

## Research Article

# Estimation for a Second-Order Jump Diffusion Model from Discrete Observations: Application to Stock Market Returns

Tianshun Yan <sup>1</sup>, Yanyong Zhao <sup>2</sup>, and Shuanghua Luo<sup>3</sup>

<sup>1</sup>School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an, Shaanxi, China

<sup>2</sup>School of Science, Nanjing Audit University, Nanjing, Jiangsu, China

<sup>3</sup>School of Science, Xi'an Polytechnic University, Xi'an, Shaanxi, China

Correspondence should be addressed to Tianshun Yan; [tianshun\\_yan@163.com](mailto:tianshun_yan@163.com)

Received 2 February 2018; Accepted 24 May 2018; Published 13 June 2018

Academic Editor: Xiaohua Ding

Copyright © 2018 Tianshun Yan et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

This paper proposes a second-order jump diffusion model to study the jump dynamics of stock market returns via adding a jump term to traditional diffusion model. We develop an appropriate maximum likelihood approach to estimate model parameters. A simulation study is conducted to evaluate the performance of the estimation method in finite samples. Furthermore, we consider a likelihood ratio test to identify the statistically significant presence of jump factor. The empirical analysis of stock market data from North America, Asia, and Europe is provided for illustration.

## 1. Introduction

Continuous-time stochastic processes have been widely used to model securities prices for option valuation. Nicolau considered a second-order diffusion process which is defined by

$$\begin{aligned} dY_t &= X_t dt, \\ dX_t &= \mu(X_t) dt + \sigma(X_t) dW_t, \end{aligned} \quad (1)$$

where  $\mu(X_t)$  and  $\sigma(X_t)$  are the drift and diffusion functions, respectively [1].  $W_t$  is a standard one-dimensional Brownian motion.  $Y_t$  is directly observable and differentiable. In this model,  $Y_t$  can also be expressed as the integrated process

$$Y_t = Y_0 + \int_0^t X_u du. \quad (2)$$

For model (1), a nonparametric approach which is based on the infinitesimal generator and Taylor series expansion has been developed to estimate the drift and diffusion functions [1]. Thereafter, Wang and Lin [2] presented the local linear estimation of the diffusion and drift functions and proved that the estimators are weakly consistent. Wang et al. [3]

proposed the reweighted estimation of the diffusion function and investigated the consistency of the estimator. Furthermore, Hanif [4] studied the nonparametric estimation of the drift and diffusion functions using an asymmetric kernel and proved that the estimators are consistent and asymptotically normal.

As pointed out by Nicolau [1, 5], model (1) is especially useful in empirical finance. First, the model accommodates nonstationary integrated stochastic process  $Y_t$  that can be made stationary by differencing. Second, in the context of stock prices,  $X_u$  represents stock return and  $Y_t$  indicates the cumulation of  $X_u$ . The model suggests directly modeling return rather than stock price and meets many general properties of stock returns such as stationarity in the mean, nonnormality of the distribution and weak autocorrelation.

In financial markets, the correct specification of drift and volatility is essential and instructive among practitioners in obtaining valid conclusions. Unfortunately, the existing economic theory generally provides little guidance about the precise specification of them. Model misspecification may lead to misleading conclusions in estimation and hypothesis testing. Therefore, much attention has been paid to the issue of specifying the functions forms for continuous-time diffusion models. On the other hand, it has the advantage that

the problem of its estimation can be reduced to the determination of some low-dimensional parameters by applying more efficient statistical methods (see [6, 7] for details). In particular, Nicolau [1, 5] pointed out that

$$\begin{aligned} dY_t &= X_t dt, \\ dX_t &= (\alpha_0 + \alpha_1 X_t) dt + \sqrt{\beta + \tau(X_t - \theta)^2} dW_t \end{aligned} \quad (3)$$

is a promising model for stock returns and possesses some interesting properties. In Nicolau's empirical studies on the American stock markets, a regular pattern in all estimators of drift and diffusion can be observed: the drift is clearly linear, the volatility is a quadratic function with a minimum in the neighborhood of zero, and the specification  $\sigma^2(X_t) = \beta + \tau(X_t - \theta)^2$  fits the nonparametric estimators very well. Furthermore, Yan and Mei [8] developed the generalized likelihood ratio test to check the empirical finding of Nicolau. The empirical analysis of real-world data sets demonstrates that it is in general reasonable to suppose that the volatility is a quadratic function in stock markets.

The empirical distribution of stock returns typically exhibits skewness and excess kurtosis, which could be induced by various macroeconomic shocks, such as the unemployment announcement, the Gulf war, and the oil crisis. Unfortunately, the standard second-order diffusion framework (3) is not tailored to capture these stylized facts. A more appropriate specification is to modify the aforementioned standard second-order diffusion model to allow for discontinuity. This is easily obtained by combining the general diffusion model with a jump factor. The popularity of the jump diffusion models stems from at least two facts. First, as distinguished from pure diffusion processes, the jump processes can affect and match high levels of kurtosis and skewness. Second, they are economically attractive because they admit that stock prices change by sudden jumps in a short time, which is a reasonable assumption for an efficient stock market. Recently, jump noises have been also applied in population models to describe the abrupt changes of population sizes (see [9, 10], etc.). For example, Zhou et al. [9] introduced a two-population mortality model with transitory jump effects and applied it to pricing catastrophic mortality securitizations.

