

# The Vehicle Routing Problem with Stochastic Demand and Duration Constraints

Alan L. Erera, Juan C. Morales and Martin Savelsbergh

*The Logistics Institute  
Georgia Institute of Technology  
School of Industrial and Systems Engineering  
Atlanta, GA 30332-0205*

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## **Abstract**

Time considerations have been largely ignored in the study of vehicle routing problems with stochastic demands even though such considerations are crucial in practice. We show that hard tour duration limits can be effectively and efficiently incorporated in solution approaches for vehicle routing problems with stochastic demands. To do so requires solving an adversarial problem that determines a demand realization that results in the maximum duration for a given delivery tour. A computational study demonstrates that enforcing hard tour duration limits leads to substantially different problem solutions, often requiring additional vehicles to ensure feasibility.

## **1 Introduction**

The vehicle routing problem with stochastic demand (VRPSD) is a fairly well-studied problem that has been a focus of research on and off since initial work by Tillman (1969). A key challenge for models that focus on serving all customer demands with very high probability using fixed (or *a priori*) tours is modeling tour feasibility. It is usually impossible, and if possible most likely extremely costly, to ensure the feasibility of a set of *a priori* vehicle tours for all possible demand realizations in the absence of a recourse policy. Thus, most models are more flexible and allow some recourse decisions. In these cases, the feasibility of the set of *a priori* tours is determined given a recourse policy that specifies what actions to take when an infeasibility, or tour failure, occurs during the execution of a tour for a given demand realization.

The most popular recourse policy studied in the literature is one that we denote *detour-to-depot*: when a vehicle capacity infeasibility arises, *i.e.*, when the demand of a customer cannot be satisfied given the current vehicle remaining capacity, the vehicle makes an out-and-back trip to the depot to restock (or unload) before resuming its tour. Various heuristic and some exact approaches for constructing a set of tours minimizing expected tour costs for this recourse policy have been proposed and analyzed in the literature.

From a practical perspective, however, there is something disconcerting about the unrestricted use of detour-to-depot recourse. Not only does an out-and-back trip to the depot result in additional costs, it also results in additional travel time and thus a longer duration of the delivery route. Since the number of hours that drivers can spend behind the wheel is often limited by regulation, and since the operating day of most pickup/delivery operations is also constrained by business practices, the duration of an actual tour usually cannot be extended beyond certain hard limits. Thus, the duration of a tour, including any additional travel time incurred during recourse actions, cannot exceed an upper bound. This additional feasibility criterion leads to a problem that we denote the Vehicle Routing Problem with Stochastic Demand and Duration Constraints (VRPSD-DC).

In this paper, we formally define the VRPSD-DC, develop a tabu search heuristic for its solution under a common class of recourse policies, and perform a computational study to analyze the impact of duration constraints on the structure of a set of near-optimal *a priori* tours. The tabu search heuristic relies heavily on solving an “adversarial” problem, which determines a customer demand realization that maximizes the actual duration of a given tour. We show that this adversarial problem can be solved efficiently for detour-to-depot policies, and specifically in polynomial time for a detour-to-depot policy that does not allow customer demand splitting. Our computational study shows that imposing and guaranteeing a route duration limit, which is important in practice, can change tour structure substantially and often necessitates an increase in fleet size to ensure feasibility.

The remainder of the paper is organized as follows. In Section 2, we briefly review related literature. In Section 3, we introduce the vehicle routing problem with stochastic demand and duration constraints. In Section 4, we discuss the adversarial problem and present a longest path algorithm for its solution. In Section 5, we discuss a tabu search heuristic for constructing a set of *a priori* tours with low expected costs. In Section 7, we present the results of the computational study. Finally, in Section 8, we make some final observations and outline plans for future research.

## 2 Related Literature

As mentioned, the vehicle routing problem with stochastic demands is a well-studied problem. For excellent surveys, see Dror et al. (1989) and Gendreau et al. (1996a).

Modeling and solution approaches for the VRPSD can be divided into three main research streams:

1. Approaches based on chance constrained models;
2. Approaches based on stochastic programming with recourse models;
3. Approaches based on Markov decision models.

An early chance-constrained models is proposed in Stewart and Golden (1983). The paper presents a model to identify minimum cost tours subject to a threshold constraint on the probability of a route failure. A similar approach is proposed in Laporte et al. (1989); their a model uses fewer variables, but requires a homogeneous fleet of vehicles. In both cases, the model can be transformed into a deterministic vehicle routing problem under reasonable assumptions. The major shortfall in these models is that although the probability of a failure is constrained, the locations of failures (and hence their costs) are ignored. Routes with the same *a priori* cost and the same failure probability can have significantly different expected costs, depending on the possible failure locations.

Two-stage stochastic programming models with recourse minimize expected tour costs, which include of course the cost of recourse actions when failures occur. Dror and Trudeau (1986) propose a model that takes into consideration the location of a failure, where recourse actions take the form of out-and-back trips to the depot (*i.e.*, detour-to-depot). The majority of papers in this area focus use models of this type, but they tend to be difficult to solve exactly for large scale problems.

Significant effort has been devoted to the development of heuristics for solving stochastic programming models with recourse for VRPSD. Stewart and Golden (1983) and Dror and Trudeau (1986) both propose algorithms inspired by the ideas underlying the savings heuristic of Clarke and Wright (1964) for the deterministic VRP. An efficient local search approach for solving this type of problems is presented in Savelsbergh and Goetschalckx (1995) and their computational experiments shows that the local search approach compares favorably to the savings-based algorithms. Gendreau et al. (1996b) extends local search ideas to a tabu search approach. The quality of the tabu search is assessed by comparison to results using the exact solution from Gendreau et al. (1995) for a common set of instances; the tabu search algorithm produced an optimal set of routes in 89.45% of the cases. Furthermore, an average deviation from optimal expected cost of only 0.38% was observed. It is noteworthy that most of the research on recourse models for the VRPSD focuses on simple recourse policies that are separable by vehicle (see for instance Bertsimas (1992) and Bertsimas and Simchi-Levi (1994)); an exception is a two-vehicle sharing recourse policy proposed in Ak and Erera (2007).

Solving the VRPSD to optimality is very hard. The first exact approach of which we are aware is given in Gendreau et al. (1995), where an integer L-shaped method capable of solving small instances is proposed. Laporte et al. (2002) presents an improved approach capable of handling larger instances. An important element of this improved approach is the use of lower bounds at the root node which helps to speed up solution times. These bounds

are calculated under the assumption that the expected value of demand on any tour is less than or equal to the vehicle capacity.

Markov decision models offer the potential for making optimal recourse decisions each time new information is revealed, rather than relying on static recourse policies; the drawback is that these models require a very large state space, and thus are intractable even for modest size instances. Dror et al. (1989) develops a single-vehicle model where a decision epoch corresponds to the moment the vehicle arrives at a customer location and its demand is revealed. At that point, two possible decisions can be made prior to serving the customer: (i) not to serve the customer and move to another location, or (ii) serve the customer and then move to another location. No solution approach or computational study is presented.

An interesting approach, based on Markov decision models, is presented in Yang et al. (2000). The authors attempt to specify an optimal restocking (or unloading) policy for the vehicle in conjunction with the routing decisions. Under such a policy, the vehicle might restock (or unload) at the depot *before* a capacity failure actually occurs. For a given tour, it is shown that the optimal restocking policy has a simple threshold form: after serving a customer if on-board inventory drops below a customer-dependent threshold, then restock, else continue to the next customer in the tour. Although the optimal policy is quite simple, solving a model that further considers routing decisions is difficult, so heuristic approaches are developed. Unfortunately, the paper fails to compare the results from the approach to those obtained using heuristics that assume traditional simple recourse actions.

### 3 The Vehicle Routing Problem with Stochastic Demand and Duration Constraints

The Vehicle Routing Problem with Stochastic Demands and Duration Constraints is defined on a directed graph  $G = (V_0, A)$  where  $V_0 = \{0\} \cup V$  with 0 representing the depot and  $V = \{1, \dots, N\}$  representing the set of customers, and  $A = \{(i, j) \mid i, j \in V_0, i \neq j\}$ . A cost  $l(i, j)$  is associated with each arc  $(i, j) \in A$ ; we assume it represents the travel time between customers  $i$  and  $j$ , and that these travel times satisfy the triangle inequality, *i.e.*,  $l(i, j) \leq l(i, k) + l(k, j)$  for all  $i, j, k \in V_0$ . The vehicle capacity is denoted by  $Q$  and we assume for exposition that vehicles carry inventory to be delivered to customers.

Customer demands are integer-valued random variables with known distributions and are denoted by vector  $\tilde{d} \in \mathbb{Z}_+^{|V|}$ . Customer demands are assumed to be independently distributed. It is further assumed that support vectors  $\underline{d}, \bar{d} \in \mathbb{Z}_+^{|V|}$  are known such that  $\underline{d} \leq \tilde{d} \leq \bar{d}$ , and  $\bar{d}(i) > 0$  and  $\bar{d}(i) < Q$  for all  $i \in V$ . If we further assume that there is a positive probability that each demand  $\tilde{d}(i)$  takes every value in the range  $[\underline{d}(i), \bar{d}(i)]$ , then the outcome space of demand realizations is

$$\mathcal{U} = \left\{ d \in \mathbb{Z}_+^{|V|} : \underline{d} \leq d \leq \bar{d} \right\}.$$

Finally, for convenience of notation, we further define  $\underline{d}(0) = \bar{d}(0) = 0$ . Uncertain customer

demands are assumed to be revealed with certainty at some time immediately prior to or within each operating day. For example, a common assumption is that customer demand becomes known upon the arrival of a vehicle.

