} \mu_L(0, J) + \mu_L(0, I)$$

$$= \sum_{Ry \leqslant I} \mu(0, Ry) + \mu_L(0, I) = \mu_L(0, I),$$

a contradiction to the fact that $\mu_L(0, I) \neq 0$ whenever the interval $I/0$ is atomistic. Hence $\mathrm{soc}({}_RR)$ is a left principal ideal of R. ∎

Combining the foregoing results we obtain the following conclusion:

COROLLARY 1.6. *For a finite ring R, a positive real number c, and a homogeneous weight $w: R \to \mathbb{R}$, $x \mapsto c\,(1 - \mu(0, Rx)/|R^\times x|)$ the following are equivalent*:

(a) $\mathrm{soc}({}_RR)$ *is left principal.*

(b) *For all nonzero $I \leqslant {}_RR$ there holds $\sum_{y \in I} w(y) = c\,|I|$.*

Proof. It is easy to check that

$$\sum_{x \in I} w(x) = c\,|I| - c \sum_{x \in I} \frac{\mu(0, Rx)}{|R^\times x|} = c\,|I| - c \sum_{Rx \leqslant I} \mu(0, Rx)$$

holds for all non-zero $I \leqslant {}_RR$, and therefore our claim is a consequence of the foregoing lemma. ∎

Remark 1.7. (a) After submission of this paper we became aware of the paper by W. Heise *et al.* [7] dealing with homogeneous weights on modules. That approach postulates the stronger condition (b) of Corollary 1.6 in the definition of the homogeneous weight, and hence yields the existence result 1.3 only for finite Frobenius rings.

(b) An Artinian ring R is a Frobenius ring if and only if ${}_R\mathrm{soc}({}_RR)$ and $\mathrm{soc}(R_R)_R$ are principal: the necessity of the cyclicity conditions directly results from the fact that an Artinian ring is Frobenius if and only if ${}_R(R/\mathrm{rad}(R)) \cong {}_R\mathrm{soc}({}_RR)$ and $(R/\mathrm{rad}(R))_R \cong \mathrm{soc}(R_R)_R$. Assuming ${}_R\mathrm{soc}({}_RR)$ to be principal we have $\mathrm{soc}({}_RR) \cong R/I$ for some left ideal $I \geqslant \mathrm{rad}(R)$. Now the decomposition of ${}_RR$ into a direct sum of indecomposable left ideals (cf. [15, Sect. 1]) shows that the lattice rank of $\mathrm{soc}({}_RR)$ is lower bounded by that of $R/\mathrm{rad}(R)$. Consequently $I = \mathrm{rad}(R)$ and by the same argument on the right side our claim follows.

Recently, T. Honold [8] informed us that in the finite case the left socle being left principal implies the right socle to be right principal, i.e. a finite ring R is Frobenius if and only if ${}_R\mathrm{soc}({}_RR)$ *or* $\mathrm{soc}(R_R)_R$ are principal.

2. THE EQUIVALENCE THEOREM FOR HOMOGENEOUS WEIGHTS

In the current section we give a characterization of linear isomorphisms which preserve homogeneous weights. On the finite ring R we fix the homogeneous weight

$$w_{\text{hom}} : R \to \mathbb{R}, \qquad x \mapsto 1 - \frac{\mu(0, Rx)}{|R^{\times}x|}.$$

As it is common in coding theory, we tacitly extend w_{hom} additively to a weight on R^n. Furthermore let π_i denote the projection of R^n onto its ith coordinate.

