Multiple Lenders, Strategic Default and Covenants

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Online Appendix

Appendix A: proofs of Propositions 4, 5, 6, and 7

Proof of Proposition 4: Define $\gamma > 0$ as

$$\gamma \equiv \max\{\frac{G}{B(G-\frac{B}{\Delta\pi})}, \frac{G}{(\pi_H G - 1)(G-\frac{B}{\Delta\pi})}, \left(\frac{\pi_H G - B}{B}\right)^2 \frac{1}{\pi_H (G-\frac{B}{\Delta\pi})}\},$$

and $U_N = U_B + 2\gamma(U(I^c, R^c, H) - U_B)/(\sqrt{N}-1)$ with $U_B = \max\{U(0), U(I^c, R^c, H) - BI^c\}$. Assume N satisfies $N > \underline{N} = (2\gamma+1)^2$. See that $U_N < U(I^c, R^c, H)$ for any $N > \underline{N}$. Note also that, when $B > \pi_H G - 1$, $U_B = U(0)$ so that $\lim_{N\longrightarrow\infty} U_N = U(0)$ as stated in Proposition 4.

We exhibit a profile of investors' strategies that supports any allocation (I^*, R^*) satisfying (6) at equilibrium. Each investor $i = \{1, 2, ..., K\}$, with $K \equiv \lfloor\sqrt{N}\rfloor$, offers $M_i^* = \{(0,0); (I^*/K, R_i^*(.))\}$, with $R_i^*(I) = G(I + A)$ for $I \notin \{I^*, I^*/K\}$, $R_i^*(I^*) = R^*/K$, and $R_i^*(I^*/K) = R^* - GI^*(K-1)/K$. Each investor $j = \{K+1, ..., N\}$ offers $M_j^* = \{(0,0); (\bar{I}/(N-K), \bar{R}_j(.))\}$ with $\bar{R}_j(I) = G(I+A)$ $\forall I \in \mathbb{R}_+$. The investment level $\bar{I}$ is such that

$$U(I^*, R^*, H) = B(I^* + \bar{I} + A) - A. \tag{OA1}$$

Choosing I^*/K in each menu and selecting $e = H$ is an optimal choice for the entrepreneur. Observe also that $R_i^*(I^*/K)$ guarantees that $U(I^*, R^*, H)$ remains available to the entrepreneur if any of the investors withdraws his offer.

Now consider investors' deviations. See first that every profitable deviation must induce the entrepreneur to choose $e = H$. Indeed, by Lemma A1, an investor may achieve a positive profit by inducing $e = L$ only if the entrepreneur trades several contracts out of equilibrium and does not default. Given equilibrium covenants, this is possible only if she takes up an aggregate loan I^* and chooses $e = L$, which yields the entrepreneur a payoff smaller than the (available) equilibrium one, and cannot be an optimal choice.

We next show that there is no unilaterally profitable deviation for investors. A deviation of investor $i \in \{1, \cdots, K\}$ can be characterized, without loss of generality, by the menu $M_i' = \{(I^*/K + I_i', R^*/K + R_i'(.)), (0,0)\}$, with $I_i' \in [-I^*/K, I^c - I^*/K]$. Similarly, a deviation of investor $j \in \{K+1, \cdots, N\}$ can be characterized by $M_j' = \{(\bar{I}/(N-K) + I_j', R_j'(.)), (0,0)\}$, with $I_j' \in [-\bar{I}/(N-K), I^c - \bar{I}/(N-K)]$. For any such deviation to be profitable when the entrepreneur chooses I and $e = H$, one needs that

$$R'_i(I) > \frac{1}{\pi_H} I'_i, \quad R'_j(I) > \frac{1}{\pi_H}\left(I'_j + \frac{\bar{I}}{N-K}\right), \tag{OA2}$$

for $i \in \{1, \cdots, K\}, j \in \{K+1, \cdots, N\}$. If, following any such deviation, the entrepreneur defaults, her payoff is $U_{sd} = U(I^*, R^*, H) + BI'_h$, with $h = i, j$. We now evaluate the entrepreneur's payoff when $e = H$ is chosen.

First, consider a deviation of an active investor i. If the entrepreneur only trades with investor i at the deviation stage, her payoff is

(OA3)

$$\pi_H\big(G(\frac{I^*}{K}+I'_i+A)-\frac{R^*}{K}-R'_i(I'_i+\frac{I^*}{K})\big)^+-A < \frac{K-1}{K}U(0)+\frac{U(I^*,R^*,H)}{K}+I'_i(\pi_H G-1),$$

where the inequality follows from (OA2). Since the equilibrium utility is available at the deviation stage, the right-hand side of (OA3) must be strictly larger than $U(I^*, R^*, H)$, which implies that $I'_i > 0$. Thus, if U_{sd} is greater than the right-hand side of (OA3), the entrepreneur finds optimal to strategically default. This is the case whenever

$$\frac{K-1}{K}\big(U(I^*, R^*, H) - U(0)\big) \geq (\pi_H G - 1 - B)I'_i. \tag{OA4}$$

We show that each $(I^*, R^*) \in \mathcal{F}$ satisfying (6) also satisfies (OA4). We consider two cases:

1. If $B > \pi_H G - 1$, the right-hand side of (OA4) is negative for each $I'_i > 0$, and (OA4) is satisfied by any $(I^*, R^*) \in \mathcal{F}$.
2. If $B \leq \pi_H G - 1$, a sufficient condition for (OA4) to hold is

$$U(I^*, R^*, H) - U_B \geq \frac{1}{K-1}(U_B - U(0)), \tag{OA5}$$

in which I'_i is replaced by I^c in (OA4). Note that, as $K - 1 \geq \sqrt{N} - 1$, we have that $2/(\sqrt{N} - 1) \geq 1/(K - 1)$. Moreover, as

$$\gamma \geq \frac{G}{B(G - \frac{B}{\Delta\pi})} > \frac{\pi_H G}{B} > \frac{\pi_H G - 1 - B}{B} = \frac{U_B - U(0)}{U(I^c, R^c, H) - U_B}, \tag{OA6}$$

condition (6) implies (OA5) thus the result.

If several contracts are traded at the deviation stage, then, given equilibrium covenants, $e = H$ is an optimal choice only if the aggregate investment is I^*. In this case, we necessarily have $I'_i = kI^*/K$, with $k \in \{1, ..., K-2\}$. The entrepreneur's payoff when $e = H$ is chosen is then

(OA7)
$$\pi_H\big(G(I^*+A)-\frac{[K-(k+1)]+1}{K}R^*-R_i'(I^*)\big)-A < U(I^*,R^*,H)+\frac{k}{K}(\pi_H R^*-I^*),$$

where the inequality follows from (OA2). The entrepreneur strategically defaults if U_{sd} is greater than the right-hand side of (OA7) which, since $\pi_H R^* - I^* = (\pi_H G-1)I^* - (U(I^*,R^*,H)-U(0))$, yields

$$\text{(OA8)} \qquad \frac{k}{K}\big(U(I^*,R^*,H)-U(0)\big) \geq \frac{k}{K}(\pi_H G - 1 - B)I^*.$$

Since each $(I^*,R^*) \in \mathcal{F}$ satisfying (6) is such that $U(I^*,R^*,H) \geq U_B$, any of these allocations also satisfies (OA8).

Suppose now that an inactive investor j deviates. If the entrepreneur only trades with investor j at the deviation stage, given (OA2), her payoff when $e = H$ is chosen is bounded above by $U(0)+(\pi_H G-1)\big(\bar{I}/(N-K)+I_j'\big)$. The entrepreneur therefore prefers to strategically default if

$$\text{(OA9)} \qquad U(I^*,R^*,H)-U(0) \geq I_j'(\pi_H G-1-B) + \frac{\bar{I}}{N-K}(\pi_H G-1).$$

From (OA1), we get

$$\text{(OA10)} \qquad B\bar{I} \leq (\pi_H G-1-B)I^* + U(0) + (1-B)A.$$

Given (OA10), a sufficient condition for (OA9) is then

$$\text{(OA11)} \qquad U(I^*,R^*,H)-U(0) \geq (\pi_H G-1-B)I_j' + \frac{1}{N-K}\frac{\pi_H G-1}{B}\big(U(0)+(1-B)A+(\pi_H G-1-B)I^*\big).$$

We show that each $(I^*,R^*) \in \mathcal{F}$ satisfying (6) also satisfies (OA11). We consider two cases:

1. If $B > \pi_H G - 1$, the inequality $U(I^*,R^*,H)-U(0) \geq (-\bar{I}/(N-K))(\pi_H G - 1 - B) + (\pi_H G-1)\bar{I}/(N-K) = B\bar{I}/(N-K)$ is weaker than (OA9). Indeed, we obtain the right-hand side by replacing I_j' by its lower bound $-\bar{I}/(N-K)$ in (OA9). Given (OA10), and since $\pi_H G - 1 - B < 0$, (OA9) is a fortiori weaker than $U(I^*,R^*,H)-U(0) \geq (U(0)+(1-B)A)/(N-K)$, which in turn implies (OA11). To show that this inequality holds for each $(I^*,R^*) \in \mathcal{F}$, observe that, since $N \geq 3$ by construction, we get $2/(\sqrt{N}-1) \geq 1/(N-\sqrt{N}-1) \geq 1/(N-K)$.

In addition, using $\pi_H G - 1 < B < 1$ we have

$$\gamma \geq \frac{G}{(\pi_H G-1)(G-\frac{B}{\Delta\pi})} > \frac{(\pi_H G - B)\left(\frac{1}{\pi_H} - (G-\frac{B}{\Delta\pi})\right)}{(\pi_H G-1)(G-\frac{B}{\Delta\pi})} = \frac{(\pi_H G - B)A}{(\pi_H G-1)I^c} = \frac{U(0)+(1-B)A}{U(I^c,R^c,H)-U(0)}. \tag{OA12}$$

Condition (6) then implies

$$\begin{aligned} U(I^*,R^*,H)-U(0) \geq & \frac{1}{N-K}\frac{U(0)+(1-B)A}{U(I^c,R^c,H)-U(0)}(U(I^c,R^c,H)-U(0)) \\ = & \frac{1}{N-K}(U(0)+(1-B)A), \end{aligned} \tag{OA13}$$

thus the result.

2. If $B \leq \pi_H G - 1$, replacing I'_j and I^* by I^c, a sufficient condition for (OA11) is

$$U(I^*,R^*,H)-U(0) \geq (U_B - U(0)) + \frac{1}{N-K}\frac{\pi_H G - 1}{B}\big(U_B + (1-B)A\big). \tag{OA14}$$

As in the former case, $2/(\sqrt{N}-1) \geq 1/(N-K)$. Moreover,

$$\begin{aligned} \gamma &\geq \left(\frac{\pi_H G - B}{B}\right)^2 \frac{1}{\pi_H(G-\frac{B}{\Delta\pi})} \geq \left(\frac{\pi_H G-1}{B}\right)\left(\frac{\pi_H G - B}{B}\right)\left(1+\frac{\frac{1}{\pi_H}-(G-\frac{B}{\Delta\pi})}{(G-\frac{B}{\Delta\pi})}\right) \\ &\geq \left(\frac{\pi_H G-1}{B}\right)\left(\frac{\pi_H G - B - 1}{B} + \frac{\pi_H G - B}{B}\frac{A}{I^c}\right) \\ &\geq \frac{\pi_H G-1}{B}\frac{(\pi_H G - 1 - B)I^c + A(\pi_H G - B)}{BI^c} = \frac{\pi_H G - 1}{B}\frac{U_B+(1-B)A}{U(I^c,R^c,H)-U_B}. \end{aligned} \tag{OA15}$$

Then, (6) implies

$$U(I^*,R^*,H)-U_B \geq \frac{1}{N-K}\frac{\pi_H G-1}{B}\frac{U_B+(1-B)A}{U(I^c,R^c,H)-U_B}(U(I^c,R^c,H)-U_B), \tag{OA16}$$

thus the result.

If several contracts are traded at the deviation stage, then, given equilibrium covenants, $e = H$ is an optimal choice only if the aggregate investment is I^*. In this case, we necessarily have $I'_j = kI^*/K - \bar{I}/(N-K)$, with $k \in \{1, ..., K-1\}$.

