Auction Design with Advised Bidders

Andrey Malenko*
MIT Sloan School of Management
Anton Tsoy
MIT, Department of Economics

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Abstract

This paper studies efficient and optimal auction design where bidders do not know their values and solicit advice from informed but biased advisors via a cheap-talk game. When advisors are biased toward overbidding, we characterize efficient equilibria of static auctions and equilibria of the English auction under the NITS condition (Chen, Kartik and Sobel (2008)). In static auctions, advisors transmit a coarsening of their information and a version of the revenue equivalence holds. In contrast, in the English auction, information is transmitted perfectly from types in the bottom of the distribution, and pooling happens only at the top. Under NITS, any equilibrium of the English auction dominates any efficient equilibrium of any static auction in terms of both efficiency and the seller's revenue. The distinguishing feature of the English auction is that information can be transmitted over time and bidders cannot submit bids below the current price of the auction. This results in a higher efficiency due to better information transmission and allows the seller to extract additional profits from the overbidding bias of advisors. When advisors are biased toward underbidding, there is an equilibrium of the Dutch auction that is more efficient than any efficient equilibrium of any static auction, however, it can bring lower expected revenue.

Keywords: auction design, cheap-talk, full revelation, English auction, communication.

*Malenko: amalenko@mit.edu. Tsoy: atsoy@mit.edu. We thank Nanin Kartik, Jean Tirole and participants of MIT Theory Lunch for comments and suggestions.

1 Introduction

In many M&A contests and auctions, potential buyers get advice from biased advisors about the value of the asset sold. For example, in M&A contests, bids are submitted and approved by the board of directors that is advised on the value of the target by the management team and investment bankers. The management is frequently prone to empire building and investment bankers get large fees for successfully closing the deal. Hence, they are more eager to acquire the asset than shareholders. Similarly, in FCC spectrum auctions, the research team preparing for the auction, and in particular producing estimates of the value of auctioned frequencies, has career concerns. Winning the auction gives a positive signal to the market and helps the research team to attract future business. The unifying theme of these examples is the misalignment of interests between the party that submits the bid (board of directors, upper management) and the party that has the information about the value of the asset (management team and investment bankers, research team).

This paper studies the auction design in the presence of this imperfect alignment of interests and answers several important questions: How does the choice of the auction format affect the informativeness of communication between bidders and advisors? What are the implications of different auction formats for allocation efficiency? Which auction format maximizes the expected revenue? Is there a trade-off between higher revenue and efficiency in the choice of the auction format?

We depart from the canonical auction model with independent private values in one aspect. Each bidder does not know her value and consults an informed advisor. The advisor is biased toward overbidding, that is, she overvalues the gains from winning the auction. The overbidding bias captures empire-building motives or career concerns in the examples above. The communication between the bidder and the advisor is modeled as a cheap-talk game. The advisor can send messages to the bidder at no cost at any stage of the auction. In this environment, there is an interesting interaction between the communication and the auction design. On the one hand, the amount of information transmitted from each advisor to her bidder affects bids submitted in an auction and through them the efficiency and revenue of each auction format. On the other hand, the auction format affects the incentives of advisors to reveal their information to their bidders.

We analyze equilibria of the model under *the NITS (for no incentive to separate) condition* adapted from Chen, Kartik and Sobel (2008). The NITS condition requires that the weakest type has the option of credibly revealing herself at any stage of the auction, which puts a lower bound on the utility that such type gets at any stage of the auction. Chen, Kartik and Sobel (2008) provides a number of justifications for NITS in general cheap-talk games including perturbations with nonstrategic players and costly lying and shows that NITS refines equilibria of the cheap-talk game which in general exhibit multiplicity. Notice that NITS need not select the unique equilibrium. Our analysis does not require any further refinement, and the NITS condition turns out to be the key property of equilibria for our efficiency and revenue comparison.

In this paper, we characterize *(bidder)-efficient equilibria* of static auctions in which the auction is won by a bidder with the highest expected value conditional on the information transmitted from the advisor. The outcome of the bidder-efficient equilibria need not be efficient from the point of view of advisors, as they may communicate only a crude information to bidders. We also characterize equilibria of the English auction satisfying NITS and show that the English auction dominates any static auction in terms of information transmission, efficiency, and revenue.

