JOURNAL OF NUMBER THEORY 38, 18–34 (1991)

On the Resolution of Index Form Equations in Biquadratic Number Fields, I

I. GAÁL* AND A. PETHÖ*

Kossuth Lajos University, Mathematical Institute, Debrecen, Pf. 12, H-4010 Hungary

AND

M. POHST†

Mathematisches Institut der Universität Düsseldorf, Universitätsstrasse 1, 4000 Düsseldorf 1, Germany

Communicated by Hans Zassenhaus

Received December 20, 1988

In this paper we develop a method for computing all small solutions (i.e. with coordinates of absolute value $<10^7$) of index form equations in totally real biquadratic number fields. If the index form equation is not solvable, this will also be recognized by our algorithm in most cases. As an application we present all such solutions in quadratic extensions K of $Q(\sqrt{5})$ of discriminant $D_{K/Q}<63000$ and of $Q(\sqrt{2})$ of discriminant $D_{K/Q}<39000$.

1. INTRODUCTION

Let K be an algebraic number field of degree n with integral basis 1, $w_2, ..., w_n$. Then

$$D_{K/Q}(w_2X_2+ \cdots + w_nX_n)=(I(X_2, ..., X_n))^2\, D_{K/Q},$$

where the left-hand side is the discriminant of the linear form $w_2X_2+ \cdots +w_nX_n$ and on the right side $D_{K/Q}$ is the discriminant of the field K. $I(X_2, ..., X_n)$ is a homogeneous polynomial of degree $n(n-1)/2$ over the rational integers which is called the *index form* of the basis 1, $w_2, ..., w_n$. The *index form equation*

$$I(x_2, ..., x_n)= \pm 1 \qquad \text{in} \quad x_2, ..., x_n \in Z \tag{1}$$

* Research supported in part by Hungarian National Foundation for Scientific Research Grant No. 273/86.

† Research supported in part by the Deutsche Forschungsgemeinschaft.

0022-314X/91 $3.00

plays an essential role in algebraic number theory since for any[1] $\theta \in Z_K$ the elements $1, \theta, ..., \theta^{n-1}$ form an integral basis of the field K if and only if

$$\theta = \pm(x_1 + w_2 x_2 + \cdots + w_n x_n),$$

where $x_1 \in Z$ and $(x_2, ..., x_n) \in Z^{n-1}$ is a solution of (1). Hence, solving (1) yields all power integral bases of K. Using Baker's method Győry [5, 6, 7], Győry and Papp [8] (see also the references given in these papers), and Trelina [18] gave effective upper bounds for all solutions of (1). These bounds imply in principle that all solutions can be determined, but in practice they are too large to check all possibilities.

Until today there exists no general computational method for the resolution of (1). Using the reduction method discovered by Baker and Davenport [1] and applying some ideas of Ellison [2], Ellison *et al.* [3], Steiner [17], Pethö and Schulenberg [13], and Pethö [12] (concerning the resolution of Thue equations), recently Gaál and Schulte [4] determined all solutions of several index form equations in cubic number fields.

The determination of all solutions of an index form equation splits into two subtasks. The search for small solutions (e.g., the coordinates being less than 10^7 in absolute value) and for large solutions is done separately since the latter in general means to prove that they do not exist. Hence, in this paper we just present an algorithm for the computation of all small solutions in the case when K is a totally real biquadratic number field. (We remark that our method is applicable also if K is not totally real, but those cases can be dealt with much easier.) In Part II we shall give a method for the complete resolution of (1) under the additional assumption that K is cyclic. This second algorithm involves the reduction method with all its advantages and disadvantages: it proves that there are no "large" solutions, but the price of it is a longer computation time.

2. Index Form Equations of Biquadratic Number Fields

Let K be a biquadratic number field. As in the paper of Pohst [14] we assume that all these fields are given in the form $K = Q(\sqrt{\mu})$ with $\mu = (e + f\sqrt{m})/2$, where $e, f, m \in Z$, m is positive and square-free and μ is totally positive and not a square in $L = Q(\sqrt{m})$. Let $w = (g + h\sqrt{m})/2$ with $g = h = 1$ if $m \equiv 1 \pmod 4$ and $g = 0$, $h = 2$ if $m \equiv 2, 3 \pmod 4$. If L has class number one, then there exist $a, b, c, d \in Z$ (cf. [14]) such that if $\psi = (a + b\sqrt{m} + (c + d\sqrt{m})\sqrt{\mu})/4$ then $\{1, w, \psi, w\psi\}$ forms an integral

[1] As usual, Z_K denotes the ring of integers of the algebraic number field K.

basis of K. In the sequel we denote by w' the conjugate of $w \in L$ over Q and by $\psi = \psi_1, \psi_2, \psi_3, \psi_4$ the conjugates of $\psi \in K$ over Q. We note that the conjugates of $\sqrt{\mu} \in K$ over Q are $\sqrt{\mu}, \sqrt{\mu'}, -\sqrt{\mu}, -\sqrt{\mu'}$ $(\mu' = (e - f\sqrt{m})/2)$.

The discriminant of the field is

$$D_{K/Q} = \begin{vmatrix} 1 & w & \psi_1 & w\psi_1 \\ 1 & w' & \psi_2 & w'\psi_2 \\ 1 & w & \psi_3 & w\psi_3 \\ 1 & w' & \psi_4 & w'\psi_4 \end{vmatrix}^2 = [(w - w')^2 (\psi_1 - \psi_3)(\psi_2 - \psi_4)]^2. \qquad (2)$$

Further, let $l_{ij}(\underline{X})$ denote the difference of the ith and jth conjugates of the linear form $l(\underline{X}) = wX_2 + \psi X_3 + w\psi X_4$ for any $1 \leqslant i, j \leqslant 4$. Then we have

$$\begin{aligned}
l_{12}(\underline{X}) &= (w - w')\, X_2 + (\psi_1 - \psi_2)\, X_3 + (w\psi_1 - w'\psi_2)\, X_4 \\
l_{23}(\underline{X}) &= -(w - w')\, X_2 + (\psi_2 - \psi_3)\, X_3 + (w'\psi_2 - w\psi_3)\, X_4 \\
l_{34}(\underline{X}) &= (w - w')\, X_2 + (\psi_3 - \psi_4)\, X_3 + (w\psi_3 - w'\psi_4)\, X_4 \\
l_{41}(\underline{X}) &= -(w - w')\, X_2 + (\psi_4 - \psi_1)\, X_3 + (w'\psi_4 - w\psi_1)\, X_4 \\
l_{13}(\underline{X}) &= (\psi_1 - \psi_3)(X_3 + wX_4) \\
l_{24}(\underline{X}) &= (\psi_2 - \psi_4)(X_3 + w'X_4).
\end{aligned} \qquad (3)$$

PROPOSITION 1. *$(x_2, x_3, x_4) \in Z^3$ is a solution of*

$$I(x_2, x_3, x_4) = \pm 1 \qquad (4)$$

if and only if

$$x_3^2 + (w + w')\, x_3 x_4 + ww' x_4^2 = \pm 1 \qquad (5)$$

and

$$l_{12}(\underline{x})\, l_{23}(\underline{x})\, l_{34}(\underline{x})\, l_{41}(\underline{x}) = \pm (w - w')^2. \qquad (6)$$