In view of the abovementioned facts, we extend the work of Nicolau [1, 5] and then concentrate on a new second-order jump diffusion model:

$$\begin{aligned} dY_t &= X_t dt, \\ dX_t &= (\alpha_0 + \alpha_1 X_t) dt + \sqrt{\beta + \tau(X_t - \theta)^2} dW_t \\ &\quad + J_t d\pi_t(\lambda), \end{aligned} \quad (4)$$

where the arrival of jumps  $J_t$  is governed by the continuous-time Poisson process  $\pi_t$  with frequency parameter  $\lambda$ , which denotes the average number of jumps per year. The jump size may be a constant or may be drawn from a probability distribution. The diffusion and Poisson process are independent of each other, and each of them is independent

of jump  $J_t$  as well. Consequently, the stock return is the sum of three components. The component  $(\alpha_0 + \alpha_1 X_t) dt$  represents the instantaneous expected return on the stock. The  $\sqrt{\beta + \tau(X_t - \theta)^2} dW_t$  part describes the instantaneous variance of the stock return due to the arrival of "normal" information, and the  $J_t d\pi_t(\lambda)$  part describes the total instantaneous stock return owing to the arrival of "abnormal" information. Next, we develop parameter estimation methodology to estimate the coefficients in the drift, diffusion, and jump factor.

## 2. Model Estimation

For model (4), the integrated process  $Y_t$  is generally observable at the time points  $\{t_i, i = 0, 1, \dots, n+1\}$ , while  $X_t$  is a nonobservable process. In fact, for the fixed sampling interval  $\Delta = t_i - t_{i-1}$ , the exact distribution of  $\{Y_{t_i}\}_{i=0}^{n+1}$  is generally not explicit. An exception is the case where  $X_t$  follows an Ornstein-Uhlenbeck process [11]. In practical applications, the time points are equally spaced. For example, when the time unit is a year, weekly data are sampled at  $t_i = t_0 + (i/52)$  ( $i = 0, 1, \dots, n+1$ ) for a given initial time point  $t_0$ . Based on  $Y_t = Y_0 + \int_0^t X_s ds$  and the discrete-time observations  $\{Y_{t_i}\}_{i=0}^{n+1}$ , we have

$$\begin{aligned} Y_{t_i} - Y_{t_{i-1}} &= \int_0^{t_i} X_s ds - \int_0^{t_{i-1}} X_s ds = \int_{t_{i-1}}^{t_i} X_s ds, \\ & \quad i = 1, 2, \dots, n. \end{aligned} \quad (5)$$

Then  $X_{t_i}$  can be approximated by

$$\bar{X}_{t_i} = \frac{Y_{t_i} - Y_{t_{i-1}}}{\Delta}, \quad i = 1, 2, \dots, n. \quad (6)$$

Naturally, the smaller the time span  $\Delta$  is, the closer  $\bar{X}_{t_i}$  is to  $X_{t_i}$ . In fact, stock prices are usually observed daily or higher frequency.

Let  $\Delta U_{t_i} = \bar{X}_{t_{i+1}} - \bar{X}_{t_i}$  and  $V_{t_i} = W_{t_{i+1}} - W_{t_i}$ . According to the independent increment property of the standard Brownian motion,  $V_{t_i}$  ( $i = 1, 2, \dots, n$ ) are independently and normally distributed with mean zero and variance  $\Delta$ . Therefore, we can write  $V_{t_i}$  as  $V_{t_i} = \sqrt{\Delta} \varepsilon_{t_i}$  ( $i = 1, 2, \dots, n$ ), where  $\{\varepsilon_{t_i}\}_{i=1}^n$  are independently and identically distributed as the standard normal distribution. Then the Euler discretization of model (4) can be expressed as

$$\begin{aligned} \Delta U_{t_i} &= \bar{X}_{t_{i+1}} - \bar{X}_{t_i} \\ &\approx (\alpha_0 + \alpha_1 \bar{X}_{t_i}) \Delta + \sqrt{\beta + \tau(\bar{X}_{t_i} - \theta)^2} \sqrt{\Delta} \varepsilon_{t_i} \\ &\quad + J_{t_i} (\mu_J, \sigma_J) \Delta \pi_{t_i}(\lambda), \quad i = 1, 2, \dots, n, \end{aligned} \quad (7)$$

where  $J_{t_i}$  is normally distributed with mean  $\mu_J$  and variance  $\sigma_J^2$ .  $\Delta \pi_{t_i}(\lambda)$  is the discrete-time Poisson increment. We set the mean jump size equal to zero,  $\mu_J = 0$ , guaranteeing a symmetric stock return distribution according to [12, 13].

It is well recognized that discretization of continuous-time diffusion models for estimation does introduce an estimation bias, but this is relatively small (see [14]). In this case the discrete-time approach (7) allows us to estimate the proposed model where the jump is normally distributed. In this model, the density functions, a mixture of Poisson–Gaussian distribution is generally used to define the jump diffusion model. Here, we employ a Bernoulli approximation, first introduced in [8], to approximate Poisson–Gaussian distribution underlying the discretized (7). We assume that in each time interval either only one jump occurs or no jump occurs. No other information arrivals over this period of time are allowed. This is tenable for short frequency data, e.g., daily stock returns, and may be debatable for data at higher frequencies. As Ball and Torous demonstrate, it provides an approximation procedure which is highly tractable, stable, and convergent [8]. Hence, the discretized (7) can be rewritten as

$$\begin{aligned} \bar{X}_{t_{i+1}} - \bar{X}_{t_i} &\approx (\alpha_0 + \alpha_1 \bar{X}_{t_i}) \Delta \\ &+ \sqrt{\beta + \tau (\bar{X}_{t_i} - \theta)^2} \sqrt{\Delta_i} \varepsilon_{t_i} + J_{t_i} I_{t_i}, \end{aligned} \quad (8)$$