As a direct consequence of Corollary 1.6 we state:

LEMMA 2.1. *Let R be a finite Frobenius ring. Then for every R-linear left code C there holds*

$$\frac{1}{|C|} \sum_{c \in C} w_{\text{hom}}(c) = |\{i \mid \pi_i(C) \neq 0\}|.$$

Proof. Let C be a left R-linear code of length n. By an application of Corollary 1.6 we obtain $|C \cap \ker(\pi_i)| \sum_{r \in \pi_i(C)} w_{\text{hom}}(r) = |C|$ provided $\pi_i(C) \neq 0$, and it follows

$$\sum_{c \in C} w_{\text{hom}}(c) = \sum_{i=1}^{n} \sum_{c \in C} w_{\text{hom}}(\pi_i(c))$$
$$= \sum_{i=1}^{n} |C \cap \ker(\pi_i)| \sum_{r \in \pi_i(C)} w_{\text{hom}}(r) = |C| \cdot |\{i \mid \pi_i(C) \neq 0\}|. \quad \blacksquare$$

DEFINITION 2.2. Let C be a left R-linear code of length n. A linear mapping $C \xrightarrow{\varphi} R^n$ is called *homogeneous isometry* if $w_{\text{hom}}\varphi(c) = w_{\text{hom}}(c)$ for all $c \in C$.

The following generalization of the *Nullspaltenlemma* in [5] will be a basic ingredient of the proof of Theorem 2.5.

LEMMA 2.3. *Let R be a finite Frobenius ring and let C be a left R-linear code of length n. Then for every homogeneous isometry $C \xrightarrow{\varphi} R^n$ one has*

$$|\{i \mid \pi_i(C) = 0\}| = |\{i \mid \pi_i\varphi(C) = 0\}|.$$

Proof. By $|C| = |\varphi(C)| \cdot |\ker(\varphi)|$ we obtain $\frac{1}{|C|} \sum_{c \in C} w_{\text{hom}}(c) = \frac{1}{|\varphi(C)|} \sum_{d \in \varphi(C)} w_{\text{hom}}(d)$, which yields our claim via Lemma 2.1. $\blacksquare$

Remark 2.4. It is known that for a (quasi-) Frobenius ring R any pair ι, κ of embeddings of an ideal $I \leqslant {}_R M$ into R differs by an automorphism of ${}_R R$,

$$\begin{array}{ccc} {}_R I & \xrightarrow{\ \iota\ } & {}_R R \\ & \searrow^{\kappa} & \downarrow \exists \\ & & {}_R R \end{array}$$

This means that every embedding is restriction of a monomial transformation of ${}_R R$ and will be crucial for the inductive proof of the next theorem.

THEOREM 2.5. *Let R be a finite Frobenius ring, C a left R-linear code of length n and $C \xrightarrow{\varphi} R^n$ an embedding. Then the following are equivalent*:

(a) *φ is a homogeneous isometry.*

(b) *φ is restriction of a monomial transformation of R^n.*

Proof. We first observe, that according to Remark 1.4 monomial transformations preserve homogeneous weights. Let, conversely, $C \xrightarrow{\varphi} R^n$ be an injective homogeneous isometry. By Lemma 2.3 we may assume that C and $D := \varphi(C)$ do not possess zero coordinates. We now choose a coordinate $i \in \{1, ..., n\}$, for which $\pi_i(C)$ is of minimal cardinality and set $C_i := C \cap \ker(\pi_i)$. Again by Lemma 2.3 the code $\varphi(C_i) \subseteq D$ possesses at least one zero coordinate, say j, and we obviously have $\varphi(C_i) \subseteq D_j$. The latter containment is even an equality, because otherwise $\varphi^{-1}(D_j)$ would be a supercode of C_i with at least one zero coordinate by Lemma 2.3 which contradicts our minimality assumption on the cardinality of $\pi_i(C)$. By homomorpy we then have

$$\pi_i(C) \cong C/C_i \overset{\varphi}{\cong} D/D_j \cong \pi_j(D),$$

and hence, by Remark 2.4 we obtain a unit $u \in R$ with $\pi_j \varphi(c) = \pi_i(c)\, u$ for all $c \in C$. Projecting C and D onto the coordinates different from i and j, respectively, φ induces a homogeneous isometry between the resulting codes (cf. Remark 1.4). Since these are of smaller length, our claim follows by induction on n. ∎

3. AN INVERSION PRINCIPLE FOR FUNCTIONS ON MODULES

In this section we introduce an inversion principle for real-valued functions on unital modules. Later this will allow an application of Theorem 2.5 to derive the corresponding equivalence theorem for Hamming isometries, stated in Theorem 4.4.

