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CHARACTERIZATIONS OF DUAL MULTIWAVELET FRAMES OF PERIODIC FUNCTIONS

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ABSTRACT. We prove characterizations of dual multiwavelet frames for $L_2[0,1)^s$ constructed from a pair of $\mathbb{Z}^s$-periodic multirefinable generators. Based on the notion of the Mixed Fundamental sequence we derive Mixed Extension Principles dictating the construction of dual wavelet masks with respect to given refinement masks and we show that these Principles characterize dual framelets.

1. Introduction

The construction of dual wavelet frames provides great flexibility in decomposing and reconstructing elements in various function spaces, including spaces of $\mathbb{Z}^s$-periodic functions. Here, we say that f is a $\mathbb{Z}^s$-periodic finction if $f = f(\cdot - k)\ \forall k \in \mathbb{Z}^s$. In practice, many signals are either $\mathbb{Z}^s$-periodic or they can be treated as $\mathbb{Z}^s$-periodic, such as dilates of time (or band)-limited signals. Periodic wavelet analysis is an important tool for analyzing such signals, with applications in noise reduction, quantization, or image restoration [3, 6]. The first prototype of a $\mathbb{Z}$-periodic orthonormal multiresolution analysis (PMRA) with the corresponding orthonormal wavelets was obtained from the $\mathbb{Z}$-periodization of a multiresolution analysis (MRA) of $L_2(\mathbb{R})$ in [26], see also [13, 18]. Later, trigonometric scaling functions and wavelets were constructed directly from a PMRA on the unit circle [8, 9, 12, 15, 34]. More general constructions of periodic orthogonal wavelets or biorthogonal multidimensional multiwavelets emerged from non-stationary PMRA's in the sense that different scaling functions and wavelets are involved at different scales [22, 27]. Constructions of tight periodic (multi)wavelet frames were obtained in [20, 21, 23, 25, 33]. In [32], a dual pair of periodic wavelet frames for $L_2[0, 2\pi]$ was derived from 2π-periodization of a pair of functions on $\mathbb{R}$ satisfying certain conditions and in [33] these conditions were relaxed. However, the construction of dual multiwavelet frames directly from a PMRA has not been studied in detail. In this work we provide characterizations of dual multiwavelet frames derived from a pair of non-stationary PMRA's related with the ability of constructing dual wavelet masks with respect to given refinement masks.

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More precisely, let $L_2^r := L_2^r(\mathbb{T}^s)$ be the Hilbert space of all $\mathbb{Z}^s$-periodic function vectors $g : \mathbb{T}^s := [0,1)^s \to \mathbb{C}^r :\ g(\gamma) = (g_1(\gamma), ..., g_r(\gamma))$ with finite norm $\|g\|_{L_2^r}^2 = \sum_{k=1}^r \int_{\mathbb{T}^s} |g_k(\gamma)|^2 d\gamma$ and A be an $s \times s$ expansive matrix with integer entries, i.e. the eigenvalues of A are bigger than one in modulus. We consider a pair

$$\Phi = \big\{\phi_j = (\phi_{j,1}, ..., \phi_{j,r})^T : j \geq 0\big\},$$
$$\Phi^d = \big\{\phi_j^d = (\phi_{j,1}^d, ..., \phi_{j,r}^d)^T : j \geq 0\big\}$$

of *multirefinable* generators in L_2^r in the sense that every element of Φ (and Φ^d) satisfies a periodic *refinement* equation

$$\widehat{\phi}_j(n) = \widehat{H}_{j+1}(n)\, \widehat{\phi}_{j+1}(n) \text{ (and } \widehat{\phi}_j^d(n) = \widehat{H}_{j+1}^d(n)\, \widehat{\phi}_{j+1}^d(n)), n \in \mathbb{Z}^s \tag{1.1}$$

for some sequence $\widehat{H} = \{\widehat{H}_{j+1}\}_{j\geq 0}$ (and $\widehat{H}^d = \{\widehat{H}_{j+1}^d\}_{j\geq 0}$) whose elements are $A^{*(j+1)} := (A^*)^{j+1}$-periodic matrix valued functions $\widehat{H}_{j+1}, \widehat{H}_{j+1}^d : \mathbb{Z}^s \to \mathbb{C}^{r\times r}$ called *refinement masks*. Here, A^* is the hermitian transpose of the matrix A and

$$\widehat{\phi}_j(n) = \int_{\mathbb{T}} \phi_j(\gamma) e^{-2\pi i n\cdot\gamma} d\gamma,\ n \in \mathbb{Z}^s$$

is the sequence of Fourier coefficients of the function vector ϕ_j. From now on, we identify by $\mathcal{L}_j$ (resp. $\mathcal{R}_j$) a full set of all coset representatives of the cyclic group $\mathbb{Z}^s/A^j\mathbb{Z}^s$ (resp. $\mathbb{Z}^s/A^{*j}\mathbb{Z}^s$) containing $|Det(A^j)| = |Det(A^{*j})| = |A^j|$ elements in $\mathbb{Z}^s \cap A^j[0,1)^s$ (resp. in $\mathbb{Z}^s \cap A^{*j}[0,1)^s$). If $j = 0$, then we set

$$\mathcal{L}_0 = \mathcal{R}_0 = \{\mathbf{0}\}.$$

Then, every element $\widehat{H}_{j+1}$ is the discrete Fourier transform of an A^{j+1}-periodic matrix valued sequence $H_{j+1} : \mathbb{Z}^s \to \mathbb{C}^{r\times r}$ such that

$$\widehat{H}_{j+1}(n) = \sum_{l\in\mathcal{L}_{j+1}} H_{j+1}(l) e^{-2\pi i n\cdot A^{-(j+1)} l},\ n \in \mathcal{R}_{j+1}.$$

The inverse discrete Fourier transform of $\widehat{H}_{j+1}$ is defined by

$$H_{j+1}(l) = \frac{1}{Det(A^{j+1})} \sum_{n\in\mathcal{R}_{j+1}} \widehat{H}_{j+1}(n) e^{2\pi i l\cdot A^{*-(j+1)} n},\ l \in \mathcal{L}_{j+1}.$$

The refinement equation (1.1) written in time domain leads to

$$\begin{cases} \phi_j = \sum_{l\in\mathcal{L}_{j+1}} H_{j+1}(l)\, \phi_{j+1}\big(\cdot - A^{-(j+1)} l\big) \\ \phi_j^d = \sum_{l\in\mathcal{L}_{j+1}} H_{j+1}^d(l)\, \phi_{j+1}^d\big(\cdot - A^{-(j+1)} l\big) \end{cases}$$

and so, every element ϕ_j (resp. ϕ_j^d) belongs in the space

$$V_{j+1} = \text{span}\Big\{\phi_{j+1,k}\big(\cdot - A^{-(j+1)} l\big) : l \in \mathcal{L}_{j+1},\ k = 1, ..., r\Big\} \text{ (resp. } V_{j+1}^d).$$

For any $j \geq 0$, the *spectrum* of the space V_j is denoted by

$$\sigma_j = \Big\{\xi \in \mathcal{R}_j : \sum_{k=1}^r \sum_{l\in\mathbb{Z}^s} \big|\widehat{\phi}_{j,k}(\xi + A^{*j} l)\big|^2 \neq 0\Big\} \tag{1.2}$$

with a similar notation for the spectrum σ_j^d. Notice that some elements of the mask $\widehat{H}$ (or $\widehat{H}^d$) may not be unique. For example, if $\widehat{\phi}_{j_0+1,k_0}(\xi_0 + A^{*(j_0+1)} l) = 0$ for some selections of j_0, k_0, ξ_0 and for all $l \in \mathbb{Z}^s$, then we deduce from (1.1) that the k_0-column of the $A^{*(j_0+1)}$-periodic mask $\widehat{H}_{j_0+1}(\xi_0)$ can be arbitrarily defined.

From now on, if $\xi_0 \in \mathcal{R}_{j_0}\backslash\sigma_{j_0}$ (or $\xi \in \mathcal{R}_{j_0}\backslash\sigma^d_{j_0}$) but $\xi_0 + A^{*j_0}\varepsilon_0 \in \sigma_{j_0+1}$ (or $\xi_0 + A^{*j_0}\varepsilon_0 \in \sigma^d_{j_0+1}$) for some $\varepsilon_0 \in \mathcal{R}_1$, then we always consider

$$\widehat{H}_{j_0+1}(\xi_0 + A^{*j_0}\varepsilon_0) = \mathbf{O} \quad (\text{or } \widehat{H}^d_{j_0+1}(\xi_0 + A^{*j_0}\varepsilon_0) = \mathbf{O}). \tag{1.3}$$

On the other hand, let

$$\Psi = \big\{\psi_j = (\psi_{j,1}, ..., \psi_{j,\rho_j})^T \in L_2^{\rho_j}(\mathbb{T}^s) : \ j \geq 0\big\},$$
$$\Psi^d = \big\{\psi^d_j = (\psi^d_{j,1}, ..., \psi^d_{j,\rho_j})^T \in L_2^{\rho_j}(\mathbb{T}^s) : \ j \geq 0\big\}$$

be two sets of *multiwavelets* such that every element of Ψ (and Ψ^d) satisfies a refinement equation of the form

$$\widehat{\psi}_j(n) = \widehat{S}_{j+1}(n)\,\widehat{\phi}_{j+1}(n) \ (\text{and } \widehat{\psi}^d_j(n) = \widehat{S}^d_{j+1}(n)\,\widehat{\phi}^d_{j+1}(n)),\ n \in \mathbb{Z}^s, \tag{1.4}$$

where $\widehat{S}_{j+1}, \widehat{S}^d_{j+1} : \mathbb{Z}^s \to \mathbb{C}^{\rho_j \times r}$ are $A^{*(j+1)}$-periodic matrix valued sequences called *wavelet masks*. With the above selection of (Φ, Φ^d) and (Ψ, Ψ^d) we define a *multiwavelet system* $X(\phi_0, \Psi)$ by

$$X(\phi_0, \Psi) = \big\{\phi_0 : \ \phi_0 \in \Phi\big\} \cup \big\{\psi_j(\cdot - A^{-j}l) : \ j \geq 0,\, l \in \mathcal{L}_j,\, \psi_j \in \Psi\big\}, \tag{1.5}$$

with a similar notation for the system $X(\phi^d_0, \Psi^d)$. If $\Big(X(\phi_0, \Psi), X(\phi^d_0, \Psi^d)\Big)$ is a pair of Bessel systems for

$$L_2 := L_2(\mathbb{T}^s),$$

the Hilbert space of all square integrable $\mathbb{Z}^s$-periodic functions with usual norm $\|\cdot\|_{L_2}$ and if for any $f \in L_2$ we have

$$f = \sum_{m=1}^{r} \big\langle f, \phi^d_{0,m}\big\rangle \phi_{0,m} + \sum_{j=0}^{\infty}\sum_{l\in\mathcal{L}_j}\sum_{m=1}^{\rho_j} \big\langle f,\, \psi^d_{j,m}(\cdot - A^{-j}l)\big\rangle\, \psi_{j,m}(\cdot - A^{-j}l) \tag{1.6}$$

in the L_2-sense, then we say that $X(\phi^d_0, \Psi^d)$ is a *dual multiwavelet frame* of $X(\phi_0, \Psi)$ for L_2, or we say that $\Big(X(\phi_0, \Psi), X(\phi^d_0, \Psi^d)\Big)$ is a pair of *dual multiwavelet frames* (or a pair of *dual framelets*) for L_2. If (1.6) holds for the self-dual selection $\Phi = \Phi^d$ and $\Psi = \Psi^d$ then we say that $X(\phi_0, \Psi)$ is a *Parseval multiwavelet frame* (or an *affine Parseval frame*) for L_2. The existence of generators satisfying (1.1) was studied in [20], see also [19] for the one-dimensional case. The construction of Parseval (multi)wavelet frames of periodic functions was carried out in [20, 21, 23, 25] by an approach based on polyphase splines [10, 11] with analysis on frequency domain. Our results are obtained from tools borrowed from the theory of shift invariant frames [7, 28, 29, 30] which we suitably adopt to the study of $\mathbb{Z}^s$-periodic multiwavelet frames. A key tool to our investigation is the notion of the Mixed Fundamental sequence which is defined iteratively in (3.1). The Mixed Fundamental sequence can be considered as the analog of the Mixed Fundamental function from which Mixed Extension Principles of dual multiwavelet frames of $L_2(\mathbb{R}^s)$ naturally arise. In our case, the Mixed Fundamental sequence facilitates the characterization of periodic dual multiwavelets via Extension Principles.

Extension Principles were first proposed by Ron and Shen [29, 30] and subsequently were extended by Daubechies *et al.* in the form of the Oblique Extension Principle [14]. Extension Principles are used for the construction of affine dual frames of $L_2(\mathbb{R}^s)$ arising from a pair of refinable functions or function vectors, see [1, 2, 4, 5, 14, 29, 30, 31] and references therein. In [21], the authors derived a

Unitary Extension Principle and a weak version of an Oblique Extension Principle for Parseval multiwavelet frames of type (1.5). Also, in [33] the authors proved that the 2π-periodization of any pair of dual wavelet frames for $L_2(\mathbb{R})$ constructed by the Mixed Oblique Extension Principle is a pair of 2π-periodic dual wavelet frames. In this work we derive Mixed (Unitary and Oblique) Extension Principles for dual multiwavelet frames (1.5) extending the results obtained in [21, 33] to two directions: from periodic Parseval wavelet frames to periodic dual wavelet frames and from periodic dual wavelet frames derived from periodization to periodic dual wavelet frames generated from a pair $\left(\Phi, \Phi^d\right)$ of $\mathbb{Z}^s$-periodic multirefinable generators. In addition, we show that under a mild assumption on the pair $\left(\Phi, \Phi^d\right)$, namely condition (C) in section 3, these Principles characterize dual framelets. These characterizations facilitate the construction of dual multiwavelet frames from given Parseval multiwavelet frames, see example 3 in section 4.