The entrepreneur's payoff when $e = H$ is chosen is then

$$\pi_H\big(G(I^*+A) - \frac{K-k}{K}R^* - R_j'(I^*)\big) - A \leq U(I^*,R^*,H) + \frac{k}{K}(\pi_H R^* - I^*), \tag{OA17}$$

where the inequality follows from (OA2). The entrepreneur prefers to strategically default if U_{sd} is greater than the right-hand side of (OA17), which is the case if

$$B\big(\frac{k}{K}I^* - \frac{\bar{I}}{N-K}\big) \geq \frac{k}{K}(\pi_H R^* - I^*).$$

Using (OA10) and $\pi_H R^* - I^* = (\pi_H G - 1)I^* - (U(I^*,R^*,H) - U(0))$, a sufficient condition for (OA17) is

$$\begin{aligned} &\frac{k}{K}\big((1+B-\pi_H G)I^* + U(I^*,R^*,H) - U(0)\big) \geq \\ &\frac{1}{N-K}\left((\pi_H G - 1 - B)I^* + U(0) + (1-B)A\right). \end{aligned} \tag{OA18}$$

The left-hand side of (OA18) is increasing in k. This is straightforward if $B > \pi_H G - 1$. If $B \leq \pi_H G - 1$, the result follows from $U(I^*,R^*,H) - U(0) \geq U_B - U(0) = (\pi_H G - 1 - B)I^c$ and $(I^c - I^*) \geq 0$. It is hence enough to verify (OA18) for $k = 1$, that is,

$$U(I^*,R^*,H) - U(0) \geq \frac{K}{N-K}\left(U(0) + (1-B)A\right) + \frac{N}{N-K}(\pi_H G - 1 - B)I^*. \tag{OA19}$$

We show that each $(I^*,R^*) \in \mathcal{F}$ satisfying (6) also satisfies (OA19). We consider two cases:

1. If $B > \pi_H G - 1$, we show that $U(I^*,R^*,H) - U(0) \geq (U(0) + A)K/(N-K)$, which is stronger than (OA19), is satisfied. Note that $(\sqrt{N}-1)^2 \geq 0$, or equivalently, $2N - 2\sqrt{N} \geq N - 1$. This implies that $2/(\sqrt{N}-1) \geq (\sqrt{N}+1)/(N-\sqrt{N}) \geq K/(N-K)$. In addition,

$$\begin{aligned} \gamma &\geq \frac{G}{(\pi_H G - 1)(G - \frac{B}{\Delta\pi})} = \frac{\pi_H G}{\pi_H G - 1}\frac{1/\pi_H}{G - \frac{B}{\Delta\pi}} \geq \frac{\pi_H G}{\pi_H G - 1}\frac{A}{I^c} \\ &= \frac{U(0) + A}{U(I^c,R^c,H) - U(0)}. \end{aligned} \tag{OA20}$$

To conclude, observe that (6) implies

$$U(I^*,R^*,H) - U(0) \geq \frac{K}{N-K}\frac{U(0)+A}{U(I^c,R^c,H) - U(0)}\left(U(I^c,R^c,H) - U(0)\right). \tag{OA21}$$

2. If $B \leq \pi_H G - 1$, we remark that

$$\begin{aligned} \text{(OA22)} \qquad & \frac{K}{N-K}(U(0)+(1-B)A) + N(\pi_H G - 1 - B)I^* \\ & \leq \frac{K}{N-K}(U(0)+(1-B)A) + N(U_B - U(0)) \\ & = U_B - U(0) + \frac{K}{N-K}(U_B + (1-B)A). \end{aligned}$$

We show that $U(I^*, R^*, H) - U_B \geq (U_B + (1-B)A)K/(N-K)$, which is stronger than (OA19). As in the former case, we have $2/(\sqrt{N}-1) \geq K/(N-K)$. Then, using (OA16), condition (6) implies that

$$\begin{aligned} \text{(OA23)} \qquad U(I^*, R^*, H) - U_B \geq & \frac{K}{N-K}\frac{U_B + (1-B)A}{U(I^c, R^c, H) - U_B}(U(I^c, R^c, H) - U_B) \\ = & \frac{K}{N-K}(U_B + (1-B)A). \end{aligned}$$

Thus, any allocation $(I^*, R^*) \in \mathcal{F}$ satisfying (6) also satisfies (OA4), (OA8), (OA11), and (OA19). This proves that any such allocation is supported at equilibrium by the investors' strategies $M_1^*, ..., M_N^*$. ■

Proof of Proposition 5: We extend the results of Propositions 3 and 4 to the case in which covenants can be contingent on the initial debt I_0. Assume $\pi_L = 0$, take any $(I^*, R^*) \in \mathcal{F}$ such that $I^* \geq I^m$ and consider the following profile of strategies. Each investor $i = 1, 2, ..., N$ offers the same menu $M^* = \{(0,0,0,0), \left(I^*/N, R^*(.), I^+(.), R^+(.)\right), \left(0, 0, \hat{I}^+(.), \hat{R}^+(.)\right)\}$. We denote the null contract $(0,0,0,0)$, the equilibrium contract $(I^*/N, R^*(.), I^+(.), R^+(.))$, and the latent contract $(0, 0, \hat{I}^+(.), \hat{R}^+(.))$. In each equilibrium contract, $R^*(.)$ is such that

$$\text{(OA24)} \qquad R^*(I_0, I^F(I_0)) = \begin{cases} R^* - \frac{N-1}{N} G I^* & \text{if } I_0 = I^F(I_0) = \frac{I^*}{N}, \\ \frac{R^*}{N} & \text{otherwise,} \end{cases}$$

where $I^F(I_0)$ is the amount ultimately invested for a given initial I_0. The additional offer $(I^+(.), R^+(.))$ is such that

$$\text{(OA25)} \qquad I^+(I_0) = \begin{cases} 0 & \text{if } I_0 = k\frac{I^*}{N}, \text{ for } k = 1, 2, ..., N, \\ I^{CL} & \text{otherwise,} \end{cases}$$

(OA26)

$$R^+(I_0, I^F(I_0)) = \begin{cases} 0 & \text{if } I_0 = I^F(I_0) = I^*, \text{or } I_0 = I^F(I_0) = \frac{I^*}{N}, \\ G(I^F(I_0) + A) & \text{if } I_0 = I^*, I^F(I_0) \neq I^*, \text{or } I_0 = \frac{I^*}{N}, I^F(I_0) \neq \frac{I^*}{N}, \\ p^* I^{CL} & \text{otherwise,} \end{cases}$$

with $p^* = R^*/I^*$. The investment I^{CL} is the additional credit line which any investor stands ready to provide against any competitors' threat to ask for an accelerated repayment. It is such that

$$U(\frac{I^*}{N} + I^{CL}, p^*(\frac{I^*}{N} + I^{CL}), L) = U(I^c, R^c, H). \tag{OA27}$$

In each latent contract, the additional offer $(\hat{I}^+(.), \hat{R}^+(.))$ is such that

$$\hat{I}^+(I_0) = \begin{cases} \frac{\hat{I}}{N} & \text{if } I_0 = 0, \\ \frac{1}{N-1}(\hat{I} - \frac{I^*}{N}) & \text{if } I_0 = \frac{I^*}{N}, \\ 0 & \text{otherwise,} \end{cases} \tag{OA28}$$

and $\hat{R}^+(I_0, I^F(I_0)) = G(I^F(I_0) + A)$ for all I_0 and $I^F(I_0)$. As in the proof of Proposition 3, the investment $\hat{I}$ is characterized by $U(I^*, R^*, H) = B(\hat{I} + A) - A$. Given these offers, the entrepreneur cannot obtain a payoff higher than $U(I^*, R^*, H)$. At equilibrium, she achieves $U(I^*, R^*, H)$ by trading the same equilibrium contract with each of the investors, receiving thereby no additional funds at the second stage, and selecting $e = H$. As in the proof of Proposition 3, none of the investors is indispensable to provide the equilibrium payoff: the entrepreneur can get $U(I^*, R^*, H)$ by trading the equilibrium contract with only one investor.[1]

Consider then investors' deviations. Without loss of generality, any unilateral deviation can be represented by a menu $M' = \{(0,0,0,0), \big(I', R'(.), I'^+(.), R'^+(.)\big)\}$. By Lemma A1, and given that $\pi_L = 0$, each profitable deviation must induce the entrepreneur to choose $e = H$. Hence, it must be that $I' \in \{0, I^*/N\}$ in any profitable deviation. Indeed, if $I' \notin \{0, I^*/N\}$, the entrepreneur can combine the deviating contract with (at least) one equilibrium contract, and obtain the line of credit $(I^{CL}, p^* I^{CL})$. This guarantees her (at least) the payoff $U(I^*/N + I^{CL}, p^*(I^*/N + I^{CL}), L) = U(I^c, R^c, H)$. The entrepreneur's strategy can therefore be constructed so that she strategically defaults when trading I' with the deviating investor. In addition, if the entrepreneur chooses $e = H$, then the aggregate initial financing induced by the deviation to M' must be $I_0 \in \{0, I^*/N, I^*\}$. Indeed, $R^+(.)$ is such that, if she trades any equilibrium contract together with the deviating one, the entrepreneur strategically defaults unless her initial outside financing is I^*/N or I^*.[2]

Assuming that $e = H$ is chosen, we distinguish two cases, depending on whether the borrower trades at least one of the equilibrium contracts, or she only trades with the deviating investor. In the first case, given $R^+(.)$, it must be that $I_0 = I^F(I_0) \leq I^*$ and $I'^+(I_0) = I^+(I_0) = 0$, otherwise she would default. It follows

[1] Alternatively, she can also get $U(I^*, R^*, H)$ by trading all latent contracts and defaulting.
[2] $\hat{R}^+(.)$ is such that trading any of the latent contracts straightforwardly leads to strategic default.

that, since $I' \in \{0, I^*/N\}$, we must have $I' = I^*/N$ for the deviation to be profitable. Thus, the corresponding entrepreneur's payoff is

$$\text{(OA29)}\quad \pi_H(G(\frac{I^*}{N} + k\frac{I^*}{N} + A) - R'(I_0, I^F(I_0)) - R'^+(I_0, I^F(I_0)) - k\frac{R^*}{N}) - A$$
$$< \pi_H G(I^* + A) - I' + \frac{I^*}{N} - \pi_H R^* - A = U(I^*, R^*, H) + (\frac{I^*}{N} - I'),$$

where $k \in \{1, ..., N-1\}$ is any number of equilibrium contracts optimally traded by the entrepreneur when $e = H$. Inequality (OA29) obtains since $\pi_H\big(R'(I_0, I^F) + R'^+(I_0, I^F)\big) - I' > (\pi_H R^*/N - I^*/N)$, which guarantees that the deviation is profitable, and by observing that $GI^*/N - R^*/N > 0$ by construction. Thus, (OA29) implies that, following the deviation, the payoff achieved by the entrepreneur when choosing $e = H$ is strictly below $U(I^*, R^*, H)$, which contradicts the fact that $U(I^*, R^*, H)$ is still available at the deviation stage.

We next consider the case in which, when choosing $e = H$, the entrepreneur only trades with the deviating investor, which implies that $I_0 = I' \in \{0, I^*/N\}$. Her corresponding payoff is

$$\text{(OA30)}\quad \pi_H\big(G(I' + I'^+(I') + A) - R'(I', I^F(I')) - R'^+(I', I^F(I'))\big) - A$$
$$< \pi_H(G(I' + A) - \frac{R^*}{N}) - A + (\pi_H G - 1)I'^+(I') + (\frac{I^*}{N} - I')$$
$$\text{(OA31)}\quad < U(I^*, R^*, H) + (\pi_H G - 1)I'^+(I').$$

Inequality (OA30) follows from $\pi_H(R'(I', I^F(I') + R'^+(I', I^F(I')) > I' + I'^+(I') + (\pi_H R^*/N - I^*/N)$, which guarantees that the deviation is profitable. Inequality (OA31) obtains because $I' \leq I^*/N$ and $\pi_H G - 1 > 0$. Since $U(I^*, R^*, H)$ is available to the entrepreneur at the deviation stage, (OA31) implies that $I'^+(I') > 0$. We now prove that, by strategically defaulting, the entrepreneur gets a payoff larger than the upper bound in (OA30). Suppose that, together with the deviating contract, she takes $N-1$ latent contracts at the deviation stage. Given $\hat{R}^+(.)$, she then finds optimal to default. Her corresponding payoff is

$$\text{(OA32)}\quad U_{sd} = B(I' + I'^+(I') + (N-1)\bar{I}^+(I') + A) - A.$$

If $I_0 = I' = I^*/N$, (OA32) yields $U_{sd} = U(I^*, R^*, H) + BI'^+(I') > U(I^*, R^*, H) + (\pi_H G - 1)I'^+(I')$, as $I'^+(I') > 0$, and $B > \pi_H G - 1$ since $\pi_L = 0$. If $I_0 = I' = 0$, (OA30) together with the fact that $U(I^*, R^*, H)$ remains available at the deviation stage imply $\pi_H(G(I'^+(0) + A) - I'^+(0) - (\pi_H R^*/N - I^*/N) > U^*(I^*, R^*, H)$. Since $U^*(I^*, R^*, H) - U(0) = (\pi_H G - 1)I^* - (\pi_H R^* - I^*)$, we get $I'^+(0) > I^*$. Thus, without loss of generality, we can write $I'^+(0) = I^*/N + I''$ with $I'' > I^*(N-1)/N > 0$. Then, (OA30) implies that the entrepreneur's payoff is bounded by $(U(0)(N-1) + U(I^*, R^*, H))/N + (\pi_H G - 1)I''$, and (OA32) can be rewritten as

$$\text{(OA33)}\quad U_{sd} = B\Big(\frac{N-1}{N}\hat{I} + \frac{I^*}{N} + I'' + A\Big) - A = U(I^*, R^*, H) + B(I'' + \frac{I^*}{N} - \frac{\hat{I}}{N}).$$

As shown in the proof of Proposition 3, we have $U_{sd} \geq (U(0)(N-1)+U(I^*, R^*, H))/N + (\pi_H G - 1)I''$ for each $I'' > 0$. This guarantees that the entrepreneur strategically defaults and reestablishes that any aggregate allocation $(I^*, R^*) \in \mathcal{F}$ satisfying $I^* \geq I^m$ is sustained at equilibrium.