Proof. In our case

$$\begin{aligned}
D_{K/Q}(wX_2 + \psi X_3 + w\psi X_4) &= \prod_{1 \leqslant i < j \leqslant 4} l_{ij}^2(\underline{X}) \\
&= (I(X_2, X_3. X_4))^2\, D_{K/Q}
\end{aligned}$$

whence by (2) the index form can be written as

$$I(X_2, X_3, X_4) = \frac{\pm 1}{(w-w')^2} (X_3^2 + (w+w') X_3 X_4 + ww' X_4^2) \times l_{12}(\underline{X})\, l_{23}(\underline{X})\, l_{34}(\underline{X})\, l_{41}(\underline{X}).$$

But $I(X_2, X_3, X_4)$, the second degree factor $(X_3^2 + (w+w') X_3 X_4 + ww' X_4^2)$, and the product of the last four factors have integral coefficients. Thus the constant $(w-w')^2 = h^2 m \in Z$ cannot divide the second degree factor (which has leading coefficient 1) only the product of the last four factors. Thus the index form equation (4) is equivalent to

$$(x_3^2 + (w+w') x_3 x_4 + ww' x_4^2)\, l_{12}(\underline{x})\, l_{23}(\underline{x})\, l_{34}(\underline{x})\, l_{41}(\underline{x}) = \pm(w-w')^2 \quad \text{in} \quad \underline{x} = (x_2, x_3, x_4) \in Z^3$$

and for any solution x_2, x_3, x_4 of it we must have (5) and (6). ∎

3. Transition to a Quadratic Subfield

In this part we reduce the problem of solving (5) and (6) to a diophantine problem over a quadratic subfield of K. This will lead us to a second degree equation (15) in x_2, x_3, x_4 with rational integral coefficients which, combined with (5), will give all solutions of (4). First we prove:

THEOREM 1. *Let c, d, e, f, h, m be as above, $(e^2 - f^2 m)/4 = m_0^2 m_1$, where m_1 is square-free, and let $u = m_0(c^2 - d^2 m)/4$. If* (5) *and* (6) *have a solution, then there exists an integer v such that*

$$v^2 - m_1 u^2 = \pm 4h^2 m. \tag{7}$$

Remark. This theorem is very useful for proving that (4) is unsolvable. In our table (at the end of the paper) one can see that in the considered cases, the solvability of (7) implies the solvability of (4), with the exception of only one example. Further we note that by (7) there are four possibilities for v corresponding to u (according to signs), and in the sequel we have to examine the occurring integral values for v separately.

Proof. Using (3) we calculate the coefficients of the product of the linear forms $l_{12}(\underline{X})$ and $l_{34}(\underline{X})$. We obtain

$$l_{12}(\underline{X})\, l_{34}(\underline{X}) = C_2 X_2^2 + (C_3 + C_3'\zeta) X_3^2 + (C_4 + C_4'\zeta) X_4^2 + C_{23} X_2 X_3 + C_{24} X_2 X_4 + (C_{34} + C_{34}'\zeta) X_3 X_4 \tag{8}$$

with $\zeta = \sqrt{(e^2 - f^2 m)/4} = m_0 \sqrt{m_1}$, where m_0, m_1 are positive integers, m_1 is square-free and

$$\begin{aligned}
C_2 &= h^2 m,\\
C_3 &= (4b^2 m - c^2 e - 2cdfm - d^2 me)/16,\\
C_4 &= \{4m(gb+ha)^2 - e[(g^2+h^2m)(c^2+d^2m) + 4ghcdm]\\
&\quad - fm[2gh(c^2+d^2m) + 2cd(g^2+h^2m)]\}/64,\\
C_3' &= (c^2 - d^2 m)/8,\\
C_4' &= (g^2 - h^2 m)(c^2 - d^2 m)/32,\\
C_{23} &= hbm,\\
C_{24} &= hm(gb+ha)/2,\\
C_{34} &= \{4bm(gb+ha) - e[g(c^2+d^2m) + 2cdhm]\\
&\quad - fm[h(c^2+d^2m) + 2cdg]\}/16,\\
C_{34}' &= g(c^2 - d^2 m)/8.
\end{aligned}$$

As we can see $l_{12}(\underline{X})\, l_{34}(\underline{X})$ has coefficients in $M = Q(\sqrt{m_1})$. Note that $l_{23}(\underline{X})\, l_{41}(\underline{X})$ is just the conjugate of $l_{12}(\underline{X})\, l_{34}(\underline{X})$ over M. In view of (6) this implies that for any solution $\underline{x} \in Z^3$ of (6) we have

$$l_{12}(\underline{x})\, l_{34}(\underline{x}) = \pm \alpha \eta^B, \tag{9}$$

where η is the fundamental unit of M, $B \in Z$ and $\alpha \in Z_M$ has norm $\pm h^2 m = \pm (w - w')^2$. Let us denote by α', η' the conjugates of $\alpha, \eta \in M$. We take conjugates in (9) and subtract the result from (9) to obtain

$$l_{12}(\underline{x})\, l_{34}(\underline{x}) - l_{23}(\underline{x})\, l_{41}(\underline{x}) = \pm(\alpha\eta^B - \alpha'(\eta')^B). \tag{10}$$

On the other hand, we take conjugates in (8) and calculate the coefficients of the expression on the left side of (10); to obtain

$$\begin{aligned}
&l_{12}(\underline{X})\, l_{34}(\underline{X}) - l_{23}(\underline{X})\, l_{41}(\underline{X})\\
&\quad = m_0 \sqrt{m_1}\, \frac{c^2 - d^2 m}{4} \left(X_3^2 + g X_3 X_4 + \frac{g^2 - h^2 m}{4} X_4^2 \right).
\end{aligned} \tag{11}$$

But $w + w' = g$ and $ww' = (g^2 - h^2 m)/4$ and thus combining (5), (11), and (10) we conclude

$$u = m_0 \frac{(c^2 - d^2 m)}{4} = \frac{\pm(\alpha\eta^B - \alpha'(\eta')^B)}{\sqrt{m_1}}. \tag{12}$$

We observe that there is a close connection between $u = \pm(\alpha\eta^B - \alpha'(\eta')B)/\sqrt{m_1}$ and $v = \pm(\alpha\eta^B + \alpha'(\eta')B)$ which makes it possible to find the value of v, since the value of u is fixed. Namely, we have

$$\begin{aligned} v^2 - m_1 u^2 &= \alpha^2\eta^{2B} + (\alpha')^2(\eta')^{2B} + 2\alpha\alpha'(\eta\eta')^B \\ &\quad - [\alpha^2\eta^{2B} + (\alpha')^2(\eta')^{2B} - 2\alpha\alpha'(\eta\eta')^B] \\ &= 4\alpha\alpha'(\eta\eta')^B = \pm 4N_{M/Q}(\alpha) = \pm 4h^2 m \end{aligned}$$

which proves our assertion. ▌

In the same way as we did above let us calculate the sum of $l_{12}(\underline{x})\, l_{34}(\underline{x})$ and $l_{23}(\underline{x})\, l_{41}(\underline{x})$. Similarly to (10), (9) implies

$$l_{12}(\underline{x})\, l_{34}(\underline{x}) + l_{23}(\underline{x})\, l_{41}(\underline{x}) = \pm(\alpha\eta^B + \alpha'(\eta')^B). \tag{13}$$