Communication in efficient equilibria of static auctions takes a partition form: all types of the advisor are partitioned into intervals and types in each interval induce the same bid. This result is in line with the communication in general cheap-talk games characterized by Crawford and Sobel (1982). Notice however that our model is not reduced to a particular case of the cheap-talk game in Crawford and Sobel (1982). In our model, the cheap-talk game between the bidder and her advisor is endogenous. The profitability of each bid depends on bidding strategies of opponents, which in turn depend on the equilibrium communication between opponent bidders and their advisors. For the same reason, the analysis of Chen, Kartik and Sobel (2008) does not immediately apply to show that there is an equilibrium of the static auction satisfying NITS. Despite the endogenous cheap-talk game, we show that efficient equilibria of the static auction still have a partition structure with an upper bound on the number of partition intervals, and that the most informative equilibrium satisfies the NITS condition.

For static auctions, a version of the revenue equivalence holds: modulo the existence of efficient equilibria in a particular static auction, all efficient equilibria of any static auction bring the same expected revenue and generate the same communication. In other words, in static auctions one cannot manipulate the rules of the auction to extract extra revenue or induce better communication and maintain bidder-efficiency. Hence, in static auctions, communication does not alter the celebrated revenue equivalence (Myerson (1981) and Riley and Samuelson (1981)).

Equilibria of the English under the NITS condition are quite different from equilibria in static auctions. The information transmission is perfect at the bottom of the type distribution and there is pooling at the top in a sense that types at the top induce the same bid. Because of the superior information transmission, the English auction generally outperforms any static auction both in terms of efficiency and expected revenue. In particular, the revenue equivalence does not hold between static and dynamic auctions.

The key distinction of the English auction is that the advisor can reveal information over time. Under the simplest communication protocol, the advisor reveals her information to the bidder right before the advisor's optimal quitting time. Under such a communication protocol, perfect information transmission is possible for types at the bottom of the distribution for the following reason. If the bidder observes her values, then it is optimal for bidders to quit the auction when the running price equals her value. Because of the overbidding bias, the advisor prefers to quit the auction later than the bidder. If the advisor perfectly reveals the value at her optimal quitting time, then it is optimal for the bidder to immediately quit the auction. Indeed, at this point the bidder is already past her break-even price and any further delay will result into a higher chance of winning at a price that brings negative profit. Because in the English auction the bidder is restricted to submit only bids higher than the current auction price, types at the bottom of the distribution are able to communicate their private information perfectly and induce the bidder to quit at their optimal price.

However, even in the English auction, information cannot be transmitted perfectly for all types when the support of the distribution of values is finite. As the price of the auction approaches the highest type, the uncertainty of the bidder about her value decreases. At some point, she can accurately predict her value as well as the fact that she will overpay for the asset if she wins, because the advisor waits until the advisor's optimal price to quit. Therefore, the bidder will always quit when her uncertainty is sufficiently reduced before types of the advisor at the top reveal themselves.

The information transmission affects the efficiency of auction formats. Because of imperfect information transmission, static auctions are necessarily inefficient, as ties occur with positive probability. At the same time, in the English auction the information transmission for types at the bottom of the distribution is perfect, hence allocation is more efficient for these types compared to static auctions. It turns out that even taking into account the pooling at the top, the English auction is still more efficient than any static auction, as no static auction makes these types at the top separate even partially.

The information transmission also affects the revenue of auction formats. Under the monotone hazard ratio, the expected revenue of the auction is higher in the English auction than in any static auction format. Hence, when advisors are biased toward overbidding, there is no trade-off between revenue and efficiency. This fact has an important practical implication. While in M&A contests, the expected revenue is the key objective of the seller, in FCC auctions and other government auctions, efficiency is the primary goal, but maximizing the revenue is also a desirable goal. Moreover, the bias for overbidding is relevant in many applications, because of the empire-building and career concerns described above. Our results suggest that the English auction is the preferred method of selling assets in this environment no matter whether the seller is concerned about efficiency, revenue or both.

The intuition for the higher revenue comes from the fact that the seller would prefer to sell directly to advisors, as they have a higher willingness to pay for the asset. However, because bidders are in control of bidding and advisors can only affect them through the information they provide, the equilibrium bids reflect a mix of interests of bidders and advisors, and so, are lower. The English auction is an auction format that allows the seller to essentially eliminate bidders and sell directly to advisors, as bids are optimal for advisors.