We now extend the result of Proposition 4. We exhibit a profile of investors' strategies that supports at equilibrium any allocation (I^*, R^*) satisfying (6). Each investor $i = \{1, 2, ..., K\}$, with $K \equiv \lfloor\sqrt{N}\rfloor$, offers

$$\text{(OA34)}\qquad M_i^* = \{(0,0,0,0); \big(\frac{I^*}{K}, R_i^*(.), I_i^+(.), R_i^+(.)\big), \big(0, 0, \bar{I}_i^+(.), \bar{R}_i^+(.)\big)\},$$

and each investor $j = \{K+1, ..., N\}$ offers

$$\text{(OA35)}\qquad M_j^* = \{(0,0,0,0); \big(\frac{\bar{I}}{N-K}, \bar{R}_j(.), 0, 0\big); \big(0, 0, \tilde{I}_j^+(.), \tilde{R}_j^+(.)\big)\}.$$

We denote the equilibrium contract $(I^*/K, R_i^*(.), I_i^+(.), R_i^+(.))$, the type-1 latent contract $(\bar{I}/(N-K), \bar{R}_j(.), 0, 0)$, the type-2 latent contract $(0, 0, \bar{I}_i^+(.), \bar{R}_i^+(.))$, and the type-3 latent contract $(0, 0, \tilde{I}_j^+(.), \tilde{R}_j^+(.))$. In each equilibrium contract, $R_i^*(.)$ is such that

$$\text{(OA36)}\qquad \begin{cases} R_i^*(I_0) = (R^* - \frac{K-1}{K}GI^*) \text{ if } I_0 = I^F(I_0) = \frac{I^*}{K}, \\ R_i^*(I_0) = \frac{R^*}{K} \text{ otherwise.} \end{cases}$$

The additional offer $(I_i^+(.), R_i^+(.))$ is such that:

(OA37)

$$I_i^+(I_0) = \begin{cases} 0 \text{ if } I_0 = k\frac{I^*}{K} + l\frac{\bar{I}}{N-K}, \text{ for } k = 1, 2, ..., K \text{ and } l = 0, ..., N-K, \\ I^{CL} \text{ otherwise,} \end{cases}$$

and

(OA38)

$$R_i^+(I_0, I^F(I_0)) = \begin{cases} 0 \text{ if } I_0 = I^* \text{ and } I^F(I_0) = I^*, \text{ or if } I_0 = \frac{I^*}{K} \text{ and } I^F(I_0) = \frac{I^*}{K}, \\ G(I^F(I_0) + A) \text{ if } I_0 = I^* \text{ and } I^F(I_0) \neq I^*, \\ \text{or if } I_0 = \frac{I^*}{K} \text{ and } I^F(I_0) \neq \frac{I^*}{K}, \text{ or if } I_0 = k\frac{I^*}{K} + l\frac{\bar{I}}{N-K}, \forall I^F(I_0) \\ \text{for } k = 1, 2, ..., K \text{ and } l = 0, ..., N-K, \text{ with } (k, l) \notin \{(K, 0), (1, 0)\}, \\ G(I^{CL} + A) \text{ otherwise,} \end{cases}$$

where I^{CL} is such that $B(I^*/K + I^{CL} + A) - A = U(I^c, R^c, H)$ and $\bar{I}$ is such that

$U(I^*, R^*, H) = B(I^* + \bar{I} + A) - A$. The additional offer $(\bar{I}_i^+(.), \bar{R}_i^+(.))$ is such that

$$\bar{I}_i^+(I_0) = \begin{cases} \frac{I^*}{K} \text{ if } I_0 = 0 \text{ or } I_0 = \frac{I^*}{K} \text{ or } I_0 = \frac{\bar{I}}{N-K}, \\ 0 \text{ otherwise,} \end{cases} \tag{OA39}$$

and $\bar{R}_i^+(I_0, I^F(I_0)) = G(I^F(I_0) + A)$ for all I_0 and $I^F(I_0)$.

For any $j \in \{K+1, 2, ..., N\}$, the repayment $\bar{R}_j(.)$ satisfies $\bar{R}_j^+(I_0, I^F(I_0)) = G(I^F(I_0) + A)$ for all I_0 and $I^F(I_0)$ and the offers $(\tilde{I}_j^+(.), \tilde{R}_j^+(.))$ are such that

$$\tilde{I}_j^+(I_0) = \begin{cases} \frac{\bar{I}}{N-K} & \text{if } I_0 = \frac{I^*}{K} \text{ or, } I_0 = 0, \\ 0 & \text{otherwise,} \end{cases} \tag{OA40}$$

and $\tilde{R}_j^+(I_0, I^F(I_0)) = G(I^F(I_0) + A)$ for all I_0 and $I^F(I_0)$.

Given these offers, the entrepreneur cannot obtain more than $U(I^*, R^*, H)$. At equilibrium, she achieves $U(I^*, R^*, H)$ by trading the equilibrium contract with each lender $i \in \{1, 2, ..., K\}$, the null contract with each lender $j \in \{K + 1, 2, ..., N\}$, obtains no additional funds in the second stage and selects $e = H$. As before, none of the investors is indispensable: the entrepreneur can get $U(I^*, R^*, H)$ by trading the equilibrium contract with only one investor.[3]

Consider then investors' deviations. Without loss of generality, a deviation by any investor is represented by $M' = \{(0, 0, 0, 0), (I', R'(.), I'^+(.), R'^+(.))\}$. Every profitable deviation must induce the entrepreneur to choose $e = H$. Indeed, given Lemma A1, a single investor may achieve a positive profit by inducing $e = L$ only if the entrepreneur trades several contracts out of equilibrium and does not default. Given the equilibrium contracts, this is only possible if she chooses initially $I_0 = I^*$, invests ultimately $I^F(I^*) = I^*$ and chooses $e = L$, which yields the entrepreneur a payoff smaller than the available equilibrium one, and thus cannot be an optimal choice.

Suppose an investor $i \in \{1, ..., K\}$ deviates. Any profitable deviation must be such that $I' \in \{0, I^*/K\}$. Indeed, if $I' \notin \{0, I^*/K\}$, the entrepreneur can combine the deviating contract with (at least) one equilibrium contract, and get access to the line of credit $(I^{CL}, G(I^{CL} + A))$, which ensures her a payoff at least equal to $B(I^*/K + I^{CL} + A) - A = U(I^c, R^c, H)$. This shows that in any profitable deviation the borrower defaults, which constitutes a contradiction. In addition, at the deviation stage, the initial investment I_0 belongs to the set $\{0, I^*/K, I^*\}$. Indeed, $R_i^+(.)$ is such that, if she trades any equilibrium contract together with the deviating one, the entrepreneur necessarily defaults if $I_0 \notin \{I^*/K, I^*\}$. Assuming that $e = H$ is chosen, we distinguish two cases, depending on whether the entrepreneur trades at least one of the equilibrium contracts, or she only trades with the deviating investor.

[3] She may also get $U(I^*, R^*, H)$ by trading the type-1 latent contract with each investor $j \in \{K + 1, 2, ..., N\}$.

Consider that the entrepreneur trades equilibrium contracts at the deviation stage. Given $R_i^+(.)$, we have $I_0 = I^F(I_0)$ which implies $I'^+(I_0) = I^+(I_0) = 0$, otherwise the entrepreneur would default. It follows that, since $I' \in \{0, I^*/K\}$, it must be that $I' = I^*/K$ for the deviation to be profitable. Thus, the entrepreneur's payoff is

$$\text{(OA41)} \quad \pi_H(G(\frac{I^*}{K} + k\frac{I^*}{K} + A) - R'(I_0, I^F(I_0)) - R'^+(I_0, I^F(I_0)) - k\frac{R^*}{K}) - A < U(I^*, R^*, H),$$

where $k \in \{1, ..., N-1\}$ is any number of equilibrium contracts optimally traded by the entrepreneur when $e = H$. The latter inequality obtains since $\pi_H\big(R'(I_0, I^F)+ R'^+(I_0, I^F)\big) - I^*/K > (\pi_H R^*/K - I^*/K)$, which guarantees that the deviation is profitable, and by observing that $GI^*/K - R^*/K > 0$ by construction. Thus, (OA41) implies that, following the deviation, the payoff achieved by the entrepreneur when choosing $e = H$ is strictly below $U(I^*, R^*, H)$, which contradicts the fact that $U(I^*, R^*, H)$ is still available.

Consider the case where the entrepreneur trades the null contract with all non deviating lenders. This implies that $I_0 = I' \in \{0, I^*/K\}$ and her payoff is

$$\text{(OA42)} \quad \pi_H\big(G(I' + I'^+(I') + A) - R'(I', I^F(I')) - R'^+(I', I^F(I'))\big) - A < \pi_H(G(I' + A) - \frac{R^*}{K}) - A + (\pi_H G - 1)I'^+(I') + (\frac{I^*}{K} - I')$$

$$\text{(OA43)} \quad < U(I^*, R^*, H) + (\pi_H G - 1)I'^+(I'),$$

where (OA42) obtains since

$$\text{(OA44)} \quad \pi_H(R'(I', I^F(I')) + R'^+(I', I^F(I')) > I' + I'^+(I') + (\pi_H\frac{R^*}{K} - \frac{I^*}{K})$$

for the deviation to be profitable. The second one obtains because $I' \leq I^*/K$ and $\pi_H G - 1 > 0$. Since the payoff $U(I^*, R^*, H)$ is available to the entrepreneur at the deviation stage, (OA43) implies that $I'^+(I^*/K) > 0$.

We prove that the upper-bound (OA42) of the entrepreneur's payoff is less than what she gets if she strategically defaults. If, following the deviation, the entrepreneur strategically defaults, she can select $\big(0, 0, \bar{I}_i^+(.), \bar{R}_i^+(.)\big)$ in the menu of each non-deviating investor $i \in \{1, ..., K\}$ and $\big(0, 0, \tilde{I}_j^+(.), \tilde{R}_j^+(.)\big)$ in the menu of each lender $j \in \{K+1, ..., N\}$. She then obtains

$$\text{(OA45)} \quad U_{sd} = B\big(I' + I'^+(I') + (K-1)\bar{I}_i^+(I') + (N-K)\tilde{I}_j^+(I') + A)\big) - A.$$

If $I' = I^*/K$, then $U_{sd} = U(I^*, R^*, H) + BI'^+(I^*/K)$. Thus, using (OA43), a sufficient condition for the entrepreneur to strategically default is $(K-1)/K(U(I^*, R^*, H)-$

$U(0)) \geq (\pi_H G - 1 - B)I'^{+}(I^*/K)$. This corresponds to (OA4), which holds from the proof of Proposition 4. If $I' = 0$, we deduce from (OA44) that

$$\pi_H(R'(0, I^F(0)) + R'^{+}(0, I^F(0))) > I'^{+}(0) + (\pi_H \frac{R^*}{K} - \frac{I^*}{K}), \tag{OA46}$$

from which it follows that $I'^{+}(0) > I^*$. Thus, without loss of generality, we write $I'^{+}(0) = I^*/K + I^{''}$ with $I^{''} > I^*(K-1)/K > 0$. Then, using again (OA42) and (OA46), we get an upper bound for the entrepreneur's payoff:

$$\begin{aligned} &\pi_H\big(G(I'^{+}(0) + A) - R'(0, I^F(0)) - R'^{+}(0, I^F(0))\big) - A \\ &< \frac{U(0)(K-1) + U(I^*, R^*, H)}{K} + (\pi_H G - 1)I^{''}. \end{aligned} \tag{OA47}$$

But (OA45) becomes $U_{sd} = B\big(I^*(K-1)/K + I^*/K + I^{''} + \bar{I} + A\big) - A = U(I^*, R^*, H) + BI^{''}$. Thus, a sufficient condition for the entrepreneur to default is $(U(I^*, R^*, H) - U(0))(K-1)/K \geq (\pi_H G - 1 - B)I^{''}$ with $I^{''} > 0$. Again, we have established this relation in the proof of Proposition 4.