Again calculating the coefficients of the left side of (13) from (8), we obtain

$$\begin{aligned} &2C_2x_2^2 + 2C_3x_3^2 + 2C_4x_4^2 + 2C_{23}x_2x_3 + 2C_{24}x_2x_4 + 2C_{34}x_3x_4 \\ &\quad = \pm(\alpha\eta^B + \alpha'(\eta')^B). \end{aligned} \tag{14}$$

If v is a possible value in (7), then using (14) we obtain

$$\begin{aligned} &2C_2x_2^2 + 2C_3x_3^2 + 2C_4x_4^2 + 2C_{23}x_2x_3 \\ &\quad + 2C_{24}x_2x_4 + 2C_{34}x_3x_4 + v = 0. \end{aligned} \tag{15}$$

Consider this equation as an equation in x_2. It can have an integral solution only if its discriminant with respect to x_2 is a full square, that is

$$A_3x_3^2 + A_4x_4^2 + A_{34}x_3x_4 - A_0 = y^2$$

for some $y \in Z$ and

$$\begin{aligned} A_3 &= 4C_{23}^2 - 16C_2C_3, \\ A_4 &= 4C_{24}^2 - 16C_2C_4, \\ A_{34} &= 8C_{23}C_{24} - 16C_2C_{34}, \\ A_0 &= 8C_2v. \end{aligned}$$

Hence we proved the following theorem.

THEOREM 2. *If $(x_2, x_3, x_4) \in Z^3$ is a solution of (4) then there exists an $y \in Z$ with*

$$x_3^2 + (w + w')\, x_3x_4 + ww'x_4^2 = \pm 1 \tag{16}$$

and

$$A_3x_3^2 + A_4x_4^2 + A_{34}x_3x_4 - A_0 = y^2. \tag{17}$$

Remark. Mordell [9, p. 59] proved, that the system of diophantine equations (16) and (17) has only finitely many solutions x_3, x_4, y. Shorey and Stewart [16] gave even an effectively computable upper bound for $\max(|x_3|, |x_4|, |y|)$ of any solution of (16) and (17).

4. Connection with Recurrence Sequences

We shall show that combining (17) with (16) our problem reduces to the searching for elements of the form $y^2 + \delta$ ($\delta \in Z$ fixed) in second order linear recurrence sequences. For the resolution of similar problems see, e.g., Pethö [10, 11]. We are interested only in small solutions so we can simply test the elements of the sequence with small index whether or not they are of the form $y^2 + \delta$.

We consider the cases $m \equiv 2, 3 \pmod 4$ and $m \equiv 1 \pmod 4$ separately.

I. If $m \equiv 2, 3 \pmod 4$ then (16) has the form

$$x_3^2 - mx_4^2 = \pm 1$$

the solutions of which are

$$x_3 = \frac{\varepsilon^n + (\varepsilon')^n}{2} \quad \text{and} \quad x_4 = \frac{\varepsilon^n - (\varepsilon')^n}{2\sqrt{m}} \quad (n \in Z), \tag{18}$$

where $\varepsilon = e_0 + e_1\sqrt{m}$ ($e_0, e_1 \in Z$) is a fundamental unit of L and ε' its conjugate. Substituting (18) into

$$A_3x_3^2 + A_4x_4^2 + A_{34}x_3x_4 = y^2 + A_0 \tag{19}$$

(cf. (17)) we obtain

$$\{(mA_3 + A_4 + \sqrt{m}A_{34})\,\varepsilon^{2n} + (mA_3 + A_4 - \sqrt{m}A_{34})(\varepsilon')^{2n}\}/(4m)$$
$$= y^2 + A_0 - (\varepsilon\varepsilon')^n \frac{mA_3 - A_4}{2m}. \tag{20}$$

By our definition A_3 is even and $2m$ divides A_4, hence the right side of (20) is an integer. Denote by G_n the left side of (20). Then $\{G_n\}$ satisfies the difference equation

$$G_{n+2} = 2(e_0^2 + me_1^2)\,G_{n+1} - G_n$$

and we can easily determine G_0 and G_1.

II. If $m \equiv 1 \pmod 4$ then (16) can be written as

$$x_3^2 + x_3 x_4 - \frac{m-1}{4} x_4^2 = \pm 1.$$

Obviously all solutions of that equation are given by

$$x_3 = \frac{-w'\varepsilon^n + w(\varepsilon')^n}{\sqrt{m}} \quad \text{and} \quad x_4 = \frac{\varepsilon^n - (\varepsilon')^n}{\sqrt{m}} \quad (n \in Z), \tag{21}$$

where $\varepsilon = f_0 + f_1((1+\sqrt{m})/2)$ $(f_0, f_1 \in Z)$ is a fundamental unit of L and ε' its conjugate. Inserting the values (21) into (19) we obtain

$$\{(A_3(w')^2 + A_4 - A_{34}w')\,\varepsilon^{2n} + (A_3 w^2 + A_4 - A_{34} w)(\varepsilon')^{2n}\}/m$$
$$= y^2 + A_0 + \frac{1}{m}\left(\frac{1-m}{2} A_3 + 2A_4 - A_{34}\right)(\varepsilon\varepsilon')^n. \tag{22}$$

By the definition of C_2, C_3, C_{23} and C_{24}, m divides A_3, A_4 and A_{34}, hence the right side of (22) is an integer. Denote by G_n the left side of (22). As one can see it satisfies the difference equation

$$G_{n+2} = \left(2f_0^2 + 2f_0 f_1 + f_1^2 \frac{1+m}{2}\right) G_{n+1} - G_n$$

and again G_0, G_1 can be easily calculated.

If in case I (resp. in case II) we find an index n such that G_n has the prescribed form (20) (resp. (22), then from (18) (resp. (21)) we can determine the corresponding x_3 and x_4. Finally, x_2 can be obtained from (15).

5. Computational Results

In the table below we determined small solutions of index form equations corresponding to all totally real biquadratic number fields with $m = 5$, $D_{K/Q} < 63000$ and $m = 2$, $D_{K/Q} < 39000$.[2] (We remark that we have $2A_3 - 2A_4 + A_{34} = 0$ in case $m = 5$ and $A_4 = 2A_3$ in case $m = 2$). All input data were taken from the paper of Pohst [15]. For every example our table contains the discriminant, the values of e and f, for each possible value of v the coefficients of (17); $A33$, $A44$, $A34$, and $A0$ denote the coefficients of x_3^2, x_4^2, $x_3 x_4$, and the constant term, respectively. The following line contains the data of the corresponding recurrence sequence: $G_0 = G0$, $G_1 = G1$ are the initializing terms and pd is such that $G_{n+2} = pd * G_{n+1} - G_n$. The sign "no values for v" means that in the actual case there

[2] Recently we extended our computations to any biquadratic number field K with $D_{K/Q} \leqslant 1000000$ satisfying the assumptions of Section 2.

are no solutions. If we found a possible v, then "no small values for x_3, x_4" means that the recurrence sequence did not have an element of the prescribed form with small index n. (In principle there may be large suitable values for n but it is very unlikely by our experience.)

This occurred two times in our computations, namely for $D_{K/Q}=2048$ and $v=0$ as well as for $D_{K/Q}=18432$ and $v=16$. It is easy to see that the corresponding system of Eqs. (16) and (17) does not have any solutions. In fact (17) implies that x_3 is even, which is impossible by (16).