$i = 1, 2, \dots, n.$

where  $I_{t_i} = 1$ , if there is a jump with probability  $q = \lambda \Delta + O(\Delta)$  and 0 otherwise with probability  $1 - q$ . All other notations have been defined previously. Since the limit of the Bernoulli process is governed by a Poisson distribution (see the Appendix for some technique details), we can approximate the likelihood function for the Poisson–Gaussian model using a Bernoulli mixture of the normal distributions. Let  $\Phi = (\alpha_0, \alpha_1, \beta, \tau, \theta, \sigma_j, q)$ . Then, the transition probabilities for the stock returns following a Poisson–Gaussian process are written as  $f(\bar{X}_{t+\Delta} | \bar{X}_t; \Phi)$ :

$$\begin{aligned} &f(\bar{X}_{t+\Delta} | \bar{X}_t; \Phi) \\ &= q \exp \left( \frac{-(\bar{X}_{t+\Delta} - \bar{X}_t - (\alpha_0 + \alpha_1 \bar{X}_t) \Delta)^2}{2 [(\beta + \tau (\bar{X}_t - \theta)^2) \Delta + \sigma_j^2]} \right) \\ &\times \frac{1}{\sqrt{2\pi [(\beta + \tau (\bar{X}_t - \theta)^2) \Delta + \sigma_j^2]}} + (1 \\ &- q) \exp \left( \frac{-(\bar{X}_{t+\Delta} - \bar{X}_t - (\alpha_0 + \alpha_1 \bar{X}_t) \Delta)^2}{2 (\beta + \tau (\bar{X}_t - \theta)^2) \Delta} \right) \\ &\times \frac{1}{\sqrt{2\pi (\beta + \tau (\bar{X}_t - \theta)^2) \Delta}}, \end{aligned} \quad (9)$$

which approximates the true Poisson–Gaussian density with a mixture of normal distributions. Then  $\Phi$  can be estimated via maximizing the profile pseudo likelihood function

$$\ln L(\Phi) = \sum_{i=1}^n \ln f(\bar{X}_{t_{i+1}} | \bar{X}_{t_i}; \Phi), \quad (10)$$

which yields estimators of  $\Phi$ . The maximization problem can be carried out in many scientific computing packages. Note that the likelihood function (10) is complex nonlinear function based on transition probability density; hence the analytical formula for the estimators could not derived and the large sample property of estimators becomes challenging. We will not discuss this aspect any further. For more details, we refer the reader to Xu and Wang [15] or Zheng and Lin [16]. The numerical technique, sequential quadratic programming algorithm (see [17, 18]) is considered to solve this maximization problem. In our implementation, the likelihood function (10) is numerically maximized with the “fmincon” routine embedded in the “Optimization Toolbox” of MATLAB using the sequential quadratic programming algorithm. Furthermore, in [19], maximum likelihood estimation allows the construction of approximate confidence intervals for the parameters of interest and these confidence intervals are asymptotically optimal.

*Remark 1.* The sequential quadratic programming algorithm has been proved to be an excellent nonlinear programming method for solving constrained optimization problems in a variety of statistical models, such as linear models with longitudinal data under inequality restrictions [15], semi-parametric regression models with censored data [16], autologistic models and exponential family models for dependent data [20], Cox–Ingersoll–Ross model for the interest rate [21], integer-valued GARCH models [22], and Gaussian stochastic process (GASP) models [23]. Our experience shows that the algorithm performs very well for moderate sample sizes. To gain more confidence in the estimates, we also try “Global Optimization Toolbox” of MATLAB based on the “fmincon” routine. We find that choosing different initial values yields identical results.

Furthermore, we also provide a formal statistical test for the presence of jumps in the stock and stock index returns. Since a pure diffusion model is nested within a combined diffusion and jump model, a likelihood ratio test can be used to test the null hypothesis  $H_0$ : stock and stock index returns are normally distributed. The corresponding likelihood ratio test statistic is

$$\Omega = 2 (\ln L(\bar{X}; \Phi^*) - \ln L(\bar{X}; \Phi^0)), \quad (11)$$

where  $\Phi^*$  denotes the maximum likelihood estimates under jump diffusion model and  $\Phi^0$  is the maximum likelihood estimates corresponding to the situation when no jump structure occurs (i.e.,  $\lambda=0$ ). Under the null hypothesis, the stock returns are consistent with a log-normal diffusion process without jump factor and  $\Omega$  is asymptotically distributed  $\chi^2$  with  $d-5$  degrees of freedom, where  $d$  denotes the number of parameters to be estimated. More details about asymptotically distribution of the likelihood ratio test statistic can be found in ([24, 25]).

### 3. Simulation Study

In this section, we conduct a simulation study for second-order jump diffusion model (4) aimed at examining the

TABLE 1: Parameters estimate, standard error, and root of mean square error for model (12), where  $\alpha_0=0.2$ ,  $\alpha_1=-20$ ,  $\beta=0.1$ ,  $\tau=10$ ,  $\theta=0.05$ ,  $\sigma_j=0.1$ , and  $q=0.5$ .