Suppose an investor $j \in \{K+1, ..., N\}$ deviates. Any profitable deviation must be such that $I' \in \{0, \bar{I}/(N-K)\}$. Indeed, if $I' \notin \{0, \bar{I}/(N-K)\}$, the entrepreneur can combine equilibrium contracts and/or latent contracts of type-1, get access to the line of credit, and earn at least the payoff $B(I^*/K + I^{CL} + A) - A = U(I^c, R^c, H)$. This shows that, in any profitable deviation, the entrepreneur defaults, which constitutes a contradiction. Below, we consider the two cases $I' = \bar{I}/(N-K)$ and $I' = 0$ and show that in each case the entrepreneur strategically defaults following the deviation.
First, consider the case $I' = \bar{I}/(N-K)$. Because $e = H$ is chosen at the deviation stage, the entrepreneur does not trade latent contracts. Furthermore, given the additional offers $(I_i^{+}(.), R_i^{+}(.))$ she trades null contracts with each lender $i \in \{1, ..., K\}$. Thus, following the deviation, the entrepreneur trades the null contract with each non-deviating investor, which implies that $I_0 = \bar{I}/(N-K)$. When the entrepreneur chooses $e = H$, her payoff is bounded above by $U(0) + (\pi_H G - 1)(\bar{I}/(N-K) + I^{+'}(\bar{I}/(N-K)))$, where $I^{+'}(\bar{I}/(N-K)) \geq 0$. Under default, the entrepreneur's payoff is bounded below by

$$\begin{aligned} U_{sd} &= B(\frac{\bar{I}}{N-K} + I^{+'}(\frac{\bar{I}}{N-K}) + \frac{N-K-1}{N-K}\bar{I} + K\bar{I}_i^{+}(\frac{\bar{I}}{N-K}) - A) - A \\ &= U^*(I^*, R^*, H) + BI^{+'}(\frac{\bar{I}}{N-K}). \end{aligned} \tag{OA48}$$

Thus, a sufficient condition for the entrepreneur to default is

$$U^*(I^*,R^*,H)-U(0)\geq(\pi_H G-1-B)I^{+'}(\frac{\bar{I}}{N-K})+(\pi_H G-1)\frac{\bar{I}}{N-K}, \tag{OA49}$$

which corresponds to (OA9) established in the proof of Proposition 4.

Second, consider the case $I'=0$. Again, because $e=H$ is chosen at the deviation stage, the entrepreneur does not trade latent contracts together with the deviating contract. Remark that we must have $I'^{+}(I_0)>0$ for the deviation to be profitable. Furthermore, given the additional offers $(I_i^+(.),R_i^+(.))$, the entrepreneur either raises $I_0=I^*/K$, or $I_0=I^*$.
If the entrepreneur raises $I_0=I^*/K$ together with $e=H$, she must trade one equilibrium contract at the deviating stage. The inequality $I'^{+}(I^*/K)>0$ implies that $I^F(I^*/K)>I^*/K$. Thus, given $R_i^+(.)$, the entrepreneur cannot get more than her reservation utility. Thus, when choosing $e=H$, she prefers not to trade the deviating contract.
The same logic applies if the entrepreneur chooses $I_0=I^*$ together with $e=H$. In that case she must trade an equilibrium contract with each investor $i\in\{1,...,K\}$. The inequality $I'^{+}(I^*)>0$ implies that $I^F(I^*)>I^*$, and, given $R_i^+(.)$, the entrepreneur cannot get more than her reservation utility. ■

Proof of Proposition 6: The proof is developed in two steps. Step 1 identifies an open set of parameters that allows to characterize the candidate equilibrium allocation (I^*,R^*). Step 2 provides our equilibrium analysis.

Step 1: Characterization of the aggregate allocation (I^*,R^*)**.** We require that π_H,π_L and G satisfy the following

ASSUMPTION OA1:

$$\pi_H>3\pi_L, \tag{OA50}$$

and

$$G\in\left(\frac{1}{\pi_H}\left(2-\frac{\pi_L}{\pi_H}\right),\min\left(\frac{2}{\pi_H},\frac{1}{\sqrt{\pi_H\pi_L}}\right)\right). \tag{OA51}$$

Note that the interval in (OA51) is nonempty whenever (OA50) is satisfied.[4] The inequalities (OA50) and (OA51) imply that

[4]Precisely, (OA50) implies that $(2-\pi_L/\pi_H)/\pi_H<1/\sqrt{\pi_H\pi_L}$. To see this point, rewrite this inequality under the form $2-\pi_L/\pi_H<\sqrt{\pi_H/\pi_L}$. Denoting $x=\sqrt{\pi_H/\pi_L}$, (OA50) implies that $x>\sqrt{3}$. Therefore, the former inequality corresponds to $x^3+1-2x^2>0$, which holds at $x=\sqrt{3}$, and which left-hand side is increasing in x for $x>\sqrt{3}$.

$$G\Delta\pi > 1, \tag{OA52}$$

$$\frac{\pi_H G(1-\pi_L G)}{\pi_H G - 1} > 1, \tag{OA53}$$

which will be used throughout the proof.[5] We first identify the competitive allocation $(I^c, I^c/\pi_H)$. The next lemma shows existence and uniqueness of this allocation under Assumption OA1. It also characterizes the comparative statics of I^c with respect to A.

LEMMA OA1: *Under Assumption OA1, there exists only one aggregate allocation* $(I^c, R^c) = (I^c, I^c/\pi_H)$ *such that* $U(I^c, R^c, H) = U(I^c, R^c, L)$ *with* $I^c > 0$. *In addition,* I^c *is increasing in* A.

Proof of Lemma OA1: Fix A and consider the function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$\psi(I; A) = (\pi_H G - 1)I - \frac{\pi_H}{\Delta\pi} B(I + A) + \pi_H G A. \tag{OA54}$$

Note that $\psi(I; A) = 0$ is equivalent to $I/\pi_H = (G - B/\Delta\pi)(I + A)$. Thus, any aggregate allocation $(I, R) = (I, I/\pi_H)$ with $I \geq 0$ such that $\psi(I; A) = 0$ makes the entrepreneur indifferent between choosing $e = L$ and $e = H$. We show that there exists only one such I, which we denote I^c. Remark that ψ is convex and decreasing in I. Indeed, we have that $\Psi_I(I; A)/(\pi_H G - 1) = 1 - \pi_H B'(I + A)/(\Delta\pi(\pi_H G - 1)) < 1 - \pi_H/\Delta\pi < 0$, with the first inequality being implied by (7), and the convexity following from the concavity of B. We have

$$\frac{\psi(0; A)}{\pi_H G A} = 1 - \frac{B(A)}{G\Delta\pi A} > 1 - \frac{B'(0)A}{G\Delta\pi A} > 1 - \frac{G\Delta\pi(\pi_H G - 1)}{G\Delta\pi} = 2 - \pi_H G > 0, \tag{OA55}$$

where the first inequality follows from the concavity of $B(\cdot)$, the second from (7)

[5]Indeed, we first observe that $(OA51)$ is equivalent to $\pi_H G - 1 \in \left(\Delta\pi/\pi_H, \min\left(1, \sqrt{\pi_H/\pi_L} - 1\right)\right)$. In addition, since that $\Delta\pi/\pi_H - (1 - \pi_L G) = (\pi_H G - 1)\pi_L/\pi_H$, it is also equivalent to $1 - \pi_L G \in \left(\max\left((\pi_H - 2\pi_L)/\pi_H, 1 - \sqrt{\pi_L/\pi_H}\right), (\Delta\pi/\pi_H)^2\right)$. Then, from $G\Delta\pi = (\pi_H G - 1) + (1 - \pi_L G)$ it follows that $G\Delta\pi > \Delta\pi/\pi_H + (\pi_H - 2\pi_L)/\pi_H = 1 + (\pi_H - 3\pi_L)/\pi_H$ which, given (OA50) implies (OA52). Finally, it follows from (OA51) that $G < 1/\sqrt{\pi_H \pi_L}$, equivalently, $\pi_H \pi_L G^2 < 1$, or, $(\pi_H G - 1) < \pi_H G(1 - \pi_L G)$. This latter inequality is equivalent to (OA53).

and the last from (OA51). Thus, we have $\psi(0) > 0$. In addition,

$$\frac{\psi(I;A)}{(\pi_H G-1)I} = 1 - \frac{\pi_H}{\Delta\pi}\frac{B(I+A)-B(A)}{(\pi_H G-1)I} + \frac{-\frac{\pi_H}{\Delta\pi}B(A)+\pi_H GA}{(\pi_H G-1)I} \tag{OA56}$$

$$< 1 - \frac{\pi_H}{\Delta\pi}\frac{B'(I+A)I}{(\pi_H G-1)I} + \frac{-\frac{\pi_H}{\Delta\pi}B(A)+\pi_H GA}{(\pi_H G-1)I} \tag{OA57}$$

$$< 1 - \frac{\pi_H}{\Delta\pi} + \frac{-\frac{\pi_H}{\Delta\pi}B(A)+\pi_H GA}{(\pi_H G-1)I}, \tag{OA58}$$

where the first inequality follows from the concavity of B and the second from (7). We deduce from (OA58) that $\lim_{I\to\infty}\psi(I;A) < 0$. Then, the existence of a unique $I^c \in (0,+\infty)$ follows from the intermediate value theorem.

Let $I^c(A)$ be the implicit function defined by $\psi(I^c;A)=0$ and denote $\dot{I}^c = \partial I^c/\partial A$. Implicit differentiation yields

$$\left(\frac{\pi_H}{\Delta\pi}B'(I^c+A)-(\pi_H G-1)\right)\dot{I}^c = \pi_H G - \frac{\pi_H}{\Delta\pi}B'(I^c+A) \tag{OA59}$$

$$= 1 + \left((\pi_H G-1) - \frac{\pi_H}{\Delta\pi}B'(I^c+A)\right), \tag{OA60}$$

and

$$\dot{I}^c = -1 + \frac{1}{\frac{\pi_H}{\Delta\pi}B'(I^c+A)-(\pi_H G-1)}. \tag{OA61}$$

From (7), we have that $B'(I^c+A)\pi_H/\Delta\pi - (\pi_H G-1) > (\pi_H G-1)\pi_L/\Delta\pi > 0$ and $\pi_H G - B'(I^c+A)\pi_H/\Delta\pi > 0$, which implies that $\dot{I}^c > 0$. ■

The next lemma characterizes a unique pair $((I^*,R^*),(I_L^*,R_L^*))$ that will be used in Step 2 of the proof to show that (I^*,R^*) is an equilibrium allocation supported by latent contracts issued at fair price $1/\pi_L$. As an illustration, (I^*,R^*) and (I_L^*,R_L^*) are represented in Figure 3.

LEMMA OA2: *Under Assumption OA1, there exists $\underline{A} > 0$, such that, for all $A \in (0,\underline{A})$ there exists one and only one pair of investments I^* and I_L^* with $0 < I^* < I_L^* < I^c$ such that the aggregate allocations $((I^*, I^*/\pi_H),(I_L^*, G(I_L^*+A))$ are connected by a line of slope $1/\pi_L$ and satisfy $U(I^*, I^*/\pi_H, H) = U(I_L^*, G(I_L^*+A), L)$.*

Proof of Lemma OA2: Take any $I_L \geq 0$. Let $I(I_L)$ be the value such that

$$I(I_L) = \frac{1}{\pi_H G-1}(B(I_L+A)-\pi_H GA). \tag{OA62}$$

For a given $I_L \geq 0$, the investment $I(I_L) \geq 0$ is such that the two allocations

$(I(I_L), I(I_L)/\pi_H)$ and $(I_L, G(I_L+A))$ satisfy $U(I(I_L), I(I_L)/\pi_H, H) = U(I_L, G(I_L+A), L)$. In the following I_L^0 denotes the investment level such that $I(I_L^0) = 0$. It follows from (OA62) that $B(I_L^0 + A) = \pi_H GA$.

We now establish the existence of $I_L^* \geq 0$ such that $\big(I(I_L^*), I(I_L^*)/\pi_H\big)$ and $(I_L^*, G(I_L^* + A))$ are connected by a line of slope $1/\pi_L$. That is,

$$\text{(OA63)} \qquad \frac{G(I_L^* + A) - \frac{1}{\pi_H}\frac{1}{\pi_H G-1}(B(I_L^* + A) - \pi_H GA)}{I_L^* - \frac{1}{\pi_H G-1}(B(I_L^* + A) - \pi_H GA)} = \frac{1}{\pi_L}.$$

Equation (OA63) is equivalent to $g(I_L^*; A) = 0$ where the function $g : \mathbb{R} \to \mathbb{R}$ is defined by

$$\text{(OA64)} \quad g(I_L; A) = -(\pi_H G - 1)(1 - \pi_L G)I_L + \frac{\Delta\pi}{\pi_H} B(I_L + A) + \pi_H GA(\pi_L G - 1),$$

for any given A. We prove that I_L^* is well defined. The concavity of B and the inequality (7) yield that g is concave and increasing. Indeed, $g_I(I; A)/(\pi_H G - 1) = -(1 - \pi_L G) + \Delta\pi B'(I_L + A)/(\pi_H(\pi_H G - 1)) > -(1 - \pi_L G) + \Delta\pi/\pi_H = \pi_L\,(G - 1/\pi_H) > 0$.