If (x_2, x_3, x_4) is a solution then also $(-x_2, -x_3, -x_4)$ is one but we print only one of them. The numbers heading the parts of the table (e.g., (2), (3)) coincide with the corresponding numbers in the table of [15].

In case $m=5$ we tested the members of the recurrence sequence with index $|n| \leqslant 37$. In case $m=2$ the bound was 20. This means that in both cases we determined all solutions of the index form equation with $\max(|x_3|, |x_4|) \leqslant 15000000$. The program was written in PASCAL on an IBM PC compatible machine and the execution time was under two seconds for every example.

Extensions of $Q(\sqrt{5})$

(2)
discr = 725 $m=5$ $e/2=7$ $f/2=2$
***** $v=7$
square: $A33=70$ $A44=155$
$A34=170$ $\pm A0=-70$
rec. seq.: $G0=70$ $G1=155$ $pd=3$
$x2=-2$ $x3=0$ $x4=1$
$x2=0$ $x3=1$ $x4=0$
$x2=0$ $x3=2$ $x4=-1$
$x2=1$ $x3=0$ $x4=1$
$x2=1$ $x3=2$ $x4=-1$
***** $v=3$
square: $A33=70$ $A44=155$
$A34=170$ $\pm A0=-30$
rec. seq.: $G0=70$ $G1=155$ $pd=3$
$x2=-1$ $x3=-1$ $x4=1$
$x2=-1$ $x3=1$ $x4=0$
$x2=-1$ $x3=5$ $x4=-3$
$x2=0$ $x3=-1$ $x4=1$
$x2=1$ $x3=1$ $x4=0$
$x2=4$ $x3=5$ $x4=-3$
discr = 2225 $m=5$ $e/2=13$ $f/2=4$
no values for v
discr = 2525 $m=5$ $e/2=11$ $f/2=2$
***** $v=11$
square: $A33=110$ $A44=215$
$A34=210$ $\pm A0=-110$
rec. seq.: $G0=110$ $G1=215$ $pd=3$
$x2=-2$ $x3=-1$ $x4=1$
$x2=0$ $x3=1$ $x4=0$
$x2=1$ $x3=-1$ $x4=1$
***** $v=9$
square: $A33=110$ $A44=215$
$A34=210$ $\pm A0=-90$
rec. seq.: $G0=110$ $G1=215$ $pd=3$
$x2=-3$ $x3=1$ $x4=1$
$x2=-1$ $x3=-1$ $x4=1$
$x2=0$ $x3=-1$ $x4=1$
$x2=2$ $x3=1$ $x4=1$
discr = 4525 $m=5$ $e/2=19$ $f/2=6$
no values for v
discr = 5725 $m=5$ $e/2=27$ $f/2=10$
no values for v
discr = 8725 $m=5$ $e/2=23$ $f/2=6$
no values for v
discr = 10025 $m=5$ $e/2=41$ $f/2=16$
no values for v
discr = 11525 $m=5$ $e/2=31$ $f/2=10$
***** $v=21$
square: $A33=310$ $A44=715$
$A34=810$ $\pm A0=-210$
rec. seq.: $G0=310$ $G1=715$ $pd=3$
$x2=-1$ $x3=1$ $x4=0$

Extensions of $Q(\sqrt{5})$

$x2 = 1 \quad x3 = 1 \quad x4 = 0$
discr = 12725 m = 5 e/2 = 23 f/2 = 2
***** $v = 23$
square: $A33 = 230 \quad A44 = 395$
$A34 = 330 \quad \pm A0 = -230$
rec. seq.: $G0 = 230 \quad G1 = 395 \quad pd = 3$
$x2 = -3 \quad x3 = 0 \quad x4 = 1$
$x2 = 0 \quad x3 = 1 \quad x4 = 0$
$x2 = 2 \quad x3 = 0 \quad x4 = 1$
discr = 13025 m = 5 e/2 = 29 f/2 = 8
no values for v
discr = 13525 m = 5 e/2 = 39 f/2 = 14
no values for v
discr = 17725 m = 5 e/2 = 27 f/2 = 2
***** $v = 27$
square: $A33 = 270 \quad A44 = 455$
$A34 = 370 \quad \pm A0 = -270$
rec. seq.: $G0 = 270 \quad G1 = 455 \quad pd = 3$
$x2 = -3 \quad x3 = -1 \quad x4 = 1$
$x2 = 0 \quad x3 = 1 \quad x4 = 0$
$x2 = 2 \quad x3 = -1 \quad x4 = 1$
discr = 19025 m = 5 e/2 = 29 f/2 = 4
no values for v
discr = 19225 m = 5 e/2 = 33 f/2 = 8
no values for v
discr = 20225 m = 5 e/2 = 53 f/2 = 20
no values for v
discr = 23525 m = 5 e/2 = 31 f/2 = 2
***** $v = 31$
square: $A33 = 310 \quad A44 = 515$
$A34 = 410 \quad \pm A0 = -310$
rec. seq.: $G0 = 310 \quad G1 = 515 \quad pd = 3$
$x2 = -2 \quad x3 = 2 \quad x4 = -1$
$x2 = 0 \quad x3 = 1 \quad x4 = 0$
$x2 = 3 \quad x3 = 2 \quad x4 = -1$
discr = 25225 m = 5 e/2 = 33 f/2 = 4
no values for v
discr = 25525 m = 5 e/2 = 39 f/2 = 10
no values for v
discr = 26225 m = 5 e/2 = 37 f/2 = 8
no values for v
discr = 26525 m = 5 e/2 = 59 f/2 = 22
no values for v
discr = 27725 m = 5 e/2 = 67 f/2 = 26
***** $v = 33$
square: $A33 = 670 \quad A44 = 1655$
$A34 = 1970 \quad \pm A0 = -330$
rec. seq.: $G0 = 670 \quad G1 = 1655 \quad pd = 3$
$x2 = -1 \quad x3 = -1 \quad x4 = 1$
$x2 = 0 \quad x3 = -1 \quad x4 = 1$
discr = 30725 m = 5 e/2 = 47 f/2 = 14
no values for v
discr = 31225 m = 5 e/2 = 57 f/2 = 20
no values for v
discr = 32225 m = 5 e/2 = 37 f/2 = 4
no values for v
discr = 34025 m = 5 e/2 = 41 f/2 = 8
no values for v
discr = 35225 m = 5 e/2 = 73 f/2 = 28
no values for v
discr = 38725 m = 5 e/2 = 63 f/2 = 22
no values for v
discr = 40025 m = 5 e/2 = 41 f/2 = 4
no values for v
discr = 40525 m = 5 e/2 = 51 f/2 = 14
no values for v
discr = 41725 m = 5 e/2 = 43 f/2 = 6
no values for v
discr = 42725 m = 5 e/2 = 47 f/2 = 10
no values for v
discr = 43025 m = 5 e/2 = 61 f/2 = 20
no values for v
discr = 43525 m = 5 e/2 = 79 f/2 = 30
no values for v
discr = 44725 m = 5 e/2 = 87 f/2 = 34
no values for v
discr = 46525 m = 5 e/2 = 59 f/2 = 18
no values for v
discr = 50725 m = 5 e/2 = 47 f/2 = 6
no values for v
discr = 51725 m = 5 e/2 = 67 f/2 = 22
no values for v
discr = 52025 m = 5 e/2 = 49 f/2 = 8
no values for v
discr = 52225 m = 5 e/2 = 53 f/2 = 12
no values for v
discr = 59725 m = 5 e/2 = 83 f/2 = 30
no values for v
discr = 61025 m = 5 e/2 = 61 f/2 = 16
no values for v