		$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\beta}$	$\hat{\tau}$	$\hat{\theta}$	$\hat{\sigma}_j$	$\hat{q}$
$n=1260$	MEAN	0.2746	-20.7403	0.0960	10.2741	0.0538	0.1086	0.4663
	SD	0.2708	2.6095	0.0264	1.4731	0.0167	0.0069	0.0473
	RMSE	0.2856	2.5985	0.0261	1.4763	0.0166	0.0067	0.0471
$n=2520$	MEAN	0.2372	-20.3026	0.1014	9.8620	0.0496	0.1024	0.4986
	SD	0.2250	2.0568	0.0132	1.2086	0.0082	0.0028	0.0222
	RMSE	0.2269	2.1638	0.0132	1.2105	0.0082	0.0028	0.0221
$n=3780$	MEAN	0.2151	-20.1906	0.0994	10.0149	0.0498	0.1006	0.5040
	SD	0.2017	1.4665	0.0114	0.8613	0.0069	0.0023	0.0204
	RMSE	0.2041	1.4679	0.0114	0.8570	0.0068	0.0023	0.0203
$n=5040$	MEAN	0.2112	-20.1625	0.0992	9.9446	0.0504	0.1001	0.5026
	SD	0.1502	1.4916	0.0093	0.7038	0.0062	0.0023	0.0181
	RMSE	0.1498	1.5089	0.0092	0.7098	0.0061	0.0023	0.0184

performance of the estimation methodology. We consider a special case of (4):

$$dY_t = X_t dt,$$

$$dX_t = (0.2 - 20X_t) dt + \sqrt{0.1 + 10(X_t - 0.05)^2} dW_t \quad (12)$$

$$+ J_t d\pi_t(\lambda),$$

where  $J_t \sim N(0, 0.1^2)$ , probability of jump  $q=0.5$ , and initial values  $X_0=0.01$  and  $Y_0=0$ . The values of the parameters except jump term are given in [1]. We generate 100 replications (paths) of  $Y_t$  and  $X_t$  according to the above design with each replication consisting of  $n$  ( $n=1260, 2520, 3780, 5040$ ) daily observations and a time step between two consecutive observation equal to  $\Delta=1/252$ . Then the coefficients  $\alpha_0$  and  $\alpha_1$  in the drift functions,  $\beta$ ,  $\tau$ , and  $\theta$ , in the diffusion function, and  $\sigma_j$  and  $q$ , in the jump factor, can be estimated by the proposed estimation methodology. The empirical sample mean (MEAN), empirical standard deviation (SD), and root mean square error (RMSE) of the coefficient estimators  $\hat{\alpha}_0$ ,  $\hat{\alpha}_1$ ,  $\hat{\beta}$ ,  $\hat{\tau}$ ,  $\hat{\theta}$ ,  $\hat{\sigma}_j$ , and  $\hat{q}$  are reported (Table 1). We can see the mean of the estimators becomes more accurate; meanwhile the empirical standard deviation and root mean square error become smaller as the observations  $n$  increases from 1260 to 5040.

## 4. Empirical Analysis

**4.1. Data Source.** In this section, we analyze stock market data based on jump diffusion model (4). The data used in this study consists of daily closing stock prices of US companies (Alcoa, American Express, Boeing, Coca Cola, Disney, General Electric, Microsoft, Johnson & Johnson and JP Morgan) between January 25, 2000 and January 25, 2005 and daily stock market indexes between January 1998 and September 2008 from North America, Asia, and Europe: USA (S&P500, NASDAQ, DJIA), UK (FTSE100), Germany (DAX), France (CAC40), Hong Kong (HANG

TABLE 2: Summary statistics.

Statistic	Panel A : $Y_t$	Panel B : $\Delta Y_t$
Mean	3.7957	0.0011
Standard deviation	0.4370	0.0314
Skewness	-0.1045	7.4655
Excess kurtosis	-1.4564	155.9618
Minimum	3.1268	-0.1788
Maximum	4.7173	0.6602
Jarque-Bera	113.2***	1283600***

\*\*\* Indicates significance at 1% level.

Note: The statistics reported are for log-price ( $Y_t$ ) and change ( $\Delta Y_t$ ). The Jarque-Bera statistic tests for normality distribution. If the data are normally distributed, the skewness is 0, whereas the kurtosis is equal to 3.

SENG), Australia (SPASX200), Switzerland (SMI), and Taiwan (TWII). The datasets are available at the Yahoo Finance website (<https://finance.yahoo.com/>). Following the conventional practice in stock market research, we take the logarithmic transformation of the selected data.

**4.2. Descriptive Statistics.** Microsoft Corporation's daily stock prices from January 25, 2000 to January 25, 2005 are plotted in Figure 1. The most striking feature in the figure is the existence of conspicuous spikes in the stock prices and returns. The descriptive statistics are reported (Table 2). The change in stock prices is clear, positively skewed, and has excess kurtosis indicating that leptokurtosis is undeniable, which also predicates the use of a jump diffusion model. The Jarque-Bera test of both dataset rejects the hypothesis of normality. Other stock prices and indexes have similar statistical characterization. We do not enumerate them here.