Given a module ${}_RM$ over the finite ring R, we are looking for a function $K: M \times M \to \mathbb{R}$ such that for arbitrary elements f, g of an as large as possible class of real-valued functions on M the following statements are equivalent:

$$\text{(i)}\quad g(x) = \frac{1}{|R|} \sum_{r \in R} f(rx) \quad \text{for all } x \in M.$$

$$\text{(ii)}\quad f(x) = \frac{1}{|R|} \sum_{r \in R} g(rx)\, K(rx, x) \quad \text{for all } x \in M.$$

Let $F({}_RM, \mathbb{R})$ denote the vector space of all functions $f: M \to \mathbb{R}$ for which $Rx = Ry$ implies $f(x) = f(y)$ for all $x, y \in M$. We define the *kernel* as $K: M \times M \to \mathbb{R}$ via

$$K(x, y) := \frac{|Rx|}{|R^{\times}x|} \cdot \frac{|Ry|}{|R^{\times}y|} \cdot \mu(Rx, Ry),$$

where again μ denotes the Möbius function on the set $\{Rx \mid x \in M\}$.

THEOREM 3.1. *The endomorphisms Σ and Δ of $F({}_RM, \mathbb{R})$ with*

$$(\Sigma f)(x) := \frac{1}{|R|} \sum_{r \in R} f(rx) \qquad \textit{and} \qquad (\Delta f)(x) := \frac{1}{|R|} \sum_{r \in R} f(rx)\, K(rx, x)$$

are mutually inverse.

Proof. For all $x \in M$ we observe

$$(\Sigma f)(x) = \frac{1}{|R|} \sum_{t \in Rx} f(t) \cdot \operatorname{Ann}_R(x)$$

$$= \sum_{t \in Rx} f(t) \cdot \frac{1}{|Rx|} = \sum_{Rt \leqslant Rx} f(t) \cdot \frac{|R^{\times}t|}{|Rx|},$$

where $\operatorname{Ann}_R(x)$ denotes the annihilator of x in ${}_RR$. Therefore $\Sigma f = g$ is equivalent to

$$g(x)\, |Rx| = \sum_{Rt \leqslant Rx} f(t)\, |R^{\times}t|$$

for all $x \in M$, and by Möbius inversion finally to

$$f(x)\,|R^{\times}x| = \sum_{Rt \leqslant Rx} g(t)\,|Rt|\,\mu(Rt, Rx)$$

for all $x \in M$. As above this equation can be rewritten as

$$\begin{aligned} f(x) &= \sum_{Rt \leqslant Rx} g(t)\,\mu(Rt, Rx) \cdot \frac{|Rt|}{|R^{\times}x|} \\ &= \sum_{t \in Rx} g(t) \cdot \frac{\mu(Rt, Rx)}{|R^{\times}t| \cdot |R^{\times}x|} \cdot |Rt| \\ &= \frac{1}{|R|} \sum_{r \in R} g(rx)\,K(rx, x) = (\Delta g)(x), \end{aligned}$$

which proves our claim. ∎

4. EXTENSION OF HAMMING ISOMETRIES

Using the previously established inversion principle we are able to clarify the connection between the homogeneous weight and the Hamming weight, and finally the connection between homogeneous isometries and Hamming isometries.

To avoid confusion in the following statement we denote by $f^{(n)}$ the additive extension of $f \in F({}_RR, \mathbb{R})$ to R^n, i.e., we set $f^{(n)}(x) := f(x_1) + \cdots + f(x_n)$ for all $x \in R^n$.

Proposition 4.1. *For all $f \in F({}_RR, \mathbb{R})$ and all $n \in \mathbb{N}$ there holds:*

(a) $(\Sigma f)^{(n)} = \Sigma f^{(n)}$ *and* $(\Delta f)^{(n)} = \Delta f^{(n)}$.