(3)
discr = 1125 m = 5 e/2 = 15 f/2 = 6
***** $v = 5$
square: $A33 = 6 \quad A44 = 15$
$A34 = 18 \quad \pm A0 = -2$

Extensions of $Q(\sqrt{5})$

rec. seq.: $G0=6$ $G1=15$ $pd=3$
$x2=-6$ $x3=1$ $x4=2$
$x2=-2$ $x3=-3$ $x4=2$
$x2=-1$ $x3=-1$ $x4=1$
$x2=-1$ $x3=1$ $x4=0$
$x2=-1$ $x3=13$ $x4=-8$
$x2=0$ $x3=-3$ $x4=2$
$x2=0$ $x3=-1$ $x4=1$
$x2=0$ $x3=2$ $x4=-1$
$x2=1$ $x3=1$ $x4=0$
$x2=1$ $x3=2$ $x4=-1$
$x2=4$ $x3=1$ $x4=2$
$x2=9$ $x3=13$ $x4=-8$
discr = 5125 $m=5$ $e/2=95$ $f/2=42$
***** $v=15$
square: $A33=38$ $A44=99$
$A34=122$ $\pm A0=-6$
rec. seq.: $G0=38$ $G1=99$ $pd=3$
$x2=-2$ $x3=-1$ $x4=1$
$x2=-1$ $x3=-3$ $x4=2$
$x2=0$ $x3=2$ $x4=-1$
$x2=0$ $x3=13$ $x4=-8$
$x2=1$ $x3=-1$ $x4=1$
$x2=1$ $x3=2$ $x4=-1$
$x2=8$ $x3=13$ $x4=-8$
discr = 5225 $m=5$ $e/2=17$ $f/2=-4$
no values for v
discr = 6125 $m=5$ $e/2=35$ $f/2=14$
***** $v=15$
square: $A33=14$ $A44=35$
$A34=42$ $\pm A0=-6$
rec. seq.: $G0=14$ $G1=35$ $pd=3$
$x2=-1$ $x3=-1$ $x4=1$
$x2=0$ $x3=-1$ $x4=1$
$x2=0$ $x3=2$ $x4=-1$
$x2=1$ $x3=2$ $x4=-1$
discr = 7625 $m=5$ $e/2=65$ $f/2=28$
no values for v
discr = 8525 $m=5$ $e/2=19$ $f/2=2$
***** $v=19$
square: $A33=190$ $A44=335$
$A34=290$ $\pm A0=-190$
rec. seq.: $G0=190$ $G1=335$ $pd=3$
$x2=-3$ $x3=1$ $x4=1$
$x2=0$ $x3=1$ $x4=0$
$x2=2$ $x3=1$ $x4=1$
discr = 9225 $m=5$ $e/2=33$ $f/2=12$
no values for v
discr = 13625 $m=5$ $e/2=145$ $f/2=64$
no values for v
discr = 13725 $m=5$ $e/2=27$ $f/2=6$
***** $v=23$
square: $A33=270$ $A44=555$
$A34=570$ $\pm A0=-230$
rec. seq.: $G0=270$ $G1=555$ $pd=3$
$x2=-1$ $x3=-1$ $x4=1$
$x2=0$ $x3=-1$ $x4=1$
discr = 14725 $m=5$ $e/2=63$ $f/2=26$
no values for v
discr = 16225 $m=5$ $e/2=37$ $f/2=12$
no values for v
discr = 18625 $m=5$ $e/2=85$ $f/2=36$
no values for v
discr = 19525 $m=5$ $e/2=31$ $f/2=-6$
no values for v
discr = 21725 $m=5$ $e/2=107$ $f/2=46$
no values for v
discr = 24525 $m=5$ $e/2=51$ $f/2=18$
***** $v=31$
square: $A33=510$ $A44=1215$
$A34=1410$ $\pm A0=-310$
rec. seq.: $G0=510$ $G1=1215$ $pd=3$
$x2=-3$ $x3=-1$ $x4=1$
$x2=2$ $x3=-1$ $x4=1$
discr = 28025 $m=5$ $e/2=49$ $f/2=16$
no values for v
discr = 30125 $m=5$ $e/2=275$ $f/2=122$
***** $v=35$
square: $A33=110$ $A44=287$
$A34=354$ $\pm A0=-14$
rec. seq.: $G0=110$ $G1=287$ $pd=3$
$x2=-1$ $x3=-3$ $x4=2$
$x2=0$ $x3=5$ $x4=-3$
$x2=3$ $x3=5$ $x4=-3$
discr = 33525 $m=5$ $e/2=39$ $f/2=6$
no values for v
discr = 33625 $m=5$ $e/2=105$ $f/2=44$
no values for v
discr = 33725 $m=5$ $e/2=187$ $f/2=82$
***** $v=37$
square: $A33=1870$ $A44=4855$
$A34=5970$ $\pm A0=-370$
rec. seq.: $G0=1870$ $G1=4855$ $pd=3$
$x2=0$ $x3=2$ $x4=-1$
$x2=1$ $x3=2$ $x4=-1$
discr = 35125 $m=5$ $e/2=135$ $f/2=58$
no values for v
discr = 36025 $m=5$ $e/2=89$ $f/2=36$