We remark that $g(-A; A) = (\pi_H G - 1)(1 - \pi_L G)A + \pi_H GA(\pi_L G - 1) = -(1 - \pi_L G)A < 0$. In addition we have that

$$\text{(OA65)} \quad \frac{g(I_L; A)}{(\pi_H G - 1)I_L} > -(1 - \pi_L G) + \frac{\Delta\pi}{\pi_H}\frac{B'(I_L + A)}{\pi_H G - 1}\frac{I_L + A}{I_L} + \frac{\pi_H G(\pi_L G - 1)}{\pi_H G - 1}\frac{A}{I_L}$$

$$\text{(OA66)} \quad > -(1 - \pi_L G) + \frac{\Delta\pi}{\pi_H}\frac{I_L + A}{I_L} + \frac{\pi_H G(\pi_L G - 1)}{\pi_H G - 1}\frac{A}{I_L}$$

$$\text{(OA67)} \quad = \frac{\pi_L}{\pi_H}(\pi_H G - 1) + \left[\frac{\Delta\pi}{\pi_H} + \frac{\pi_H G(\pi_L G - 1)}{\pi_H G - 1}\right]\frac{A}{I_L}.$$

The first inequality follows from the concavity of B, while the second follows from (7). We deduce that $\lim_{I_L \to +\infty} g(I_L; A) = +\infty$. The intermediate value theorem guarantees the existence and uniqueness of $I_L^* \in (-A, +\infty)$ such that $g(I_L^*; A) = 0$.

We write $I^* \equiv I(I_L^*)$ and we show that $I^* < I_L^*$. Using (OA62), this amounts to showing that $B(I_L^* + A) < I_L^*(\pi_H G - 1) + \pi_H GA$. Using the relation $g(I_L^*; A) = 0$, we re-write the latter inequality under the form $(\pi_H G - 1)(1 - \pi_L G)I_L^*\pi_H/\Delta\pi - GA(\pi_L G - 1)\pi_H^2/\Delta\pi < I_L^*(\pi_H G - 1) + \pi_H GA$. An easy computation yields $I_L^*(1 - \pi_H G) < \pi_H GA$ which is always satisfied since $G > 1/\pi_H$ and $I_L^* > -A$.

We prove that, for any A in some interval $(0, \underline{A})$ with $\underline{A} > 0$, we have $0 < I^*$ and $I_L^* < I^c$. Let us consider the investment levels I_L^0 and I_L^* as functions of A. Specifically, the equality $g(I_L^*; A) = 0$ defines $I_L^*(A)$ and the equality $B(I_L^0 + A) - \pi_H GA = 0$ defines $I_L^0(A)$. We note $\dot{I}_L^* = \partial I_L^*/\partial A$ and $\dot{I}_L^0 = \partial I_L^0/\partial A$.

Now, if $A = 0$, we get from (OA62) and (OA64) that $I_L^0(0) = 0$ and $I_L^*(0) = 0$ (recalling that $B(0) = 0$). Moreover, $\dot{I}_L^*|_{A=0} > \dot{I}_L^0|_{A=0} > 0$,[6] thus there exists an open interval $(0, A_1)$ with $A_1 > 0$ such that for all $A \in (0, A_1)$ we have $I_L^*(A) > I_L^0(A) > 0$. Then, because $I(I_L)$ (defined in (OA62)) is increasing in I_L and that, by definition, $I(I_L^0(A)) = 0$, we get $I^* \equiv I(I_L^*(A)) > I(I_L^0(A)) = 0$.

We last show that $I_L^* < I^c$. Again, we consider I^c and I_L^* as functions of A. We have $I_L^*(0) = 0$ and, from (OA54), $I^c(0) = 0$. Thus, if $\dot{I}^c|_{A=0} > \dot{I}_L^*|_{A=0}$, there exists an open interval $(0, A_2)$ with $A_2 > 0$ such that whenever $A \in (0, A_2)$, $I_L^*(A) < I^c(A)$. Using (OA61), we get

(OA68)
$$\dot{I}^c|_{A=0} > \dot{I}_L^*|_{A=0} \;\Leftrightarrow\; \frac{1}{\frac{\pi_H}{\Delta\pi}B' - (\pi_H G - 1)} > \frac{1 - \pi_L G}{\frac{\Delta\pi}{\pi_H}B' - (\pi_H G - 1)(1 - \pi_L G)},$$

which can be rewritten under the form $\Delta\pi/\pi_H > \pi_H(1 - \pi_L G)/\Delta\pi$. The last inequality is satisfied given (OA51). Indeed, using the lower bound on G, one gets $(1 - \pi_L G)\pi_H/\Delta\pi < (1 - (2 - \pi_L/\pi_H)\pi_L/\pi_H)\pi_H/\Delta\pi = \Delta\pi/\pi_H$. To conclude the proof, it is sufficient to take $\underline{A} = \min(A_1, A_2)$. ■

Step 2: Proof that (I^*, R^*) is supported by latent contracts at a fair price $1/\pi_L$. To establish our result, we additionally assume that the number of investors is sufficiently large.

ASSUMPTION OA2: *The number N of investors satisfies $N - 2 > I_L^*/I^*$, where the investment levels I_L^* and I^* are defined in Lemma OA2.*

We are now able to provide an explicit formulation of Proposition 6.

PROPOSITION OA1: *Under Assumptions OA1-OA2, and for each $A \in (0, \underline{A})$ the following holds:*

1) *The aggregate allocation $(I^*, R^*) = (I(I_L^*), I(I_L^*)/\pi_H)$ characterized in Lemma OA2 is supported at equilibrium by investors' strategies such that each non-traded contract has a constant price $1/\pi_L$.*

2) *The corresponding allocation is Pareto-dominated in the set of equilibrium allocations.*

[6]Indeed, differentiating (OA62) yields $B'(I_L^0 + A)(\dot{I}_L^0 + 1) = \pi_H G$, thus, $\dot{I}_L^0 = (\pi_H G - B'(I_L^0 + A))/(B'(I_L^0 + A)) > 0$, the inequality coming from (7). Recalling that $I_L^0(0) = 0$, we get $\dot{I}_L^0|_{A=0} = \pi_H G/B'(0) - 1$. Differentiating $g(I_L^*(A); A) = 0$ with respect to A, we get $\left((\pi_H G - 1)(1 - \pi_L G) - B'(I_L^* + A)\Delta\pi/\pi_H\right)\dot{I}_L^* = B'(I_L^* + A)\Delta\pi/\pi_H - (1 - \pi_L G)\pi_H G = \left(B'(I_L^* + A)\Delta\pi/\pi_H - (\pi_H G - 1)(1 - \pi_L G)\right) - (1 - \pi_L G)$ and then, $\dot{I}_L^* = -1 + (1 - \pi_L G)/\left((I_L^* + A)B'\Delta\pi/\pi_H - (\pi_H G - 1)(1 - \pi_L G)\right)$. In particular, since $I_L^* = 0$ for $A = 0$, we have $\dot{I}_L^*|_{A=0} = -1 + (1 - \pi_L G)/\left(B'(0)\Delta\pi/\pi_H - (\pi_H G - 1)(1 - \pi_L G)\right)$. Thus, $\dot{I}_L^*|_{A=0} > \dot{I}_L^0|_{A=0} \Leftrightarrow (1 - \pi_L G)/(B'(0)\Delta\pi/\pi_H - (\pi_H G - 1)(1 - \pi_L G)) > \pi_H G/B'(0)$, or equivalently, $B'(0)(1 - \pi_L G) > G\Delta\pi B'(0) - \pi_H G(\pi_H G - 1)(1 - \pi_L G)$, that is $B'(0)/(\pi_H G - 1) < \pi_H G(1 - \pi_L G)/(\pi_H G - 1)$, which is satisfied given (7).

Proof of Proposition OA1: Consider the following profile of strategies. Each investor $i = \{1,2\}$ offers $M_i^* = \{(0,0); \big(I^*/2, R_i^*(.)\big)\}$, with $R_i^*(I) = G(I+A)$ for $I \notin \{I^*/2, I^*, I_L^*\}$, $R_i^*(I^*/2) = (R^* - GI^*/2)$, and $R_i^*(I^*) = R_i^*(I_L^*) = R^*/2$, where $I^* = I(I_L^*)$ and I_L^* have been characterized in Lemma OA2. Observe also that $R_i^*(I^*/2)$, for $i = 1,2$, guarantees that $U(I^*, R^*, H)$ is available to the entrepreneur if any of the investors withdraws his offer. Each investor $j = \{3, ..., N\}$ offers $M_j^* = \{(0,0); \big((I_L^* - I^*)/(N-2), R_j^*\big)\}$, with $R_j^* = \left(G(I_L^* + A) - I^*/\pi_H\right)/(N-2) = (I_L^* - I^*)/((N-2)\pi_L)$. We denote $\bar{I} = I_L^* - I^* > 0$.

Given these offers, it is optimal for the entrepreneur to choose $I^*/2$ in M_1^* and M_2^*, $(0,0)$ otherwise, and to select $e = H$. Her corresponding payoff is $U(I^*, R^*, H) = B(I_L^* + A) - A = B(I^* + \bar{I} + A) - A$.

Remark: See that (I^*, R^*) is not the unique entrepreneur's optimal choice. She could alternatively trade with all investors, get the aggregate investment I_L^* and choose $e = L$. See that each of such inactive contracts is issued at the corresponding fair price $1/\pi_L$.

Consider now investors' deviations. We first show that there is no profitable deviation inducing the entrepreneur to choose $e = H$. A deviation of investor $i \in \{1,2\}$ can be characterized, without loss of generality, by the menu $M_i' = \{\big(I_i' + I^*/2, R_i'(.) + R^*/2\big), (0,0)\}$, with $I_i' \in [-I^*/2, I^c - I^*/2]$. Similarly, a deviation of any investor $j \in \{3, \cdots, N\}$ can be characterized by the menu $M_j' = \{\big(I_j' + \bar{I}/(N-2), R_j'(.)\big), (0,0)\}$, with $I_j' \in \big[-\bar{I}/(N-2), I^c - \bar{I}/(N-2)\big]$. For any such deviation to be profitable when the entrepreneur chooses aggregate investment I and effort $e = H$, one needs that

$$\text{(OA69)} \qquad R_i'(I) > \frac{1}{\pi_H} I_i', \;\; R_j'(I) > \frac{1}{\pi_H}\left(I_j' + \frac{\bar{I}}{N-2}\right),$$

for $i \in \{1,2\}, j \in \{3, \cdots, N\}$.

We first evaluate the entrepreneur's payoff when $e = H$ is chosen. Consider first a deviation of an active investor i. If the entrepreneur only trades with investor i at the deviation stage, her payoff is

$$\text{(OA70)} \quad \pi_H (G(\frac{I^*}{2} + I_i' + A) - \frac{R^*}{2} - R_i'(I_i' + \frac{I^*}{2})) - A < \frac{U(0) + U(I^*, R^*, H)}{2} + I_i'(\pi_H G - 1),$$

where the inequality follows from (OA69). Since the equilibrium utility is available at the deviation stage, the right-hand side of (OA70) must be strictly greater than $U(I^*, R^*, H)$, which implies that $I_i' > 0$. If, following any such deviation, the entrepreneur defaults, her payoff is

$$\text{(OA71)} \qquad U_{sd} = B(I_i' + I_L^* + A) - A > B(I_L^* + A) - A + B'(I_j' + I_L^* + A) I_j',$$

by concavity of B, and $I_i' > 0$. Thus, if the right-hand side of (OA71) is greater than the right-hand side of (OA70), the entrepreneur finds optimal to strategically default. This is the case whenever

$$\text{(OA72)} \qquad \frac{1}{2}\big(U(I^*,R^*,H) - U(0)\big) \geq (\pi_H G - 1 - B'(I_i' + I_L^* + A))I_i',$$

which is always satisfied, since the right-hand side is negative given (7). If several contracts are traded at the deviation stage, then, given equilibrium covenants, $e = H$ is an optimal choice only if aggregate investment is I^*. In this case, since $\pi_H R^* = I^*$, there cannot be any profitable deviation since the entrepreneur must at least receive $U(I^*,R^*,H)$.