A tour specifies an *a priori* sequence in which some subset of customers is visited by a single vehicle each day; all tours start and end at the depot. Each customer is assumed to be served by one and only one vehicle. Let  $\mathcal{T}_k = \{i_1, i_2, \dots, i_{n_k}\}$  denote the tour of vehicle  $k$ , where  $i_j \in V$ . Observe that we do not include the depot at the beginning and at the end of this tour representation, but it is implied.

Each day when a vehicle traverses its tour, all customer demands must be satisfied, but this may not be possible if the vehicle were to simply serve each customer in its tour in sequence. In this problem definition, we will consider *recourse policies* under which each vehicle serves customer in its tour according to the *a priori* ordering, but returns to the depot to restock to ensure that all customer demands are satisfied. We will denote as *recourse actions* these restocking detours.

Let  $L(\mathcal{T}_k)$  be the total travel time required by vehicle  $k$  to complete its *a priori* tour if no recourse actions were required, and let  $\phi(\mathcal{T}_k, \mathcal{P}, d)$  be the total additional travel time due to recourse actions specified by policy  $\mathcal{P}$  and demand realization  $d \in \mathcal{U}$ . Then, the maximum duration of tour  $k$  is

$$\mathcal{L}(\mathcal{T}_k, \mathcal{P}) = L(\mathcal{T}_k) + \max_{d \in \mathcal{U}} \phi(\mathcal{T}_k, \mathcal{P}, d).$$

Since the additional travel time incurred due to recourse actions depends on the demand realization, determining the maximum additional travel time is an optimization problem that we denote the *adversarial problem*.

Next, we introduce the function

$$\mathcal{L}_E(\mathcal{T}_k, \mathcal{P}) = L(\mathcal{T}_k) + \mathbf{E}[\phi(\mathcal{T}_k, \mathcal{P}, d)],$$

where  $\mathbf{E}$  denotes the expectation operator with respect to the outcomes in  $\mathcal{U}$ . Thus  $\mathbf{E}[\phi(\mathcal{T}_k, \mathcal{P}, d)]$  denotes the expected additional travel time incurred by vehicle  $k$  due to recourse actions under recourse policy  $\mathcal{P}$ . The Vehicle Routing Problem with Stochastic Demands and Duration Constraints then is to find a set of tours with minimum total expected duration subject to a hard constraint on individual vehicle duration:

$$\begin{aligned} \text{VRPSD} - \text{DC} \quad & \min_{\{\mathcal{T}_k\}} \sum_k \mathcal{L}_E(\mathcal{T}_k, \mathcal{P}) & (1) \\ \text{s.t.} \quad & \mathcal{L}(\mathcal{T}_k, \mathcal{P}) \leq D \quad \forall k, \end{aligned}$$

where  $D$  denotes the maximum travel time duration allowed for a tour, and the customer set  $V$  is partitioned among the tour set  $\{\mathcal{T}_k\}$ . In the remainder of the paper, we use  $\Phi(\mathcal{T}_k, \mathcal{P})$  to denote  $\max_{d \in \mathcal{U}} \phi(\mathcal{T}_k, \mathcal{P}, d)$ .

## 4 The Adversarial Problem

Consider a single vehicle and its tour  $\mathcal{T} = \{1, 2, \dots, n\}$ . The adversarial problem seeks to determine a demand realization  $d \in \mathcal{U}$  that maximizes  $\phi(\mathcal{T}, \mathcal{P}, d)$ . If the function  $\phi$  is non-decreasing in  $d(i)$  for all  $i \in \mathcal{T}$  for  $\mathcal{P}$ , it is clear that an optimal solution is to set  $d$  equal to  $\bar{d}$ . As we will show, this is not the case for all recourse policies; that is, unlike many robust optimization problems, the worst-case scenario for the adversarial problem is not always found at an extreme point of  $\mathcal{U}$ . Therefore, we investigate solution approaches for the adversarial problem. Since the size of  $\mathcal{U}$  for a given tour  $\mathcal{T}$  is given by

$$\prod_{i \in \mathcal{T}} (\bar{d}(i) - \underline{d}(i) + 1),$$

simple enumerative approaches may require excessive computational burden that hopefully can be avoided.

In Morales (2006), it is shown how the adversarial problem can be solved for several different recourse policies. In this paper, we present these ideas specifically for two simple detour-to-depot recourse policies.

### 4.1 Detour-to-depot recourse

Suppose that the demand of customer  $i$  is revealed only upon arrival of the vehicle to  $i$ . We now define two recourse policies, one which allows individual customer demands to be split and one which does not.

**Definition 1 (Splittable detour-to-depot recourse)**  $\mathcal{P}^S$  is used to denote the following recourse policy: given tour  $\mathcal{T}$ , a recourse action is initiated at customer  $i \in \mathcal{T}$  if and only if the arriving vehicle observes  $d(i)$  which is strictly greater than on-board inventory. After satisfying as much of  $d(i)$  as possible with on-board inventory, the vehicle restocks at the depot, then returns to  $i$  and finishes satisfying the rest of the demand before proceeding.

**Definition 2 (Non-splittable detour-to-depot recourse)**  $\mathcal{P}^{NS}$  is used to denote the following recourse policy: given tour  $\mathcal{T}$ , a recourse action is initiated at customer  $i \in \mathcal{T}$  if and only if the arriving vehicle observes  $d(i)$  which is strictly greater than the on-board inventory. Before satisfying any part of  $d(i)$ , the vehicle restocks at the depot, returns to  $i$ , and satisfies the entire demand.

Note that under each policy a vehicle never performs more than one recourse action at a single customer  $i$ , since  $d(i) < Q$ ; furthermore, the first recourse action will never take place at the first customer in the tour. These policies also have the characteristics that for each  $d \in \mathcal{U}$ , the customers at which recourse actions are initiated are uniquely determined, and that the number of recourse actions is a non-decreasing function of total tour demand  $\sum_{i \in \mathcal{T}} d(i)$ .

## 4.2 Solving the adversarial problem

Let  $\mathcal{P}$  be either the splittable policy  $\mathcal{P}^S$  or the non-splittable policy  $\mathcal{P}^{NS}$ . The execution of tour  $\mathcal{T}$  under policy  $\mathcal{P}$  for demand realization  $d$  can be described in terms of state variables  $(r, i, I_i)$  defined for each customer  $i$  in the tour, where  $r = k$  if the  $k$ -th recourse action is initiated at customer  $i$  and  $r = 0$  if no recourse action is initiated at this customer, and where  $I_i$  denotes the on-board inventory of the vehicle when it departs  $i$  after delivery.

It is easy to see that possible set of all states  $(r, i, I_i)$  is fairly compact. For each customer  $i$  in the tour,  $0 \leq I_i \leq Q$  and  $i \leq r \leq R$ , where  $R$  denotes the maximum number of recourse actions that can occur. Furthermore, the value of  $R$  can be determined by executing the tour under  $\mathcal{P}$  for demand realization  $\bar{d}$ ;  $R$  is also clearly bounded from above by  $n$ , the number of customers in the tour. Note further that the values of the state variables are uniquely determined for each realization  $d$ .

To solve the adversarial problem, we seek to characterize the set of states  $S$  for which there exists a demand realization  $d \in \mathcal{U}$  that visits that state. Given  $\underline{d}$  and  $\bar{d}$ , it is possible to determine the set of customers at which the first recourse action may be initiated, and for each such customer a corresponding demand realization and the resulting vehicle load at the departure from that customer. Let  $C^1(i, I_i)$  for  $i \in \mathcal{T}$  denote a set of necessary and sufficient conditions that ensures the existence of a demand realization  $d \in \mathcal{U}$  for which the *first* recourse action is initiated at customer  $i$  and such that the vehicle load at the departure from customer  $i$  is  $I_i$ . Furthermore, let  $C^{r,r+1}(i, I_i, k, I_k)$  for  $i, k \in \mathcal{T}$  such that  $i < k$  and for  $r = 1, \dots, R - 1$  denote a set of necessary and sufficient conditions that ensures the existence of a demand realization  $d \in \mathcal{U}$  for which the  $(r + 1)$ -th recourse action is initiated at customer  $k$  and such that the vehicle load at the departure from customer  $k$  is  $I_k$  given that the  $r$ -th recourse action is initiated at customer  $i$  with vehicle load  $I_i$  at departure. Conditions  $C^1(i, I_i)$  and  $C^{r,r+1}(i, I_i, k, I_k)$  are referred to as *recourse conditions*. We will show subsequently that such conditions can be developed for policies  $\mathcal{P}^S$  and  $\mathcal{P}^{NS}$ .