Extensions of $Q(\sqrt{5})$

no values for v
discr = 37525 m = 5 e/2 = 39 f/2 = −2
***** $v = 39$
square: $A33 = 390$ $A44 = 535$
$A34 = 290$ $\pm A0 = -390$
rec. seq.: $G0 = 390$ $G1 = 535$ $pd = 3$
$x2 = 0$ $x3 = 1$ $x4 = 0$
discr = 38225 m = 5 e/2 = 53 f/2 = 16
no values for v
discr = 41525 m = 5 e/2 = 71 f/2 = 26
***** $v = 41$
square: $A33 = 710$ $A44 = 1715$
$A34 = 2010$ $\pm A0 = -410$
rec. seq.: $G0 = 710$ $G1 = 1715$ $pd = 3$
$x2 = -5$ $x3 = -3$ $x4 = 2$
$x2 = 3$ $x3 = -3$ $x4 = 2$
discr = 45725 m = 5 e/2 = 43 f/2 = 2
***** $v = 43$
square: $A33 = 430$ $A44 = 695$
$A34 = 530$ $\pm A0 = -430$
rec. seq.: $G0 = 430$ $G1 = 695$ $pd = 3$
$x2 = -4$ $x3 = 1$ $x4 = 1$
$x2 = 0$ $x3 = 1$ $x4 = 0$
$x2 = 3$ $x3 = 1$ $x4 = 1$
discr = 48625 m = 5 e/2 = 325 f/2 = 144
no values for v
discr = 49225 m = 5 e/2 = 57 f/2 = −16
no values for v
discr = 50225 m = 5 e/2 = 77 f/2 = 28
no values for v
discr = 51125 m = 5 e/2 = 215 f/2 = 94
***** $v = 45$
square: $A33 = 86$ $A44 = 223$
$A34 = 274$ $\pm A0 = -18$
rec. seq.: $G0 = 86$ $G1 = 223$ $pd = 3$
$x2 = -2$ $x3 = -3$ $x4 = 2$
$x2 = 0$ $x3 = -3$ $x4 = 2$
$x2 = 0$ $x3 = 2$ $x4 = -1$
$x2 = 1$ $x3 = 2$ $x4 = -1$
discr = 52525 m = 5 e/2 = 51 f/2 = 10
no values for v
discr = 52625 m = 5 e/2 = 125 f/2 = 52
no values for v
discr = 54225 m = 5 e/2 = 93 f/2 = 36
no values for v
discr = 54725 m = 5 e/2 = 47 f/2 = −2
***** $v = 47$
square: $A33 = 470$ $A44 = 655$
$A34 = 370$ $\pm A0 = -470$
rec. seq.: $G0 = 470$ $G1 = 655$ $pd = 3$
$x2 = -4$ $x3 = -1$ $x4 = 1$
$x2 = 0$ $x3 = 1$ $x4 = 0$
$x2 = 3$ $x3 = -1$ $x4 = 1$
discr = 55025 m = 5 e/2 = 109 f/2 = 44
no values for v
discr = 56125 m = 5 e/2 = 155 f/2 = 66
no values for v
discr = 58025 m = 5 e/2 = 49 f/2 = 4
no values for v
discr = 60525 m = 5 e/2 = 51 f/2 = 6
**** $v = 49$
square: $A33 = 510$ $A44 = 915$
$A34 = 810$ $\pm A0 = -490$
rec. seq.: $G0 = 510$ $G1 = 915$ $pd = 3$
$x2 = -14$ $x3 = 2$ $x4 = 3$
$x2 = 11$ $x3 = 2$ $x4 = 3$
discr = 61225 m = 5 e/2 = 73 f/2 = −24
no values for v
discr = 62225 m = 5 e/2 = 53 f/2 = 8
no values for v
discr = 62525 m = 5 e/2 = 139 f/2 = 58
no values for v
discr = 62525 m = 5 e/2 = 59 f/2 = 14
no values for v

(4)
discr = 15125 m = 5 e/2 = 55 f/2 = 22
***** $v = 25$
square: $A33 = 22$ $A44 = 55$
$A34 = 66$ $\pm A0 = -10$
rec. seq.: $G0 = 22$ $G1 = 55$ $pd = 3$
$x2 = -1$ $x3 = -1$ $x4 = 1$
$x2 = 0$ $x3 = -1$ $x4 = 1$
$x2 = 0$ $x3 = 2$ $x4 = -1$
$x2 = 1$ $x3 = 2$ $x4 = -1$
discr = 26125 m = 5 e/2 = 435 f/2 = 194
no values for v
discr = 32625 m = 5 e/2 = 165 f/2 = 72
no values for v
discr = 42625 m = 5 e/2 = 245 f/2 = 108
no values for v
discr = 45125 m = 5 e/2 = 95 f/2 = 38
no values for v
discr = 47025 m = 5 e/2 = 141 f/2 = 60
no values for v

Extensions of $Q(\sqrt{5})$

(5)

discr = 2000 m = 5 e/2 = 5 f/2 = 2

***** $v = 10$

square: $A33 = 2$ $A44 = 5$

$A34 = 6$ $\pm A0 = -1$

rec. seq.: $G0 = 2$ $G1 = 5$ $pd = 3$

$x2 = -2$ $x3 = 0$ $x4 = 1$

$x2 = -2$ $x3 = 5$ $x4 = -3$

$x2 = -1$ $x3 = -3$ $x4 = 2$

$x2 = -1$ $x3 = 1$ $x4 = 0$

$x2 = 0$ $x3 = -1$ $x4 = 1$

$x2 = 0$ $x3 = 2$ $x4 = -1$

$x2 = 1$ $x3 = -3$ $x4 = 2$

$x2 = 1$ $x3 = 1$ $x4 = 0$

$x2 = 2$ $x3 = 0$ $x4 = 1$

$x2 = 2$ $x3 = 5$ $x4 = -3$

discr = 4400 m = 5 e/2 = 4 f/2 = 1

***** $v = 14$

square: $A33 = 40$ $A44 = 85$

$A34 = 90$ $\pm A0 = -35$

rec. seq.: $G0 = 40$ $G1 = 85$ $pd = 3$

$x2 = -2$ $x3 = 2$ $x4 = -1$

$x2 = 0$ $x3 = -1$ $x4 = 1$

$x2 = 2$ $x3 = 2$ $x4 = -1$

discr = 7600 m = 5 e/2 = 8 f/2 = 3

***** $v = 18$

square: $A33 = 80$ $A44 = 195$

$A34 = 230$ $\pm A0 = -45$

rec. seq.: $G0 = 80$ $G1 = 195$ $pd = 3$

$x2 = -2$ $x3 = 2$ $x4 = -1$

$x2 = 0$ $x3 = -1$ $x4 = 1$

$x2 = 2$ $x3 = 2$ $x4 = -1$

discr = 12400 m = 5 e/2 = 6 f/2 = 1

no values for v

discr = 16400 m = 5 e/2 = 11 f/2 = 4

***** $v = 26$

square: $A33 = 110$ $A44 = 265$

$A34 = 310$ $\pm A0 = -65$

rec. seq.: $G0 = 110$ $G1 = 265$ $pd = 3$

$x2 = 0$ $x3 = -1$ $x4 = 1$

discr = 23600 m = 5 e/2 = 8 f/2 = 1

no values for v

discr = 24400 m = 5 e/2 = 9 f/2 = 2

no values for v

discr = 28400 m = 5 e/2 = 14 f/2 = 5

***** $v = 34$

square: $A33 = 140$ $A44 = 335$

$A34 = 390$ $\pm A0 = -85$

rec. seq.: $G0 = 140$ $G1 = 335$ $pd = 3$

$x2 = -3$ $x3 = 1$ $x4 = 0$

$x2 = 0$ $x3 = -1$ $x4 = 1$

$x2 = 3$ $x3 = 1$ $x4 = 0$

discr = 31600 m = 5 e/2 = 18 f/2 = 7

no values for v

discr = 43600 m = 5 e/2 = 17 f/2 = 6

***** $v = 42$

square: $A33 = 170$ $A44 = 405$

$A34 = 470$ $\pm A0 = -105$

rec. seq.: $G0 = 170$ $G1 = 405$ $pd = 3$

$x2 = -3$ $x3 = -3$ $x4 = 2$

$x2 = 0$ $x3 = -1$ $x4 = 1$

$x2 = 3$ $x3 = -3$ $x4 = 2$

discr = 52400 m = 5 e/2 = 16 f/2 = 5

***** $v = 46$

square: $A33 = 160$ $A44 = 365$

$A34 = 410$ $\pm A0 = -115$

rec. seq.: $G0 = 160$ $G1 = 365$ $pd = 3$

$x2 = 0$ $x3 = -1$ $x4 = 1$

discr = 55600 m = 5 e/2 = 12 f/2 = 1

no values for v

discr = 59600 m = 5 e/2 = 13 f/2 = 2

no values for v

discr = 60400 m = 5 e/2 = 14 f/2 = 3

no values for v

(6)