The paper is organized in the following sections:

In section 2 we prove characterizations of dual shift invariant frames for L_2 and of wavelet type dual frames for L_2 in a sense we define later. In theorems 1 and 2 of section 3 we state and prove our main characterizations of periodic dual multiwavelet frames related with Mixed Extension Principles. Finally, in section 4 we present some constructions of periodic dual framelets.

2. Periodic dual shift invariant frames for L_2

Let A be an $s \times s$ expansive matrix, I be a countable subset of $\mathbb{N}$ and

$$\mathcal{G} = \left\{ g_k = (g_{k,1}, ..., g_{k,r})^T : k \in I \right\},$$

$$\mathcal{G}^d = \left\{ g_k^d = (g_{k,1}^d, ..., g_{k,r}^d)^T : k \in I \right\}$$

be two sequences of $\mathbb{Z}^s$-periodic function vectors in L_2^r. For any non-negative integer j, we consider a shift invariant subset of the space L_2 (of all square integrable $\mathbb{Z}^s$-periodic functions) produced from all A^{-j} shifts of $\mathcal{G}$ by

$$E_{\mathcal{G}}^j = \left\{ g_{k,m}(\cdot - A^{-j} l) : k \in I,\, l \in \mathcal{L}_j,\, m = 1, ..., r \right\}.$$

The corresponding notation for the set $E_{\mathcal{G}^d}^j$ is similar. If $E_{\mathcal{G}}^j$ is a Bessel system for L_2, then there exists a positive constant C such that for any $f \in L_2$ we have

$$\sum_{k \in I} \sum_{l \in \mathcal{L}_j} \sum_{m=1}^{r} \left| \langle f,\, g_{k,m}(\cdot - A^{-j} l) \rangle \right|^2 \le C \|f\|_{L_2}^2,$$

or equivalently in the Fourier domain:

$$|A^j| \sum_{k \in I} \sum_{\xi \in \mathcal{R}_j} \sum_{m=1}^{r} \Big| \sum_{n \in \mathbb{Z}^s} \widehat{f}(A^{*j} n + \xi) \overline{\widehat{g}_{k,m}(A^{*j} n + \xi)} \Big|^2 \le C \|\widehat{f}\|_{\ell_2}^2.$$

Here, $\|\cdot\|_{\ell_2}$ is the usual inner product on the space $\ell_2 := \ell_2(\mathbb{Z}^s)$ of all square summable complex valued sequences on $\mathbb{Z}^s$. Since any integer p can be uniquely written by $p = A^{*j} n + \xi$ for some pair $(n, \xi) \in \mathbb{Z}^s \times \mathcal{R}_j$, by selecting $\widehat{f}(k) = \delta_{p,k}$ $(k \in \mathbb{Z}^s)$ in the above inequality we immediately obtain the following:

Lemma 1. *If $E^j_{\mathcal{G}}$ is a Bessel system for L_2 for some $j \geq 0$, then*

$$|A^j| \sum_{k\in I} \sum_{m=1}^{r} |\widehat{g}_{k,m}(p)|^2 \leq C$$

for some positive constant C independent of $p \in \mathbb{Z}^s$.

Let $\big(E^j_{\mathcal{G}}, E^j_{\mathcal{G}^d}\big)$ be a pair of Bessel systems of L_2 for some $j \geq 0$. If for any $f \in L_2$ we have

$$f = \sum_{k\in I} \sum_{l\in\mathcal{L}_j} \sum_{m=1}^{r} \big\langle f,\, g^d_{k,m}(\cdot - A^{-j}l)\big\rangle\, g_{k,m}(\cdot - A^{-j}l) \tag{2.1}$$

in the L_2-sense, then we say that $E^j_{\mathcal{G}^d}$ is a *dual shift invariant frame* of $E^j_{\mathcal{G}}$ for L_2 under all A^{-j} shifts. If (2.1) holds for the special case $\mathcal{G} = \mathcal{G}^d$, then we say that $E^j_{\mathcal{G}}$ is a *Parseval shift invariant frame* of L_2 under all A^{-j} shifts.
Also, for any $\xi \in \mathcal{R}_j$, the operators

$$\widetilde{G}^j_\xi : \ell_2 \to \ell_2 :\ \widetilde{G}^j_\xi(\nu,\mu) = |A|^j \sum_{k\in I} \sum_{m=1}^{r} \overline{\widehat{g}^d_{k,m}(A^{*j}\nu + \xi)}\,\widehat{g}_{k,m}(A^{*j}\mu + \xi) \tag{2.2}$$

are well defined on ℓ_2 as a result of lemma 1 and an application of the Cauchy-Schwarz inequality. Now we have:

Proposition 1. *Let $\big(E^j_{\mathcal{G}}, E^j_{\mathcal{G}^d}\big)$ be a pair of Bessel systems for some $j \geq 0$ and the operators $\widetilde{G}^j_\xi$ be defined in (2.2). Then $\big(E^j_{\mathcal{G}}, E^j_{\mathcal{G}^d}\big)$ is a pair of dual shift invariant frames for L_2 under all A^{-j} shifts if and only if, for any $\xi \in \mathcal{R}_j$ we have*

$$\widetilde{G}^j_\xi(\nu,\mu) = \delta_{\mu,\nu}. \tag{2.3}$$

If $j = 0$, then we simply write $\widetilde{G}^0(\nu,\mu) = \delta_{\mu,\nu}$.

Proof. If $\big(E^j_{\mathcal{G}}, E^j_{\mathcal{G}^d}\big)$ is a pair of dual frames for L_2, then for any $f \in L_2$ we have

$$\|f\|_2^2 = \sum_{k\in I} \sum_{l\in\mathcal{L}_j} \sum_{m=1}^{r} \big\langle f,\, g^d_{km}(\cdot - A^{-j}l)\,\big\rangle \big\langle g_{km}(\cdot - A^{-j}l),\, f\big\rangle,$$

or equivalently in the Fourier domain

$$\|\widehat{f}\|^2_{\ell_2} = \sum_{\xi\in\mathcal{R}_j} \big\langle \widetilde{G}^j_\xi\, \widehat{\mathbf{f}}^j_\xi,\, \widehat{\mathbf{f}}^j_\xi \big\rangle_{\ell_2}, \tag{2.4}$$

where $\widehat{\mathbf{f}}^j_\xi = \big\{\widehat{f}(A^{*j}\nu + \xi) \,:\, \nu \in \mathbb{Z}^s\big\}$ is a $\mathbb{Z}^s$-column vector and $\widetilde{G}^j_\xi$ is defined in (2.2). For any fixed $\xi_0 \in \mathcal{R}_j$ and for every $m \in \mathbb{Z}^s$ we consider the functions $h_m = e^{2\pi i(A^{*j}m+\xi_0)\cdot}$ whose Fourier coefficients satisfy $\widehat{h}_m(l) = \delta_{l,A^{*j}m+\xi_0}$. Let $\widehat{\mathbf{h}}^j_{m,\xi} = \big\{\widehat{h}_m(A^{*j}\nu + \xi) \,:\, \nu \in \mathbb{Z}^s\big\}$ and $m' \in \mathbb{Z}^s$. By substituting $\widehat{\mathbf{f}}^j_\xi = \widehat{\mathbf{h}}^j_{m,\xi} \pm \widehat{\mathbf{h}}^j_{m',\xi}$, or $\widehat{\mathbf{f}}^j_\xi = \widehat{\mathbf{h}}^j_{m,\xi} \pm i\,\widehat{\mathbf{h}}^j_{m',\xi}$ in (2.4) we obtain

$$\begin{cases} \big\langle \widetilde{G}^j_{\xi_0}\big(\widehat{\mathbf{h}}^j_{m,\xi_0} \pm \widehat{\mathbf{h}}^j_{m',\xi_0}\big),\, \widehat{\mathbf{h}}^j_{m,\xi_0} \pm \widehat{\mathbf{h}}^j_{m',\xi_0}\big\rangle_{\ell_2} = \|\widehat{\mathbf{h}}^j_{m,\xi_0} \pm \widehat{\mathbf{h}}^j_{m',\xi_0}\|^2_{\ell_2} \\ \big\langle \widetilde{G}^j_{\xi_0}\big(\widehat{\mathbf{h}}^j_{m,\xi_0} \pm i\widehat{\mathbf{h}}^j_{m',\xi_0}\big),\, \widehat{\mathbf{h}}^j_{m,\xi_0} \pm i\widehat{\mathbf{h}}^j_{m',\xi_0}\big\rangle_{\ell_2} = \|\widehat{\mathbf{h}}^j_{m,\xi_0} \pm i\widehat{\mathbf{h}}^j_{m',\xi_0}\|^2_{\ell_2} \end{cases},$$

because $\widehat{\mathbf{h}}^j_{m,\xi}=\mathbf{0}$ and $\widehat{\mathbf{h}}^j_{m',\xi}=\mathbf{0}$ whenever $\xi\neq\xi_0$. Now we combine the polarization identity with the above two equalities and we have

$$\begin{aligned}4\big\langle\widehat{\mathbf{h}}^j_{m,\xi_0},\widehat{\mathbf{h}}^j_{m',\xi_0}\big\rangle_{\ell_2} &= \|\widehat{\mathbf{h}}^j_{m,\xi_0}+\widehat{\mathbf{h}}^j_{m',\xi_0}\|^2_{\ell_2}-\|\widehat{\mathbf{h}}^j_{m,\xi_0}-\widehat{\mathbf{h}}^j_{m',\xi_0}\|^2_{\ell_2}\\ &+ i\|\widehat{\mathbf{h}}^j_{m,\xi_0}+i\widehat{\mathbf{h}}^j_{m',\xi_0}\|^2_{\ell_2}-i\|\widehat{\mathbf{h}}^j_{m,\xi_0}-i\widehat{\mathbf{h}}^j_{m',\xi_0}\|^2_{\ell_2}\\ &= \big\langle\widetilde{G}^j_{\xi_0}\big(\widehat{\mathbf{h}}^j_{m,\xi_0}+\widehat{\mathbf{h}}^j_{m',\xi_0}\big),\ \widehat{\mathbf{h}}^j_{m,\xi_0}+\widehat{\mathbf{h}}^j_{m',\xi_0}\big\rangle_{\ell_2}\\ &- \big\langle\widetilde{G}^j_{\xi_0}\big(\widehat{\mathbf{h}}^j_{m,\xi_0}-\widehat{\mathbf{h}}^j_{m',\xi_0}\big),\ \widehat{\mathbf{h}}^j_{m,\xi_0}-\widehat{\mathbf{h}}^j_{m',\xi_0}\big\rangle_{\ell_2}\\ &+ i\big\langle\widetilde{G}^j_{\xi_0}\big(\widehat{\mathbf{h}}^j_{m,\xi_0}+i\widehat{\mathbf{h}}^j_{m',\xi_0}\big),\ \widehat{\mathbf{h}}^j_{m,\xi_0}+i\widehat{\mathbf{h}}^j_{m',\xi_0}\big\rangle_{\ell_2}\\ &- i\big\langle\widetilde{G}^j_{\xi_0}\big(\widehat{\mathbf{h}}^j_{m,\xi_0}-i\widehat{\mathbf{h}}^j_{m',\xi_0}\big),\ \widehat{\mathbf{h}}^j_{m,\xi_0}-i\widehat{\mathbf{h}}^j_{m',\xi_0}\big\rangle_{\ell_2}\\ &= 4\big\langle\widetilde{G}^j_{\xi_0}\widehat{\mathbf{h}}^j_{m,\xi_0},\ \widehat{\mathbf{h}}^j_{m',\xi_0}\big\rangle_{\ell_2}.\end{aligned}$$

Therefore

$$\big\langle\big(\widetilde{G}^j_{\xi_0}-I\big)\widehat{\mathbf{h}}^j_{m,\xi_0},\ \widehat{\mathbf{h}}^j_{m',\xi_0}\big\rangle_{\ell_2}=0,$$

where I is the identity operator on ℓ_2 and so

$$\sum_{\mu,\nu\in\mathbb{Z}^s}\overline{\widehat{h}_{m'}(A^{*j}\nu+\xi_0)}\Big(\widetilde{G}^j_{\xi_0}-I\Big)_{\nu,\mu}\widehat{h}_m(A^{*j}\mu+\xi_0)=0$$

or

$$\sum_{\mu,\nu\in\mathbb{Z}^s}\delta_{m',\nu}\big(\widetilde{G}^j_{\xi_0}-I\big)_{\nu,\mu}\delta_{\mu,m}=0\Rightarrow\widetilde{G}^j_{\xi_0}=I.$$

Since $\xi_0\in\mathcal{R}_j$ is arbitrarily selected the proof is complete. On the other hand, if $\widetilde{G}^j_\xi=I$ for any $\xi\in\mathcal{R}_j$, then by applying this in the right hand side of (2.4) and its equivalent part in time domain the conclusion follows. □

Corollary 1. *The set $E^j_{\mathcal{G}}$ is a Parseval shift invariant frame of L_2 for some $j\geq 0$ if and only if, for any $\xi\in\mathcal{R}_j$ we have*

$$G^j_\xi(\nu,\mu)=|A|^j\sum_{k\in I}\sum_{m=1}^r\overline{\widehat{g}_{k,m}(A^{*j}\nu+\xi)}\widehat{g}_{k,m}(A^{*j}\mu+\xi)=\delta_{\mu,\nu}\quad\mu,\nu\in\mathbb{Z}^s.$$

If $j=0$, then we simply write $G^0(\nu,\mu)=\delta_{\mu,\nu}$.