**4.3. Parameter Estimation and Testing Results.** We shall summarize and compare the parameter estimates between second-order diffusion model (3) and second-order jump diffusion model (4) for the stocks and stock indexes (Tables 3 and 4). In addition to the 7 parameters to be estimated ( $\alpha_0$  and  $\alpha_1$  in instantaneous mean and  $\beta$ ,  $\tau$ , and  $\theta$  in the

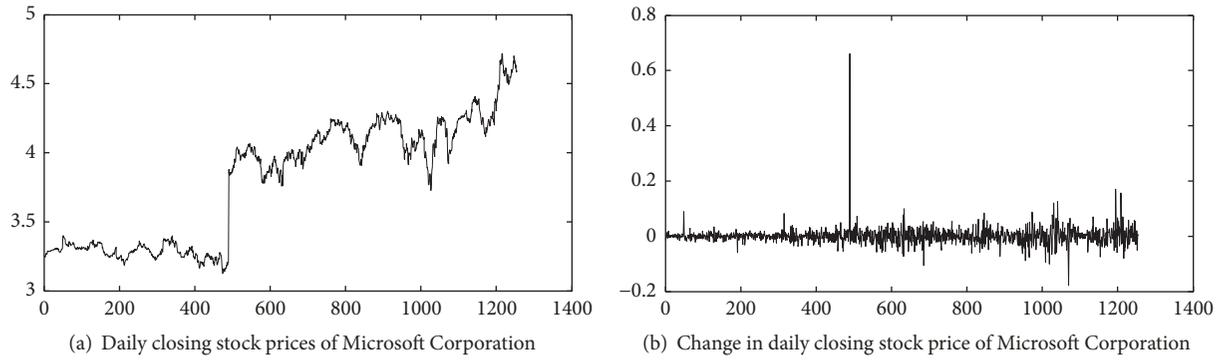


FIGURE 1: Daily closing stock price of Microsoft Corporation from January 25, 2000 to January 25, 2005.

TABLE 3: Parameter estimates for stocks.

Stock	N	$\alpha_0$	$\alpha_1$	$\beta$	$\tau$	$\theta$	$\sigma_J$	$q$	ln L	$\Omega$
Alcoa	1257	0.1713	-257.9342	0.6380	23.9529	-0.0156	0.0972	0.1135	1718.1	56.18***
		(0.3494)	(7.7857)	(0.0588)	(3.9444)	(0.0167)	(0.0164)	(0.0540)		
		0.0504	-258.9853	0.9154	23.2125	-0.0056			1690.0	
American Express	1251	-0.2182	-256.5039	0.1810	75.6013	-0.0038	0.0646	0.4051	1873.8	87.21***
		(0.1785)	(8.6131)	(0.0514)	(16.0989)	(0.0054)	(0.0051)	(0.1017)		
		-0.3313	-258.0656	0.5884	83.6826	0.00458			1830.2	
Boeing	1257	-0.2355	-265.9633	0.4276	52.2001	-0.0069	0.0885	0.1011	1899.7	48.24***
		(0.4224)	(7.7640)	(0.0655)	(5.9056)	(0.0067)	(0.0257)	(0.0760)		
		0.0139	-263.8557	0.5745	72.6873	-0.0086			1875.6	
Coca Cola	1257	0.0434	-249.5004	0.1210	48.5000	-0.0062	0.0580	0.3086	2263.1	135.99***
		(0.2459)	(7.9359)	(0.0210)	(6.1083)	(0.0050)	(0.0048)	(0.0611)		
		-0.0280	-248.9476	0.3378	79.3847	-0.0103			2195.1	
Disney	1257	-0.2449	-260.6646	0.5405	57.6146	0.0107	0.1658	0.0312	1812.0	140.31***
		(0.9548)	(7.7072)	(0.0377)	(10.0298)	(0.0064)	(0.0326)	(0.0141)		
		-0.2363	-254.5632	0.7399	50.7316	0.0287			1741.8	
General Electric	1257	0.0495	-257.2053	0.2935	47.9844	0.0058	0.0779	0.1955	1969.8	110.54***
		(0.1662)	(8.1046)	(0.0431)	(7.8468)	(0.0070)	(0.0092)	(0.0664)		
		-0.2494	-260.3366	0.5816	52.8035	0.0048			1914.5	
Johnson & Johnson	1257	-0.3607	-257.3816	0.2195	69.4946	0.0032	0.1207	0.0244	2368.4	106.95***
		(0.0753)	(7.2543)	(0.0176)	(4.0190)	(0.0041)	(0.0356)	(0.0161)		
		-0.4890	-255.5323	0.2457	112.3180	0.0110			2314.9	
JP Morgan	1256	0.3209	-257.6896	0.4855	42.1887	0.0162	0.1640	0.0892	1722.0	335.67***
		(0.2897)	(6.8796)	(0.0443)	(7.8642)	(0.0088)	(0.0180)	(0.0252)		
		0.1524	-260.0101	1.0547	42.8909	-0.0307			1554.2	
Microsoft	1257	0.4073	-267.0635	0.5217	52.5552	0.0041	0.3059	0.0370	1761.8	570.82***
		(0.3472)	(8.1481)	(0.0452)	(11.3793)	(0.0069)	(0.0491)	(0.0119)		
		0.6737	-267.9022	0.6690	264.2013	-0.0316			1476.4	
		(0.1681)	(11.8308)	(0.0409)	(31.7660)	(0.0036)				

\*\*\* Indicates significance at 1% level.

<sup>a</sup> Standard errors of maximum likelihood estimates are in parentheses.

TABLE 4: Parameter estimates for stock indexes.