Suppose now that an inactive investor j deviates. If the entrepreneur only trades with investor j at the deviation stage, given (OA69), her payoff when $e = H$ is chosen is bounded above by $U(0) + (\pi_H G - 1)\big(I_j' + \bar{I}/(N-2)\big)$. In addition, given equilibrium covenants, any such deviation must guarantee the entrepreneur at least the equilibrium utility $U(0) + (\pi_H G - 1)I^*$. Thus, for the deviation to be profitable, it must be $I_j' > I^* - (I_L^* - I^*)/(N-2) > (I_L^* - I^*)/(N-2) - (I_L^* - I^*)/(N-2) = 0$, where the second inequality follows from Assumption OA2. If, following this deviation, the entrepreneur defaults, then, given the concavity of B, her payoff is

$$\text{(OA73)} \qquad U_{sd} = B(I_j' + I_L^* + A) - A > B(I_L^* + A) - A + B'(I_i' + I_L^* + A)I_i'.$$

The entrepreneur finds therefore optimal to strategically default if

$$\text{(OA74)} \quad U(I^*,R^*,H) - U(0) - \frac{I_L^* - I^*}{N-2}(\pi_H G - 1) \geq I_j'\big(\pi_H G - 1 - B(I_j' + I_L^* + A)\big),$$

which right-hand side is negative by (7). The left-hand side can be written as $\big(\pi_H G - 1\big)(I^* - (I_L^* - I^*)/(N-2))$ which is positive since $N - 2 \geq (I_L^* - I^*)/I^*$ by Assumption OA2.

To complete the proof we need to show that no investor can profitably deviate inducing the entrepreneur to choose $e = L$. Given equilibrium covenants, an investor may get a positive profit under $e = L$ only if the entrepreneur strategically defaults. We show that, in that case, any unilateral deviation yields a strictly negative profit.

Consider first a deviation of an active investor i, and let I_i' be the investment level traded with the entrepreneur when she strategically defaults. Given that the equilibrium utility is available at the deviation stage, it must be that $B(I_i' + I^*/2 + (I_L^* - I^*) + A) \geq B(I_L^* + A) = U(I^*,R^*,H) - A$, which implies $I_i' \geq I^*/2$.

The corresponding profit to investor i can be written as

$$V_i' = \pi_L G(I_i' + \frac{I^*}{2} + (I_L^* - I^*) + A)\frac{I_i'}{I_i' + \frac{I^*}{2} + (I_L^* - I^*)} - I_i' \tag{OA75}$$

$$= I_i'\big(\pi_L G - 1 + \frac{\pi_L GA}{I_i' + \frac{I^*}{2} + (I_L^* - I^*)}\big)$$

$$\leq I_i'\big(\pi_L G - 1 + \frac{\pi_L GA}{I^* + (I_L^* - I^*)}\big)$$

$$= I_i'\big(\pi_L G - 1 + \frac{\pi_L GA}{I_L^*}\big) = I_i'\frac{I^*}{I_L^*}(\frac{\pi_L}{\pi_H} - 1) < 0$$

where the first inequality comes from $I_i' \geq I^*/2$ and the last equality comes from the relation $(G(I_L^* + A) - I^*/\pi_H)/(I_L^* - I^*) = 1/\pi_L$. Thus, we have $V_i' < 0$.

Consider now a deviation of an inactive investor j and let I_j' be the investment level traded by the entrepreneur when she strategically defaults. Given that the entrepreneur's equilibrium payoff is available at the deviation stage, it must be that $B(I_j'+I^*+(I_L^*-I^*)(N-3)/(N-2)+A) = B(I_j'+(I^*-I_L^*)/(N-2)+I_L^*+A) \geq B(I_L^* + A)$, which implies that $I_j' \geq (I_L^* - I^*)/(N-2)$. The corresponding profit to investor j can be written as

$$V_j' = \pi_L G(I_j' + I^* + \frac{N-3}{N-2}(I_L^* - I^*) + A)\frac{I_j'}{I_j' + I^* + \frac{N-3}{N-2}(I_L^* - I^*)} - I_j' \tag{OA76}$$

$$= I_j'\big(\pi_L G - 1 + \frac{\pi_L GA}{I_j' + I^* + \frac{N-3}{N-2}(I_L^* - I^*)}\big) \tag{OA77}$$

$$\leq I_j'\left(\pi_L G - 1 + \frac{\pi_L GA}{\frac{1}{N-2}(I_L^* - I^*) + I^* + \frac{N-3}{N-2}(I_L^* - I^*)}\right) \tag{OA78}$$

$$= I_j'\big(\pi_L G - 1 + \frac{\pi_L GA}{I_L^*}\big) = I_j'\frac{I^*}{I_L^*}(\frac{\pi_L}{\pi_H} - 1) < 0, \tag{OA79}$$

the first inequality coming from $I_j' \geq (I_L^* - I^*)/(N-2)$. Again we get $V_j' < 0$. This completes the proof of the first part of Proposition OA1.

We now show that the above aggregate allocation (I^*, R^*) is *inefficient*. Specifically, we prove that the competitive allocation (I^c, R^c), which Pareto-dominates it, can also be supported at equilibrium. To this extent, we denote I_L^c the investment level such that $U(I^c, R^c, H) = B(I_L^c + A) - A$. Consider the following profile of strategies. Each investor $i = \{1, 2\}$ offers $M_i^* = \{(0,0); \big(I^c/2, R_i^*(.)\big)\}$, with $R_i^*(I) = G(I + A)$ for $I \notin \{I^c/2, I^c\}$, $R_i^*(I^c/2) = (I^c/\pi_H - GI^c/2)$, and $R_i^*(I^c) = I^c/(2\pi_H)$. Each investor $j = \{3, ..., N\}$ offers $M_j^* = \{(0,0); \big((I_L^c - I^c)/(N-2), R_j^*)\big)\}$, with $R_j^* = (G(I_L^c + A) - I^c/\pi_H)/(N-2)$.

Given these offers, and recalling that $U(I^c, R^c, H) = B(I_L^c + A) - A$, it is optimal for the entrepreneur to choose $I^*/2$ in M_1^* and M_2^*, $(0,0)$ otherwise, and to select $e = H$. It is immediate to check that, given the equilibrium covenants, none of the investors can profitably deviate inducing $e = H$. It remains to show that no investor can profitably deviate by inducing the entrepreneur to strategically default. Following the proof of the first part of the proposition, the profit at the deviation of an active (or inactive) investor i is bounded above by $\bar{V}_i' = \big(\pi_L G - 1 + \pi_L GA/I_L^c\big)$. We remark that, for any $i \in \{1, ..., N\}$, $\bar{V}_i'$ has the same sign as $\sum_{k=1}^N \bar{V}_k' = I_L^c\big(\pi_L G - 1 + \pi_L GA/I_L^c\big) = \big(\pi_L G - 1\big)I_L^c + \pi_L GA$.[7] We show that $\sum_{k=1}^N \bar{V}_k' < 0$. Indeed, one can write

$$\text{(OA80)} \qquad \begin{aligned} \sum_{k=1}^N \bar{V}_k' &= \big(\pi_L G - 1\big)I_L^* + \pi_L GA + (\pi_L G - 1)(I_L^c - I_L^*) \\ &= (\frac{\pi_L}{\pi_H} - 1)I^* + \big(\pi_L G - 1\big)(I_L^c - I_L^*) \\ &< (\pi_L G - 1)(I_L^c - I_L^*) < 0. \end{aligned}$$

The second equality follows from the relation $G(I_L^* + A) - I^*/\pi_H = (I_L^* - I^*)/\pi_L$ which holds by construction. The first inequality in (OA80) comes from $\pi_L < \pi_H$ and the second from the fact that $\pi_L G < 1$ and $I_L^* < I^c < I_L^c$. This completes the proof of the second part of Proposition OA1. ■

Proof of Proposition 7:

The Mechanism. A mechanism consists in a system of transfers from the investors to the entrepreneur and a randomizing device. Both the transfers and the device are contingent on the observable investment levels $(I_1, I_2, ...I_N)$ chosen by the entrepreneur, and on the investors' decisions to participate. The device selects a "pivotal" investor who makes no payment to the entrepreneur after investors' menus have been posted, and participation and effort decisions have been made. Specifically, letting $K \leq N$ be the number of investors who provide a loan to the entrepreneur, any investor j such that $I_j > 0$ is pivotal with probability $1/K$. The pivotal investor will be denoted by κ. Clearly, if the entrepreneur raises funds from one investor only, the latter is pivotal with probability one. Investors learn who is pivotal after all relevant decisions are made, and before payments occur.

The schedule of transfers from investor i to the entrepreneur when the project succeeds is denoted $T_i(.)$. Transfers are equal to zero in case of failure. If the entrepreneur raises $I = \sum_{i=1}^N I_i$, then the contribution of each non pivotal investor

[7] Observe that the allocations $(I_L^c, G(I_L^c + A))$ and $(I^c, I^c/\pi_H)$ are not connected by a line of slope $1/\pi_L$. Indeed, the function g defined by (OA64) satisfies $g(I_L^c; A) > 0$. It follows that we cannot directly conclude that $\pi_L G - 1 + \pi_L GA/I_L^c < 0$ as in the proof of Proposition OA1. We overcome this difficulty by studying $\sum_{k=1}^N \bar{V}_k'$.

i is

$$\text{(OA81)} \qquad T_i(I) = \begin{cases} \frac{1}{\pi_H}\frac{1}{N}BI & \text{if } I \in [0, I^c), \\ 0 & \text{if } I \geq I^c, \end{cases}$$

whenever all investors agree to participate. In that case, if the project succeeds, the entrepreneur receives $T(I) = (N-1)BI/(\pi_H N)$. The schedule $T_i(.)$ is set equal to zero in all other cases.

No Transfer at Equilibrium. We first establish that the entrepreneur receives no transfer at equilibrium. To show this result, consider an equilibrium aggregate allocation (I^*, R^*) with $I^* > 0$, and the equilibrium surplus $(\pi_H G - 1)(I^* + A)$. The entrepreneur's equilibrium payoff cannot be lower than $(\pi_H G - 1)A$, which implies that investors' aggregate profit cannot be greater than $(\pi_H G - 1)I^*$. Suppose, by contradiction, that investors decide to pay a transfer at equilibrium. Given that $B > \pi_H G - 1$, the aggregate profit earned by the pivotal investors is bounded above by $(N-1)(\pi_H G - 1)I^*/N < (N-1)BI^*/N = \pi_H T(I^*)$, as equilibrium is symmetric. That is, total transfers of non pivotal investors exceed their maximal equilibrium aggregate profit. Therefore, at least one investor refuses to participate. This contradicts the assumption that investors pay a positive transfer at equilibrium.

We now show that, given the subsidy mechanism, only the competitive allocation is supported at equilibrium. For future reference, we denote $\hat{I} \geq I^*$ the maximal available investment at equilibrium.The proof is developed in three steps.

Constructing a profitable deviation. Given $(T_1(.), T_2(.), ..., T_N(.))$, we construct a deviation from each symmetric equilibrium supporting an aggregate allocation $(I^*, R^*) \in \mathcal{F} - \{I^c, R^c\}$. We denote

$$\text{(OA82)} \qquad \alpha = \min\left(G - \frac{1}{\pi_H}, \frac{N-1}{N}\frac{I^c - I^*}{B(I^c + A)}A, \frac{A}{\pi_H G - 1}, \frac{1-B}{B}I^*\right).$$

Thus, in any equilibrium with $I^* > 0$ and $(I^*, R^*) \in \mathcal{F} - \{I^c, R^c\}$, implying that $I^* < I^c$, we have $\alpha > 0$.[8]

Suppose that investor k deviates to $M'_k = ((0,0), (I'_k, R'_k(.)))$ with $I'_k = I^* + \varepsilon - (N-1)(\pi_H R^* - I^*)/(N(\pi_H G - 1))$, and $\varepsilon \in (0, \alpha)$. The repayment $R'_k(.)$ is such that

$$\text{(OA83)} \qquad R'_k(I) = \begin{cases} R^* + \dfrac{\varepsilon}{\pi_H} + \varepsilon^2 - G\frac{N-1}{\pi_H G - 1}\dfrac{\pi_H R^* - I^*}{N} & \text{if } I = I'_k, \\ G(I + A) & \text{if } I \neq I'_k. \end{cases}$$

[8]Note that $I^* = 0$ *cannot* be supported in a symmetric equilibrium. If it could, one would have $U(0,0,H) \geq B(\hat{I} + A) - A$. Any investor j could then deviate to $(I'_j, R'_j(.))$ such that $U(I'_j, R'_j(I'_j), H) > \max\{U(0,0,H), B((N-1)\hat{I}/N + I'_j + A) - A\}$ and $\pi_H R'_j(I'_j) - I'_j > 0$. This deviation would induce the entrepreneur to choose I'_j, and to select $e = H$, guaranteeing a positive profit to investor j.