Proof. It is a minor modification of proposition 1 for the particular selection $\mathcal{G}=\mathcal{G}^d$. Notice that in this case we don't need to assume a priori a Bessel assumption on $E^j_{\mathcal{G}}$, because once $E^j_{\mathcal{G}}$ is a Parseval frame for L_2 then the Bessel property of $E^j_{\mathcal{G}}$ is automatically satisfied. If $\mathcal{G}\neq\mathcal{G}^d$ instead, the absence of a Bessel assumption on the sets $E^j_{\mathcal{G}}$ and $E^j_{\mathcal{G}^d}$ may lead to conditional convergence of the series (2.1). □

We consider now two sets of multiwavelets Ψ and Ψ^d satisfying (1.4) which are derived from a pair of multirefinable generators Φ and Φ^d satisfying (1.1). For any non-negative integer j_0 we define the following set

$$\begin{aligned}X(\phi_{j_0},\Psi_{j_0}) &= \big\{\phi_{j_0,k}(\cdot-A^{-j_0}l):\,l\in\mathcal{L}_{j_0},\,k=1,...,r\big\}\\ &\bigcup\ \big\{\psi_{j,k}(\cdot-A^{-j}l):\,j\geq j_0,\,l\in\mathcal{L}_j,\,k=1,...,\rho_j\big\},\end{aligned}\qquad(2.5)$$

where $\phi_{j_0}\in\Phi$ and $\Psi_{j_0}=\big\{\psi_j\in\Psi:\ \ j\geq j_0\big\}$. The set $X(\phi^d_{j_0},\Psi^d_{j_0})$ is defined in a similar manner. If $j_0=0$, then we obtain the multiwavelet system $X(\phi_0,\Psi)$ as in eq. (1.5), otherwise we talk about a *truncated multiwavelet system*. Now we have:

Proposition 2. *Let* $\Big(X(\phi_{j_0},\Psi_{j_0}), X(\phi^d_{j_0},\Psi^d_{j_0})\Big)$ *be a pair of Bessel systems of* L_2 *for some* $j_0 \geq 0$ *as in* (2.5). *Then* $X(\phi^d_{j_0},\Psi^d_{j_0})$ *is a dual frame of* $X(\phi_{j_0},\Psi_{j_0})$ *for* L_2 *if and only if, for any* $\xi \in \mathcal{R}_{j_0}$ *we have*

$$|A|^{j_0}\ \widehat{\phi}^{d*}_{j_0}(A^{*j_0}\nu+\xi)\widehat{\phi}_{j_0}(A^{*j_0}\mu+\xi) + \sum_{j=j_0}^{j_0+\kappa_0(\nu-\mu)} |A|^j\ \widehat{\psi}^{d*}_j(A^{*j_0}\nu+\xi)\widehat{\psi}_j(A^{*j_0}\mu+\xi) = \delta_{\mu,\nu},\ \mu,\nu\in\mathbb{Z}^s, \quad (2.6)$$

where $\kappa_0 : \mathbb{Z}^s \to \mathbb{N} :\ \kappa_0(n) = sup\{j \geq 0 :\ A^{*-j}n \in \mathbb{Z}^s\}$. *Here and hereafter we use the notation* $\widehat{\phi}^{d*}_{j_0}(k)\widehat{\phi}_{j_0}(n) = \sum_{m=1}^r \overline{\widehat{\phi}^d_{j_0,m}(k)}\widehat{\phi}_{j_0,m}(n)$ *with a similar notation for* $\widehat{\psi}^{d*}_j(k)\widehat{\psi}_j(n),\ k,n\in\mathbb{Z}^s$.

Proof. Let $j_0 \geq 0$. We observe that the set

$$\mathcal{G} = \{\phi_{j_0}\} \cup \{\psi_j(\cdot - A^{-j}p) :\ j \geq j_0,\ p \in \mathcal{L}_{j-j_0},\ \psi_j \in \Psi\}$$

is a shift invariant generator of the set $X(\phi_{j_0},\Psi_{j_0})$ under all A^{-j_0} shifts. A similar argument holds for the set $X(\phi^d_{j_0},\Psi^d_{j_0})$ as well. Hence, for this selection of $\mathcal{G}$ we apply (2.3) and we derive that $\Big(X(\phi_{j_0},\Psi_{j_0}), X(\phi^d_{j_0},\Psi^d_{j_0})\Big)$ is a pair of dual frames for L_2 if and only if, for any $\xi \in \mathcal{R}_{j_0}$ we have

$$|A|^{j_0}\Big(\widehat{\phi}^{d*}_{j_0}(A^{*j_0}\nu+\xi)\widehat{\phi}_{j_0}(A^{*j_0}\mu+\xi) + \sum_{j=j_0}^{\infty}\sum_{p\in\mathcal{L}_{j-j_0}} \widehat{\psi}^{d*}_j(A^{*j_0}\nu+\xi)\widehat{\psi}_j(A^{*j_0}\mu+\xi)e^{2\pi i A^{*-(j-j_0)}(\nu-\mu)\cdot p}\Big) = \delta_{\mu,\nu}.$$

Let $\kappa_0(n) = \sup\{j \geq 0 :\ A^{*-j}n \in \mathbb{Z}^s\}$. Since

$$\sum_{p\in\mathcal{L}_{j-j_0}} e^{2\pi i A^{*-(j-j_0)}(\nu-\mu)\cdot p} = \begin{cases} |A|^{j-j_0} & \text{if } j-j_0 \leq \kappa_0(\nu-\mu) \\ 0 & \text{if } j-j_0 > \kappa_0(\nu-\mu) \end{cases},$$

we substitute this in the above equality and we obtain the result. □

3. The Mixed Fundamental sequence and Extension Principles

In this section we provide characterizations of periodic dual multiwavelet frames related with Mixed Extension Principles. For $j \geq 0$, let $\widehat{H}_{j+1}$ and $\widehat{H}^d_{j+1}$ be $A^{*(j+1)}$-periodic refinement masks as in (1.1) and $\widehat{S}_{j+1}$, $\widehat{S}^d_{j+1}$ be $A^{*(j+1)}$-periodic wavelet masks as in (1.4) with respect to an $s \times s$ expansive matrix A. We define their corresponding *Mixed Fundamental sequence*

$$\Theta^{\mathrm{M}} = \{\Theta^{\mathrm{M}}_j :\ j \geq -1\}$$

such that:

- **(i)** Every element $\Theta^{\mathrm{M}}_j : \mathbb{Z}^s \to \mathbb{C}^{r\times r}$ is an $A^{*(j+1)}$-periodic matrix valued sequence, i.e. $\Theta^{\mathrm{M}}_j(n) = \Theta^{\mathrm{M}}_j\big(n + A^{*(j+1)}l\big)\ \forall n,l \in \mathbb{Z}^s$,
- **(ii)** $\Theta^{\mathrm{M}}_{-1}(n) = I_r\ \ \forall n \in \mathbb{Z}^s$ and

(iii) for any $j \geq 0$ and $n \in \mathcal{R}_{j+1}$ such that $n = A^{*j}\varepsilon_0 + \zeta$ ($\zeta \in \mathcal{R}_j,\ \varepsilon_0 \in \mathcal{R}_1$), the values $\Theta_j^{\mathrm{M}}(n)$ are defined recursively by

$$|A|\ \Theta_j^{\mathrm{M}}(A^{*j}\varepsilon_0 + \zeta) = \widehat{H}_{j+1}^{d*}(A^{*j}\varepsilon_0 + \zeta)\Theta_{j-1}^{\mathrm{M}}(\zeta)\widehat{H}_{j+1}(A^{*j}\varepsilon_0 + \zeta)$$
$$+ \quad \widehat{S}_{j+1}^{d*}(A^{*j}\varepsilon_0 + \zeta)\widehat{S}_{j+1}(A^{*j}\varepsilon_0 + \zeta). \tag{3.1}$$

Notice that (3.1) combined with the periodicity requirement (i) determine completely the values of Θ_j^{M} on $\mathbb{Z}^s$. If $\widehat{H} = \widehat{H}^d$ and $\widehat{S} = \widehat{S}^d$ then we simply talk about the *Fundamental sequence* with respect to the pair $(\widehat{H}, \widehat{S})$ denoted by

$$\Theta = \{\Theta_j :\ j \geq -1\}.$$

We aim to prove that the Mixed Fundamental sequence characterizes dual multiwavelet frames of type (1.5). To do that we need to impose the following condition on the pair (Φ, Φ^d) of multirefinable generators whose corresponding pair of masks is $(\widehat{H}, \widehat{H}^d)$:

Condition (C): For any $j \geq 0$ and $\xi \in \sigma_j \cap \sigma_j^d$ (see (1.2)), each one of the sets $\Big\{\big\{\widehat{\phi}_{j,k}(\xi + A^{*j}l) :\ l \in \mathbb{Z}^s\big\} :\ k = 1, ..., r\Big\}$ and $\Big\{\big\{\widehat{\phi}_{j,k}^d(\xi + A^{*j}l) :\ l \in \mathbb{Z}^s\big\} :\ k = 1, ..., r\Big\}$ forms a basis of an r-dimensional subspace of ℓ_2. Then there exist two sets $I_{j,\xi} \subset \mathbb{Z}^s$ and $I_{j,\xi}^d \subset \mathbb{Z}^s$ both of length r such that the $r \times r$ matrices $W_j(\xi) = \big\{\widehat{\phi}_{j,k}(\xi + A^{*j}l) :\ k = 1, ..., r,\ l \in I_{j,\xi}\big\}$ and $W_j^d(\xi) = \big\{\widehat{\phi}_{j,k}^d(\xi + A^{*j}l) : k = 1, ..., r,\ l \in I_{j,\xi}^d\big\}$ are invertible.

Remark 1. If $\sigma_j \cap \sigma_j^d = \mathcal{R}_j$ for some j, then the validity of condition (C) implies that the sets $\big\{\phi_{j,k}\big(\cdot - A^{-j}l\big) :\ l \in \mathcal{L}_j,\ k = 1, ..., r\big\}$ and $\big\{\phi_{j,k}^d\big(\cdot - A^{-j}l\big) :\ l \in \mathcal{L}_j,\ k = 1, ..., r\big\}$) are bases of the spaces V_j and V_j^d, otherwise either one or both of them are simply spanning sets for their corresponding spaces V_j and V_j^d. Another implication of this condition is the following: If $\xi \notin \sigma_j \cap \sigma_j^d$ but $\xi + A^{*j}\varepsilon_0 \in \sigma_{j+1} \cap \sigma_{j+1}^d$ for some $\varepsilon_0 \in \mathcal{R}_1$ (and fixed j), then either

$$\widehat{H}_{j+1}(\xi + A^{*j}\varepsilon_0) = \mathbf{O}, \tag{3.2}$$

or

$$\widehat{H}_{j+1}^d(\xi + A^{*j}\varepsilon_0) = \mathbf{O}. \tag{3.3}$$

For example, if $\xi \notin \sigma_j$, then for every $l \in \mathbb{Z}^s$ we have

$$\mathbf{0} = \widehat{\phi}_j\big(\xi + A^{*j}\varepsilon_0 + A^{*(j+1)}l\big) = \widehat{H}_{j+1}(\xi + A^{*j}\varepsilon_0)\widehat{\phi}_{j+1}\big(\xi + A^{*j}\varepsilon_0 + A^{*(j+1)}l\big).$$

Let $W_{j+1}(\xi + A^{*j}\varepsilon_0)$ be an $r \times r$ invertible matrix defined in the statement of condition (C). From the last equality repeatedly applied for every $l \in I_{j+1,\xi+A^{*j}\varepsilon_0}$ we obtain

$$\mathbf{O} = \widehat{H}_{j+1}(\xi + A^{*j}\varepsilon_0)W_{j+1}(\xi + A^{*j}\varepsilon_0).$$

Then the invertibility of $W_{j+1}(\xi + A^{*j}\varepsilon_0)$ leads to (3.2). We use similar arguments to show (3.3). We also note that the difference of (3.2) with (1.3) is that in this case there are no other possible definitions for the mask $\widehat{H}_{j+1}(\xi_0 + A^{*j_0}\varepsilon_0)$ (or $\widehat{H}_{j+1}^d(\xi_0 + A^{*j_0}\varepsilon_0)$) than the zero matrix.