Stock index	N	$\alpha_0$	$\alpha_1$	$\beta$	$\tau$	$\theta$	$\sigma_j$	$q$	ln L	$\Omega$
CAC40	2798	0.2383	-257.6798	0.1330	54.8009	0.0116	0.0627	0.1645	5376.7	344.59***
		(0.1155)	(4.9044)	(0.0103)	(5.0972)	(0.0029)	(0.0043)	(0.0285)		
		-0.0615	-257.9421	0.2687	82.9046	0.0093			5204.4	
DAX	2785	0.2784	-255.8185	0.1555	57.6903	0.0127	0.0645	0.1652	5164.5	280.85***
		(0.2202)	(24.7216)	(0.0167)	(17.3980)	(0.0031)	(0.0039)	(0.0300)		
		-0.0118	-252.6514	0.2919	86.8931	0.0124			5024.1	
DJIA	2703	0.1067	-210.7317	0.0590	47.0208	0.0082	0.0430	0.1481	6373.6	317.71***
		(0.0605)	(5.4284)	(0.0048)	(8.5147)	(0.0021)	(0.0035)	(0.0263)		
		0.0699	-211.6276	0.1050	66.2194	0.0168			6214.8	
FTSE100	2713	0.1136	(5.6852)	(0.0041)	(8.1033)	(0.0022)				
		-0.1870	-260.5673	0.0649	61.1198	-0.0018	0.0379	0.3053	5841.7	222.60***
		(0.3935)	(9.5712)	(0.0387)	(31.0472)	(0.0030)	(0.0084)	(0.2009)		
HANGSENG	2702	-0.1749	-264.5221	0.1641	79.7363	-0.0028			5730.4	
		(0.4693)	(18.1376)	(0.0456)	(65.8072)	(0.0018)				
		0.1166	-245.2607	0.1369	41.1688	0.0186	0.0655	0.2412	4863.4	366.22***
NASDAQ	2767	(1.5002)	(7.6386)	(0.1075)	(8.8111)	(0.0167)	(0.0125)	(0.1701)		
		-0.0341	-238.4197	0.3550	81.9699	0.0035			4683.4	
		(0.1558)	(6.8947)	(0.0132)	(12.5799)	(0.0028)				
SP500	2804	0.4242	-253.6221	0.1810	66.7837	0.0047	0.0658	0.2204	4740.6	238.63***
		(0.1197)	(6.3600)	(0.0143)	(9.0416)	(0.0033)	(0.0043)	(0.0312)		
		-0.3926	-255.9448	0.3824	91.8324	0.0070			4621.3	
SPASX200	2870	(0.0814)	(7.9904)	(0.0177)	(19.2416)	(0.0028)				
		0.1679	-261.8900	0.0441	24.8916	0.0235	0.0360	0.4098	6122.3	278.23***
		(0.1935)	(5.3790)	(0.0108)	(4.8198)	(0.0050)	(0.0022)	(0.0706)		
SMI	2704	-0.0194	-261.4696	0.1745	43.5322	0.0093			5983.2	
		(0.1217)	(5.5881)	(0.0072)	(4.1993)	(0.0031)				
		0.2102	-257.0073	0.0570	71.0473	0.0061	0.0514	0.0785	6951.4	389.95***
TWII	2639	(0.1332)	(6.5243)	(0.0035)	(6.9430)	(0.0014)	(0.0052)	(0.0180)		
		0.0966	-253.5373	0.0908	112.8947	0.0093			6756.5	
		(0.0567)	(6.6217)	(0.0034)	(8.9553)	(0.0011)				
TWII	2639	0.1878	-255.8562	0.0000	0.1737	0.8342	0.0526	0.1877	5684.6	39.95***
		(0.1344)	(4.5107)	(0.0386)	(0.0315)	(0.1503)	(0.0037)	(0.0266)		
		0.0394	-247.5534	0.1492	113.4965	0.0079			5664.6	
TWII	2639	(0.1479)	(6.3341)	(0.0062)	(12.8102)	(0.0014)				
		-0.0621	-240.4190	0.1094	21.5185	0.0008	0.0508	0.4276	4836.2	238.07***
		(0.1073)	(5.0817)	(0.0153)	(3.9170)	(0.0065)	(0.0023)	(0.0457)		
TWII	2639	-0.0273	-245.3496	0.3676	31.7291	0.0037			4717.1	
		(0.4595)	(5.6751)	(0.0111)	(2.9732)	(0.0056)				

\*\*\* Indicates significance at 1% level.

<sup>a</sup> Standard errors of maximum likelihood estimates are in parentheses.

volatility component, the variance  $\sigma_j^2$  and probability  $q$  of jump factor), the standard error, the log-likelihood value (ln L), and the likelihood ratio test statistic  $\Omega$  are also reported. For each sampled stock and stock index, the first row and the second row present maximum likelihood estimates and corresponding standard errors of model (4), respectively. The third row and the fourth row present maximum likelihood estimates and corresponding standard errors of model (3).

Note that the estimates of standard errors of  $\Phi^*$  and  $\Phi^0$  for the maximum likelihood estimation are obtained from the main diagonal of the inverse of the Hessian matrix, respectively.

It can be seen that the standard errors of parameter estimates for jump diffusion model (4) are extremely small, confirming the preferability of the estimation methodology, and are mostly smaller than that of pure diffusion model (3) (Tables 3 and 4). For all the stocks and stock indexes,

the corresponding log-likelihood values,  $\ln L$ , of model (4) are also higher than that of model (3). Furthermore, the likelihood ratio test demonstrates the presence of jumps at the 1 percent significance level for each stock and stock index. These empirical results show that the second-order jump diffusion model (4) is more suitable for modeling the stock market returns from North America, Asia, and Europe than second-order pure diffusion model (3).