The pair $(I'_k, R'_k(I'_k))$ satisfies $U(I'_k, R'_k(I'_k), H) = U(I^*, R^*, H) + (\pi_H G - 1)\varepsilon - \pi_H \varepsilon^2 > U(I^*, R^*, H)$ and $\pi_H R'_k(I'_k) - I'_k = [N - (\pi_H G - 1)(N-1)/(\pi_H G - 1)] (\pi_H R^* - I^*)/N + \pi_H \varepsilon^2 = (\pi_H R^* - I^*)/N + \pi_H \varepsilon^2$. Observe that, when $\varepsilon = 0$, $(I'_k, R'_k(I'_k))$ lies at the intersection of the entrepreneur's equilibrium indifference curve with the investor k's equilibrium isoprofit line. In the relevant case $\varepsilon \in (0, \alpha)$, we have

$$\text{(OA84)} \qquad \begin{aligned} R'_k(I'_k) - (G - \frac{B}{\pi_H})(I'_k + A) &= R^* - (G - \frac{B}{\pi_H})(I^* + A) \\ &- \frac{B}{\pi_H} \frac{N-1}{\pi_H G - 1} \frac{\pi_H R^* - I^*}{N} + \frac{\varepsilon}{\pi_H}(B - (\pi_H G - 1) + \pi_H \varepsilon) < 0, \end{aligned}$$

for every $(I^*, R^*) \in \mathcal{F} - \{I^c, R^c\}$.[9] Given α, (OA84) guarantees that $(I'_k, R'_k(I'_k)) \in int(\mathcal{F})$.

The deviation M'_k is designed to induce the entrepreneur to exclusively trade with investor k and choose $e = H$. We now show that, upon receiving a subsidy, these choices are optimal for the entrepreneur.

Observe first that, if $e = H$ is chosen, and given the covenants in M'_k, then the entrepreneur's (unique) optimal choice is to select $(I'_k, R'_k(.))$ only, therefore trading with investor k alone. By doing so, she achieves a payoff strictly above the equilibrium one. To establish the optimality of $e = H$, we need to show that

$$\text{(OA85)} \qquad U(I'_k, R'_k(I'_k), H) + \pi_H T(I'_k) > B(\hat{I} + A) - A + B(I'_k - \frac{1}{N}\hat{I}),$$

which takes into account the fact that, since $\pi_L = 0$, the entrepreneur receives no subsidy under strategic default. Since $T(I'_k) = (N-1)BI'_k/(\pi_H N)$ the above inequality can be rewritten as

$$\text{(OA86)} \qquad U(I'_k, R'_k(I'_k), H) > \frac{1}{N}\big[B(I'_k + A) - A\big] + \frac{N-1}{N}\big[B(\hat{I} + A) - A\big],$$

which is satisfied, since $U(I'_k, R'_k(I'_k), H) > \max\big(B(I'_k + A) - A; B(\hat{I} + A) - A\big)$; indeed, $(I'_k, R'_k(I'_k)) \in int(\mathcal{F})$ hence $U(I'_k, R'_k(I'_k), H) > B(I'_k + A) - A$, and $U(I'_k, R'_k(I'_k), H) > U^* \geq B(\hat{I} + A) - A$ for each $\varepsilon \in (0, \alpha)$.

[9] To get the result, it is sufficient to show that (OA84) is satisfied in the boundary cases in which $(I^*, R^*) \in \Psi$, or $(I^*, R^*) \in \{\pi_H I^* - R^* = 0\}$ with $(I^*, R^*) \neq (I^c, R^c)$. By continuity, (OA84) holds for every $(I^*, R^*) \in \mathcal{F} - \{I^c, R^c\}$. Observe first that $(B - (\pi_H G - 1) + \pi_H \varepsilon)\,\varepsilon/\pi_H \leq B\varepsilon/\pi_H \leq (N-1)(I^c - I^*)A/(\pi_H N(I^c + A))$. With $\pi_L = 0$, it follows that:

- If $(I^*, R^*) \in \Psi$, then $\pi_H R^* - I^* = \pi_H(G - B/\pi_H)(I^* + A) - I^* = \big(-B + (\pi_H G - 1)\big)(I^* + A) + A = -A(I^* + A)/(I^c + A) + A = A(I^c - I^*)/(I^c + A)$, implying that $R'_k(I'_k) - (G - B/\pi_H)(I'_k + A) \leq \big(1 - B/(\pi_H G - 1)\big)(N-1)(I^* + A)A/(\pi_H N(I^c + A)) < 0$
- If $\pi_H R^* - I^* = 0$, then, $R^* - (G - B/\pi_H)(I^* + A) = I^*/\pi_H - (G - B/\pi_H)(I^* + A) = \big(B - (\pi_H G - 1)\big)(I^* + A)/\pi_H - A/\pi_H = -(I^c - I^*)A/(\pi_H(I^c + A))$, implying that $R'_k(I'_k) - (G - B/\pi_H)(I'_k + A) \leq -(1 - (N-1)/N)(I^c - I^*)A/(\pi_H(I^c + A)) < 0$ when $I^* \neq I^c$.

The entrepreneur's choices with no subsidy. To fully characterize the entrepreneur's optimal behavior, we also need to consider subgames induced by the deviation to M'_k in which she does *not* receive a subsidy. In any such situation, given the covenants associated to $(I'_k, R'_k(.))$, she chooses $e = H$ whenever

$$U(I'_k, R'_k(I'_k), H) > B(\hat{I} + A) - A + B(I'_k - \frac{1}{N}\hat{I}), \tag{OA87}$$

in which case she only selects $(I'_k, R'_k(.))$. If, on the contrary, one has

$$U(I'_k, R'_k(I'_k), H) < B(\hat{I} + A) - A + B(I'_k - \frac{1}{N}\hat{I}), \tag{OA88}$$

then she strategically defaults on the total loan $I'_k + \hat{I}(N-1)/N$.[10]

In summary, the entrepreneur always selects $(I'_k, R'_k(.))$ and chooses $e = H$, except if both condition (OA88) and the decision *not* to pay the transfer to the entrepreneur hold.

Investors' decisions and participation subgame. We now characterize investors' equilibrium decisions to pay transfers, given the posted menus (M'_k, M^*_{-j}). See first that, if (OA87) is satisfied, i.e. the entrepreneur chooses $e = H$ even in the absence of a subsidy, then posting M'_k and choosing *not* to pay any transfer is a profitable deviation for investor k. We henceforth consider the case in which, following the deviation to M'_k, (OA88) is satisfied.

Investors participate sequentially, and the history of such decisions is perfectly observable. Thus, given transfers (OA81), if an investor j observes that at least one of his predecessors chose not to pay the transfer, he anticipates that no subsidy will be provided to the entrepreneur. In this case, given (M'_k, M^*_{-j}), the entrepreneur strategically defaults, which hinders investor k's deviation. We now solve the game backward, starting with investor N.

We first consider the case $N \neq k$. Investor N takes as given the mechanism, the menus $(M^*_1, ..., M'_k, ..., M^*_N)$ and the entire history of participation decisions of investors $1, 2, ..., N-1$. In addition, he anticipates that, as shown in step 1, the entrepreneur chooses $e = H$ and trades only with investor k if the subsidy is provided, and that, as pointed out above, (OA88) is satisfied in the absence of transfers. We then show that investor N finds optimal to finance the transfer if all other investors have already done so. In that case, investor N compares his loss if strategic default occurs with the transfer he has to pay to the entrepreneur.[11] Paying is therefore the unique optimal choice if $-\pi_H T_N(I'_k) > -\hat{I}_N$. Using (OA81), together with the fact any investor offers at most $\hat{I}/N$ in a symmetric equilibrium,

[10] We do not consider the non generic case in which (OA87) holds as equality. Indeed for any $\bar{\varepsilon}$ which induces this equality, one can find $\varepsilon < \bar{\varepsilon}$, such that $\varepsilon \in (0, \alpha)$ that satisfies (OA88).

[11] Since $N \neq k$, investor N perfectly anticipates that he will not be pivotal: If $(I'_k, R'_k(.))$ is the only contract selected by the entrepreneur, then investor k ends up being pivotal (i.e. $\kappa = k$).

the inequality can be rewritten as

$$\frac{B}{N} I'_k < \frac{\hat{I}}{N}, \tag{OA89}$$

which, recalling that $\hat{I} \geq I^*$, holds if

$$I'_k < \frac{1}{B} I^*. \tag{OA90}$$

To show that (OA90) is satisfied, it is enough to observe that

$$I'_k - \frac{1}{B} I^* = \varepsilon + I^* \big(1 - \frac{1}{B}\big) - \frac{N-1}{\pi_H G - 1} \frac{\pi_H R^* - I^*}{N} \leq \varepsilon - I^* \big(\frac{1-B}{B}\big) < 0, \tag{OA91}$$

with the last holding because $\varepsilon < \alpha$.

If $N = k$, then investor N anticipates that he will not pay any transfer, and participating is his (unique) optimal choice when all investors $j < N$ decided to participate. In this case, the subsidy is provided to the entrepreneur.

Using backward induction, one can iterate this reasoning to show that it is the unique optimal choice for each investor $j \in (2, ..., N-1)$ to pay the transfer if every investor $j' < j$ paid. If at least one investor j' refused to pay, then everyone anticipates that the entrepreneur will default and each investor $j > j'$ is indifferent between participating or not, since no payment will be required. Thus, given (OA90), investor 1 strictly prefers to pay, which shows that providing the subsidy is the unique equilibrium outcome of the continuation game.

Given (OA86), the deviation to M'_k then induces the entrepreneur to trade $I'_k, R'_k(.)$ only, and to choose $e = H$, so that investor k is pivotal. The deviation is profitable since

$$\pi_H R'_k(I'_k) - I'_k = \frac{\pi_H R^* - I^*}{N} + \pi_H \varepsilon^2, \tag{OA92}$$

and investor k's profit increases by $\pi_H \varepsilon^2$.

To show that the competitive allocation (I^c, R^c) can be supported at equilibrium, it is enough to consider strategies exhibited in the proof of Proposition 3, and to see that no investor can exploit the subsidy mechanism to construct a profitable deviation. ■

The proof extends to the case in which investors can write covenants contingent on the initial debt I. The system of transfers to the entrepreneur and the randomizing device are unchanged, and, again, there is no transfer on the equilibrium path. Then, the proof follows the same logic as above. We only need to adapt some of the main inequalities.

Take any aggregate allocation $(I^*, R^*) \neq (I^c, R^c)$ supported in a symmetric equilibrium. Let $(I/N, R(I, I^*)/N, (I^+(I))/N, (R^+(I, I^*))/N)$ be the equilibrium trade with each investor with $I + I^+(I) = I^*$ and $R(I, I^*) + R^+(I, I^*) = R^*$.

Let investor k deviate to the menu $M_k' = ((0,0,0,0), (I_k', R_k'(.), 0, 0))$ where I_k' and $R_k'(.)$ are defined as above, that is, $I_k' = I^* + \varepsilon - (N-1)(\pi_H R^* - I^*)/(N(\pi_H G - 1))$ with $\varepsilon \in (0, \alpha)$, and

$$R_k'(I) = \begin{cases} R^* + \dfrac{\varepsilon}{\pi_H} + \varepsilon^2 - G\frac{N-1}{\pi_H G - 1}\dfrac{\pi_H R^* - I^*}{N} & \text{if } I = I_k', \\ G(I + A) & \text{if } I \neq I_k'. \end{cases} \tag{OA93}$$

As in the first part of the proof, M_k' is such that investor k has a profitable deviation whenever $e = H$ is chosen. In this case, $(I_k', R_k'(.), 0, 0)$ is the only contract selected by the entrepreneur, which guarantees that investor k is pivotal and that his profit increases by $\pi_H \varepsilon^2$.

It remains to check that, following the deviation to M_k', the entrepreneur chooses $e = H$ provided that all investors agree to pay transfers. As above, this leads to consider the inequality

$$U(I_k', R_k', H) > \frac{1}{N}\big[B(I_k' + A) - A\big] + \frac{N-1}{N}\big[B(\hat{I} + \hat{I}^+(\frac{N-1}{N}\hat{I} + I_k') + A) - A\big] \tag{OA94}$$

which corresponds to (OA86) derived in the first part of the proof. In (OA94), $\hat{I}/N$ and $\hat{I}^+((N-1)\hat{I}/N + I_k')/N$ denote respectively the largest initial and additional investment provided by any equilibrium menu for an initial I. The same argument as in the first part of the proof applies: following the deviation, the entrepreneur does *not* choose $e = H$ when she receives no subsidy from the entrepreneur, in which case the condition

$$\begin{aligned} &U(I_k', R_k'(I_k'), H) \\ &< B(\hat{I} + \hat{I}^+(\frac{N-1}{N}\hat{I} + I_k') + A) - A + B(I_k' - \frac{1}{N}(\hat{I} + \hat{I}^+(\frac{N-1}{N}\hat{I} + I_k'))), \end{aligned} \tag{OA95}$$

analogous to (OA88), holds.

To show that paying the transfer is the unique (subgame perfect) equilibrium of the participation game, the condition

$$I_k' < \frac{1}{B}\big(\hat{I} + \hat{I}^+(\frac{N-1}{N}\hat{I} + I_k')\big), \tag{OA96}$$

analogous to (OA90), should be satisfied. The result follows, since $BI_k' < I^*$. ■

Online Appendix B: Equilibria with all investors active.