Given recourse conditions, the adversarial problem can be solved by finding a longest path on an acyclic digraph  $\mathcal{G}(\mathcal{T}, \mathcal{P}) = (\mathcal{N}, \mathcal{A})$ , with node set

$$\mathcal{N} = \{s\} \cup \{t\} \cup \{(r, i, I_i) \mid i \in \mathcal{T}, r \in \{i, \dots, R\}, I_i \in \{0, 1, \dots, Q\}\}. \quad (2)$$

The arc set  $\mathcal{A}$  is defined by the recourse conditions:

- $(s, (1, i, I_i)) \in \mathcal{A}$  for  $i$  and  $I_i$  when conditions  $C^1(i, I_i)$  are satisfied. The cost of such arcs is the additional travel time of a recourse action at  $i$ ,  $l_{i0} + l_{0i}$ .
- $((r, i, I_i), (r + 1, k, I_k)) \in \mathcal{A}$  for  $r = 1, \dots, R - 1$  and for  $i, k \in \mathcal{T}$ ,  $i < k$ , and  $I_i, I_k$  when conditions  $C^{r,r+1}(i, I_i, k, I_k)$  are satisfied. The cost of such arcs is the additional travel time of a recourse action at  $k$ ,  $l_{k0} + l_{0k}$ .
- $((r, i, I_i), t) \in \mathcal{A}$  for all  $(r, i, I_i) \in \mathcal{N}$  such that  $\text{indeg}(r, i, I_i) > 0$  and  $\text{outdeg}(r, i, I_i) = 0$ . The cost of such arcs is 0.

Figure 1 shows an example digraph  $\mathcal{G}(\mathcal{P}) = (\mathcal{N}, \mathcal{A})$  for an instance with  $Q = 1$  and  $n = R = 4$ .

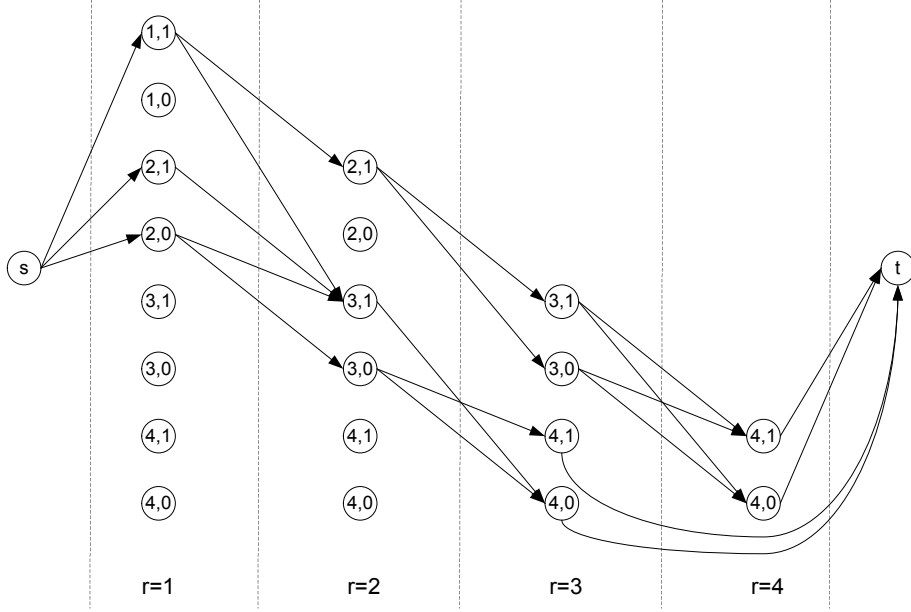


Figure 1:  $\mathcal{G}(\mathcal{T}, \mathcal{P})$  for an instance with  $Q = 1$  and  $n = R = 4$ , with  $(i, I_i)$  in each node.

Let  $L(\mathcal{G}(\mathcal{T}, \mathcal{P}))$  be the length of the longest  $s - t$  path in the graph. If there does not exist an  $s - t$  path, then let  $L(\mathcal{G}(\mathcal{T}, \mathcal{P})) = 0$ .

**Lemma 1** For a tour  $\mathcal{T}$  operated using recourse policy  $\mathcal{P}$ ,  $\Phi(\mathcal{T}, \mathcal{P}) = L(\mathcal{G}(\mathcal{T}, \mathcal{P}))$ .

*Proof.* For any demand realization  $d$ , there is a unique set of customers in  $\mathcal{T}$  where recourse actions occur. From the construction of network  $\mathcal{G}(\mathcal{T}, \mathcal{P})$ , it is clear that every  $d \in \mathcal{U}$  is associated with one and only one  $s - t$  path in  $\mathcal{G}(\mathcal{T}, \mathcal{P})$ , and also, that every  $s - t$  path in the network is associated with one and only one demand realization. To evaluate  $\Phi(\mathcal{T}, \mathcal{P})$ , the longest path in the network is identified because this path is associated with a demand realization with maximum additional travel time.  $\square$

**Lemma 2**  $L(\mathcal{G}(\mathcal{T}, \mathcal{P}))$  can be calculated in  $O(n^3 Q^2)$ .

*Proof.* Evaluating  $L(\mathcal{G}(\mathcal{T}, \mathcal{P}))$  requires  $O(|\mathcal{A}|)$  steps since  $\mathcal{G}(\mathcal{T}, \mathcal{P})$  is acyclic. Observe that instances where  $\underline{d}(i) = 0$  and  $\bar{d}(i) = Q$  for all  $i \in V$  must yield the largest value for  $|\mathcal{A}|$  since the adversary in such cases has the most flexibility. For such instances, the number of nodes  $(i, r_i, I_i)$  with  $r_i = r$  is clearly bounded by  $(n - r + 1)(Q + 1)$ . Observe then that node  $(i, r, I_i)$  in  $\mathcal{N}$  can be connected directly with any other node  $(k, r + 1, I_k)$  such that  $k > i$ ; hence, the number of arcs with tail  $(i, r, I_i)$  is  $O((n - i)Q)$ . For a fixed  $i$ , there can



be  $Q + 1$  of such nodes, therefore the number of arcs with tail  $(i, r, I_i)$  for  $I_i \in \{0, \dots, Q\}$  is  $O((n - i)Q^2)$ . For a given  $r$ ,  $i$  is bounded between  $r$  and  $R$ , so the total number of arcs with tail  $(i, r, I_i)$  for all  $i$  and  $I_i$  is  $O(\sum_{i=r}^R (n - i)Q^2)$ . Summing over  $r$ , the number of arcs connecting the nodes associated with the  $r^{\text{th}}$  and  $(r + 1)^{\text{th}}$  recourse actions for all  $r$  is then  $O(\sum_{r=1}^{R-1} \sum_{i=r}^R (n - i)Q^2)$ , and since  $R$  is bounded by  $n$ , this expression reduces to  $O(n^3Q^2)$ . The number of nodes connected to both  $s$  and  $t$  is bounded by  $nQ$  in each case, so the total number of arcs in  $\mathcal{G}(\mathcal{T}, \mathcal{P})$  is  $O(n^3Q^2)$ .  $\square$

We now focus on developing recourse conditions, and consider the splittable policy  $\mathcal{P}^S$ . We first prove a useful lemma.

**Lemma 3** Consider vectors  $\underline{d}, \bar{d} \in \mathbb{Z}_+^n$  such that  $\underline{d} \leq \bar{d}$  and  $\bar{d} > \mathbf{0}$ . If

$$\sum_{\ell=1}^{n-1} \underline{d}(\ell) \leq M \quad \text{and} \quad \sum_{\ell=1}^n \bar{d}(\ell) \geq M + 1,$$

where  $M < \infty$  is an integer constant, then there exists a vector  $d \in \mathbb{Z}_+^n$  such that  $\underline{d} \leq d \leq \bar{d}$  that satisfies

$$\sum_{\ell=1}^{n-1} d(\ell) \leq M \quad \text{and} \quad \sum_{\ell=1}^n d(\ell) \geq M + 1.$$

*Proof.* A constructive proof is provided. Initialize  $d$  as  $d(\ell) = \underline{d}(\ell)$  for  $\ell = 1, \dots, n - 1$  and  $d(n) = \bar{d}(n)$ . Clearly  $\underline{d} \leq d \leq \bar{d}$  and  $\sum_{\ell=1}^{n-1} d(\ell) \leq M$ . If  $d$  also satisfies  $\sum_{\ell=1}^n d(\ell) \geq M + 1$ , then the desired vector is obtained. Else, since  $\bar{d} > \mathbf{0}$ , it follows that  $\sum_{\ell=1}^{n-1} d(\ell) < M$  and that

$$\sum_{\ell=1}^{n-1} d(\ell) < \sum_{\ell=1}^n \bar{d}(\ell). \quad (3)$$

Take any  $d(\ell)$  such that  $d(\ell) < \bar{d}(\ell)$  for  $1 \leq \ell \leq n - 1$  and perform the following update:  $d(\ell) \leftarrow d(\ell) + 1$ . Such an  $\ell$  exists, otherwise  $\sum_{\ell=1}^n \bar{d}(\ell) \geq M + 1$  would be contradicted. After such an update, inequality (3) remains true, and furthermore  $\underline{d} \leq d \leq \bar{d}$  and  $\sum_{\ell=1}^{n-1} d(\ell) \leq M$ . Therefore, if  $\sum_{\ell=1}^n d(\ell) \geq M + 1$  stop with the desired  $d$ ; else, take any  $d(\ell)$  such that  $d(\ell) < \bar{d}(\ell)$  for  $1 \leq \ell \leq n - 1$  and repeat the process described in this paragraph. At every step the difference between  $M + 1$  and  $\sum_{\ell=1}^n d(\ell)$  is reduced by 1, and condition  $\sum_{\ell=1}^{n-1} d(\ell) \leq M$  remains true; hence, in a finite number of steps the desired vector is obtained.  $\square$

Now, recourse conditions can be developed.