Perhaps surprisingly, under the bias toward underbidding, the comparison of dynamic and static auctions is ambiguous. In this case, if the bidder knew her value, she would submit a bid that wins with higher probability than an optimal bid of the advisor. Hence, with bias toward underbidding, the Dutch auction that restricts bidders to submit bids not higher than the current price of the auction allows for a better information transmission. We construct an equilibrium of the Dutch auction that exhibits pooling at the bottom and perfect information transmission at the top of the distribution. This equilibrium is more efficient than any efficient equilibrium of any static auction, but it can bring lower expected revenue to the seller. The reason for this is that when the advisor is biased toward underbidding, selling directly to advisors no longer guarantees the highest expected revenue, as advisors have lower willingness to pay. Because of that, it is possible that the seller benefits from imperfect communication between the advisor and the bidder, as it results into an upward bias of bids relative to the bids submitted directly by advisors.

Literature Review This paper is related to the literature on cheap-talk, information acquisition in auctions, and comparison of auction format.

This paper is related to the literature on auctions with information acquisition in which bidders learn additional information about the values in the process of bidding. This literature focuses on either exogenous or endogenous information acquisition. With exogenous information acquisition, the information is revealed to bidders by some exogenous process (refs??). With endogenous information acquisition, bidders optimally choose how much to invest into the information acquisition. The information acquisition technology is fixed in this case. In contrast, in our paper, the information acquisition technology is endogenous: the choice of the auction format, shapes the incentives of advisors to transmit information to bidders. To the best of our knowledge, our paper is the first to study the auction design with endogenous technology of information acquisition.

Compte and Jehiel (2007) shows that the English auction brings higher revenue than static auctions when bidders can acquire information about their values. The key conditions for this result are an asymmetry of bidders, i.e. some bidders may be initially informed, a sufficiently large number of bidders, and the availability of the information about the number of bidders. Under these conditions, as bidders drop out of the English auction, information acquisition may becomes profitable even when it is not profitable in the beginning of the auction. Hence, the English auction brings higher revenue because it induces more information acquisition, and makes uninformed bidders stay longer pushing up the price paid by informed bidders.[1] The results of our paper also highlight the possibility of the communication over the course of the auction as the key factor in improving the performance of the auction, but the mechanism is quite different. In particular, our results hold for any number of bidders that do not know their values initially irrespective of whether the number of remaining bidders is observable or not. Given that the average number of participants in M&A contests is relatively small, .

There is an extensive literature on cheap-talk game started by Crawford and Sobel (1982) (see Sobel (2010) for review). This literature models in a reduced form the utility of parties from the decision made by the bidder. This paper considers the cheap-talk game in which utilities of advisors and bidders from bids are derived from the auction game. In this respect, our paper is related to Grenadier, Malenko and Malenko (2015) who study a cheap-talk game in which payoffs are derived from exercising a real option.

[1]Compte and Jehiel (2004) studies auction efficiency in this environment. Rezende (2005) shows that under different information acquisition technology, even in secret auctions, dynamic format bring higher revenues than static formats.

The main difference from their model is that they study a single-advisor decision problem (exercise of the option), while this paper studies how communication affects bidding behavior in auctions which is a game. In this respect, the cheap-talk game is endogenous in this paper, as the communication between opponent bidders and their advisors affects the profitability of different bids in the auction.

There are several papers that study dynamic aspects of communication. Aumann and Hart (2003) and Krishna and Morgan (2004) show that multiple rounds of communication can attain a better information transmission. Their results rely on the finite state space, infinite exchange of messages and simultaneous actions of both parties. Golosov, Skreta, Tsyvinski, and Wilson (2014) show that with repeated actions a perfect information transmission is possible even with an infinite state space and finite horizon.[2] Sobel (1985), Morris (2001), Ottaviani and Sorensen (2006a, 2006b) study the role of reputation in communication. In our paper, we provide a novel mechanism of the partially perfect information transmission when receivers' actions are bids in an auction.