This Appendix provides an equilibrium characterization which differs to that derived in Proposition 4 by letting all investors be active.

PROPOSITION OB1: *Equilibria with active investors only. If* $\pi_H G - 1 > 2\sqrt{\pi_L/\pi_H}$*, there exists an investment level* $\underline{I} < I^c$ *such that any aggregate allocation* $(I^*, R^*) \in \mathcal{F}$ *with* $I^* \geq \underline{I}$ *can be sustained with equilibrium strategies such that each investor is active.*

COROLLARY OB1: *Each* $(I^*, R^*) \in \mathcal{F}$ *such that* $I^* \geq \underline{I}$ *can be supported at equilibrium with any number* $N \geq 2$ *of investors and all of them being active.*

Proof of Proposition OB1 and Corollary OB1: Assume

$$\text{(OB1)} \qquad \pi_H G - 1 > 2\sqrt{\frac{\pi_L}{\pi_H}}.$$

We first establish a set of relationships that will be used throughout the proof. First, from (1) we get

$$\text{(OB2)} \qquad G - \frac{B}{\Delta\pi} > G + \frac{\pi_L G - 1}{\Delta\pi} = \frac{\pi_H G - 1}{\Delta\pi} > \frac{2}{\Delta\pi}\sqrt{\frac{\pi_L}{\pi_H}},$$

where the last inequality follows from (OB1).

Second, given (OB1), (1) and (4) together imply[12] $1 - \pi_H B/\Delta\pi > 2\pi_L\sqrt{\pi_L/\pi_H}/\Delta\pi$ and $\pi_H B/\Delta\pi > 2\sqrt{\pi_L/\pi_H}$. Adding the two conditions, one gets $1 > 2\sqrt{\pi_L \pi_H}/\Delta\pi \Leftrightarrow \sqrt{\pi_H/\pi_L} - \sqrt{\pi_L/\pi_H} > 2 \Leftrightarrow \left(\sqrt{\pi_H/\pi_L} - 1\right)^2 > 2$, which yields

$$\text{(OB3)} \qquad \sqrt{\frac{\pi_H}{\pi_L}} > 1 + \sqrt{2}, \qquad \sqrt{\frac{\pi_L}{\pi_H}} < \sqrt{2} - 1 \qquad \text{and} \qquad \pi_H > (3 + 2\sqrt{2})\pi_L.$$

We now turn to the proof of Proposition OB1. It is useful to characterize equilibrium allocations in terms of two parameters, which we denote ε and η. Precisely, let $(\varepsilon, \eta) \in [0, \underline{\varepsilon}] \times [\pi_H (G - B/\Delta\pi), 1]$, with

$$\text{(OB4)} \qquad \underline{\varepsilon} = \frac{1}{N}\min\left(\frac{\Delta\pi}{B}(G - \frac{1}{\pi_H}), \frac{\pi_H G - 1}{B} - \frac{\pi_L}{\pi_H \Delta\pi}\frac{1}{G - \frac{B}{\Delta\pi}}, \left(1 - \frac{\pi_L}{\Delta\pi}\frac{\frac{1}{\pi_H}}{G - \frac{B}{\Delta\pi}}\right)\frac{B}{2B + (1 - \pi_H G)}\right).$$

[12] It is useful to rewrite (1) as $\pi_H G - 1 < \Delta\pi(1 - \pi_H B/\Delta\pi)/\pi_L$, and (4) as $\pi_H G - 1 < \pi_H B/\Delta\pi$.

Observe that (1) and (4) imply that $\Delta\pi(G-1/\pi_H)/B \in (0,1)$, and, given (OB2), both the second and third terms are strictly positive. Thus, we get $0 < N\underline{\varepsilon} < 1$. Consider now the aggregate allocation $(I^\varepsilon, R^\varepsilon_\eta) = (I^c(1-\varepsilon), I^c(1-\eta\varepsilon)/\pi_H)$. It is immediate to check that, if $e = H$, the aggregate profit $\pi_H R^\varepsilon_\eta - I^\varepsilon$ is strictly decreasing in η. See also that $\forall\varepsilon \in [0, \underline{\varepsilon}]$, $I^\varepsilon > I^m$.[13] Letting $\underline{I} = I^{\underline{\varepsilon}} > I^m$, and considering all $(\varepsilon, \eta) \in [0, \underline{\varepsilon}] \times [\pi_H(G - B/\Delta\pi), 1]$, one can hence generate a closed subset of $\mathcal{F}$ including all aggregate allocations $(I^\varepsilon, R^\varepsilon_\eta) \in \mathcal{F}$ such that $I^\varepsilon \geq \underline{I}$. The subset is represented in the dashed area in Figure OB1.

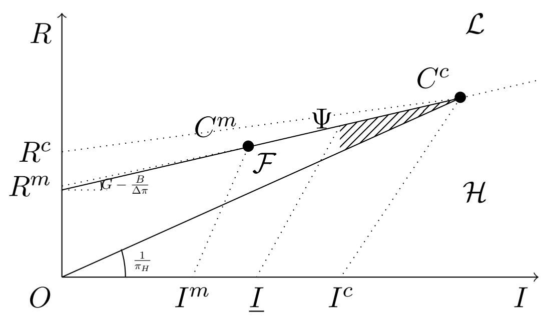

FIGURE OB1. SET OF AGGREGATE ALLOCATIONS $(I^\varepsilon, R^\varepsilon_\eta)$

Consider now any allocation $(I^\varepsilon, R^\varepsilon_\eta)$ such that $I^\varepsilon \geq \underline{I}$, and denote it (I^*, R^*). We show that it is supported at equilibrium by the following strategies, which are similar to those used in the proof of Proposition 3. Each investor $i = 1, 2, ..., N$ offers the same menu $M^* = \{(0,0), (I^*/N, R^*(.)), (\hat{I}/N, \hat{R}(.))\}$, with $R^*(I) = G(I + A)$ for $I \notin \{I^*, I^*/N\}$, and

$$R^*(I) = \begin{cases} \frac{R^*}{N} & \text{if } I = I^*, \\ R^* - \frac{N-1}{N} G I^* & \text{if } I = \frac{I^*}{N}. \end{cases} \quad \text{(OB5)}$$

The investment level $\hat{I}$ is such that

$$B(\hat{I} + A) - A = U(I^*, R^*, H) = U^c - \varepsilon I^c(\pi_H G - \eta), \quad \text{(OB6)}$$

which guarantees, given that $U(I^*, R^*, H) \geq B(I^* + A) - A$ whenever $(I^*, R^*) \in \mathcal{F}$, that $\hat{I} > I^* \geq 0$.

As in the proof of Proposition 3, one can check that choosing the investment I^*/N in each menu and selecting $e = H$ is an optimal choice for the entrepreneur. Consider now investors' deviations. Without loss of generality, any unilateral

[13] Indeed, $I^c(1 - \Delta\pi(G - 1/\pi_H)/B) = \Delta\pi(G - B/\Delta\pi)A/B = I^m$ and $\underline{\varepsilon} < \Delta\pi(G - 1/\pi_H)/B$.

deviation can be represented by a menu $M' = \{\big(I' + I^*/N, R'(.) + R^*/N\big), (0,0)\}$. A profitable deviation must necessarily induces $e = H$,[14] and one must have that $R'(I) > I'/\pi_H$, with I being the aggregate investment traded at the deviation stage, and that $I' \leq I^c - I^*/N$. If, following the deviation, the entrepreneur chooses $e = L$ and strategically defaults, then, given (OB6), she gets $U_{sd} = U(I^*, R^*, H) + B\big(I' + I^*/N - \hat{I}/N\big)$.

Assume first that the entrepreneur only trades with the deviating investor. In this case, choosing $e = H$ yields her the payoff

$$\text{(OB7)} \qquad \pi_H\big(G(\frac{I^*}{N} + I' + A) - \frac{R^*}{N} - R'(\frac{I^*}{N} + I')\big)^+ - A < \frac{U(0)(N-1) + U(I^*, R^*, H)}{N} + (\pi_H G - 1)I'.$$

Since the entrepreneur's equilibrium utility remains available at the deviation stage, the right-hand side of (OB7) must be strictly greater than $U(I^*, R^*, H)$, which implies that $I' > 0$. The entrepreneur therefore finds optimal to strategically default if

$$\text{(OB8)} \qquad \frac{N-1}{N}\big(U(I^*, R^*, H) - U(0)\big) - \frac{B}{N}(\hat{I} - I^*) \geq (\pi_H G - 1 - B)I'.$$

The left-hand side being increasing with N, a sufficient condition for (OB8) obtains with $N = 2$. That is, after rearranging: $B(I^* + A) - \pi_H GA \geq 2(\pi_H G - 1 - B)I'$ or $B(I^c + A) - \pi_H GA \geq B\varepsilon I^c + 2(\pi_H G - 1 - B)I'$. See that $B(I^c + A) - \pi_H GA = -\pi_L B(I^c + A)/\Delta\pi + (\pi_H G - 1)I^c$ which leads to the condition:

$$\text{(OB9)} \qquad \pi_H G - 1 \geq f(I', \varepsilon),$$

with $f(I', \varepsilon) = B\left(\pi_L(I^c + A)/(\Delta\pi I^c) + \varepsilon\right) + 2(\pi_H G - 1 - B)I'/I^c$. Given the linearity of f, we have that

$$\text{(OB10)} \qquad f(I', \varepsilon) = f\left(\frac{I'}{I^c}(I^c, 0) + (1 - \frac{I'}{I^c})\left(0, \frac{\varepsilon}{1 - \frac{I'}{I^c}}\right)\right)$$

$$\text{(OB11)} \qquad = \frac{I'}{I^c} f(I^c, 0) + (1 - \frac{I'}{I^c}) f\left(0, \frac{\varepsilon}{1 - \frac{I'}{I^c}}\right).$$

[14]Indeed, given Lemma A1, a deviating investor may achieve a positive profit by inducing $e = L$ only if the entrepreneur trades several contracts out of equilibrium and does not default. Given equilibrium menus, this is only possible if the entrepreneur invests $I = I^*$ and selects $e = L$; this in turn provides her a payoff smaller than the equilibrium one, which guarantees that this is not an optimal choice.

To prove (OB9), using $0 < I' \leq (N-1)I^c/N < I^c$, we simply need to show that

$$\pi_H G - 1 \geq f(I^c, 0), \tag{OB12}$$

$$\pi_H G - 1 \geq f\left(0, \frac{\varepsilon}{1-\frac{I'}{I^c}}\right). \tag{OB13}$$

Condition (OB12) is equivalent to $B\left(2 - \pi_L/(\pi_H(\Delta\pi G - B))\right) \geq \pi_H G - 1$, which holds since $B\left(2 - \pi_L/(\pi_H(\Delta\pi G - B))\right) > B\left(2 - \sqrt{\pi_L/\pi_H}/2\right) > B(2 - (\sqrt{2} - 1)/2) > B/(1-(3-2\sqrt{2})) > \pi_H B/\Delta\pi \geq \pi_H G - 1$. The first inequality comes from (OB2), the second and the fourth from (OB3) and the last one is (4). To prove (OB13), first remark that (OB9) holds for any couple $(0, \varepsilon)$ with $\varepsilon \leq N\underline{\varepsilon}$. Indeed, by definition of $\underline{\varepsilon}$, we have $\varepsilon \leq (\pi_H G - 1)/B - \pi_L/(\pi_H(\Delta\pi G - B))$ for any $\varepsilon \leq N\underline{\varepsilon}$. To complete the proof of (OB13), observe that, since $I' \leq (N-1)I^c/N$ and $\varepsilon \leq \underline{\varepsilon}$, we have $(\varepsilon)/(1 - I'/I^c) \leq N\underline{\varepsilon}/(N(1 - I'/I^c)) \leq N\underline{\varepsilon}/(N(1-(N-1)/N)) = N\underline{\varepsilon}$.

Assume next that several contracts are traded at the deviation stage. Going back to the proof of Proposition 3, following any unilateral deviation, the entrepreneur strategically defaults if (A30) holds. Again, the left-hand side of (A30) is increasing in k. A sufficient condition for (A30) is therefore, using (OB6) and $I^* = I^c(1-\varepsilon)$, $BI^c\left(1 - \pi_L/(\pi_H(\Delta\pi G - B))\right) \geq \varepsilon I^c(2B + 1 - \pi_H G)$, which is equivalent to $\varepsilon \leq \left(1 - \pi_L/(\pi_H(\Delta\pi G - B))\right) B/(2B + (1 - \pi_H G))$. The inequality holds by definition of $\underline{\varepsilon}$, which concludes the proof that (I^*, R^*) is supported at equilibrium. Corollary OB1 follows immediately from the fact that all conditions above hold for $N \geq 2$. ∎