**Proposition 1 (Recourse condition  $C^{r,r+1}(i, I_i, k, I_k)$  for  $\mathcal{P}^S$ )** Assume that there exists  $d \in \mathcal{U}$  such that the  $r$ -th recourse action occurs at customer  $i$ , and the vehicle departs  $i$  with

inventory  $I_i$ . The  $(r + 1)$ -th recourse can occur at customer  $k > i$  with corresponding  $I_k$  if and only if

$$\sum_{\ell=i+1}^{k-1} \underline{d}(\ell) \leq I_i \quad \text{and} \quad \sum_{\ell=i+1}^k \bar{d}(\ell) \geq I_i + 1;$$

and furthermore, the possible remaining inventory  $I_k$  is bounded by:

$$Q + I_i - \min \left\{ I_i, \sum_{\ell=i+1}^{k-1} \bar{d}(\ell) \right\} - \bar{d}(k) \leq I_k \leq Q + I_i - \sum_{\ell=i+1}^k \underline{d}(\ell).$$

*Proof.*

( $\Rightarrow$ ) Let  $d$  be a realization where the  $r$ -th recourse occurs at  $i$  with remaining inventory  $I_i$  and the  $(r + 1)$ -th occurs at  $k$  with  $I_k$ . Such a  $d$  must satisfy

$$\sum_{\ell=i+1}^{k-1} d(\ell) \leq I_i \quad \text{and} \quad \sum_{\ell=i+1}^k d(\ell) \geq I_i + 1;$$

with corresponding

$$I_k = I_i - \sum_{\ell=i+1}^k d(\ell) + Q.$$

Since  $d \in \mathcal{U}$ , this implies that

$$\sum_{\ell=i+1}^{k-1} \underline{d}(\ell) \leq \sum_{\ell=i+1}^{k-1} d(\ell) \leq I_i \quad \text{and} \quad \sum_{\ell=i+1}^k \bar{d}(\ell) \geq \sum_{\ell=i+1}^k d(\ell) \geq I_i + 1;$$

and that

$$Q + I_i - \sum_{\ell=i+1}^k \bar{d}(\ell) \leq I_k \leq Q + I_i - \sum_{\ell=i+1}^k \underline{d}(\ell).$$

Since no recourse occurs between  $i$  and  $k$ ,  $\sum_{\ell=i+1}^{k-1} d(\ell) \leq I_i$  and thus

$$Q + I_i - \min \left\{ I_i, \sum_{\ell=i+1}^{k-1} \bar{d}(\ell) \right\} - \bar{d}(k) \leq I_k \leq Q + I_i - \sum_{\ell=i+1}^k \underline{d}(\ell).$$

( $\Leftarrow$ ) The existence of a demand realization in  $\mathcal{U}$  such that the  $r + 1$  recourse takes place for  $k$  given that  $r^{\text{th}}$  takes place for  $i$  with on-board inventory  $I_i$  follows directly from Lemma 3;  $I_k$  just needs to be bounded accordingly.  $\square$

When the vehicle departs from the depot for the first time, system conditions are equivalent to having a recourse at customer 0 (*i.e.*, the depot) with  $I_0 = Q$ ; the following result then follows directly from Proposition 1.

**Proposition 2 (Recourse condition  $C^1(i, I_i)$  for  $\mathcal{P}^S$ )** *Recourse conditions  $C^1(i, I_i)$  are equivalent to  $C^{r, r+1}(0, Q, i, I_i)$ .*

Propositions 1 and 2 clearly allow the construction of  $\mathcal{G}$  recursively in pseudopolynomial time, beginning with customers at which the first recourse may occur.

### 4.3 Solving the adversarial problem for policy $\mathcal{P}^{NS}$

It is not difficult to extend the ideas from the previous section to develop a pseudopolynomial approach to solve the adversarial problem for a tour  $\mathcal{T}$  operated under recourse policy  $\mathcal{P}^{NS}$ . However, we can take advantage of an additional feature of this policy to develop a polynomial approach. Consider the following simple but useful observation.

**Observation 1** *Consider a tour  $\mathcal{T}$  operated using policy  $\mathcal{P}^{NS}$ , and a demand realization  $d$  such that a recourse action occurs at customer  $i \in \mathcal{T}$ . Then, after  $i$  is served the on-board inventory is always  $Q - d(i)$ .*

Next, we proof a useful lemma.

**Lemma 4** *Consider a tour  $\mathcal{T}$  operated using policy  $\mathcal{P}^{NS}$  and a demand realization  $d$  where the  $(r - 1)$ -th recourse action occurs at  $j$ . If the  $r$ -th recourse action occurs at  $i > j$ , then*

$$d(i) \geq \underline{d}(i/j) \equiv \max \left\{ 1, Q + 1 - \sum_{\ell=j}^{i-1} \bar{d}(\ell) \right\}.$$

*Proof.* Under  $\mathcal{P}^{NS}$ , the vehicle satisfies the demand of customer  $j$  before proceeding to  $i$ . Such a demand realization  $d$  that initiates recourse at  $i$  must satisfy

$$\sum_{\ell=j}^{i-1} d(\ell) \leq Q, \text{ and} \tag{4}$$

$$\sum_{\ell=j}^i d(\ell) \geq Q + 1. \tag{5}$$

Since  $d \in \mathcal{U}$ , we have  $\underline{d}(\ell) \leq d(\ell) \leq \bar{d}(\ell)$  and summing over  $\ell$  gives

$$\sum_{\ell=j}^{i-1} d(\ell) \leq \sum_{\ell=j}^{i-1} \bar{d}(\ell).$$

The above expression together with (4) implies

$$\sum_{\ell=j}^{i-1} d(\ell) \leq \min \left\{ Q, \sum_{\ell=j}^{i-1} \bar{d}(\ell) \right\}.$$

From (5), we get

$$\begin{aligned} d(i) &\geq Q + 1 - \sum_{\ell=j}^{i-1} d(\ell) \\ &\geq Q + 1 - \min \left\{ Q, \sum_{\ell=j}^{i-1} \bar{d}(\ell) \right\} \\ &\geq \max \left\{ 1, Q + 1 - \sum_{\ell=j}^{i-1} \bar{d}(\ell) \right\} \end{aligned}$$

□

Thus,  $\underline{d}(i/j)$  is a lower bound on the demand of customer  $i$  if a recourse action is to occur at customer  $i$  given that the preceding recourse action occurred at customer  $j$ .

We can now develop a somewhat simpler set of recourse conditions. To begin, we show that the first recourse action ( $r = 1$ ) occurs at node  $i$  via conditions identical to those for the splittable policy:

**Proposition 3 (Recourse condition  $C^1(i)$  for  $\mathcal{P}^{NS}$ )** *There exists a demand realization  $d \in \mathcal{U}$  such that the first recourse occurs at  $i$  if and only if*

$$\sum_{\ell=1}^{i-1} \underline{d}(\ell) \leq Q \quad \text{and} \quad \sum_{\ell=1}^i \bar{d}(\ell) \geq Q + 1.$$

*Proof.*

( $\Rightarrow$ ) Parallel to the ( $\Rightarrow$ ) proof for Proposition 1.

( $\Leftarrow$ ) Follows from Lemma 3 by taking  $n = i$  and  $M = Q$ . □

Next, we develop conditions that characterize where the  $(r + 1)$ -th recourse action may occur, given the locations of actions  $(r - 1)$  and  $r$ :

**Proposition 4 (Recourse condition  $C^{r-1,r,r+1}(j, i, k)$  for  $\mathcal{P}^{NS}$ )** *Assume there exists a demand realization  $d \in \mathcal{U}$  such that the  $(r - 1)$ -th recourse occurs at  $j$  and the  $r$ -th recourse occurs at  $i > j$  for  $r > 0$ . The  $(r + 1)$ -th recourse can occur at  $k > i$  if and only if*

$$\max\{\underline{d}(i/j), \underline{d}(i)\} + \sum_{\ell=i+1}^{k-1} \underline{d}(\ell) \leq Q \quad \text{and} \quad \sum_{\ell=i}^k \bar{d}(\ell) \geq Q + 1.$$

*Proof.*

( $\Rightarrow$ ) Since a recourse action occurs at  $i$ , by Observation 1 the on-board inventory after  $i$  is served is  $Q - d(i)$ . Hence, such a demand realization  $d$  must satisfy

$$\sum_{\ell=i}^{k-1} d(\ell) \leq Q \quad \text{and} \quad \sum_{\ell=i}^k d(\ell) \geq Q + 1.$$

Since  $d \in \mathcal{U}$ , we have  $Q + 1 \leq \sum_{\ell=i}^k d(\ell) \leq \sum_{\ell=i}^k \bar{d}(\ell)$ , which corresponds to the second inequality in the proposition. Also, since  $\underline{d}(\ell) \leq d(\ell)$  and, by Lemma 4,  $\underline{d}(i/j) \leq d(i)$ , we have  $\max\{\underline{d}(i/j), \underline{d}(i)\} + \sum_{\ell=i+1}^{k-1} \underline{d}(\ell) \leq \sum_{\ell=i}^{k-1} d(\ell) \leq Q$ .

( $\Leftarrow$ ) Follows from Lemma 3 by taking  $M = Q$ ,  $n = k - i + 1$ , and associating  $d(1)$  in Lemma 3 with  $d(i)$  in the proposition,  $d(n)$  in Lemma 3 with  $d(k)$  in the Proposition, and shifting all other entries accordingly. Additionally, one must replace  $\underline{d}(i)$  with the tighter expression  $\max\{\underline{d}(i/j), \underline{d}(i)\}$ , which follows from Lemma 4. □

Observe that Proposition 4 can be used with  $r = 1$  to determine the customers where the second recourse may occur given that the first recourse occurred at  $i$  by setting  $j = 0$ , since we assume that  $\underline{d}(0) = \bar{d}(0) = 0$ .