Our paper is somewhat related to partial and full separation in cheap-talk with lying costs studied by Kartik (2009) and Kartik, Ottaviani, and Squintani (2007). In order to ensure a partially separation, one of the sides needs a commitment not to change her action (inflate message or choose lower action). With lying costs, such commitment is on the side of the sender. Costly lying ensures that the sender does not inflate her message too much. As a result, the sender can perfectly reveal her type. In our paper, such commitment is on the sider of the receiver. In the English auction, the bidder has a commitment not to decrease her bid below the current running price, which in turn, ensures that the advisor reveals her information truthfully. Because of the commitment being on different sides, in Kartik (2009) and Kartik, Ottaviani, and Squintani (2007), the receiver chooses her optimal action, while in our paper, the sender induces her optimal message. Similarly to their work, the full separation is possible when the type space is unbounded, while there is necessarily pooling at the top when state space is bounded. However, the mechanism is quite different. In their work, the sender runs out of messages at the top, which leads to pooling, while in our paper it is the receiver (bidder) who stops listening to the advisor once her uncertainty is sufficiently reduced. In particular, if the bidder were to deviate and continue following the advisor's recommendation, the advisor

[2] Ivanov (2015) shows that perfect information transmission is possible with two rounds of communication, when the uninformed party can control the signal of the informed party.

would continue waiting until her optimal price.[3]

Another related paper is Burkett (2014) that studies a principal-agent relationship in the auction context where the principal decides on the budget of the biased agent who submits the bid. Similarly to Burkett (2014), we study the interaction between the principal-agent relationship and the auction design. Our revenue-equivalence result in static auctions is related to his result of equivalence between the first- and second- price auctions with endogenous budget constraints. Kos (2012) studies efficient and optimal auction design when bidders can use only a finite set of messages to communicate with the seller. Our model is effectively a mechanism design problem of allocating to informed advisors with a restriction on the set of mechanisms arising from the fact that bids are submitted by biased bidders. In static auctions, the restriction on the message space arises endogenously from the communication between the bidder and the advisor. Inderst and Ottaviani (2013) studies the interaction between mechanism design and communication. In their model, by committing to a return policy, the seller can credibly convey information about the value of the product to the buyer.

Our paper is related to the auction literature that shows that the revenue-equivalence result of Myerson (1981) can fail for a number of reasons: correlation of values (Milgrom and Weber (1982)), risk-aversion (Holt (1980)), asymmetry of bidders (Maskin and Riley (2000)), budget constraints (Che and Gale (1998)).[4] Milgrom and Weber (1982) shows that in the model with affiliated values, the English auction brings higher revenue than the first- and second-price auctions. We provide a novel explanation for why even strategically equivalent auctions, such as the English auction and the second-price auction or the Dutch auction and the first-price auction, can bring different revenue.

The structure of the paper is the following. Section 2 introduces the model and illustrates our main findings with a simple example. Section 3 characterizes equilibria of static auctions and establishes a version of the revenue equivalence for static auctions. Section 4 characterizes equilibria of the English auction under the NITS condition when bidders have overbidding bias, and shows that the English auction outperforms any static auction. Section 6 analyzes the case of advisors' preferences for underbidding. Section 7 concludes. Key proofs are provided in the text and the rest

[3]A number of papers obtains a perfect information transmission in static model, but require multiple senders (Battaglini (2002), Eso and Fong (2008), and Ambrus and Lu (2010), Rubanov (2015)), or certifiable information (Mathis (2008)), which represent another form of commitment on the sender's side.

[4]See Milgrom (2004) for a survey.

of the proofs are relegated to Appendix.

2 Model

Consider the standard auction model with independent private values. There is a single indivisible asset for sale. There are N ex-ante identical bidders. The valuation of bidder i, v_i, is an i.i.d. draw from distribution with c.d.f. F and p.d.f. f. The distribution F has full support on $[\underline{v}, \overline{v}]$ with $0 \leq \underline{v} < \overline{v} \leq \infty$. In the analysis, we will frequently refer to the distribution of the value of the strongest opponent of a bidder. We denote by $\hat{v}$ the maximum of $N-1$ i.i.d. random variables distributed according to F and its c.d.f. by G.[5] The seller's value is common knowledge and is below $\underline{v}$. It is normalized to zero.

The novelty of our approach is that each bidder i does not observe v_i directly, but can consult advisor i. Advisor i knows v_i, but is biased relative to the bidder. Specifically, the payoffs from acquiring the asset are

$$\begin{aligned} \text{Bidder} &\ : \quad v_i - p, & (1) \\ \text{Advisor} &\ : \quad v_i + b - p, & (2) \end{aligned}$$

where b is the bias of the advisor. The value that all players get from not acquiring the asset is zero. Bias b is commonly known.[6] Our main focus is on the case of the preference of advisors for overbidding, $b > 0$, as it is most prominent in applications. Later, we will also consider the case $b < 0$ which shares several similarities with $b > 0$, but also differs from it in a number of important aspects.