Now we are ready to prove our first characterization. We have the following:

Theorem 1. *Let (Φ, Φ^d) be a pair of $\mathbb{Z}^s$-periodic multirefinable generators satisfying the above condition (C), $\Big(X(\phi_0, \Psi), X(\phi_0^d, \Psi^d)\Big)$ be a pair of multiwavelet Bessel systems as in* (1.5) *and Θ^{M} be the Mixed Fundamental sequence as in* (3.1)*. Then the following conditions are equivalent:*

(i) $\Big(X(\phi_0, \Psi), X(\phi_0^d, \Psi^d)\Big)$ *is a pair of dual multiwavelet frames for L_2.*

(ii) *The Mixed Fundamental sequence Θ^{M} satisfies the following conditions:*

(a) $\lim_{N\to+\infty} \Big(|A|^N \widehat{\phi}_N^{d*}(\mu)\Theta_{N-1}^{\mathrm{M}}(\mu)\widehat{\phi}_N(\mu)\Big) = 1,$ *pointwise on $\mu \in \mathbb{Z}^s$.*

(b) *For any $j \geq 0$, $\zeta \in \mathcal{R}_j$ and $\varepsilon_0 \neq \varepsilon_0'$ $(\varepsilon_0, \varepsilon_0' \in \mathcal{R}_1)$ such that $A^{*j}\varepsilon_0' + \zeta, A^{*j}\varepsilon_0 + \zeta \in \sigma_{j+1} \cap \sigma_{j+1}^d$ we have*

$$\begin{aligned} &\widehat{H}_{j+1}^{d*}(A^{*j}\varepsilon_0' + \zeta)\Theta_{j-1}^{\mathrm{M}}(\zeta)\widehat{H}_{j+1}(A^{*j}\varepsilon_0 + \zeta) \\ + \quad &\widehat{S}_{j+1}^{d*}(A^{*j}\varepsilon_0' + \zeta)\widehat{S}_{j+1}(A^{*j}\varepsilon_0 + \zeta) = \mathbf{O}. \end{aligned} \tag{3.4}$$

(iii) *There exists a sequence $\theta = \{\theta_j : \ j \geq -1\}$ whose elements are $A^{*(j+1)}$-periodic matrix valued functions $\theta_j : \mathbb{Z}^s \to \mathbb{C}^{r\times r}$ such that $\theta_{-1}(n) = I_r$ $\forall n \in \mathbb{Z}^s$ and the following conditions are satisfied:*

(a) $\lim_{N\to+\infty} \Big(|A|^N \widehat{\phi}_N^{d*}(\mu)\theta_{N-1}(\mu)\widehat{\phi}_N(\mu)\Big) = 1,$ *pointwise on $\mu \in \mathbb{Z}^s$.*

(b) *For any $j \geq 0$, $\zeta \in \mathcal{R}_j$ and $\varepsilon_0, \varepsilon_0' \in \mathcal{R}_1$ such that $A^{*j}\varepsilon_0' + \zeta, A^{*j}\varepsilon_0 + \zeta \in \sigma_{j+1} \cap \sigma_{j+1}^d$ we have*

$$\begin{aligned} &\widehat{H}_{j+1}^{d*}(A^{*j}\varepsilon_0' + \zeta)\theta_{j-1}(\zeta)\widehat{H}_{j+1}(A^{*j}\varepsilon_0 + \zeta) \\ + \quad &\widehat{S}_{j+1}^{d*}(A^{*j}\varepsilon_0' + \zeta)\widehat{S}_{j+1}(A^{*j}\varepsilon_0 + \zeta) = |A|\delta_{\varepsilon_0,\varepsilon_0'}\theta_j(A^{*j}\varepsilon_0 + \zeta). \end{aligned} \tag{3.5}$$

Proof. **(i)** $\Leftrightarrow$ **(ii)**: By applying (2.6) for $j_0 = 0$ we conclude that $X(\phi_0^d, \Psi^d)$ is a dual multiwavelet frame of $X(\phi_0, \Psi)$ for L_2 if and only if

$$\widehat{\phi}_0^{d*}(\nu)\widehat{\phi}_0(\mu) + \sum_{j=0}^{\kappa_0(\nu-\mu)} |A|^j \widehat{\psi}_j^{d*}(\nu)\widehat{\psi}_j(\mu) = \delta_{\mu,\nu}, \ \ \forall \mu, \nu \in \mathbb{Z}^s. \tag{3.6}$$

Therefore, it suffices to prove that (3.6) is equivalent to the above conditions (a) and (b) of part (ii) of this theorem. We consider two cases:

Case I: Take $\mu = \nu$ in (3.6). Then $\kappa_0(0) = +\infty$ and we shall prove that the equality

$$\widehat{\phi}_0^{d*}(\mu)\widehat{\phi}_0(\mu) + \sum_{j=0}^{\infty} |A|^j \widehat{\psi}_j^{d*}(\mu)\widehat{\psi}_j(\mu) = 1, \ \ \mu \in \mathbb{Z}^s \tag{3.7}$$

is equivalent to the first condition of part (ii) of this theorem. The assumption that $\Big(X(\phi_0, \Psi), X(\phi_0^d, \Psi^d)\Big)$ is a pair of Bessel systems implies that

$$\sum_{j=0}^{\infty} |A|^j \sum_{m=1}^{r} |\widehat{\psi}_{j,m}^d(\nu)|\,|\widehat{\psi}_{j,m}(\mu)| \leq (C\, C^d)^{\frac{1}{2}},$$

where C and C^d are positive constants independent of μ and ν derived from lemma 1 and an application of the Cauchy-Schwartz inequality. Hence, for each $\mu \in \mathbb{Z}^s$, the series $\sum_{j=0}^{\infty} |A|^j \widehat{\psi}_j^{d*}(\mu)\widehat{\psi}_j(\mu)$ is absolutely convergent and so, given a sufficiently small $\varepsilon > 0$ there exists a sufficiently large natural number $M := M(\mu, \varepsilon)$ such that for any $N > M$ we have

$$\Big| \sum_{j=N}^{\infty} |A|^j \widehat{\psi}_j^{d*}(\mu) \widehat{\psi}_j(\mu) \Big| < \varepsilon. \tag{3.8}$$

For μ, ε and M as above and for each $N > M$ we write

$$\mu = A^{*N} l + \xi \tag{3.9}$$

for some pair $(l, \xi) \in \mathbb{Z}^s \times \mathcal{R}_N$, or

$$\mu = A^{*N} l + \underbrace{A^{*(N-1)} \varepsilon_{N-1} + A^{*(N-2)} \varepsilon_{N-2} + ... + \varepsilon_0}_{\xi}$$

for some $\varepsilon_0, ..., \varepsilon_{N-1} \in \mathcal{R}_1$. Then the partial sum of the left hand side of (3.7) becomes

$$\begin{aligned} & \widehat{\phi}_0^{d*}(\mu) \widehat{\phi}_0(\mu) + \sum_{j=0}^{N-1} |A|^j \widehat{\psi}_j^{d*}(\mu) \widehat{\psi}_j(\mu) \\ = \;& \widehat{\phi}_1^{d*}(\mu) \Big(\widehat{H}_1^{d*}(\varepsilon_0) \widehat{H}_1(\varepsilon_0) + \widehat{S}_1^{d*}(\varepsilon_0) \widehat{S}_1(\varepsilon_0) \Big) \widehat{\phi}_1(\mu) + \sum_{j=1}^{N-1} |A|^j \widehat{\psi}_j^{d*}(\mu) \widehat{\psi}_j(\mu) \\ = \;& |A| \widehat{\phi}_1^{d*}(\mu) \Theta_0^{\mathrm{M}}(\varepsilon_0) \widehat{\phi}_1(\mu) + \sum_{j=1}^{N-1} |A|^j \widehat{\psi}_j^{d*}(\mu) \widehat{\psi}_j(\mu) = ... = |A|^N \widehat{\phi}_N^{d*}(\mu) \\ & \Theta_{N-1}^{\mathrm{M}} \big(A^{*(N-1)} \varepsilon_{N-1} + ... + \varepsilon_0 \big) \widehat{\phi}_N(\mu) = |A|^N \widehat{\phi}_N^{d*}(\mu) \Theta_{N-1}^{\mathrm{M}}(\xi) \widehat{\phi}_N(\mu), \end{aligned} \tag{3.10}$$

as a result of the refinement equations (1.1), (1.4) and the definition of the Mixed Fundamental function in (3.1). Taking into account (3.8) we write (3.7) by

$$\Big| \widehat{\phi}_0^{d*}(\mu) \widehat{\phi}_0(\mu) + \sum_{j=0}^{N-1} |A|^j \widehat{\psi}_j^{d*}(\mu) \widehat{\psi}_j(\mu) - 1 \Big| < \varepsilon$$

or

$$\big| |A|^N \widehat{\phi}_N^{d*}(\mu) \Theta_{N-1}^{\mathrm{M}}(\xi) \widehat{\phi}_N(\mu) - 1 \big| < \varepsilon$$

as a result of (3.10). But $\Theta_{N-1}^{\mathrm{M}}$ is A^{*N}-periodic, i.e. $\Theta_{N-1}^{\mathrm{M}}(\xi) = \Theta_{N-1}^{\mathrm{M}}(A^{*N} l + \xi) = \Theta_{N-1}^{\mathrm{M}}(\mu)$ (see (3.9)), so

$$\big| |A|^N \widehat{\phi}_N^{d*}(\mu) \Theta_{N-1}^{\mathrm{M}}(\mu) \widehat{\phi}_N(\mu) - 1 \big| < \varepsilon.$$

The last inequality is equivalent to condition (a) of part (ii) of this theorem.

Case II: Let $\mu \neq \nu$. Then (3.6) becomes

$$\widehat{\phi}_0^{d*}(\nu) \widehat{\phi}_0(\mu) + \sum_{j=0}^{k_0(\nu-\mu)} |A|^j \widehat{\psi}_j^{d*}(\nu) \widehat{\psi}_j(\mu) = 0 \;\; \forall \nu \neq \mu, \tag{3.11}$$

where we recall that $\kappa_0(n) = \sup\{j \geq 0 : \; A^{*-j} n \in \mathbb{Z}^s\}$. It suffices to prove that (3.11) is equivalent to the second condition of part (ii) of this theorem. First we observe that the set of all pairs of integers μ, ν such that $\mu \neq \nu$ can be written by the following disjoint union

$$\{(\mu,\nu):\ \mu \neq \nu\} = \bigcup_{j_0 \geq 0} \{(\mu,\nu):\ \kappa_0(\nu-\mu) = j_0\}. \tag{3.12}$$

Consider now any pair (μ, ν) such that $\kappa_0(\nu-\mu) = j_0$ for some non-negative integer j_0. Then μ and ν must have the form

$$\begin{cases} \mu = A^{*(j_0+1)}l + A^{*j_0}\varepsilon_0 + \zeta \\ \nu = A^{*(j_0+1)}k + A^{*j_0}\varepsilon_0' + \zeta \end{cases}$$

where $k, l \in \mathbb{Z}^s$, $\zeta \in \mathcal{R}_{j_0}$ and $\varepsilon_0, \varepsilon_0' \in \mathcal{R}_1$ such that $\varepsilon_0 \neq \varepsilon_0'$. For this selection of μ and ν, eq. (3.11) can be written by

$$\begin{aligned} & |A|\widehat{\phi}_1^{d*}(\nu)\Theta_0^{\mathrm{M}}(\zeta)\widehat{\phi}_1(\mu) + \sum_{j=1}^{j_0}|A|^j\widehat{\psi}_j^{d*}(\nu)\widehat{\psi}_j(\mu) = 0 \\ \Leftrightarrow\quad & |A|^2\widehat{\phi}_2^{d*}(\nu)\Theta_1^{\mathrm{M}}(\zeta)\widehat{\phi}_2(\mu) + \sum_{j=2}^{j_0}|A|^j\widehat{\psi}_j^{d*}(\nu)\widehat{\psi}_j(\mu) = 0 \\ \Leftrightarrow\quad & \dots\ \Leftrightarrow \widehat{\phi}_{j_0+1}^{d*}\big(A^{*(j_0+1)}k + \xi_{\varepsilon_0',\zeta}\big)\Big(\widehat{H}_{j_0+1}^{d*}(\xi_{\varepsilon_0',\zeta})\Theta_{j_0-1}^{\mathrm{M}}(\zeta)\widehat{H}_{j_0+1}(\xi_{\varepsilon_0,\zeta}) \\ +\quad & \widehat{S}_{j_0+1}^{d*}(\xi_{\varepsilon_0',\zeta})\widehat{S}_{j_0+1}(\xi_{\varepsilon_0,\zeta})\Big)\widehat{\phi}_{j_0+1}\big(A^{*(j_0+1)}l + \xi_{\varepsilon_0,\zeta}\big) = 0, \end{aligned} \tag{3.13}$$

where $\xi_{\varepsilon_0,\zeta} = A^{*j_0}\varepsilon_0 + \zeta$, $\xi_{\varepsilon_0',\zeta} = A^{*j_0}\varepsilon_0' + \zeta$ and $\xi_{\varepsilon_0,\zeta}, \xi_{\varepsilon_0',\zeta} \in \mathcal{R}_{j_0+1}$. If $\xi_{\varepsilon_0,\zeta} \notin \sigma_{j_0+1} \cap \sigma_{j_0+1}^d$ or $\xi_{\varepsilon_0',\zeta} \notin \sigma_{j_0+1} \cap \sigma_{j_0+1}^d$, then (3.13) is obviously satisfied. It remains to examine the case where $\xi_{\varepsilon_0,\zeta}, \xi_{\varepsilon_0',\zeta} \in \sigma_{j_0+1} \cap \sigma_{j_0+1}^d$. Here we use condition (C) and so there exist two sets $I_{j_0+1,\xi_{\varepsilon_0,\zeta}} \subset \mathbb{Z}^s$ and $I_{j_0+1,\xi_{\varepsilon_0',\zeta}}^d \subset \mathbb{Z}^s$, both of length r such that the $r \times r$ matrices

$$\begin{cases} W_{j_0+1}(\xi_{\varepsilon_0,\zeta}) = \Big\{\widehat{\phi}_{j_0+1,m}\big(\xi_{\varepsilon_0,\zeta} + A^{*(j_0+1)}l\big):\ m = 1,...,r,\ l \in I_{j_0+1,\xi_{\varepsilon_0,\zeta}}\Big\} \\ W_{j_0+1}^d(\xi_{\varepsilon_0',\zeta}) = \Big\{\widehat{\phi}_{j_0+1,m}^d\big(\xi_{\varepsilon_0',\zeta} + A^{*(j_0+1)}k\big):\ m = 1,...,r,\ k \in I_{j_0+1,\xi_{\varepsilon_0',\zeta}}^d\Big\} \end{cases} \tag{3.14}$$

are non singular. Since (3.11) holds for every $\mu \neq \nu$, eq. (3.13) holds for any $k, l \in \mathbb{Z}^s$, $\zeta \in \mathcal{R}_{j_0}$ and $\varepsilon_0 \neq \varepsilon_0'$ such that $\xi_{\varepsilon_0,\zeta}, \xi_{\varepsilon_0',\zeta} \in \sigma_{j_0+1} \cap \sigma_{j_0+1}^d$. Therefore, we may apply (3.13) for every pair $(k,l) \in I_{j_0+1,\xi_{\varepsilon_0',\zeta}}^d \times I_{j_0+1,\xi_{\varepsilon_0,\zeta}}$ and we obtain

$$\begin{aligned} & W_{j_0+1}^{d*}(\xi_{\varepsilon_0',\zeta})\Big(\widehat{H}_{j_0+1}^{d*}(\xi_{\varepsilon_0',\zeta})\Theta_{j_0-1}^{\mathrm{M}}(\zeta)\widehat{H}_{j_0+1}(\xi_{\varepsilon_0,\zeta}) \\ +\quad & \widehat{S}_{j_0+1}^{d*}(\xi_{\varepsilon_0',\zeta})\widehat{S}_{j_0+1}(\xi_{\varepsilon_0,\zeta})\Big)W_{j_0+1}(\xi_{\varepsilon_0,\zeta}) = \mathbf{O}. \end{aligned}$$