discr = 22000 m = 5 e/2 = 30, f/2 = 13

***** $v = 30$

square: $A33 = 12$ $A44 = 31$

$A34 = 38$ $\pm A0 = -3$

rec. seq.: $G0 = 12$ $G1 = 31$ $pd = 3$

$x2 = -3$ $x3 = 1$ $x4 = 0$

$x2 = -1$ $x3 = -3$ $x4 = 2$

$x2 = 0$ $x3 = 2$ $x4 = -1$

$x2 = 1$ $x3 = -3$ $x4 = 2$

$x2 = 3$ $x3 = 1$ $x4 = 0$

discr = 38000 m = 5 e/2 = 70 f/2 = 31

no values for v

discr = 39600 m = 5 e/2 = 12 f/2 = 3

no values for v

discr = 58000 m = 5 e/2 = 55 f/2 = 24

no values for v

Extensions of $Q(\sqrt{5})$

discr = 62000 $m = 5$ $e/2 = 40$ $f/2 = 17$
***** $v = 50$
square: $A33 = 16$ $A44 = 41$
$A34 = 50$ $\pm A0 = -5$
rec. seq.: $G0 = 16$ $G1 = 41$ $pd = 3$
$x2 = -6$ $x3 = 0$ $x4 = 1$
$x2 = 0$ $x3 = 2$ $x4 = -1$
$x2 = 6$ $x3 = 0$ $x4 = 1$

(7)
discr = 8000 $m = 5$ $e/2 = 10$ $f/2 = 4$
no values for v
discr = 17600 $m = 5$ $e/2 = 8$ $f/2 = 2$
no values for v
discr = 30400 $m = 5$ $e/2 = 16$ $f/2 = 6$
no values for v
discr = 46600 $m = 5$ $e/2 = 14$ $f/2 = 4$
no values for v
discr = 49600 $m = 5$ $e/2 = 12$ $f/2 = 2$
no values for v

Extensions of $Q(\sqrt{2})$

(2)
discr = 2624 $m = 2$ $e/2 = 7$ $f/2 = 2$
***** $v = 3$
square: $A33 = 7$ $A44 = 14$
$A34 = 8$ $\pm A0 = -3$
rec. seq.: $G0 = 7$ $G1 = 29$ $pd = 6$
$x2 = -1$ $x3 = -1$ $x4 = 1$
$x2 = -1$ $x3 = 1$ $x4 = 0$
$x2 = 0$ $x3 = 1$ $x4 = 0$
$x2 = 1$ $x3 = -1$ $x4 = 1$
discr = 7232 $m = 2$ $e/2 = 11$ $f/2 = 2$
***** $v = 9$
square: $A33 = 11$ $A44 = 22$
$A34 = 8$ $\pm A0 = -9$
rec. seq.: $G0 = 11$ $G1 = 41$ $pd = 6$
$x2 = -1$ $x3 = -1$ $x4 = 1$
$x2 = 1$ $x3 = -1$ $x4 = 1$
discr = 8768 $m = 2$ $e/2 = 13$ $f/2 = 4$
***** $v = 13$
square: $A33 = 13$ $A44 = 26$
$A34 = 16$ $\pm A0 = -13$
rec. seq.: $G0 = 13$ $G1 = 55$ $pd = 6$
$x2 = -2$ $x3 = -1$ $x4 = 1$
$x2 = 0$ $x3 = 1$ $x4 = 0$
$x2 = 1$ $x3 = -1$ $x4 = 1$
discr = 16448 $m = 2$ $e/2 = 17$ $f/2 = 4$
***** $v = 17$
square: $A33 = 17$ $A44 = 34$
$A34 = 16$ $\pm A0 = -17$
rec. seq.: $G0 = 17$ $G1 = 67$ $pd = 6$
$x2 = 0$ $x3 = 1$ $x4 = 0$
***** $v = 15$
square: $A33 = 17$ $A44 = 34$
$A34 = 16$ $\pm A0 = -15$
rec. seq.: $G0 = 17$ $G1 = 67$ $pd = 6$
$x2 = -6$ $x3 = 3$ $x4 = 2$
$x2 = 4$ $x3 = 3$ $x4 = 2$
discr = 20032 $m = 2$ $e/2 = 21$ $f/2 = 8$
no values for v
discr = 21568 $m = 2$ $e/2 = 25$ $f/2 = 12$
no values for v
discr = 22592 $m = 2$ $e/2 = 19$ $f/2 = 2$
no values for v
discr = 26176 $m = 2$ $e/2 = 21$ $f/2 = 4$
***** $v = 21$
square: $A33 = 21$ $A44 = 42$
$A34 = 16$ $\pm A0 = -21$
rec. seq.: $G0 = 21$ $G1 = 79$ $pd = 6$
$x2 = -3$ $x3 = 1$ $x4 = 1$
$x2 = 0$ $x3 = 1$ $x4 = 0$
$x2 = 2$ $x3 = 1$ $x4 = 1$
discr = 29248 $m = 2$ $e/2 = 23$ $f/2 = 6$
no values for v
discr = 33344 $m = 2$ $e/2 = 23$ $f/2 = 2$
no values for v
discr = 36414 $m = 2$ $e/2 = 31$ $f/2 = 14$
no values for v
discr = 36928 $m = 2$ $e/2 = 33$ $f/2 = 16$
no values for v
discr = 37952 $m = 2$ $e/2 = 25$ $f/2 = 4$
***** $v = 25$
square: $A33 = 25$ $A44 = 50$
$A34 = 16$ $\pm A0 = -25$
rec. seq.: $G0 = 25$ $G1 = 91$ $pd = 6$
$x2 = 0$ $x3 = 1$ $x4 = 0$

Extensions of $Q(\sqrt{2})$

(3)

discr = 9792 $m=2$ $e/2=15$ $f/2=6$

***** $v=11$

square: $A33=15$ $A44=30$
$A34=24$ $\pm A0=-11$

rec. seq.: $G0=15$ $G1=69$ $pd=6$
$x2=-3$ $x3=3$ $x4=-2$
$x2=-1$ $x3=1$ $x4=0$
$x2=0$ $x3=1$ $x4=0$
$x2=2$ $x3=3$ $x4=-2$

discr = 10304 $m=2$ $e/2=17$ $f/2=8$

no values for v

discr = 13888 $m=2$ $e/2=15$ $f/2=-2$

no values for v

discr = 21056 $m=2$ $e/2=23$ $f/2=10$

***** $v=19$

square: $A33=23$ $A44=46$
$A34=40$ $\pm A0=-19$

rec. seq.: $G0=23$ $G1=109$ $pd=6$
$x2=-1$ $x3=1$ $x4=0$
$x2=0$ $x3=1$ $x4=0$

discr = 31808 $m=2$ $e/2=43$ $f/2=26$

***** $v=23$

square: $A33=43$ $A44=86$
$A34=104$ $\pm A0=-23$

rec. seq.: $G0=43$ $G1=233$ $pd=6$
$x2=-5$ $x3=1$ $x4=1$
$x2=3$ $x3=1$ $x4=1$

discr = 35392 $m=2$ $e/22=29$ $f/2=12$

no values for v

(4)

discr = 4352 $m=2$ $e/2=5$ $f/2=2$

***** $v=10$

square: $A33=10$ $A44=20$
$A34=16$ $\pm A0=-10$

rec. seq.: $G0=10$ $G1=46$ $pd=6$
$x2=-2$ $x3=1$ $x4=1$
$x2=0$ $x3=-1$ $x4=1$
$x2=1$ $x3=-1$ $x4=1$
$x2=1$ $x3=1$ $x4=1$