Our formulation (1)−(2) captures the empire building motives and career concerns described in the introduction. For example, in the M&A contest, suppose that the CEO is compensated by a share in the profit α and has private benefits B from managing a larger company. Then the payoff of the CEO is $\alpha(v_i - p) + B$ and the shareholders retain $(1-\alpha)(v_i - p)$. Normalizing payoffs by α and $1-\alpha$, respectively, and denoting $b = \frac{B}{\alpha}$ we obtain formulation (1) − (2).

In this paper, we compare how different auction formats affect the seller's expected revenue and the allocative efficiency. Several auction formats are commonly used in practice and extensively studied in the academic liter-

[5] That is, $G(\hat{v}) = F(\hat{v})^{N-1}$.

[6] For most of our results it is sufficient to assume that b is commonly known by bidders and advisors, but the seller knows only the sign of the bias.

ature. In all of these auction formats, if ties occur, the winner is drawn randomly from the set of tied bidders.

1. **Second-price auction**. All bidders submit bids simultaneously and the bidder submitting the highest bid wins the auction and pays the second-highest bid.

2. **First-price auction**. All bidders submit bids simultaneously and the bidder submitting the highest bid wins the auction and pays her bid.

3. **English auction**. The seller continuously increases the price p, which we refer to as the *running price,* starting from $\underline{v}$. Bidders choose at what price to *quit* the auction. Once a bidder quits, she cannot re-enter the auction. The bidder that quits last gets allocated the asset at a price at which the last of her opponents quit the auction.

4. **Dutch auction**. The seller continuously decreases the price p, which we refer to as the *running price,* starting from $\overline{v}$. Bidders choose the time at which they *stop* the auction. The first bidder who stops the auction wins the assets at the price at the current running price.

We study a rich class of static auctions formally described in Section 3, but restrict attention to the English and Dutch auctions among dynamic auctions.

The communication between the bidder and the advisor is modeled as the cheap-talk game. If the auction consists of a single-round bidding (e.g., a first-price or a second-price auction), the timing of the game is as follows:

1. Advisor i sends a message $\tilde{m}_i \in M$ to bidder i where M is some infinite set. The message $\tilde{m}_i$ is not observed by anyone except bidder i.

2. Having observed message $\tilde{m}_i$, bidder i submits bid b_i.

3. Given all bids b_1, ..., b_N, the asset is allocated and payments are made according to the rule specified by the auction.

If the auction consists of multiple stages of bidding, the advisor sends a message to the bidder before each stage of bidding. In dynamic auctions, we index stages of the auction by corresponding running prices p.

A *private history* at stage p consists of all actions taken by bidders and the seller, and all messages sent by the advisor in the previous bidding stages. In static auctions, there is only an empty private history. A private history in the English auction consists of the set of bidders remaining in the game, the current running price of the auction and messages sent by the advisor up to stage p. A private history in the Dutch auction consists of simply the current running price of the auction and messages sent by the advisor up to stage p. Denote by $\mathcal{H}$ the set of all histories.

We focus on pure strategies in dynamic auctions, but allow for mixing by bidders in static auctions. A strategy of the advisor i is a measurable mapping $m_i : [\underline{v}, \overline{v}] \times \mathcal{H} \to M$ from the type of advisor and histories into a message sent after each history. A strategy of the bidder is a mapping $a_i : \mathcal{H} \times M \to A$ from the history and current message into the action chosen by the bidder. In static auctions, A consists of all possible mixtures over bids. In the English/Dutch auction, $A = \{0, 1\}$ consists of a decision to quit/stop the auction or continue. Posterior belief $\mu_i : \mathcal{H} \times M \to \Delta([\underline{v}, \overline{v}]))^N$ is a measurable mapping from histories and current messages into the posterior distribution over types of all advisors. We denote by μ_{ij} the posterior belief of bidder i about the value of bidder j. For $i \neq j$, μ_{ij} also represents posterior belief of advisor i about bidder j's value.