From this we conclude that

$$\widehat{H}_{j_0+1}^{d*}(\xi_{\varepsilon_0',\zeta})\Theta_{j_0-1}^{\mathrm{M}}(\zeta)\widehat{H}_{j_0+1}(\xi_{\varepsilon_0,\zeta}) + \widehat{S}_{j_0+1}^{d*}(\xi_{\varepsilon_0',\zeta})\widehat{S}_{j_0+1}(\xi_{\varepsilon_0,\zeta}) = \mathbf{O}$$

for any $\zeta \in \mathcal{R}_{j_0}$ and $\varepsilon_0 \neq \varepsilon_0'$ such that $\xi_{\varepsilon_0,\zeta}, \xi_{\varepsilon_0',\zeta} \in \sigma_{j_0+1} \cap \sigma_{j_0+1}^d$. By combining (3.11) with (3.12) we deduce that the last matrix equality must hold for any $j_0 \geq 0$ and this completes the proof.

(ii) $\Leftrightarrow$ **(iii)**: For the proof of the "$\Rightarrow$" part we select $\theta = \Theta^{\mathrm{M}}$ and we establish the result. For the proof of the "$\Leftarrow$" part first we show that for any $j \geq 0$, $\zeta \in \mathcal{R}_j$ and

$\varepsilon_0 \in \mathcal{R}_1$ such that $\xi = \zeta + A^{*j}\varepsilon_0 \in \sigma_{j+1} \cap \sigma^d_{j+1} \subseteq \mathcal{R}_{j+1}$ we have

$$\theta_j(\xi) = \Theta^{\mathrm{M}}_j(\xi). \tag{3.15}$$

To do that we consider $\varepsilon_0 = \varepsilon_0'$ in (3.5) and then we subtract (3.5) from (3.1). Therefore we obtain the equality

$$\Big(\theta_j - \Theta^{\mathrm{M}}_j\Big)(\zeta + A^{*j}\varepsilon_0) = \widehat{H}^{d*}_{j+1}(A^{*j}\varepsilon_0 + \zeta)\Big(\theta_{j-1} - \Theta^{\mathrm{M}}_{j-1}\Big)(\zeta)\widehat{H}_{j+1}(A^{*j}\varepsilon_0 + \zeta), \tag{3.16}$$

which is valid for any $j \geq 0$, $\zeta \in \mathcal{R}_j$ and $\varepsilon_0 \in \mathcal{R}_1$ such that $\zeta + A^{*j}\varepsilon_0 \in \sigma_{j+1} \cap \sigma^d_{j+1}$. We proceed by induction. Since $\theta_{-1}(n) = \Theta^{\mathrm{M}}_{-1}(n) = I_r \ \forall n \in \mathbb{Z}^s$, we apply (3.16) for $j = 0$ and we conclude that

$$\theta_0(\xi) = \Theta^{\mathrm{M}}_0(\xi), \ \xi \in \sigma_1 \cap \sigma^d_1.$$

Hence, (3.15) is true for $j = 0$. Assume that (3.15) is true for $j = k - 1$. We consider $j = k$ and $\zeta \in \mathcal{R}_k, \varepsilon_0 \in \mathcal{R}_1$ such that $\xi = A^{*k}\varepsilon_0 + \zeta \in \sigma_{k+1} \cap \sigma^d_{k+1}$. We examine the following two cases:

Case A: $\zeta \in \sigma_k \cap \sigma^d_k$. Then $\theta_{k-1}(\zeta) = \Theta^{\mathrm{M}}_{k-1}(\zeta)$ and so for such ζ's and $j = k$, the right hand side of (3.16) is equal to the zero matrix. Therefore $\theta_k(\zeta + A^{*k}\varepsilon_0) = \Theta^{\mathrm{M}}_k(\zeta + A^{*k}\varepsilon_0)$.

Case B: $\zeta \notin \sigma_k \cap \sigma^d_k$. Since $A^{*k}\varepsilon_0 + \zeta \in \sigma_{k+1} \cap \sigma^d_{k+1}$ we recall (1.3) and so for $j = k$ the right hand side of (3.16) is equal to the $r \times r$ zero matrix once again. Therefore $\theta_k(\zeta + A^{*k}\varepsilon_0) = \Theta^{\mathrm{M}}_k(\zeta + A^{*k}\varepsilon_0)$.

By combining these two cases (3.15) follows.

Now we are ready to prove both conditions of part (ii) of this theorem. The validity of assumption (a) of part (iii) of this theorem implies that for each $\mu \in \mathbb{Z}^s$ and for sufficiently small $\varepsilon > 0$ there exists a sufficiently large natural number $M := M(\mu, \varepsilon)$ such that for any natural number $N > M$ we have $\big||A|^N \widehat{\phi}^{d*}_N(\mu)\theta_{N-1}(\mu)\widehat{\phi}_N(\mu) - 1\big| < \varepsilon$. This inequality implies that $\widehat{\phi}_N(\mu) \neq \mathbf{0}$ and $\widehat{\phi}^d_N(\mu) \neq \mathbf{0}$ for each $N > M$. This means that μ belongs in a coset in $\sigma_N \cap \sigma^d_N$. But then $\theta_{N-1}(\mu) = \Theta^{\mathrm{M}}_{N-1}(\mu)$ as we showed above (apply (3.15) for $j = N - 1$ and recall that θ_{N-1} is A^{*N}-periodic) and so the above inequality becomes $\big||A|^N \widehat{\phi}^{d*}_N(\mu)\Theta^{\mathrm{M}}_{N-1}(\mu)\widehat{\phi}_N(\mu) - 1\big| < \varepsilon$. This establishes condition (a) of part (ii) of this theorem.

It remains to prove condition (b) of part (ii) of this theorem. To do that we consider $\varepsilon_0 \neq \varepsilon_0'$ and then we apply (3.5) in the left hand side of (3.4). Therefore for any $j \geq 0$, $\zeta \in \mathcal{R}_j$ and $\varepsilon_0 \neq \varepsilon_0'$ such that $\zeta + A^{*j}\varepsilon_0, \zeta + A^{*j}\varepsilon_0' \in \sigma_{j+1} \cap \sigma^d_{j+1}$ we have

$$\begin{aligned} & \widehat{H}^{d*}_{j+1}(A^{*j}\varepsilon_0' + \zeta)\Theta^{\mathrm{M}}_{j-1}(\zeta)\widehat{H}_{j+1}(A^{*j}\varepsilon_0 + \zeta) \\ + \ & \widehat{S}^{d*}_{j+1}(A^{*j}\varepsilon_0' + \zeta)\widehat{S}_{j+1}(A^{*j}\varepsilon_0 + \zeta) \\ = \ & \widehat{H}^{d*}_{j+1}(A^{*j}\varepsilon_0' + \zeta)\Big(\theta_{j-1} - \Theta^{\mathrm{M}}_{j-1}\Big)(\zeta)\widehat{H}_{j+1}(A^{*j}\varepsilon_0 + \zeta). \end{aligned} \tag{3.17}$$

If $\zeta \in \sigma_j \cap \sigma^d_j$, then $\theta_{j-1}(\zeta) = \Theta^{\mathrm{M}}_{j-1}(\zeta)$ as we showed in (3.15) and so the right hand side of (3.17) is equal to the $r \times r$ zero matrix. If $\zeta \notin \sigma_j \cap \sigma^d_j$ then we use (1.3) to show that the right hand side of (3.17) is equal to the $r \times r$ zero matrix once again. Therefore condition (b) of part (ii) of this theorem is satisfied and the proof is complete. □

Remark 2. (i) By a careful look in the above proof we can see that condition (C) can be omitted in the proof of *(ii)* $\Rightarrow$ *(i)* and *(iii)* $\Rightarrow$ *(i)* parts of theorem 1.

(ii) From the proof of the *(iii)* $\Rightarrow$ *(ii)* part of theorem 1 we deduce that for $j \geq 0$, the values of θ_j coincide with the values of Θ_j^{M} on $\sigma_j \cap \sigma_j^d$ and they can be arbitrarily selected on $\mathcal{R}_j \backslash (\sigma_j \cap \sigma_j^d)$.

The particular case $\Theta_j^{\mathrm{M}}(\xi) = I_r$ satisfying (3.4) ensures that not only the pair $\Big(X(\phi_0, \Psi), X(\phi_0^d, \Psi^d)\Big)$ forms a pair of dual multiwavelet frames for L_2, but every pair of truncated multiwavelet systems $\Big(X(\phi_j, \Psi_j), X(\phi_j^d, \Psi_j^d)\Big)$ (see (2.5)) forms a pair of dual frames for L_2 as well. Indeed we have:

Theorem 2. *Let (Φ, Φ^d) be a pair of $\mathbb{Z}^s$-periodic multirefinable generators satisfying the above condition (C) and $\Big(X(\phi_j, \Psi_j), X(\phi_j^d, \Psi_j^d)\Big)$ be pairs of multiwavelet Bessel systems for any $j \geq 0$ as in* (2.5)*. Then the following conditions are equivalent:*

(i) *For any $j \geq 0$, $X(\phi_j^d, \Psi_j^d)$ is a dual frame of $X(\phi_j, \Psi_j)$ for L_2 .*

(ii) *Both conditions of part (ii) of theorem 1 are satisfied by the Mixed Fundamental sequence Θ^{M} such that $\Theta_{j-1}^{\mathrm{M}}(\xi) = I_r$ $\forall j \geq 0$ and $\xi \in \sigma_j \cap \sigma_j^d$.*

(iii) *The following conditions hold:*

(a) $\lim\limits_{j \to +\infty} \Big(|A|^j \widehat{\phi}_j^{d*}(\mu) \widehat{\phi}_j(\mu)\Big) = 1$ *pointwise on $\mu \in \mathbb{Z}^s$.*

(b) *For any $j \geq 0$, $\zeta \in \mathcal{R}_j$ and $\varepsilon_0, \varepsilon_0' \in \mathcal{R}_1$ such that $A^{*j}\varepsilon_0' + \zeta, A^{*j}\varepsilon_0 + \zeta \in \sigma_{j+1} \cap \sigma_{j+1}^d$ we have*

$$\begin{aligned} & \widehat{H}_{j+1}^{d*}(A^{*j}\varepsilon_0' + \zeta) \widehat{H}_{j+1}(A^{*j}\varepsilon_0 + \zeta) \\ + \quad & \widehat{S}_{j+1}^{d*}(A^{*j}\varepsilon_0' + \zeta) \widehat{S}_{j+1}(A^{*j}\varepsilon_0 + \zeta) = |A| \delta_{\varepsilon_0, \varepsilon_0'} I_r. \end{aligned} \tag{3.18}$$

Proof. **(i)** $\Rightarrow$ **(ii)**: Let $\Big(X(\phi_j, \Psi_j), X(\phi_j^d, \Psi_j^d)\Big)$ be a pair of dual frames for L_2 for any $j \geq 0$. We consider the case $j = 0$. Then, $\Big(X(\phi_0, \Psi), X(\phi_0^d, \Psi^d)\Big)$ is a pair of dual frames for L_2 and so both conditions of part (ii) of theorem 1 hold. It suffices to prove that $\Theta_{j-1}^{\mathrm{M}}(\xi) = I_r$, $\forall j \geq 1$ and $\xi \in \sigma_j \cap \sigma_j^d$. To do that we recall (3.6). For each $j \geq 1$ we set $\mu = A^{*j}l + \xi, \nu = A^{*j}k + \xi$, $k, l \in \mathbb{Z}^s$, $\xi \in \mathcal{R}_j$ in (3.6) and we obtain

$$\widehat{\phi}_0^{d*}(A^{*j}k + \xi) \widehat{\phi}_0(A^{*j}l + \xi) + \sum_{n=0}^{\kappa_0(A^{*j}(k-l))} |A|^n \widehat{\psi}_n^{d*}(A^{*j}k + \xi) \widehat{\psi}_n(A^{*j}l + \xi) = \delta_{k,l}.$$