### 5. Conclusion and Discussion

In this paper, a new second-order jump diffusion model is proposed to investigate the dynamic characteristic of stock market returns from North America, Asia, and Europe. The maximum likelihood approach is employed to estimate parameters of the model. The simulation study demonstrates that the estimation approach works satisfactorily. By analyzing stock market data from North America, Asia, and Europe, we find that the second-order jump diffusion model can capture high levels of kurtosis and skewness of stock return distributions, while the second-order diffusion model that ignores the jump factor is misspecified. The likelihood ratio test also confirms the statistically significant presence of jump factor in the selected stock market returns.

It should be pointed out that there are some topics that can be further considered. This paper mainly focuses on the Poisson jump. In fact, there are some random perturbations, such as the telephone noise which can be further studied [26]. Besides, stochastic differential equations have been recently applied into the SIS and SIRS epidemic models with Markov-switching [27–29], the budworm growth model with regime switching [30], and stochastic regime switching predator-prey model [31, 32]. It is very promising to incorporate the jump factors into these stochastic models.

### Appendix

#### Technical Details

The Bernoulli approximation here can be achieved as follows. Let the random variable  $N$  denote the number of events that occur in a time interval of length  $t$ . Define  $\Delta = t/n$  for an arbitrary integer  $n$  and subdivide the interval  $(0, t)$  into  $n$  equal subintervals each of length  $\Delta$ . Let  $Y_i$  denote the number of events that occur in subinterval  $i$  and by the stationary independent increment assumption,

$$N = \sum_{i=1}^n Y_i \tag{A.1}$$

is the sum of  $n$  independent identically distributed random variables such that

$$\begin{aligned} \text{Prob}[Y_i = 0] &= 1 - \lambda\Delta + O(\Delta), \\ \text{Prob}[Y_i = 1] &= \lambda\Delta + O(\Delta), \quad \text{for } i = 1, 2, \dots, n \\ \text{Prob}[Y_i > 1] &= O(\Delta). \end{aligned} \tag{A.2}$$

For large  $n$ , each  $Y_i$  has approximately the Bernoulli distribution with parameter  $\lambda\Delta$ . Consequently, being the sum

of  $n$  independent identically distributed Bernoulli random variables,  $N$  has the binomial distribution. For  $k$  occurrences,

$$\begin{aligned} \text{Prob}[N = k] &= {}^n C_k \left(\frac{\lambda t}{n}\right)^k \left(1 - \frac{\lambda t}{n}\right)^{n-k}, \\ & \qquad \qquad \qquad k = 0, 1, 2, \dots, n \\ \text{then, } \lim_{n \rightarrow \infty} \text{Prob}[N = k] &= \frac{e^{-\lambda t} (\lambda t)^k}{k!}. \\ & \qquad \qquad \qquad k = 0, 1, 2, \dots, n \end{aligned} \tag{A.3}$$

This is a standard construction of the Poisson process. When  $\Delta$  is very small, it follows then that we can satisfactorily approximate  $N$  by the Bernoulli variate  $Y$  defined by

$$\begin{aligned} P[Y = 0] &= 1 - \lambda\Delta, \\ P[Y = 1] &= \lambda\Delta. \end{aligned} \tag{A.4}$$

The distinguishing characteristic of the Bernoulli approximation is that over a short period of time,  $\Delta$ , either no piece of information impacts the interest rate or one relevant piece of information occurs with probability  $\lambda\Delta$ . No other information arrivals over the period of time are allowed.

### Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

### Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

### Acknowledgments

The research work is supported by the National Natural Science Foundation of China under Grants nos. 11601409 and 11701286, the Natural Science Foundation of Jiangsu Province of China under Grant no. BK20171073, the University Natural Science Foundation of Education Department of Jiangsu Province of China under Grant no. 17KJB110006, and the University Philosophy and Social Science Foundation of Education Department of Jiangsu Province of China under Grants no. 2017SJB0350.

### References

- [1] J. Nicolau, “Nonparametric estimation of second-order stochastic differential equations,” *Econometric Theory*, vol. 23, no. 5, pp. 880–898, 2007.
- [2] H. Wang and Z. Lin, “Local linear estimation of second-order diffusion models,” *Communications in Statistics—Theory and Methods*, vol. 40, no. 3, pp. 394–407, 2011.
- [3] Y. Wang, L. Zhang, and M. Tang, “Re-weighted functional estimation of second-order diffusion processes,” *Metrika. International Journal for Theoretical and Applied Statistics*, vol. 75, no. 8, pp. 1129–1151, 2012.