When the running price in the English auction changes continuously, the outcome of the auction may be indeterminate, which is a common problem of formulating games in continuous time (see Simon and Stinchcombe (1989)). For example, consider the following strategy profile. Each advisor sends message "quit" when $p = v + b$ and each bidder quits immediately after she receives message "quit". If there is no such message and there are $N - 1$ bidders remaining in the game, the bidder does not wait for a recommendation and quits immediately, but she continues to wait for the advisor's recommendation for any other number of remaining bidders. Take any two bidders i and i' and consider an outcome in which bidders i and i' quit at some price p and other bidders remain in the auction. Such outcome is consistent with our strategy profile for any choice of i and i' which leads to indeterminacy when $N \geq 3$. Intuitively, once a first bidder quits, there is an indeterminacy about who will be the second bidder to quit.

To circumvent this indeterminacy, we focus on equilibria in *stationary* strategies in the English auction, i.e. strategies that condition only on the running price and messages for bidders and types for advisors, but not on the number of remaining bidders. When all players follow stationary strategies and at most one player deviates, the outcome of the auction is determined unambiguously. While stationary strategies are very restrictive in many environments, such as auctions with interdependent values, with independent values, the number of remaining bidders is not informative about the bidder's value and can only affect her payoff through the distribution over auction outcomes. As we will show further, in many instances, this effect of the number of bidders (and hence, the way we resolve the indeterminacy) is not important for the equilibrium analysis, so the restriction to stationary equilibria is without loss of generality.

Generally, an auction format is a mapping from bidders' actions into the probability of allocation for each bidder and the transfer from each bidder: $\eta : A^N \to [0,1]^N \times \mathbb{R}^N$. We refer to the distribution over probabilities of allocation and transfers given auction format η and strategies $m_i, i \in N$ and $a_i, i \in N$ as the *outcome* of strategies $m_i, i \in N$ and $a_i, i \in N$ in auction η. The equilibrium concept is the perfect Bayesian equilibrium defined as follows.

Definition 1. *Stationary strategies $m_i, i \in N$ and $a_i, i \in N$ and beliefs $\mu_i, i \in N$ constitute a* stationary equilibrium*, if and only if the following hold:*

1. *strategies are rational given players' beliefs at any history;*
2. *beliefs are updated by Bayes rule whenever possible;*
3. *beliefs of bidders i and i' about bidder $j \neq i, i'$ coincide at any history;*
4. *beliefs of bidder i about bidders j and j' are independent at every history;*
5. *beliefs of bidders do not change after their own actions.*

Conditions 1-5 are standard conditions of the perfect Bayesian equilibrium (see Fudenberg and Tirole (1991)). Since all bidders are symmetric, we focus on symmetric equilibria in which strategies m_i and a_i do not depend on i and we suppress index i in the notation. We write μ^p for a posterior belief of the bidder about her value at stage p.[7]

We restrict attention to equilibria in which the advisor gives a real-time action recommendation to the bidder defined as follows.

Definition 2. *An equilibrium in the dynamic auction is* in online strategies *if $m : [\underline{v}, \overline{v}] \times \mathcal{H} \to A$ and $a(h, \tilde{a}) = \tilde{a}$ for all $h \in \mathcal{H}$ and all $\tilde{a}$ in the image of $m(\cdot, h)$.*

We want to stress that action recommendations in online strategies happen in real time. In particular, the strategy $m : [\underline{v}, \overline{v}] \to A^{\mathcal{H}}$ in which the advisor makes her recommendation in the beginning of the game is not an online strategy. The following lemma states that it is without loss of generality to consider equilibria in online strategies.

[7] Corresponding history is omitted from the notation.

Lemma 1. *For any equilibrium of the static/dynamic auction there is another outcome-equivalent equilibrium in online strategies.*

Lemma 1 states that any equilibrium in which the advisor gradually reveals information over time is outcome-equivalent to an equilibrium in which all relevant information is revealed right before the bidder takes an action. The proof builds on a simple fact that if the action is optimal for a decision maker in every state of the world, then it is an optimal action without conditioning on the state. For any history h, let $\chi(h)$ be the history which contains only actions of bidders and the seller, but does not contain any messages of the bidder. For any equilibrium strategies m and a, we specify corresponding online strategies, in which the advisor makes an action recommendation $a(h, m(v, h))$ after any history $\chi(h)$. Under online strategies, the bidder has cruder information about her value, as she can extract additional information about her value from the history of messages and the current message. However, even given this additional information, she optimally chooses action $a(h, m(v, h))$. Hence, it is optimal for her to choose action $a(h, m(v, h))$ even without knowing a particular history of messages and current message. Moreover, the advisor can induce the same set of actions using online strategies as in the original equilibrium and so, she does not have incentives to deviate from new strategies.[8]

Equilibrium Refinement There is in general a multiplicity of equilibria both in cheap-talk and auction games. We introduce an equilibrium refinement in order to make meaningful comparison of auction formats. Without such a refinement, one can immediately see that completely uninformative messages (m does not vary with v) are always consistent with an equilibrium irrespective of the auction format.