By working exactly as in (3.10) the above equality becomes

$$\begin{aligned} & |A|^j \widehat{\phi}_j^{d*}(A^{*j}k + \xi) \Theta_{j-1}^{\mathrm{M}}(\xi) \widehat{\phi}_j(A^{*j}l + \xi) \\ + \quad & \sum_{n=j}^{j+\kappa_0(k-l)} |A|^n \widehat{\psi}_n^{d*}(A^{*j}k + \xi) \widehat{\psi}_n(A^{*j}l + \xi) = \delta_{k,l}. \end{aligned} \tag{3.19}$$

On the other hand, the fact that $\Big(X(\phi_j,\Psi_j),X(\phi_j^d,\Psi_j^d)\Big)$ is a pair of dual frames for L_2 implies that (2.6) holds and so for any $j\geq 1$ and $\xi\in\mathcal{R}_j$ we have

$$\begin{aligned}&|A|^j\widehat{\phi}_j^{d*}(A^{*j}k+\xi)\widehat{\phi}_j(A^{*j}l+\xi)\\ +&\sum_{n=j}^{j+\kappa_0(k-l)}|A|^n\widehat{\psi}_n^{d*}(A^{*j}k+\xi)\widehat{\psi}_n(A^{*j}l+\xi)=\delta_{k,l}\ \forall k,l\in\mathbb{Z}^s. \quad (3.20)\end{aligned}$$

We subtract (3.20) from (3.19) and we obtain

$$\widehat{\phi}_j^{d*}(A^{*j}k+\xi)\big(\Theta_{j-1}^{\mathrm{M}}(\xi)-I_r\big)\widehat{\phi}_j(A^{*j}l+\xi)=0$$

for any $\xi\in\mathcal{R}_j$ and $k,l\in\mathbb{Z}^s$. If $\xi\notin\sigma_j\cap\sigma_j^d$, then the above equality is obviously satisfied. For the remaining values of $\xi\in\sigma_j\cap\sigma_j^d$ we use condition (C). Let $W_j(\xi)$ and $W_j^d(\xi)$ be $r\times r$ non-singular matrices as in (3.14). Then the previous equality repeatedly applied for any $(k,l)\in I_{j,\xi}^d\times I_{j,\xi}$ yields that

$$W_j^{d*}(\xi)\big(\Theta_{j-1}^{\mathrm{M}}(\xi)-I_r\big)W_j(\xi)=\mathbf{O}$$

and from this we obtain the result.

(ii) $\Rightarrow$ **(i)**: In this case, for any $j\geq 0$, $\xi\in\mathcal{R}_j$ and $p,p'\in\mathcal{R}_1$ such that $A^{*j}p+\xi, A^{*j}p'+\xi\in\sigma_{j+1}\cap\sigma_{j+1}^d$ the following equality holds

$$\begin{aligned}&\widehat{H}_{j+1}^{d*}(A^{*j}p'+\xi)\Theta_{j-1}^{\mathrm{M}}(\xi)\widehat{H}_{j+1}(A^{*j}p+\xi)+\widehat{S}_{j+1}^{d*}(A^{*j}p'+\xi)\widehat{S}_{j+1}(A^{*j}p+\xi)\\ =\ &|A|\delta_{p,p'}I_r. \quad (3.21)\end{aligned}$$

Also, by the $(ii)\Rightarrow(i)$ part of theorem 1, $\Big(X(\phi_0,\Psi),X(\phi_0^d,\Psi^d)\Big)$ is a pair of dual multiwavelet frames for L_2 and so

$$\begin{aligned}\|f\|_{L_2}^2 &= \sum_{m=1}^{r}\big\langle f,\phi_{0,m}^d\,\big\rangle\big\langle\phi_{0,m},f\big\rangle\\ &+\sum_{n=0}^{\infty}\sum_{l\in\mathcal{L}_n}\sum_{m=1}^{\rho_n}\big\langle f,\psi_{n,m}^d(\cdot-A^{-n}l)\,\big\rangle\big\langle\psi_{n,m}(\cdot-A^{-n}l),f\big\rangle. \quad (3.22)\end{aligned}$$

Below we shall prove that for any $j\geq 1$, eq. (3.22) is equivalent to

$$\begin{aligned}\|f\|_{L_2}^2=&\sum_{l\in\mathcal{L}_j}\sum_{m=1}^{r}\big\langle f,\phi_{j,m}^d(\cdot-A^{-j}l)\,\big\rangle\big\langle\phi_{j,m}(\cdot-A^{-j}l),f\big\rangle\\ +&\sum_{n=j}^{\infty}\sum_{l\in\mathcal{L}_n}\sum_{m=1}^{\rho_n}\big\langle f,\psi_{n,m}^d(\cdot-A^{-n}l)\,\big\rangle\big\langle\psi_{n,m}(\cdot-A^{-n}l),f\big\rangle, \quad (3.23)\end{aligned}$$

which implies that $\Big(X(\phi_j,\Psi_j), X(\phi_j^d,\Psi_j^d)\Big)$ is a pair of dual frames for L_2. In order to prove (3.23) it suffices to prove that for each $j \geq 1$ the following equality holds:

$$\begin{aligned} & \sum_{l\in\mathcal{L}_j}\sum_{m=1}^{r}\big\langle f,\phi_{j,m}^d(\cdot - A^{-j}l)\big\rangle\big\langle\phi_{j,m}(\cdot - A^{-j}l), f\big\rangle \\ + & \sum_{l\in\mathcal{L}_j}\sum_{m=1}^{\rho_j}\big\langle f,\psi_{j,m}^d(\cdot - A^{-j}l)\big\rangle\big\langle\psi_{j,m}(\cdot - A^{-j}l), f\big\rangle \\ = & \sum_{l\in\mathcal{L}_{j+1}}\sum_{m=1}^{r}\big\langle f,\phi_{j+1,m}^d(\cdot - A^{-(j+1)}l)\big\rangle\big\langle\phi_{j+1,m}(\cdot - A^{-(j+1)}l), f\big\rangle. \end{aligned} \tag{3.24}$$

By recalling the notation established in the statement of proposition 2 and by working in the Fourier domain, the left hand side of (3.24) is equal to

$$\begin{aligned} & |A|^j\sum_{\mu,\nu\in\mathbb{Z}^s}\sum_{\xi\in\mathcal{R}_j}\widehat{f}(A^{*j}\mu+\xi)\Big(\widehat{\phi}_j^{d*}(A^{*j}\mu+\xi)\widehat{\phi}_j(A^{*j}\nu+\xi) \\ + & \widehat{\psi}_j^{d*}(A^{*j}\mu+\xi)\widehat{\psi}_j(A^{*j}\nu+\xi)\Big)\overline{\widehat{f}(A^{*j}\nu+\xi)}. \end{aligned}$$

Since $\Theta_{j-1}^{\mathrm{M}}(\xi) = I_r\ \forall\xi\in\sigma_j\cap\sigma_j^d$ and $\widehat{\phi}_j^{d*}(A^{*j}\mu+\xi) = \mathbf{0}$ or $\widehat{\phi}_j(A^{*j}\nu+\xi) = \mathbf{0}$ $\forall\xi\in\mathcal{R}_j\backslash(\sigma_j\cap\sigma_j^d)$, the above sum can be written in the form

$$\begin{aligned} & |A|^j\sum_{\mu,\nu\in\mathbb{Z}^s}\sum_{\xi\in\mathcal{R}_j}\widehat{f}(A^{*j}\mu+\xi)\Big(\widehat{\phi}_j^{d*}(A^{*j}\mu+\xi)\Theta_{j-1}^{\mathrm{M}}(\xi)\widehat{\phi}_j(A^{*j}\nu+\xi) \\ + & \widehat{\psi}_j^{d*}(A^{*j}\mu+\xi)\widehat{\psi}_j(A^{*j}\nu+\xi)\Big)\overline{\widehat{f}(A^{*j}\nu+\xi)} \\ = & |A|^j\sum_{\mu,\nu\in\mathbb{Z}^s}\sum_{p,p'\in\mathcal{R}_1}\sum_{\xi\in\mathcal{R}_j}\widehat{f}(A^{*(j+1)}\mu+A^{*j}p'+\xi)\widehat{\phi}_{j+1}^{d*}(A^{*(j+1)}\mu+A^{*j}p'+\xi) \\ & \Big(\widehat{H}_{j+1}^{d*}(A^{*j}p'+\xi)\Theta_{j-1}^{\mathrm{M}}(\xi)\widehat{H}_{j+1}(A^{*j}p+\xi)+\widehat{S}_{j+1}^{d*}(A^{*j}p'+\xi)\widehat{S}_{j+1}(A^{*j}p+\xi)\Big) \\ & \widehat{\phi}_{j+1}(A^{*(j+1)}\nu+A^{*j}p+\xi)\overline{\widehat{f}(A^{*(j+1)}\nu+A^{*j}p+\xi)}. \end{aligned}$$

For any $\xi\in\mathcal{R}_j$ and $p,p'\in\mathcal{R}_1$ such that $A^{*j}p+\xi, A^{*j}p'+\xi\in\sigma_{j+1}\cap\sigma_{j+1}^d$ we apply (3.21) and we note that all other summands in the right hand side of the previous equality corresponding to the case $A^{*j}p+\xi, A^{*j}p'+\xi\notin\sigma_{j+1}\cap\sigma_{j+1}^d$ are equal to zero. Then, without loss, the last triple sum can be written by

$$
\begin{aligned}
& |A|^{j+1} \sum_{\mu,\nu\in\mathbb{Z}^s} \sum_{p,p'\in\mathcal{R}_1} \sum_{\xi\in\mathcal{R}_j} \widehat{f}(A^{*(j+1)}\mu + A^{*j}p' + \xi)\widehat{\phi}^{d*}_{j+1}(A^{*(j+1)}\mu + A^{*j}p' + \xi) \\
& \delta_{p,p'} I_r \widehat{\phi}_{j+1}(A^{*(j+1)}\nu + A^{*j}p + \xi)\overline{\widehat{f}(A^{*(j+1)}\nu + A^{*j}p + \xi)} \\
= \; & |A|^{j+1} \sum_{\mu,\nu\in\mathbb{Z}^s} \sum_{\xi\in\mathcal{R}_{j+1}} \widehat{f}(A^{*(j+1)}\mu + \xi)\widehat{\phi}^{d*}_{j+1}(A^{*(j+1)}\mu + \xi) \\
& \widehat{\phi}_{j+1}(A^{*(j+1)}\nu + \xi)\overline{\widehat{f}(A^{*(j+1)}\nu + \xi)} \\
= \; & \sum_{\mu,\nu\in\mathbb{Z}^s} \sum_{\zeta,\xi\in\mathcal{R}_{j+1}} \widehat{f}(A^{*(j+1)}\mu + \zeta)\widehat{\phi}^{d*}_{j+1}(A^{*(j+1)}\mu + \zeta)\Big(\sum_{l\in\mathcal{L}_{j+1}} e^{2\pi i(\zeta-\xi)\cdot A^{-(j+1)}l}\Big) \\
& \widehat{\phi}_{j+1}(A^{*(j+1)}\nu + \xi)\overline{\widehat{f}(A^{*(j+1)}\nu + \xi)} \\
= \; & \sum_{l\in\mathcal{L}_{j+1}} \sum_{m=1}^{r} \big\langle f, \phi^d_{j+1,m}(\cdot - A^{-(j+1)}l)\big\rangle\big\langle \phi_{j+1,m}(\cdot - A^{-(j+1)}l), f\big\rangle.
\end{aligned}
$$

Since we established (3.24) and since (3.22) holds, we derive (3.23) iteratively and we complete the proof.

(ii) $\Rightarrow$ **(iii)**: We recall (3.21). For $j \geq 0$, $\xi \in \mathcal{R}_j$ and $p, p' \in \mathcal{R}_1$ such that $A^{*j}p + \xi, A^{*j}p' + \xi \in \sigma_{j+1} \cap \sigma^d_{j+1}$, this equation can be written by

$$
\begin{aligned}
& \widehat{H}^{d*}_{j+1}(A^{*j}p' + \xi)\Big(\Theta^{\mathrm{M}}_{j-1}(\xi) - I_r\Big)\widehat{H}_{j+1}(A^{*j}p + \xi) \\
+ \; & \Big(\widehat{H}^{d*}_{j+1}(A^{*j}p' + \xi)\widehat{H}_{j+1}(A^{*j}p + \xi) + \widehat{S}^{d*}_{j+1}(A^{*j}p' + \xi)\widehat{S}_{j+1}(A^{*j}p + \xi)\Big) \\
= \; & |A|\delta_{p,p'} I_r.
\end{aligned}
$$

The first term of the left hand side of the above equality is equal to the zero matrix, because if $\xi \in \sigma_j \cap \sigma^d_j$ then $\Theta^{\mathrm{M}}_{j-1}(\xi) = I_r$ (by assumption), otherwise at least one of the matrices $\widehat{H}^d_{j+1}(A^{*j}p' + \xi)$ or $\widehat{H}_{j+1}(A^{*j}p + \xi)$ is equal to the zero matrix due to (1.3). Hence, condition (b) of part (iii) of this theorem is proved. Finally, the validity of assumption (a) of part (ii) of this theorem implies that for for each μ and for sufficiently small $\varepsilon > 0$ there exists a sufficiently large $M := M(\mu, \varepsilon)$ such that for any $N > M$ we have $\big||A|^N \widehat{\phi}^{d*}_N(\mu)\Theta^{\mathrm{M}}_{N-1}(\mu)\widehat{\phi}_N(\mu) - 1\big| < \varepsilon$. Thus μ belongs in a particular coset in $\sigma_N \cap \sigma^d_N$ and so $\Theta^{\mathrm{M}}_{N-1}(\mu) = I_r$ by assumption and the fact that $\Theta^{\mathrm{M}}_{N-1}$ is A^{*N}-periodic. Therefore, the above inequality becomes $\big||A|^N \widehat{\phi}^{d*}_N(\mu)\widehat{\phi}_N(\mu) - 1\big| < \varepsilon$. This establishes condition (a) of part (iii) of this theorem.