We now show that Propositions 3 and 4 allow us to define a new network for solving the adversarial problem for policy  $\mathcal{P}^{NS}$ . Observe that in order to determine at which customer recourse action  $r + 1$  may occur, we need only know where recourse actions  $r$  and  $r - 1$  occurred; importantly, we need not explicitly track vehicle inventory levels, since they are implied at each recourse action. Thus, the adversarial problem for  $\mathcal{P}^{NS}$  can be solved using a digraph where node  $(r, i/j)$  represents that recourse action  $r$  occurs at customer  $i$  preceded by recourse  $(r - 1)$  at customer  $j$ . An arc between two nodes represents the existence of a demand realization that results in *consecutive* recourse actions at the corresponding customers. Let  $\mathcal{G}_1(\mathcal{T}, \mathcal{P}^{NS}) = (\mathcal{N}, \mathcal{A})$  denote this network, which is illustrated in Figure 3 for the example below. The set of nodes is

$$\mathcal{N} = \{s\} \cup \{t\} \cup \{(1, i/0) \mid i \in \mathcal{T} \setminus \{1\}\} \cup \mathcal{N}',$$

where  $(1, i/0)$  indicates that the first recourse action occurs at customer  $i$ , and

$$\mathcal{N}' = \{(r, i/j) \mid r \in \{2, \dots, R\}, i \in \{r + 1, \dots, n\}, j \in \{r, \dots, i - 1\}\},$$

where  $(r, i/j)$  is as defined earlier. The arc set  $\mathcal{A}$  is:

1.  $(s, (1, i/0)) \in \mathcal{A}$  for every  $i \in \mathcal{T}$  that satisfies Proposition 3. The cost of such arcs is the additional travel time of a recourse action at  $i$ ,  $l_{i0} + l_{0i}$ .
2.  $((r, i/j), (r + 1, k/i)) \in \mathcal{A}$  for  $r = 1, \dots, R - 1$  and for  $i, k \in \mathcal{T}$  such that  $j < i < k$  that satisfy Proposition 4; note that  $j \in \{0, 1, \dots, n - 2\}$ . The cost of such arcs is the additional travel time of a recourse action at  $k$ ,  $l_{k0} + l_{0k}$ .
3.  $((r, i/j), t) \in \mathcal{A}$  for all  $(r, i/j) \in \mathcal{N}$  such that  $\text{indeg}(r, i/j) > 0$  and  $\text{outdeg}(r, i/j) = 0$ . The cost of such arcs is 0.

Let  $L(\mathcal{G}_1(\mathcal{T}, \mathcal{P}^{NS}))$  be the length of the longest  $s$ - $t$  path in the graph. If there does not exist an  $s - t$  path, then by definition  $L(\mathcal{G}_1(\mathcal{T}, \mathcal{P}^{NS})) = 0$ .

**Theorem 1** *For a tour  $\mathcal{T}$  operated using policy  $\mathcal{P}^{NS}$ ,  $\Phi(\mathcal{T}, \mathcal{P}^{NS}) = L(\mathcal{G}_1(\mathcal{T}, \mathcal{P}^{NS}))$ . Furthermore,  $L(\mathcal{G}_1(\mathcal{T}, \mathcal{P}^{NS}))$  can be computed in  $O(n^4)$  time by solving a longest  $s$ - $t$  path problem on  $\mathcal{G}_1(\mathcal{T}, \mathcal{P}^{NS})$ .*

*Proof.* By construction  $\Phi(\mathcal{T}, \mathcal{P}^{NS})$  is equal to the length of the longest  $s - t$  path in  $\mathcal{G}_1(\mathcal{T}, \mathcal{P}^{NS})$ . Observe that the number of nodes in the  $r$ -th layer of the graph for  $r > 1$  is bounded by  $\sum_{i=1}^{n-r} i = \frac{(n-r)(n-r+1)}{2}$ ; each node  $(r, i/j)$  is connected to at most  $n - i$  nodes in layer  $r + 1$ . Therefore, the number of arcs between layer  $r$  and layer  $r + 1$  is  $O(n^3)$ . The number of layers  $R$  is bounded by  $n$ , so the total number of arcs between layers is  $O(n^4)$ . The

number of arcs with tail  $s$  is bounded by  $n$ . The number of nodes in the graph is bounded by  $n^3$ , so the number of arcs with head  $t$  is also bounded by  $n^3$ . Therefore, the total number of arcs in the graph is  $O(n^4)$  and the result follows because the digraph is acyclic.  $\square$

**Example:** Consider the example illustrated in Figure 2 for a vehicle with capacity  $Q = 4$ . The following table illustrates the values of the state variables (defined in Section 4) for demand realization  $\bar{d}$ , which has an associated value  $\phi(\mathcal{T}, \mathcal{P}^{NS}, \bar{d}) = 10$ . Observe that  $R = 3$ .

$i$	1	2	3	4	5	6	7
$d(i)$	3	2	2	1	3	3	1
$r$	0	1	0	2	0	3	0
$I_i$	1	2	0	3	0	1	0

Table 1: Values of the state variables for demand realization  $\bar{d}$  when the tour is operated using recourse policy  $\mathcal{P}^{dtd}$ .

Figure 3 illustrates  $\mathcal{G}_1(\mathcal{T}, \mathcal{P}^{NS})$ . The optimal solution for the adversarial problem is to have recourse actions at customers 3, 5 and 7, with an associated value  $\phi(\mathcal{T}, \mathcal{P}^{NS}, \bar{d}) = 20$ .

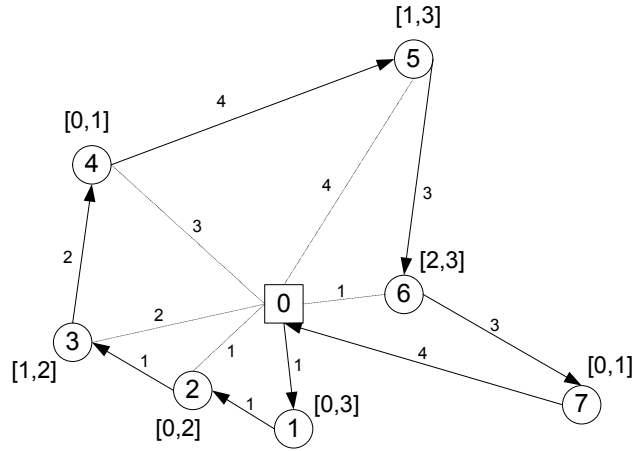


Figure 2: Example.

## 5 A Tabu Search Heuristic

We have designed and implemented a tabu search heuristic for solving instances of VRPSD-DC with detour-to-depot recourse policies. The tabu search heuristic is similar to the one

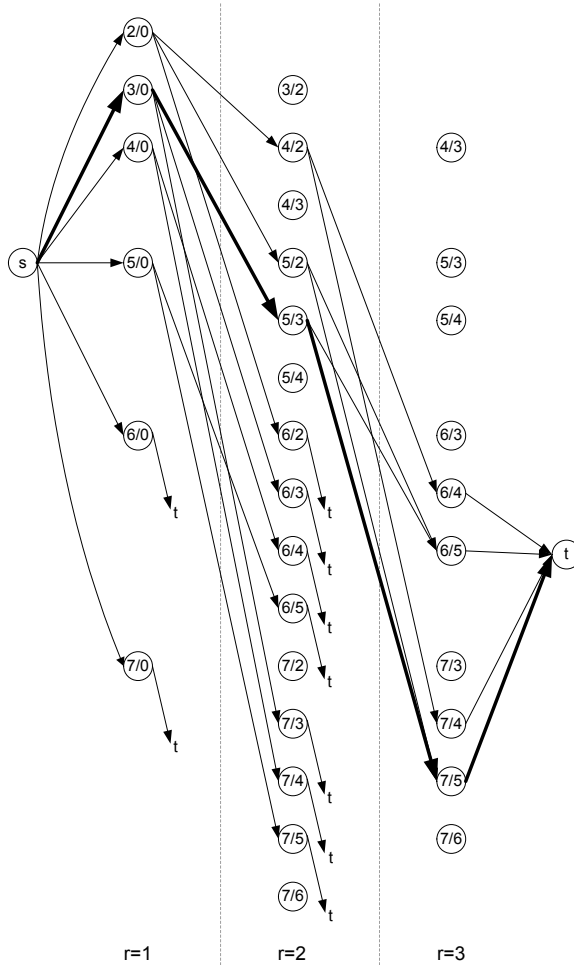


Figure 3:  $\mathcal{G}_1(\mathcal{T}, \mathcal{P}^{NS})$  for the instance in the example.

discussed in Gendreau et al. (1996b), which has been shown to perform well for vehicle routing problems with stochastic demands with a total expected cost objective. Below we briefly discuss the main components of our heuristic.

*Initial solution.* In the initial solution, we simply place each customer in its own tour.