(b) $\Sigma w_{\text{hom}}^{(n)} = w_H^{(n)}$ *and* $\Delta w_H^{(n)} = w_{\text{hom}}^{(n)}$.

Proof. From the definitions one easily derives $(\Sigma f)^{(n)} = \Sigma f^{(n)}$ and $\Sigma w_{\text{hom}} = w_H$, and hence $\Sigma w_{\text{hom}}^{(n)} = w_H^{(n)}$. Then Theorem 3.1 implies $(\Delta f)^{(n)} = \Delta f^{(n)}$ and $\Delta w_H^{(n)} = w_{\text{hom}}^{(n)}$. ∎

Definition 4.2. Let C be a left R-linear code of length n and let $f \in F({}_RR^n, \mathbb{R})$ be given. A linear mapping $C \xrightarrow{\varphi} R^n$ is called an *f-isometry* if $f\varphi(c) = f(c)$ for all $c \in C$.

Proposition 4.3. *Let $C \leqslant {}_RR^n$ and $f \in F({}_RR^n, \mathbb{R})$. A linear mapping $C \xrightarrow{\varphi} R^n$ is an f-isometry if and only if it is an (Σf)-isometry.*

Proof. We compute

$$(\Sigma f)\,\varphi(x) = \frac{1}{|R|}\sum_{r\in R} f(r\varphi(x)) = \frac{1}{|R|}\sum_{r\in R} f\varphi(rx) = \Sigma(f\varphi)(x)$$

for all $x \in C$, and hence obtain our claim by Theorem 3.1. ▮

The foregoing statement together with Proposition 4.1 shows in particular that a linear mapping is a homogeneous isometry if and only if it is a Hamming isometry. Since Hamming isometries are trivially injective, we obtain by combination of Theorem 2.5 with Propositions 4.1 and 4.3 a monomial extension of Hamming isometries.

THEOREM 4.4. *For a finite Frobenius ring R, a left R-linear code C of length n and an R-linear mapping $C \xrightarrow{\varphi} R^n$ the following are equivalent:*

(a) *φ is a Hamming isometry.*

(b) *φ is restriction of a monomial transformation of ${}_RR^n$.*

Clearly this result generalizes MacWilliams' equivalence theorem to Frobenius rings. While the original proof of the latter by J. Wood [15] employs character theory, we emphasize again that the paper at hand provides combinatorial methods.

5. A COUNTER-EXAMPLE

J. Wood [16, Remark 2] gives an example of a finite commutative local ring which violates the equivalence theorem. This ring is not a Frobenius ring nor is it a QF-ring as these classes coincide for local rings. Since QF-rings only slightly generalize Frobenius rings, one might conjecture that MacWilliams' equivalence theorem carries over to finite QF-rings. We finish our paper by disproving the latter.

EXAMPLE 5.1. Let F be a finite field and $F[\varepsilon]$ with $\varepsilon^2 = 0$ the ring of dual numbers over F. Within the matrix ring $M_3(F[\varepsilon])$ we consider the subring

$$S := \left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13}\varepsilon \\ a_{21} & a_{22} & a_{23}\varepsilon \\ a_{31}\varepsilon & a_{32}\varepsilon & a_{33} \end{bmatrix} \,\middle|\, a_{ij} \in F \text{ for all } i, j \in \{1, 2, 3\} \right\}.$$

(a) According to [15, Example 1.4] we know that S is a QF-ring but not a Frobenius ring. Indeed, for

$$\alpha := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \varepsilon & 0 & 0 \end{bmatrix} \quad \text{and} \quad \beta := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \varepsilon & 0 \end{bmatrix}$$

the left ideal $S\alpha + S\beta$ is a non-principal ideal contained in $\mathrm{soc}({}_S S)$.