- [4] M. Hanif, "Non parametric estimation of second-order diffusion equation by using asymmetric kernels," *Communications in Statistics—Theory and Methods*, vol. 44, no. 9, pp. 1896–1910, 2015.
- [5] J. Nicolau, "Modeling financial time series through second-order stochastic differential equations," *Statistics & Probability Letters*, vol. 78, no. 16, pp. 2700–2704, 2008.
- [6] H. Dette and M. Podolskij, "Testing the parametric form of the volatility in continuous time diffusion models—a stochastic process approach," *Journal of Econometrics*, vol. 143, no. 1, pp. 56–73, 2008.
- [7] J. Fan, J. Jiang, C. Zhang, and Z. Zhou, "Time-dependent diffusion models for term structure dynamics," *Statistica Sinica*, vol. 13, no. 4, pp. 965–992, 2003.
- [8] T. Yan and C. Mei, "A test for a parametric form of the volatility in second-order diffusion models," *Computational Statistics*, vol. 32, no. 4, pp. 1583–1596, 2017.
- [9] R. Zhou, J. S.-H. Li, and K. S. Tan, "Pricing standardized mortality securitizations: A two-population model with transitory jump effects," *Journal of Risk and Insurance*, vol. 80, no. 3, pp. 733–774, 2013.
- [10] Y. Liu and J. S. Li, "The age pattern of transitory mortality jumps and its impact on the pricing of catastrophic mortality bonds," *Insurance: Mathematics and Economics*, vol. 64, pp. 135–150, 2015.
- [11] A. Gloter, "Parameter estimation for a discrete sampling of an integrated Ornstein-Uhlenbeck process," *Statistics*, vol. 35, no. 3, pp. 225–243, 2001.
- [12] C. A. Ball and W. N. Torous, "A Simplified Jump Process for Common Stock Returns," *Journal of Financial and Quantitative Analysis*, vol. 18, no. 1, pp. 53–65, 1983.
- [13] S. Beckers, "A note on estimating the parameters of the diffusion-jump model of stock returns," *Journal of Financial and Quantitative Analysis*, vol. 16, no. 1, pp. 127–140, 1981.
- [14] A. R. Bergstrom, "The history of continuous-time econometric models," *Econometric Theory*, vol. 4, no. 3, pp. 365–383, 1988.
- [15] J. Xu and J. Wang, "Maximum likelihood estimation of linear models for longitudinal data with inequality constraints," *Communications in Statistics—Theory and Methods*, vol. 37, no. 6-7, pp. 931–946, 2008.
- [16] D. Zeng and D. Y. Lin, "Maximum likelihood estimation in semiparametric regression models with censored data," *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, vol. 69, no. 4, pp. 507–564, 2007.
- [17] T. Coleman, M. A. Branch, and A. Grace, *Matlab Optimization Toolbox User's Guide*, Natick, 3rd edition, 1999.
- [18] A. Gilat, *MATLAB: An Introduction with Applications*, Wiley, Hoboken, 2nd edition, 2004.
- [19] S. S. Wilks, "Shortest Average Confidence Intervals from Large Samples," *The Annals of Mathematical Statistics*, vol. 9, no. 3, pp. 166–175, 1938.
- [20] C. J. Geyer and E. A. Thompson, "Constrained Monte Carlo maximum likelihood for dependent data," *Journal of the Royal Statistical Society: Series B (Methodological)*, vol. 54, no. 3, pp. 657–699, 1992.
- [21] K. Kladvko, *Maximum likelihood estimation of the Cox-Ingersoll-Ross process: the Matlab implementation*, Technical Computing Prague, 2007.
- [22] F. Zhu, "A negative binomial integer-valued GARCH model," *Journal of Time Series Analysis*, vol. 32, no. 1, pp. 54–67, 2011.
- [23] P. Ranjan, D. Bingham, and G. Michailidis, "Sequential experiment design for contour estimation from complex computer codes," *Technometrics. A Journal of Statistics for the Physical, Chemical and Engineering Sciences*, vol. 50, no. 4, pp. 527–541, 2008.
- [24] C. Gouriéroux, A. Holly, and A. Monfort, "Likelihood ratio test, Wald test, and Kuhn-Tucker test in linear models with inequality constraints on the regression parameters," *Econometrica*, vol. 50, no. 1, pp. 63–80, 1982.
- [25] D. A. Dickey and W. A. Fuller, "Likelihood ratio statistics for autoregressive time series with a unit root," *Econometrica*, vol. 49, no. 4, pp. 1057–1072, 1981.
- [26] B. W. Stuck and B. Kleiner, "A statistical analysis of telephone noise," *Bell System Technical Journal*, vol. 53, no. 7, pp. 1263–1320, 1974.
- [27] A. Gray, D. Greenhalgh, X. Mao, and J. Pan, "The SIS epidemic model with Markovian switching," *Journal of Mathematical Analysis and Applications*, vol. 394, no. 2, pp. 496–516, 2012.
- [28] A. Lahrouz and A. Settati, "Asymptotic properties of switching diffusion epidemic model with varying population size," *Applied Mathematics and Computation*, vol. 219, no. 24, pp. 11134–11148, 2013.
- [29] A. Lahrouz and L. Omari, "Extinction and stationary distribution of a stochastic SIRS epidemic model with non-linear incidence," *Statistics and Probability Letters*, vol. 83, no. 4, pp. 960–968, 2013.
- [30] W. P. Kemp, B. Dennis, and R. C. Beckwith, "Stochastic Phenology Model for the Western Spruce Budworm (Lepidoptera: Tortricidae)," *Environmental Entomology*, vol. 15, no. 3, pp. 547–554, 1986.
- [31] L. Zu, D. Jiang, and D. O'Regan, "Conditions for persistence and ergodicity of a stochastic Lotka-Volterra predator-prey model with regime switching," *Communications in Nonlinear Science and Numerical Simulation*, vol. 29, no. 1-3, pp. 1–11, 2015.
- [32] M. Liu and K. Wang, "Global stability of a nonlinear stochastic predator-prey system with Beddington-DeAngelis functional response," *Communications in Nonlinear Science and Numerical Simulation*, vol. 16, no. 3, pp. 1114–1121, 2011.



**Hindawi**

Submit your manuscripts at  
[www.hindawi.com](http://www.hindawi.com)