(iii) $\Rightarrow$ **(ii)**: It is a minor modification of the proof of the (iii) $\Rightarrow$ (ii) part of theorem 1 for the case $\theta_j(n) = I_r,\ j \geq 0$ and $n \in \sigma_j \cap \sigma^d_j$. □

Remark 3. Condition (C) can be omitted in the proof of $(ii) \Rightarrow (i)$ and $(iii) \Rightarrow (i)$ parts of theorem 2.

We mention that if the conditions of theorem 2 hold, then we talk about the *Mixed Unitary Extension Principle* associated with the dual pairs $\big(X(\phi_j, \Psi_j), X(\phi^d_j, \Psi^d_j)\big)$, otherwise we talk about the *Mixed Oblique Extension Principle* associated with the dual pair $\big(X(\phi_0,\Psi), X(\phi_0^d,\Psi^d)\big)$. For the Oblique Extension Principle with respect to Parseval multiwavelet frames we have:

Corollary 2. *Let Φ be a sequence of multirefinable generators satisfying condition (C). Then the following conditions are equivalent:*

(i) $X(\phi_0,\Psi)$ *is a Parseval multiwavelet frame for* L_2.

(ii) *The Fundamental sequence $\Theta=\{\Theta_j\}$ satisfies the following conditions:*

(a) $\lim_{N\to+\infty}\Big(|A|^N\widehat{\phi}_N^*(\mu)\Theta_{N-1}(\mu)\widehat{\phi}_N(\mu)\Big)=1$ *pointwise on* $\mu\in\mathbb{Z}^s$.

(b) *For any $j\ge 0$, $\zeta\in\mathcal{R}_j$ and $\varepsilon_0\neq\varepsilon_0'$ $(\varepsilon_0,\varepsilon_0'\in\mathcal{R}_1)$ such that $A^{*j}\varepsilon_0'+\zeta, A^{*j}\varepsilon_0+\zeta\in\sigma_{j+1}$ we have*

$$\begin{aligned}&\widehat{H}_{j+1}^*(A^{*j}\varepsilon_0'+\zeta)\Theta_{j-1}(\zeta)\widehat{H}_{j+1}(A^{*j}\varepsilon_0+\zeta)\\ +\quad&\widehat{S}_{j+1}^*(A^{*j}\varepsilon_0'+\zeta)\widehat{S}_{j+1}(A^{*j}\varepsilon_0+\zeta)=\mathbf{O}.\end{aligned}$$

(iii) *There exists a sequence $\theta=\{\theta_j:\ j\ge -1\}$ whose elements $\theta_j:\mathbb{Z}^s\to\mathbb{C}^{r\times r}$ are $A^{*(j+1)}$-periodic matrix valued functions such that $\theta_{-1}(n)=I_r\ \forall n\in\mathbb{Z}^s$ and the following conditions are satisfied:*

(a) $\lim_{N\to+\infty}\Big(|A|^k\widehat{\phi}_N^*(\mu)\theta_{N-1}(\mu)\widehat{\phi}_N(\mu)\Big)=1$ *pointwise on* $\mu\in\mathbb{Z}^s$.

(b) *For any $j\ge 0$, $\zeta\in\mathcal{R}_j$ and $\varepsilon_0,\varepsilon_0'\in\mathcal{R}_1$ such that $A^{*j}\varepsilon_0'+\zeta, A^{*j}\varepsilon_0+\zeta\in\sigma_{j+1}$ we have*

$$\begin{aligned}&\widehat{H}_{j+1}^*(A^{*j}\varepsilon_0'+\zeta)\theta_{j-1}(\zeta)\widehat{H}_{j+1}(A^{*j}\varepsilon_0+\zeta)\\ +\quad&\widehat{S}_{j+1}^*(A^{*j}\varepsilon_0'+\zeta)\widehat{S}_{j+1}(A^{*j}\varepsilon_0+\zeta)=|A|\delta_{\varepsilon_0,\varepsilon_0'}\theta_j(A^{*j}\varepsilon_0+\zeta).\end{aligned}$$

Proof. It is a minor modification of the proof of theorem 1 for the case $\Phi=\Phi^d$, $\Psi=\Psi^d$ and $\Theta^{\mathrm{M}}=\Theta$. The only difference is that a Bessel assumption on $X(\phi_0,\Psi)$ is not a-priori required. ☐

Remark 4. A weaker version of the $(iii)\Rightarrow(i)$ part of corollary 2 was proved in [21, theorem 3.1].

Finally, for the Unitary Extension Principle with respect to Parseval (truncated) multiwavelet frames we have the following:

Corollary 3. *Under the above assumptions on Φ the following conditions are equivalent:*

(i) $X\big(\phi_j,\Psi_j\big)$ *is a Parseval frame of* L_2 *for any* $j\ge 0$.

(ii) *Both conditions of part (ii) of corollary 2 are satisfied by the Fundamental sequence Θ such that $\Theta_{j-1}(\xi)=I_r,\ \forall\xi\in\sigma_j$.*

(iii) *The following conditions hold:*

(a) $\lim_{N\to+\infty}\Big(|A|^N\widehat{\phi}_N^*(\mu)\widehat{\phi}_N(\mu)\Big)=1$ *pointwise on* $\mu\in\mathbb{Z}^s$.

(b) *For any $j\ge 0$, $\zeta\in\mathcal{R}_j$ and $\varepsilon_0,\varepsilon_0'\in\mathcal{R}_1$ such that $A^{*j}\varepsilon_0'+\zeta, A^{*j}\varepsilon_0+\zeta\in\sigma_{j+1}$ we have*

$$\begin{aligned}&\widehat{H}_{j+1}^*(A^{*j}\varepsilon_0'+\zeta)\widehat{H}_{j+1}(A^{*j}\varepsilon_0+\zeta)\\ +\quad&\widehat{S}_{j+1}^*(A^{*j}\varepsilon_0'+\zeta)\widehat{S}_{j+1}(A^{*j}\varepsilon_0+\zeta)=|A|\delta_{\varepsilon_0,\varepsilon_0'}I_r.\end{aligned}\tag{3.25}$$

Proof. **(i)** $\Leftrightarrow$ **(ii)**: It is a minor modification of the proof of the equivalence $(i) \Leftrightarrow (ii)$ of theorem 2 for the case $\Phi = \Phi^d$, $\Psi = \Psi^d$ and $\Theta^{\mathrm{M}} = \Theta$, but without any Bessel assumption imposed on $X\big(\phi_j, \Psi_j\big)$. In fact, the Bessel assumption imposed at the statement of theorem 2 is used implicitly in the proof of the "$\Leftarrow$" part to establish the unconditional convergence of the series (3.23). If $\Phi = \Phi^d$ and $\Psi = \Psi^d$, then the assumptions of part (ii) of theorem 2 ensure that $X(\phi_0, \Psi)$ is a Parseval frame of L_2, hence it is also a Bessel system. Then, by following exactly the same steps as in the proof of the $(ii) \Rightarrow (i)$ of theorem 2, eq. (3.23) becomes

$$\begin{aligned} \|f\|^2_{L_2} &= \sum_{l\in\mathcal{L}_j}\sum_{m=1}^{r}\big|\big\langle f, \phi_{j,m}(\cdot - A^{-j}l)\big\rangle\big|^2 \\ &+ \sum_{n=j}^{\infty}\sum_{l\in\mathcal{L}_n}\sum_{m=1}^{\rho_n}\big|\big\langle f, \psi_{n,m}(\cdot - A^{-n}l)\big\rangle\big|^2,\ \forall j\geq 1 \end{aligned}$$

and so $X(\phi_j, \Psi_j)$ is a Bessel system and of course a Parseval frame. The rest follow easily.

(ii) $\Leftrightarrow$ **(iii)**: It is a minor modification of the proof of the equivalence $(ii) \Leftrightarrow (iii)$ of theorem 2. □

Remark 5. The $(ii) \Rightarrow (i)$ part of corollary 3 was proved in [21, theorem 2.2] for the case $j = 0$ and $\sigma_j = \mathcal{R}_j\ \forall j \geq 0$.

4. Examples

We close with some examples illustrating applications of theorems 1 and 2.

Example 1. Let $\psi = (\psi_1, ..., \psi_m)$ and $\psi^d = (\psi^d_1, ..., \psi^d_m)$ be two sets of mother wavelets in $L^m_2(\mathbb{R}^2)$ whose dyadic dilations and translations form a pair of dual wavelet frames in $L_2(\mathbb{R}^2)$. In addition, we assume that the pair (ψ, ψ^d) is derived from a pair (ϕ, ϕ^d) of refinable functions in $L_1 \cap L_2(\mathbb{R}^2)$ satisfying the following UEP conditions (recall [14], with a variety of examples therein):

(i) $\widehat{\phi}(\gamma)$ and $\widehat{\phi}^d(\gamma)$ are continuous in a neighborhood of the origin and

$$\widehat{\phi}(0) = \widehat{\phi}^d(0) = 1. \tag{4.1}$$

(ii) $$\overline{h^d_0(\gamma + q_i)}h_0(\gamma) + h^{d*}_1(\gamma + q_i)h_1(\gamma) = \delta_{i,0}\ (i = 0, ..., 3),\ \gamma \in \mathbb{T}^2. \tag{4.2}$$

Here, $\widehat{\phi}$ (resp. $\widehat{\phi}^d$) is the Fourier transform of ϕ (resp. ϕ^d) on $\mathbb{R}^2$. Also, $h_0, h^d_0 : \mathbb{T}^2 \to \mathbb{C}$ are continuous $\mathbb{Z}^2$-periodic refinement masks of ϕ, ϕ^d and $h_1, h^d_1 : \mathbb{T}^2 \to \mathbb{C}^{m\times 1}$ are continuous $\mathbb{Z}^s$-periodic wavelet masks of ψ, ψ^d respectively. Finally, $q_i \in \{(0,0), (0, \frac{1}{2}), (\frac{1}{2}, 0), (\frac{1}{2}, \frac{1}{2})\}$. For the above selection of the pair (ϕ, ϕ^d) we define two sequences $P = \{P_j\}_{j\geq 0}$ and $P^d = \{P^d_j\}_{j\geq 0}$ of trigonometric polynomials in $L_2(\mathbb{T}^2)$ by

$$\begin{cases} P_j(\gamma) = \sum_{\mu\in\mathbb{Z}^2\cap[-2^{j-1}+1, 2^{j-1}]^2} \frac{\widehat{\phi}(2^{-j}\mu)}{2^j} e^{2\pi i\mu\gamma} \\ P^d_j(\gamma) = \sum_{\mu\in\mathbb{Z}^2\cap[-2^{j-1}+1, 2^{j-1}]^2} \frac{\widehat{\phi}^d(2^{-j}\mu)}{2^j} e^{2\pi i\mu\gamma} \end{cases}, \gamma \in \mathbb{T}^2.$$

Then, for any $j \geq 0$ and $\mu \in \mathbb{Z}^2 \cap [-2^{j-1} + 1, 2^{j-1}]^2$ we have

$$\widehat{P}_j(\mu) = \frac{\widehat{\phi}(2^{-j}\mu)}{2^j} = \frac{h_0\big(2^{-(j+1)}\mu\big)}{2^j}\widehat{\phi}\big(2^{-(j+1)}\mu\big) = 2h_0\big(2^{-(j+1)}\mu\big)\widehat{P}_{j+1}(\mu)$$

and so

$$\widehat{P}_j(\mu) = \widehat{H}_{j+1}(\mu)\widehat{P}_{j+1}(\mu),$$

where $\widehat{H}_{j+1} : \mathbb{Z}^2 \to \mathbb{C}$ is a $2^{j+1}I_2$-periodic refinement mask defined on $\mathcal{R}_{j+1}$ by

$$\widehat{H}_{j+1}(2^j\varepsilon_0 + \zeta) = \begin{cases} 2h_0\big(2^{-(j+1)}\zeta\big) & \varepsilon_0 = 0 \\ 0 & \varepsilon_0 \neq 0 \end{cases}, \ \varepsilon_0 \in \mathcal{R}_1, \ \zeta \in \mathcal{R}_j.$$