(b) Defining $F := \mathbb{Z}_2$ and $\gamma := \alpha + \beta$ the set

$$C := \{(\alpha, \beta, \gamma, 0), (0, \beta, \alpha, \gamma), (\alpha, 0, \beta, \gamma), (0, 0, 0, 0)\}$$

is a left S-linear code. The S-linear mapping $S^4 \xrightarrow{\varphi} S^4$, $(x, y, z, t) \mapsto (x + t, y + t, z + t, 0)$, maps C onto the code

$$\varphi(C) = \{(\alpha, \beta, \gamma, 0), (\gamma, \alpha, \beta, 0), (\beta, \gamma, \alpha, 0), (0, 0, 0, 0)\}.$$

Clearly φ preserves the Hamming weight whereas the conclusion in Lemma 2.3 does not hold for φ. However, the latter is necessary for the extendability of the mapping in question, and hence there does not exist a monomial extension for this isometry.

(c) We finally note that $\varphi_{|C}$ even preserves the complete composition which assigns each code word its multi-set of entries.

REFERENCES

1. M. Aigner, "Combinatorial Theory," Springer-Verlag, Berlin/Heidelberg/New York, 1997.
2. K. P. Bogart, D. Goldberg, and J. Gordon, An elementary proof of the MacWilliams theorem on equivalence of codes, *Inform. and Control* **37** (1978), 19–22.
3. I. Constantinescu, "Lineare Codes über Restklassenringen ganzer Zahlen und ihre Automorphismen bezüglich einer verallgemeinerten Hamming-Metrik," Ph.D. thesis, Technische Universität, München, 1995.
4. I. Constantinescu and W. Heise, A metric for codes over residue class rings of integers, *Problemy Peredachi Informatsii* **33**, No. 3 (1997), 22–28.
5. I. Constantinescu, W. Heise, and T. Honold, Monomial extensions of isometries between codes over $\mathbb{Z}_m$, *in* "Proceedings of the 5th International Workshop on Algebraic and Combinatorial Coding Theory (ACCT '96), Unicorn Shumen, 1996," pp. 98–104.
6. A. R. Hammons, P. V. Kumar, A. R. Calderbank, N. J. A. Sloane, and P. Solé, The $\mathbb{Z}_4$-linearity of Kerdock, Preparata, Goethals and related codes, *IEEE Trans. Inform. Theory* **40** (1994), 301–319.
7. W. Heise, T. Honold, and A. A. Nechaev, Weighted modules and representations of codes, *in* "Proceedings of the ACCT 6, Pskov, Russia, 1998," pp. 123–129.
8. T. Honold, private communication, 1999.

9. F. J. MacWilliams, "Combinatorial Properties of Elementary Abelian Groups," Ph.D. thesis, Radcliffe College, Cambridge, MA, 1962.
10. A. A. Nechaev, Kerdock codes in a cyclic form, *Discrete Math. Appl.* **1** (1991), 365–384.
11. G.-C. Rota, On the foundations of combinatorial theory. I. Theory of Möbius functions, *Z. Wahrsch. Verw. Gebiete* **2** (1964), 340–368.
12. R. P. Stanley, "Enumerative Combinatorics," Vol. 1, (foreword by Gian-Carlo Rota, Corrected reprint of the 1986 original), Cambridge Univ. Press, Cambridge, UK, 1997.
13. H. N. Ward and J. A. Wood, Characters and the equivalence of codes, *J. Combin. Theory. Ser. A* **73** (1996), 348–352.
14. R. Wiegandt, On the general theory of Möbius inversion formula and Möbius product, *Acta Sci. Math.* (*Szeged*) **20** (1959), 164–180.
15. J. A. Wood, Duality for modules over finite rings and applications to coding theory, preprint, 1997.
16. J. A. Wood, Extension theorems for linear codes over finite rings, *in* "Applied Algebra, Algebraic Algorithms and Error-Correcting Codes" (T. Mora and H. Mattson, Eds.), pp. 329–340, Springer-Verlag, New York/Berlin, 1997.
17. J. A. Woods, Weight functions and the extension theorem for linear codes over finite rings, preprint, 1997.