***** $v=6$

square: $A33=10$ $A44=20$
$A34=16$ $\pm A0=-6$

rec. seq.: $G0=10$ $G1=46$ $pd=6$
$x2=-21$ $x3=17$ $x4=-12$
$x2=-1$ $x3=1$ $x4=0$
$x2=0$ $x3=1$ $x4=0$
$x2=4$ $x3=17$ $x4=-12$

discr = 18688 $m=2$ $e/2=9$ $f/2=2$

***** $v=18$

square: $A33=18$ $A44=36$
$A34=16$ $\pm A0=-18$

rec. seq.: $G0=18$ $G1=70$ $pd=6$
$x2=-2$ $x3=1$ $x4=0$
$x2=1$ $x3=1$ $x4=0$

discr = 22784 $m=2$ $e/2=11$ $f/2=4$

***** $v=18$

square: $A33=22$ $A44=44$
$A34=32$ $\pm A0=-18$

rec. seq.: $G0=22$ $G1=98$ $pd=6$
$x2=-1$ $x3=1$ $x4=0$
$x2=0$ $x3=1$ $x4=0$

discr = 24832 $m=2$ $e/2=13$ $f/2=6$

no values for v

(5)

discr = 7168 $m=2$ $e/2=3$ $f/2=1$

***** $v=12$

square: $A33=3$ $A44=6$
$A34=4$ $\pm A0=-3$

rec. seq.: $G0=3$ $G1=13$ $pd=6$
$x2=-2$ $x3=1$ $x4=1$
$x2=0$ $x3=1$ $x4=0$
$x2=2$ $x3=1$ $x4=1$

discr = 23552 $m=2$ $e/2=5$ $f/2=1$

***** $v=20$

square: $A33=5$ $A44=10$
$A34=4$ $\pm A0=-5$

rec. seq.: $G0=5$ $G1=19$ $pd=6$
$x2=-2$ $x3=-1$ $x4=1$
$x2=0$ $x3=1$ $x4=0$
$x2=2$ $x3=-1$ $x4=1$

discr = 31744 $m=2$ $e/2=7$ $f/2=3$

no values for v

Extensions of $Q(\sqrt{2})$

(6)

discr = 2048 $m=2$ $e/2=2$ $f/2=1$

***** $v=8$

square: $A33=2$ $A44=4$

$A34=4$ $\pm A0=-2$

rec. seq.: $G0=2$ $G1=10$ $pd=6$

$x2=-1$ $x3=-1$ $x4=1$

$x2=-1$ $x3=1$ $x4=0$

$x2=0$ $x3=-1$ $x4=1$

$x2=0$ $x3=1$ $x4=0$

$x2=1$ $x3=-1$ $x4=1$

$x2=1$ $x3=1$ $x4=0$

***** $v=0$

square: $A33=2$ $A44=4$

$A34=4$ $\pm A0=0$

rec. seq.: $G0=2$ $G1=10$ $pd=6$

no small values for $x3$, $x4$

discr = 14336 $m=2$ $e/2=8$ $f/2=5$

***** $v=16$

square: $A33=2$ $A44=4$

$A34=5$ $\pm A0=-1$

rec. seq.: $G0=2$ $G1=11$ $pd=6$

$x2=-1$ $x3=1$ $x4=0$

$x2=0$ $x3=-1$ $x4=1$

$x2=1$ $x3=1$ $x4=0$

discr = 18432 $m=2$ $e/2=6$ $f/2=3$

***** $v=16$

square: $A33=6$ $A44=12$

$A34=12$ $\pm A0=-4$

rec. seq.: $G0=6$ $G1=30$ $pd=6$

no small values for $x3$, $x4$

discr = 34816 $m=2$ $e/2=14$ $f/2=9$

***** $v=24$

square: $A33=14$ $A44=28$

$A34=36$ $\pm A0=-6$

rec. seq.: $G0=14$ $G1=78$ $pd=6$

$x2=-2$ $x3=3$ $x4=-2$

$x2=0$ $x3=-1$ $x4=1$

$x2=2$ $x3=3$ $x4=-2$

References

1. A. Baker and H. Davenport, The equations $3x^2-2=y^2$ and $8x^2-7=z^2$, *Quart. J. Math. Oxford* **20** (1969), 129–137.
2. W. J. Ellison, Recipes for solving diophantine problems by Baker's method, *in* "Sém. Théorie Nombres, Année 1970–1971," exp. no. 11, "Théorie Nombres," Lab. C.N.R.S., Talence, 1971.
3. W. J. Ellison, F. Ellison, J. Pesek, C. E. Stahl, and D. S. Stall, The diophantine equation $y^2+k=x^3$, *J. Number Theory* **4** (1972), 107–117.
4. I. Gaál and N. Schulte, Computing all power integral bases of cubic number fields, *Math. Comp.* **53** (1989), 689–696.
5. K. Győry, Sur les polynômes à coefficients entiers et de discriminant donné, III, *Publ. Math. Debrecen* **23** (1976), 141–165.
6. K. Győry, On the representation of integers by decomposable forms in several variables, *Publ. Math. Debrecen* **28** (1981), 89–98.
7. K. Győry, Bounds for the solutions of norm form, discriminant form and index form equations in finitely generated integral domains, *Acta Math. Hungar.* **42** (1983), 45–80.
8. K. Győry and Z. Z. Papp, On discriminant form and index form equations, *Studia Sci. Math. Hungar.* **12** (1977), 47–60.
9. L. J. Mordell, "Diophantine Equations," Academic Press, London/New York, 1969.
10. A. Pethő, Perfect powers in second order recurrences, *in* "Topics in Classical Number Theory," Colloq. Math. Soc. János Bolyai, 34, pp. 1217–1227, North-Holland, New York, 1981.
11. A. Pethő, Full cubes in the Fibonacci sequences, *Publ. Math. Debrecen* **30** (1983), 117–127.

12. A. PETHŐ, On the resolution of Thue inequalities, *J. Symbolic Comput.* **4** (1987), 103–109.
13. A. PETHŐ AND R. SCHULENBERG, Effektives Lösen von Thue Gleichungen, *Publ. Math. Debrecen* **34** (1987), 189–196.
14. M. POHST, Berechnung unabhängiger Einheiten und Klassenzahlen in total reellen biquadratischen Zahlkörpern, *Computing* **14** (1975), 67–78.
15. M. POHST, Berechnung kleiner Diskriminanten total reeller algebraischer Zahlkörper, *J. Reine Angew. Math.* **278/279** (1975), 278–300.
16. T. N. SHOREY AND C. L. STEWART, On the diophantine equation $ax^{2t}+bx^ty+cy^2=d$ and pure powers in recurrence sequences, *Math. Scand.* **52** (1983), 24–36.
17. R. P. STEINER, On Mordell's equation $y^2-k=x^3$. A problem of Stolarsky, *Math. Comp.* **46** (1986), 703–714.
18. L. A. TRELINA, On the greatest prime factor of an index form, *Dokl. Akad. Nauk SSSR* **21** (1977), 975–976. [Russian]