*Neighborhood structure.* We use slightly expanded version of a typical  $N(p, q, x)$  neighborhood. For a given solution  $x$ , this neighborhood includes all duration-feasible tours that are obtained by removing, in turn, one of  $q$  randomly selected customers (denoted  $q'$ ), and reinserting each either immediately after or immediately before each of its  $p$  nearest neighbors (denoted  $p'$ ). Given that  $x$  is duration-feasible, a neighbor can be checked for duration-feasibility by simply checking the maximum duration constraint by solving the adversarial problem for the tour  $\mathcal{T}_k$  that receives customer  $q'$ :  $\mathcal{L}(\mathcal{T}_k, \mathcal{P}) \leq D$ .

When both customers, say  $q'$  and  $p'$ , are in the same current tour and  $q'$  precedes  $p'$ , we

consider two alternative ways of inserting  $q'$  before  $p'$ . For tour

$$\mathcal{T}_k = \{i_1, i_2, \dots, q', i_\ell, i_{\ell+1}, \dots, i_{\ell+\beta}, p', \dots, i_{n_k}\}$$

we consider both

$$\mathcal{T}_k = \{i_1, i_2, \dots, i_\ell, i_{\ell+1}, \dots, i_{\ell+\beta}, q', p', \dots, i_{n_k}\}$$

and

$$\mathcal{T}_k = \{i_1, i_2, \dots, i_{\ell+\beta}, \dots, i_{\ell+1}, i_\ell, q', p', \dots, i_{n_k}\}.$$

If both are duration-feasible and result in an improvement, we randomly select one with equal probability. Finally, for each potential move, we evaluate the tour including  $q'$  by traversing it in both directions and select the one with largest improvement, if any. We denote this expanded neighborhood of feasible solutions  $N_r(p, q, x)$ .

*Tabu moves.* If a customer is moved in iteration  $\nu$ , then any move involving that customer, either if it is among the  $q$  randomly selected or if it is among the  $p$  nearest neighbors, is tabu until iteration  $\nu + \theta$  where  $\theta$  is randomly selected in interval  $[N - 5, N]$ .

*Aspiration criteria.* The search process moves from one iteration to the next considering only non-tabu solutions in the neighborhood of the current solution, unless a tabu solution improves the best solution found thus far.

*Move evaluation.* A move is considered to be improving if the total expected duration of the resulting set of tours is less than the total expected duration of the current set of tours.

The expected duration of tour  $\mathcal{T} = \{1, 2, \dots, n\}$  is computed differently, but similarly, for recourse policies  $\mathcal{P}^S$  and  $\mathcal{P}^{NS}$ . For  $\mathcal{P}^{NS}$ , the computation is as follows. Let  $p_i(\delta)$  be the probability that the  $i$ -th customer in the tour has a demand value equal to  $\delta$ . Let  $\beta(i, s, q)$  be the probability of having on-board inventory equal to  $q$  after serving the  $i$ -th customer, given recourse states  $s$ , where  $s = 1$  if a recourse action occurs at  $i$  and  $s = 0$  if not. Furthermore, for ease of notation, define  $\mathcal{Q}(j, k) = \{j, j + 1, \dots, k - 1, k\}$ , for  $j$  and  $k$  integers such that  $0 \leq j \leq k \leq Q$ .

We can now recursively compute the probabilities  $\beta$ . First, for all  $q \in \mathcal{Q}(0, Q)$ :

$$\beta(1, s, q) = \begin{cases} p_1(Q - q) & \text{if } s = 0 \\ 0 & \text{if } s = 1 \end{cases}$$

For  $i \geq 2$  we calculate  $\beta(i, s, q)$  by conditioning on  $s$  and  $q$  for customer  $i - 1$ . For  $s = 0$ , to calculate  $\beta(i, 0, q)$  assume an on-board inventory value of  $\bar{q}$  after serving customer  $i - 1$ , which implies a demand realization of  $\bar{q} - q \geq 0$  at customer  $i$ . Therefore,

$$\beta(i, 0, q) = \sum_{s \in \{0, 1\}} \sum_{\bar{q} \in \mathcal{Q}(q, Q)} \beta(i - 1, s, \bar{q}) p_i(\bar{q} - q) \quad \text{for } i \in \{2, \dots, n\}, q \in \mathcal{Q}(0, Q).$$



To calculate  $\beta(i, s, q)$  for  $s = 1$ , again assume on-board inventory  $\bar{q}$  after serving customer  $i - 1$ . By Observation 1, the demand at customer  $i$  is  $Q - q > \bar{q}$ . Therefore,

$$\beta(i, 1, q) = \sum_{s \in \{0,1\}} \sum_{\bar{q} \in \mathcal{Q}(0, Q-q-1)} \beta(i-1, s, \bar{q}) p_i(Q-q) \text{ for } i \in \{2, \dots, n\}, q \in \mathcal{Q}(1, Q-1),$$

and  $\beta(i, 1, Q) = 0$  since no recourse is possible at  $i$  if the demand observed at  $i$  is zero.

Now let  $\pi_i$  be the probability of a recourse at customer  $i$ . Then

$$\pi_i = \sum_{q \in \mathcal{Q}(0, Q)} \beta(i, 1, q)$$

and the expected additional duration due to taking recourse actions can be calculated as

$$\mathbf{E}[\phi(\mathcal{T}, \mathcal{P}^{NS}, d)] = \sum_{i \in \mathcal{T}} \pi_i (l(i, 0) + l(0, i))$$

Note that the total cost  $C(w)$  of the set of tours for a neighbor  $w \in N_r(p, q, x)$  can be determined by recalculating only the expected cost of the one or two tours that differ from those in  $x$ . Contrary to the approach in Gendreau et al. (1996b), we compute the exact expected cost of the complete solution for each feasible potential move.

### Tabu Search Heuristic

#### STEP 1: (*Initialization*)

Construct solution  $x$  with each customer on its own tour; let  $C^*$  be the cost of  $x$  and let  $x^* = x$ . Set  $p = \min\{N - 1, 5\}$  and  $q = \min\{N - 1, 5\}$ . Set  $t_0 = 0$ ,  $t_1 = 0$  and  $t_2 = 50N$ , where  $t_0$  is the iteration counter,  $t_1$  is the number of iterations in which the best solution found thus far has not improved, and  $t_2$  is the maximum number of iterations allowed without one of such improvements.

#### STEP 2: (*Neighborhood search*)

Set  $t_0 = t_0 + 1$  and  $t_1 = t_1 + 1$ . Consider all moves in  $N_r(p, q, x)$  and build a LIST where all moves are sorted in nondecreasing order of their cost. Let  $y$  be the first move in the list, if it is not a tabu move or it improves the best solution found thus far then let  $x = y$ ; else continue examining LIST until such solution is found, and then make the corresponding assignment to  $x$ .

#### STEP 3: (*Incumbent update*)

If  $C(x) < T^*$ , set  $C^* = C(x)$ ,  $x^* = x$  and  $t_1 = 0$ . Else, set  $t_1 = t_1 + 1$ . If  $t_1 < t_2$ , go to STEP 2; otherwise go to STEP 4.

#### STEP 4: (*Intensification or termination*)

If  $t_2 = 50N$ , set  $t_1 = 0$ ,  $t_2 = 20N$ ,  $p = \min\{N - 1, 10\}$ ,  $q = N - 1$  and go to STEP 2. Otherwise; stop,  $x^*$  is the best solution found.

## 6 Detour-to-Depot Revisited

It has been frequently observed in the literature that the detour-to-depot recourse policy  $\mathcal{P}^{NS}$  as we have defined it above has the undesirable feature that, if the onboard inventory after a delivery at customer  $i - 1$  is zero, the vehicle travels to customer  $i$  before recognizing that a recourse action at customer  $i$  is necessary. The policy below remedies this short coming.

**Definition 3**  $\mathcal{P}^{NS-B}$  is used to denote the following non-splitting recourse policy on a fixed sequence  $\mathcal{T}$  with two types of recourse actions:

Type I : take a recourse action at customer  $i$  restocking at the depot and then going back to  $i$  if and only if  $d(i)$  is strictly greater than onboard inventory. The recourse action is taken before satisfying any of the demand of  $i$ .

Type II : take a recourse action for customer  $i$  immediately after serving customer  $i - 1$ , restocking at the depot before traveling to  $i$  if and only if onboard inventory after servicing  $i - 1$  is zero.

Recourse conditions for this policy can easily be derived (although it is more cumbersome because various cases have to be considered), and an adversarial problem can again be modeled using a similar longest path problem on a digraph.

For ease of presentation in this paper, we have used the basic version of the detour-to-depot policy when introducing the adversarial problem and the tabu search heuristic. However, in our computational study, to be presented next, we instead use the more sophisticated and lower cost policy  $\mathcal{P}^{NS-B}$ .

## 7 Computational Results

In this section, we present a computational study of VRPSD-DC. The objective of this study is to assess the effect of including tour duration constraints in the vehicle routing problem with stochastic demands. We are primarily interested in understanding how routing solutions change when tour duration constraints are enforced.

For a given instance, our starting point, which we refer to as the unconstrained version, is a solution in which there are no pre-planned detours to the depot and where the customers are served with the minimum number of vehicle tours  $m$  necessary to satisfy their total expected demand. To accomplish this, we use a slightly different objective function. Instead of seeking only to minimize total expected travel time, we add a penalty term to discourage using more vehicles, i.e.,

$$M(m(x) - m), \tag{6}$$

where  $m(x)$  is the number of vehicles used in the solution  $x$ , and  $M$  is a large constant conversion factor that relates vehicles to travel time. Then, the instance is resolved with constraints on the maximum duration of each individual vehicle tour. The value of the

constrained maximum duration  $D$  (*i.e.*, the right-hand side of the duration constraint) is determined separately for each instance by identifying the actual maximum tour duration in the solution to the unconstrained version, and then multiplying this value by a reduction factor  $0 < \alpha < 1$ . Because of the penalty term in the objective function, the fleet size increases only when it is no longer feasible to satisfy the tour duration constraints with  $m$  vehicle tours (without pre-planned detours to the depot).