Hence P is refinable and by using the same arguments we can show that P^d is refinable too. Let Ψ and Ψ^d be two sets of multiwavelets derived from the pair (P, P^d) by (1.4). We aim to construct $2^{j+1}I_2$-periodic wavelet masks $\widehat{S}_{j+1}, \widehat{S}^d_{j+1} : \mathbb{Z}^2 \to \mathbb{C}^{\rho_j \times 1}$ so that $\Big(X(P_0, \Psi), X(P^d_0, \Psi^d)\Big)$ is a pair of dual multiwavelet frames for $L_2(\mathbb{T}^2)$. To do that we use part (iii) of theorem 2. First, we observe that

$$\lim_{j\to+\infty} 2^{2j}\,\overline{\widehat{P}^d_j(\mu)}\widehat{P}_j(\mu) = \lim_{j\to+\infty} \overline{\widehat{\phi}^d(2^{-j}\mu)}\widehat{\phi}(2^{-j}\mu) = 1, \text{ pointwise on } \mu \in \mathbb{Z}^2$$

due to (4.1). Therefore condition (a) of part (iii) of theorem 2 is satisfied. It suffices to show (3.18) for some suitable choice of the above wavelet masks. For any $j \geq 0$, $\zeta \in \mathcal{R}_j$ and $\varepsilon_i \in \mathcal{R}_1 = \{(0,0),(1,0),(0,1),(1,1)\}$ $(i = 0, ..., 3)$, we define the following $\rho_j \times 1$ $(\rho_j \geq m)$ wavelet masks:

$$\widehat{S}_{j+1}(2^j\varepsilon_i + \zeta) = \begin{cases} \begin{pmatrix} 2\ h_1\big(2^{-(j+1)}\zeta\big) \\ ------ \\ \mathbf{O}_{(\rho_j - m)\times 1} \end{pmatrix} & i = 0 \\ 2\big(A_{j+1}\big)_{\cdot,i} & i = 1,2,3 \end{cases}$$

and

$$\widehat{S}^d_{j+1}(2^j\varepsilon_i + \zeta) = \begin{cases} \begin{pmatrix} 2\ h^d_1\big(2^{-(j+1)}\zeta\big) \\ ------ \\ \mathbf{O}_{(\rho_j - m)\times 1} \end{pmatrix} & i = 0 \\ 2\big(A^d_{j+1}\big)_{\cdot,i} & i = 1,2,3 \end{cases},$$

where A_{j+1} and A^d_{j+1} are any $\rho_j \times 3$ real matrices satisfying

$$A^d_{j+1}\, A_{j+1} = I_3 \ \forall j \geq 0, \tag{4.3}$$

provided that $\rho_j \geq 3$. Then (3.18) becomes

$$\overline{\widehat{H}^d_{j+1}(2^j\varepsilon_m + \zeta)}\widehat{H}_{j+1}(2^j\varepsilon_n + \zeta) + \widehat{S}^{d*}_{j+1}(2^j\varepsilon_m + \zeta)\widehat{S}_{j+1}(2^j\varepsilon_n + \zeta)$$

$$= \begin{cases} 4\overline{h^d_0\big(2^{-(j+1)}\zeta\big)}h_0\big(2^{-(j+1)}\zeta\big) + 4h^{d*}_1\big(2^{-(j+1)}\zeta\big)h_1\big(2^{-(j+1)}\zeta\big) = 4 & m = n = 0 \\ 4\sum_{l=1}^{\rho_j}\big(A^d_{j+1}\big)_{m,l}\big(A_{j+1}\big)_{l,n} = 4, & m = n \neq 0 \\ 4\sum_{l=1}^{\rho_j}\big(A^d_{j+1}\big)_{m,l}\big(A_{j+1}\big)_{l,n} = 0 & m \neq n \end{cases},$$

as a result of (4.2) and (4.3). Therefore condition (b) of part (iii) of theorem 2 is also satisfied. Let $\|A_{j+1}\|_{\mathcal{F}} \leq C$ and $\|A^d_{j+1}\|_{\mathcal{F}} \leq C^d$ for some absolute positive constants C and C^d, where $\|\cdot\|_{\mathcal{F}}$ is the usual Frobenious norm of the above matrices A_{j+1} and A^d_{j+1}. This condition combined with the Bessel property of the quasi-affine wavelet systems $X(\phi, \Psi)$ and $X(\phi^d, \Psi^d)$, (see [1]), ensures that $X(P_0, \Psi)$ and $X(P^d_0, \Psi^d)$ are Bessel systems for $L_2(\mathbb{T}^2)$. Then all assumptions of part (iii) of theorem 2 are satisfied and a class of dual multiwavelet frames for $L_2(\mathbb{T}^2)$ is produced.

Example 2. Let $\psi = (\psi_1, ..., \psi_m)$ be a Parseval multiwavelet frame for $L_2(\mathbb{R}^s)$ (with respect to the usual dyadic dilation) derived from a refinable function vector $\varphi = (\varphi_1, ...\varphi_r)^T \in L_1 \cap L_2^{r\times 1}(\mathbb{R}^s)$. We assume that the following UEP conditions are satisfied

$$\textbf{(i)} \quad \lim_{j\to+\infty} \widehat{\varphi}^*(2^{-j}\gamma)\widehat{\varphi}(2^{-j}\gamma) = 1,\ \gamma \in \mathbb{R}^s, \tag{4.4}$$

and

$$\textbf{(ii)} \quad h_0^*(\gamma + q_i)h_0(\gamma) + h_1^*(\gamma + q_i)h_1(\gamma) = \delta_{i,0}I_r,\ \gamma \in \mathbb{T}^s, \tag{4.5}$$

where $q_i \in \{(0,0), (0, \frac{1}{2}), (\frac{1}{2}, 0), (\frac{1}{2}, \frac{1}{2})\}$. Here, $\widehat{\varphi}$ is the Fourier transform of the function vector φ on $\mathbb{R}^s$ and $h_0 : \mathbb{T}^s \to \mathbb{C}^{r\times r}$ is a continuous $\mathbb{Z}^s$-periodic refinement mask, whereas $h_1 : \mathbb{T}^s \to \mathbb{C}^{m\times r}$ is a continuous $\mathbb{Z}^s$-periodic wavelet mask. Recall also that the Gram matrix G_φ must be non singular on $\mathbb{T}^s$ (see [2, Theorem 2], where a variety of such examples for the two-dimensional case is presented). For the above selection of φ we consider a sequence $\Phi = \{\phi_j = (\phi_{j,1}, ..., \phi_{j,r})\}_{j\geq 0}$ in $L_2^{r\times 1}(\mathbb{T}^s)$ defined by

$$\phi_{j,k}(\gamma) = 2^{js/2} \sum_{l\in\mathbb{Z}^s} \varphi_k\big(2^j(\gamma - l)\big).$$

Then for any $j \geq 0$ and $n \in \mathbb{Z}^s$ we have

$$\widehat{\phi}_{j,k}(n) = \frac{\widehat{\varphi}_k(2^{-j}n)}{2^{js/2}}$$

and so

$$\widehat{\phi}_j(n) = \frac{\widehat{\varphi}(2^{-j}n)}{2^{js/2}} = 2^{s/2}h_0\big(2^{-(j+1)}n\big)\widehat{\varphi}\big(2^{-(j+1)}n\big) = \widehat{H}_{j+1}(n)\widehat{\phi}_{j+1}(n),$$

where

$$\widehat{H}_{j+1}(n) = 2^{s/2}h_0\big(2^{-(j+1)}n\big).$$

Therefore Φ is refinable. Let Ψ be a set of multiwavelets constructed from Φ by (1.4). We aim to construct wavelet masks $\widehat{S}_{j+1} : \mathbb{Z}^s \to \mathbb{C}^{m\times r}$ so that $X(\phi_0, \Psi)$ is a Parseval multiwavelet frame for $L_2(\mathbb{T}^s)$. To do that we use part (iii) of corollary 3. First, we observe that

$$\lim_{j\to+\infty} 2^{sj}\widehat{\phi}_j^*(n)\widehat{\phi}_j(n) = \lim_{j\to+\infty} \widehat{\varphi}^*(2^{-j}n)\widehat{\varphi}(2^{-j}n) = 1, \text{ pointwise on } n \in \mathbb{Z}^s$$

due to (4.4). Therefore condition (a) of part (iii) of corollary 3 is satisfied. It suffices to show (3.25) for some suitable choice of the above wavelet filters. For $j \geq 0$, $\zeta \in \mathcal{R}_j$ and $\varepsilon_i \in \mathcal{R}_1 = \{(0,0), (1,0), (0,1), (1,1)\}$ $(i = 0, ..., 3)$ we define the following $2^{j+1}I_2$-periodic $m \times r$ wavelet mask:

$$\widehat{S}_{j+1}(2^j\varepsilon_i + \zeta) = 2\ h_1\big(\frac{\zeta}{2^{j+1}} + q_i\big),\ i = 0, ..., 3$$

where h_1 is the $\mathbb{Z}^s$-periodic $m \times r$ wavelet mask of ψ and $q_i \in \frac{1}{2}\mathcal{R}_1$. Then (3.25) is satisfied as a direct result of (4.5) and so a class of two-dimensional Parseval multiwavelet frames for $L_2(\mathbb{T}^2)$ is produced.

Example 3. Let $X(\phi_0,\Psi)$ be a Parseval multiwavelet frame of $L_2(\mathbb{T}^s)$ generated from a $\mathbb{Z}^s$-periodic multirefinable sequence $\Phi=\{\phi_j:\ j\geq 0\}$ in $L_2^r(\mathbb{T}^s)$ satisfying condition (C). Assume that $\widehat{h}=\{\widehat{h}_j\}_{j\geq 1}$ is the sequence of refinement masks of Φ. Then the UEP conditions of corollary 3 are satisfied. Consider two sequences $U=\{U_j:\ j\geq 0\}$ and $U^d=\{U_j^d:\ j\geq 0\}$ of $r\times r$ invertible real matrices and define

$$\begin{cases} \widehat{\widetilde{\phi}}_j(n)=U_j\ \widehat{\phi}_j(n) \\ \widehat{\widetilde{\phi}_j^d}(n)=U_j^d\ \widehat{\phi}_j(n)\end{cases},\ n\in\mathbb{Z}^s.$$

Then

$$\widehat{\widetilde{\phi}}_j(n)=U_j\ \widehat{\phi}_j(n)=U_j\ \widehat{h}_{j+1}(n)\ \widehat{\phi}_{j+1}(n)=U_j\ \widehat{h}_{j+1}(n)\ U_{j+1}^{-1}\ \widehat{\widetilde{\phi}}_{j+1}(n)$$

and so

$$\widehat{\widetilde{\phi}}_j(n)=\widehat{H}_{j+1}(n)\widehat{\widetilde{\phi}}_{j+1}(n),$$

where

$$\widehat{H}_{j+1}(n)=U_j\ \widehat{h}_{j+1}(n)\ U_{j+1}^{-1}.$$

Therefore $\widetilde{\Phi}=\{\widetilde{\phi}_j:\ j\geq 0\}$ is a multirefinable generator with mask $\{\widehat{H}_{j+1}\}$. We can show that $\widetilde{\Phi}^d$ is a multirefinable generator with mask $\{\widehat{H}_{j+1}^d\}$ in a similar manner. We define now a family $\theta=\{\theta_j:\ j\geq -1\}$ whose elements are $A^{*(j+1)}$-periodic matrix valued sequences $\theta_j:\mathbb{Z}^s\to\mathbb{C}^{r\times r}$ such that $\theta_{-1}(n)=I_r\ \forall n\in\mathbb{Z}^s$ and

$$\theta_{j-1}(n)=\left(U_j^{d*}\right)^{-1}U_j^{-1},\ n\in\mathbb{Z}^s.$$

For this definition of θ we use part (iii) of theorem 1 for constructing wavelet masks $\widehat{\widetilde{S}}_{j+1},\widehat{\widetilde{S}_{j+1}^d}:\mathbb{Z}^s\to\mathbb{C}^{\rho_j\times r}$, so that $X(\widetilde{\phi}_0^d,\widetilde{\Psi}^d)$ is a dual multiwavelet frame of $X(\widetilde{\phi}_0,\widetilde{\Psi})$ for $L_2(\mathbb{T}^s)$. We have

$$\lim_{j\to+\infty}|A|^j\widehat{\widetilde{\phi}_j^{d*}}(n)\theta_{j-1}(n)\widehat{\widetilde{\phi}}_j(n)=\lim_{j\to+\infty}|A|^j\widehat{\phi}_j^*(n)\widehat{\phi}_j(n)=1,\text{pointwise on } n\in\mathbb{Z}^s,$$

due to the validity of part (ii) of corollary 3 for Φ. Thus, condition (a) of part (iii) of theorem 1 is satisfied. It suffices to show (3.5) for some suitable choice of the above wavelet filters. Indeed, let

$$\begin{cases} \widehat{\widetilde{S}}_{j+1}(n)=G_{j+1}^*\ \widehat{S}_{j+1}(n)\ U_{j+1}^{-1} \\ \widehat{\widetilde{S}_{j+1}^d}(n)=G_{j+1}^{-1}\ \widehat{S}_{j+1}(n)\ \left(U_{j+1}^d\right)^{-1}\end{cases},$$

for some sequence $G=\{G_{j+1}:\ j\geq 0\}$ whose elements are $\rho_j\times\rho_j$ invertible real matrices. Then equation (3.5) is satisfied as a result of (3.25). Under the assumptions $\|U_{j+1}\|_{\mathcal{F}}\leq B$, $\|U_{j+1}^d\|_{\mathcal{F}}\leq B^d$, $\|G_{j+1}\|_{\mathcal{F}}\leq C$ and $\|G_{j+1}^{-1}\|_{\mathcal{F}}\leq D$ for some absolute positive constants B,B^d,C,D, we can show that $X(\widetilde{\phi}_j,\widetilde{\Psi}_j)$ and $X(\widetilde{\phi}_j^d,\widetilde{\Psi}_j^d)$ are Bessel systems for L_2 for any $j\geq 0$. Then all conditions of part (iii) of theorem 1 are satisfied and a class of dual multiwavelet frames for $L_2(\mathbb{T}^s)$ is produced.

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