First, we assume that bidders play weakly dominant strategies if such strategies exist. This is a standard refinement in the auction literature and it guarantees in particular, that in the second-price auction bidders submit their expected values.

Second, we impose the NITS (no incentive to separate) condition adapted from Chen, Kartik, and Sobel (2008). Define the *weakest type* v_w of advisor as follows:

$$v_w = \begin{cases} \underline{v} & \text{for } b > 0, \\ \overline{v} & \text{for } b < 0. \end{cases}$$

[8]In Appendix, we specify online strategies off-path that guarantee that the advisor cannot induce more actions.

If the bidder asked value v directly and thought that the advisor tells the truth, then when $b > 0$ all types of the advisor would tell a value higher than v to induce a higher probability of winning. In particular, nobody would prefer to tell $\underline{v}$. Symmetrically, when $b < 0$, all types of the advisor would bias their reports downwards and nobody would prefer to tell $\overline{v}$. In this sense, it is natural to think of type v_w as weak.

Similarly, we can define the weakest types at every stage of the game. Let v_w^p be the weakest type of advisor on the equilibrium path at stage p:

$$v_w^p = \begin{cases} \inf\{v|v \in supp(\mu^p)\} & \text{for } b > 0, \\ \sup\{v|v \in supp(\mu^p)\} & \text{for } b < 0. \end{cases}$$

For $b > 0$, v_w^p is the lowest type remaining in the game at stage p, and for $b < 0$, v_w^p is the highest type remaining in the game at stage p.

Chen, Kartik and Sobel (2008) introduce the NITS condition in cheap-talk games that requires that weak types can always separate themselves from the rest of the types, and hence, in equilibrium they should get a payoff no lower than the utility they receive from such separation. We additionally require that this condition holds at every stage of the game.

Definition 3. *An equilibrium satisfies* the NITS condition *if for any p, type v_w^p of the advisor weakly prefers her equilibrium strategy to the action optimally chosen by the bidder at stage p who knows that her value is v_w^p.*

Chen, Kartik and Sobel (2008) shows that NITS can be justified by perturbations of the cheap-talk game with non-strategic players and costs of lying.

We refer to an equilibrium of the static auction as *the most informative* if it induces the largest number of actions.[9] Call an equilibrium of any auction *babbling,* if it induces a single action. We show that the most informative equilibrium satisfies NITS and NITS may rule out the babbling equilibrium. However, NITS need not select a unique equilibrium. For our results, the NITS condition is the key property of equilibria and we do not require a selection beyond NITS.

2.1 Example

We next illustrate main results of the paper and intuition for them with a simple example. Suppose that advisors are biased toward overbidding, i.e.

[9]There is a slight abuse of terminology here, as it is not guaranteed that there is a unique equilibrium with the largest number of actions. Our results hold for any selection among these equilibria.

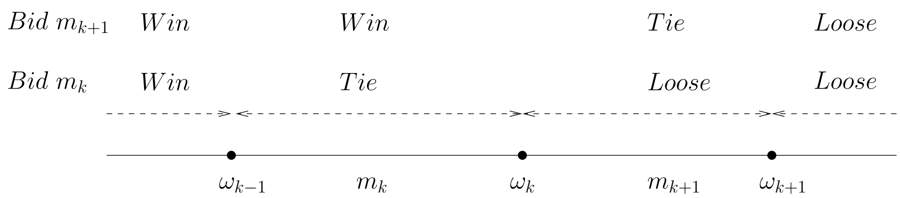


Figure 1: Thresholds in the partition equilibrium of the second-price auctions. Type ω_k is indifferent between pooling with types in $[\omega_{k-1}, \omega_k)$ by bidding m_k and types in $[\omega_k, \omega_{k+1})$ by bidding m_{k+1}. We depict types against which strategies m_k and m_{k+1} win, loose or tie. The difference between strategies m_k and m_{k+1} is that m_{k+1} wins for sure against types in $[\omega_{k-1}, \omega_k)$ and ties against types in $[\omega_{k-1}, \omega_k)$, while m_k ties with types in $[\omega_{k-1}, \omega_k)$ and looses against types in $[\omega_{k-1}, \omega_k)$.