Note that typical unconstrained VRPSD problems should always use a penalty term of this type in the objective, since in cases with no time constraints, it is always feasible to serve all customers with a single vehicle tour. Each *a priori* “tour” could be operated in sequence by a single vehicle, making preplanned return trips to the depot at the end of each tour.

In our computational study four factors were varied:

1. *Number of customers*: three problem sizes were considered:  $n = 20, 60,$  and  $100$  customers.
2. *Demand variability* ( $\sigma$ ): customer demand is assumed to be discrete uniform, independent, and identically distributed for all customers; hence the demand of any customer  $i$  takes values over the same interval  $[\underline{d}(i), \bar{d}(i)]$ . We consider three levels of variability, referred to as low, medium, and high:

	low	medium	high
$\underline{d}(i)$	4	3	2
$\bar{d}(i)$	8	9	10

Observe that the expected demand is constant across all levels of demand variability.

3. *Vehicle capacity* ( $Q$ ): two levels of vehicle capacity were considered:  $Q_1$  and  $Q_2$ , determined using expression

$$Q = \frac{n}{m} \frac{\underline{d}(i) + \bar{d}(i)}{2}.$$

The value of  $m$  is the pre-specified fleet size, and therefore this expression sets  $Q$  such that the average demand should be feasibly served by  $m$  deterministic tours not constrained by duration. When we modify the values of  $n$ ,  $\underline{d}(i)$  and  $\bar{d}(i)$ ,  $Q$  is determined by this expression:  $Q_1$  is determined by additionally setting  $m = 6$ , and  $Q_2$  by setting  $m = 3$ . Observe then that  $Q_1 < Q_2$ .

4. *Reduction factor* ( $\alpha$ ): three levels are considered:  $0.75, 0.85,$  and  $0.95$ .

We use Solomon’s uniform VRP instances to provide geographical locations of the customers and the depot in the usual way, by selecting the first  $n$  customers in the instance.

Travel times between points are determined using the Euclidean distance function. All combinations of  $n$ ,  $\sigma$ ,  $Q$ , and  $\alpha$  were considered. Thus, a total of 72 different instances are solved (18 unconstrained and 54 constrained instances).

Before presenting the results, we present some specific examples to build up intuition about the problem. Consider an instance with 20 customers,  $Q = 40$  (*i.e.*,  $Q_2$ ),  $\underline{d}(i) = 3$  and  $\bar{d}(i) = 9$ . Table 2 gives the details of the solution for the unconstrained version, which has a total expected travel time of 345.84. Observe that the maximum duration in any tour is 173.14, so in our experimental design we would apply reduction factor ( $\alpha$ ) to this value. Consider  $\alpha = 0.95$ , which yields a RHS value ( $D$ ) of 164.4; observe that two out of the three tours violate the duration constraint. Figure 4 provides a graphical representation of both

Table 2: Metrics on the best solution found for the unconstrained version (with a total expected travel time of 345.84)

Tour	{4,12,3,9,20,1}	{10,11,19,7,8,18}	{ 2,15,14,16,17,5,6,13}
Fixed Time	109.90	93.39	107.42
Exp. Time	115.73	99.34	130.77
Max. Time	173.14	145.88	168.25

the unconstrained solution as well as the constrained solution. It is easy to see that in this case the solution to the adversarial problem is to force a recourse action at customers 20, 8, and 17 (one in each of the three tours).

Table 3 gives the detailed results for the constrained solution for this example. Observe that just by reversing the direction of tour {4,12,3,9,20,1}, the maximum duration (now achieved by the adversary by a recourse action at customer 4) drops to 159.90 with a very slight increase in total expected travel time. By adding customer 6 after customer

Table 3: Metrics for the best solution found to the constrained example problem (with a total expected travel time of 352.32)

Tour	{1,20,9,3,12,4}	{10,11,19,7,8,18,6}	{5,17,16,14,15,2,13}
Fixed Time	109.90	99.93	102.40
Exp. Time	118.73	115.03	118.56
Max. Time	159.90	152.43	163.23

18 in tour {10,11,19,7,8,18}, the maximum duration (still achieved by the adversary with a recourse action at customer 8) remains the same. Finally, by reversing the sequence in tour {2,15,14,16,17,5,6,13} and dropping customer 6, the solution to the adversarial problem now forces a recourse action at customer 15 with the same increase in duration, but with a reduction in both the total and the expected travel times.

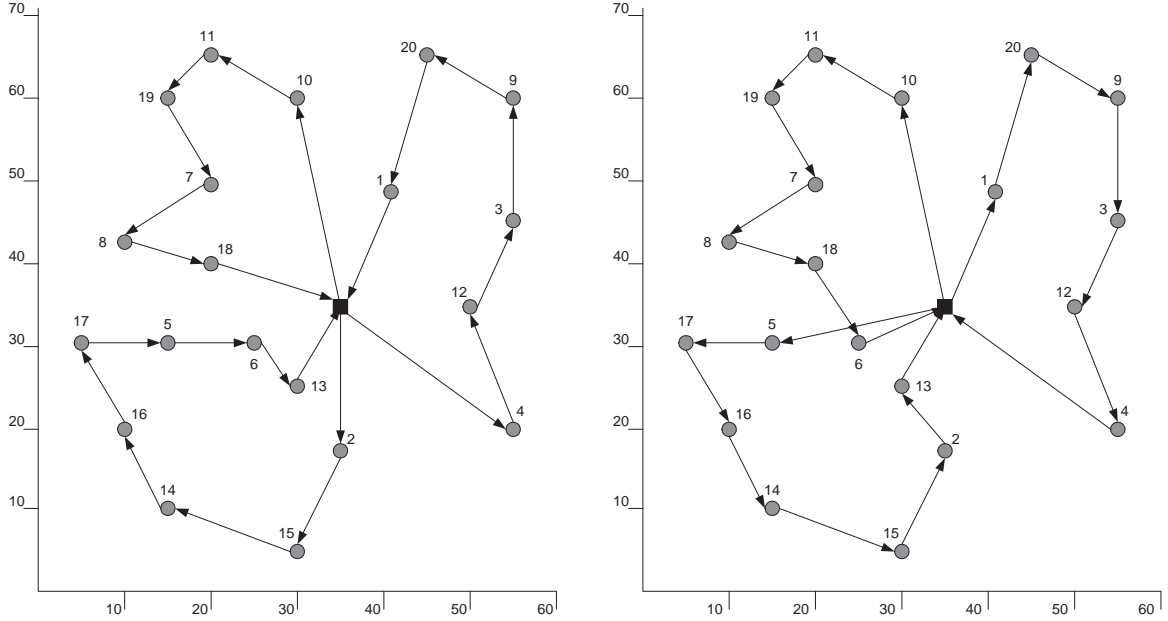


Figure 4: Best solutions found for the unconstrained and constrained example problem with duration limit  $D = 164.48$  (*i.e.*,  $\alpha = 0.95$ ).

Another interesting situation frequently encountered in the solutions found in this computational study is discussed below. It concerns a tour built for an instance with 100 customers,  $Q = Q_1 = 100$ ,  $\underline{d}(i) = 2$  and  $\bar{d}(i) = 10$ . Figure 5 shows the tour produced when there were no duration constraints as well as the tour produced when a duration limit was in place. The solution to adversarial forces recourse actions at customers 16 and 91 in the tour on the left in the figure. Observe that the adversary is taking advantage of the tour structure, which begins to get closer to the depot starting at customer 16 (the first opportunity for a recourse action) and then moves away from the depot again after customer 6. When a duration limit is enforced the structure of the tour is altered slightly to ensure that the limit cannot be violated.

Figure 6 shows the recourse probabilities for the customers in the tour in the unconstrained setting as well as for the tour in the constrained setting. We see that the unconstrained tour has high failure probabilities when the tour is close to the depot, which is to be expected given that the objective is to minimize expected duration. On the other hand, the duration-constrained tour must balance maximum duration and expected duration. Therefore, it retains some of the characteristics of the unconstrained tour, but it also sequences customers in such a way that the adversary cannot obtain consecutive failures points when the vehicle is far away from the depot; note that it does so by serving customers 91 and 100 soon after customer 16.

The maximum duration of the tour on the left in Figure 5 is 276.30 (with  $\Phi(\mathcal{T}_k, \mathcal{P}^{NS-B}) = 109.53$ ), while the maximum duration for the tour on the right is 268.30 (with  $\Phi(\mathcal{T}_k, \mathcal{P}^{NS-B}) =$

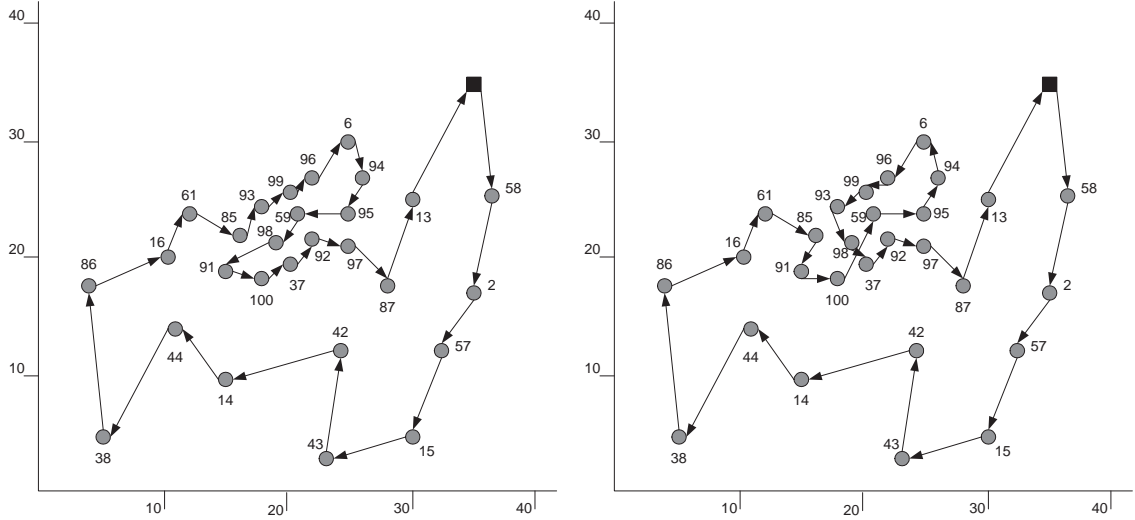


Figure 5: Two tours for an instance with 100 customers

100.83 and recourse actions at customers 16 and 98). So despite an increase in the length of the tour itself, by carefully selecting the order in which customers are visited the tour hedges against the demand uncertainty by changing the space of feasible solutions of the adversarial problem, cutting out solutions that result in large increase in duration. The expected travel time for the tour on the left is 196.03 and 199.56 for the tour on the right, so the increase in expected cost, which is the price we pay for having a tour that is more robust with respect to maximum duration, is not very significant. These slight increases in expected costs were consistently found in our computational study.

All instances in the study were solved using the tabu search heuristic described in Section 5. Each of the instances was solved ten times, each time using a different seed for the random number generator. Only the solution with the minimum total expected travel time among the ten generated solutions was used in the subsequent analysis. The results are provided in Table 4, where we report for each constrained instance the change in total expected travel time ( $\Delta E$ ) and the number of required vehicles ( $m(x)$ ) from the corresponding unconstrained instance.

We observe that for 22 out of the 54 constrained instances, the number of vehicles required to serve the customers increases. This shows that duration limits have an important effect on required fleet size, as expected. Of course, an increase in the number of required vehicles is more likely when the duration limit is more restrictive (*i.e.*, when  $\alpha$  is smaller). This is clearly reflected in the results, because for  $\alpha = 0.75$  the number of required vehicles increases for 83.30% of the instances, for  $\alpha = 0.75$  the number of required vehicles increases for 38.90% of the instances, but for  $\alpha = 0.95$  none of the instances require more vehicles.

We also observe that the price that must be paid for guaranteeing a maximum tour duration, in terms of the total expected travel time, is relatively small. There are a few exceptions. When the number of customers is small (20), the vehicle capacity is large ( $Q_2$ ),

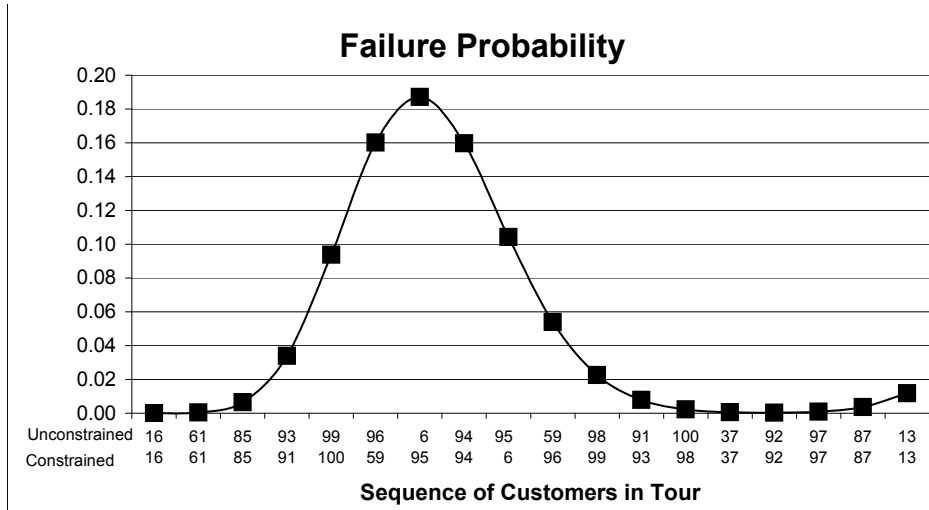


Figure 6: Recourse probabilities for the two tours illustrated in Figure 5.

and the tour duration is severely restricted ( $\alpha = 0.75$ ), then the total expected travel time increases by more than 8% regardless of whether the demand variability is low, medium, or high. Note also that these are the only instances where the number of required vehicles increases by two. In a few cases, the necessity to increase the fleet size can actually lead to a decrease in the total expected travel time. The chance of this happening is greater when the number of customers is large (60 or 100) and the vehicle capacity is small ( $Q_1$ ). Note that in such cases, it would have been cost-effective (from the perspective of only minimizing total expected cost) to introduce preplanned detours to the depot in the solution to the unconstrained problem.

Enforcing tour duration constraints in instances with a small number of customers has a stronger effect (both in terms of the required number of vehicles and the total expected travel time) than in instances with a large number of customers. This suggests that delivery (or pickup) environments in which customers are geographically more concentrated will be less affected by enforcing duration constraints than systems with more geographically dispersed customers.

Similarly, enforcing tour duration constraints in instances with high-capacity vehicles (*i.e.*,  $Q_2$ ) has a more pronounced impact than in instances with low-capacity vehicles (*i.e.*,  $Q_1$ ). For low-capacity instances, 74.1% of the constrained versions require the same number of vehicles as their unconstrained counterparts, while only 44.5% of the high-capacity instances require the same number of vehicles. Interestingly, though, for instances with a large number of customers ( $n = 100$ ) it does seem to be beneficial to have a fleet of high-capacity vehicles. The explanation may be simple. Although tour duration constraints tend to reduce the benefit of high-capacity vehicles, when customers are highly concentrated more of them can be served even in short duration tours. Thus, high-capacity vehicles may still offer advantages.

Finally, and not surprisingly, increasing demand variability tends to increase the required number of vehicles and the total expected travel time. However, the impact of demand variability seems to be less than the impact of other factors.

## 8 Conclusion

We have shown how to enforce maximum duration constraints on a set of delivery tours in the vehicle routing problem with stochastic demands, which is an important idea in practice. Furthermore, our computational study shows that while enforcing duration constraints may come only at a small increase in expected duration, it is certainly more difficult to serve customers with the same number of vehicles. Duration constraints, while important, are only one important timing consideration for stochastic vehicle routing problems. Additional work is needed to also understand how to handle other important practical time considerations, such as customer time windows.

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Table 4: Metrics for the solutions found for the constrained instances

$(n, Q, \sigma)$	$\alpha$	$\Delta \mathbf{E}$	$m(x)$
(20, $Q_1$ , High)	0.95	1.35%	3
	0.85	1.96%	4
	0.75	1.96%	4
(20, $Q_1$ , Med)	0.95	1.87%	3
	0.85	2.21%	4
	0.75	2.65%	4
(20, $Q_1$ , Med)	0.95	1.50%	3
	0.85	2.55%	4
	0.75	3.71%	4
(20, $Q_2$ , High)	0.95	2.34%	6
	0.85	4.57%	7
	0.75	8.82%	8
(20, $Q_2$ , Med)	0.95	0.38%	6
	0.85	2.58%	6
	0.75	7.02%	8
(20, $Q_2$ , Low)	0.95	1.06%	6
	0.85	4.10%	6
	0.75	8.96%	8
(60, $Q_1$ , High)	0.95	0.18%	3
	0.85	-1.03%	4
	0.75	-0.75%	4
(60, $Q_1$ , Med)	0.95	4.88%	3
	0.85	-0.34%	4
	0.75	0.73%	4
(60, $Q_1$ , Low)	0.95	1.12%	3
	0.85	0.87%	4
	0.75	4.26%	4
(60, $Q_2$ , High)	0.95	0.74%	6
	0.85	1.35%	6
	0.75	1.35%	6
(60, $Q_2$ , Med)	0.95	0.83%	6
	0.85	0.83%	6
	0.75	0.83%	6
(60, $Q_2$ , Low)	0.95	0.44%	6
	0.85	0.44%	6
	0.75	0.44%	6
(100, $Q_1$ , High)	0.95	0.67%	3
	0.85	1.28%	3
	0.75	-1.66%	4
(100, $Q_1$ , Med)	0.95	0.17%	3
	0.85	0.61%	3
	0.75	-1.15%	4
(100, $Q_1$ , Low)	0.95	0.17%	3
	0.85	0.89%	3
	0.75	-0.74%	4
(100, $Q_2$ , High)	0.95	0.69%	6
	0.85	1.60%	6
	0.75	-0.31%	7
(100, $Q_2$ , Med)	0.95	1.01%	6
	0.85	2.86%	6
	0.75	-0.41%	7
(100, $Q_2$ , Low)	0.95	1.18%	6
	0.85	1.18%	6
	0.75	5.22%	7