$b > 0$. There are two bidders ($N = 2$) and F is exponential with parameter λ, i.e. $F(v) = 1 - e^{-\lambda v}$ for all $v \geq 0$. As we will see the exponential distribution is in some sense a knife-edge case.

We start with the characterization of the most informative equilibrium of the second-price auction that will be shown later to always satisfy the NITS condition. The unique equilibrium in weakly dominant strategies in the second-price auction is that bidder i submits the bid equal to her expected value. Hence, by Lemma 1 we can restrict attention to messages which tell bidders their expected values, i.e. $m = \mathbb{E}[v|m]$. The following propositions characterizes the most informative equilibria of the second-price auction.

Proposition 1. *Under F exponential and $b > 0$, the following strategies constitute the most informative equilibrium of the second-price auction. There exists a sequence $(\omega_k)_{k=0}^K$ with $\omega_0 = 0$ and $K < \infty$ such that*

- *for all $k = 1, \ldots, K < \infty$, the advisor with type $v \in [\omega_{k-1}, \omega_k)$ sends message $m_k = \mathbb{E}[v|v \in [\omega_{k-1}, \omega_k)]$;*
- *the bidder submits a bid equal to the message received.*

For $\frac{1}{\lambda} > b > 0$, $(\omega_k)_{k=0}^K$ are given by the following recursion

$$\frac{\omega_{k-1}e^{-\lambda\omega_{k-1}} - \omega_{k+1}e^{-\lambda\omega_{k+1}}}{e^{-\lambda\omega_{k-1}} - e^{-\lambda\omega_{k+1}}} = \omega_k + b - \frac{1}{\lambda}$$

with the terminal condition $\omega_{K+1} = \infty$ where K is the maximal length of recursion possible so that $\omega_1 > 0$. For $b \geq \frac{1}{\lambda}$, there is only a babbling equilibrium, i.e. $\omega_1 = \infty$.

Proof. Theorem 1 shows generally that advisor's strategy in the static auction takes a partition form as described in the proposition. Here, we simply derive this strategy. Given the exponential assumption, we can compute messages m_k explicitly as functions of thresholds ω_{k-1} and ω_k

$$m_k = \mathbb{E}\left[v|v \in [\omega_{k-1}, \omega_k)\right] = \frac{1}{\lambda} + \frac{\omega_{k-1}e^{-\lambda\omega_{k-1}} - \omega_k e^{-\lambda\omega_k}}{e^{-\lambda\omega_{k-1}} - e^{-\lambda\omega_k}}. \tag{3}$$

Threshold types ω_k of advisor should be indifferent between sending messages m_k and m_{k+1}. The probability of winning against types below ω_{k-1} and corresponding prices paid are the same for messages m_k and m_{k+1} (see Figure 1) The only difference is when the opponent's type is above ω_{k-1}. When the advisor sends message m_k, she may tie with the opponent sending message m_k. When the advisor sends message m_{k+1}, she wins against bid m_k for sure, but can tie with an opponent submitting m_{k+1}. For indifference of type ω_k, the benefits of strategy m_{k+1} from increasing the chance of winning against types in $[\omega_{k-1}, \omega_k)$ from $\frac{1}{2}$ to 1 and paying m_k should be equal to the costs from paying higher price m_{k+1} in case the opponent's type is in $[\omega_{k-1}, \omega_k)$ and the bidder wins the tie:

$$\frac{1}{2}\mathbb{P}(v \in [\omega_{k-1}, \omega_k))(\omega_k + b - m_k) = -\frac{1}{2}\mathbb{P}(v \in [\omega_k, \omega_{k+1}))(\omega_k + b - m_{k+1}), \tag{4}$$

which implies the following equation

$$-\frac{1}{2}\left(e^{-\lambda\omega_{k-1}} - e^{-\lambda\omega_k}\right)(\omega_k + b - m_k) = \frac{1}{2}\left(e^{-\lambda\omega_k} - e^{-\lambda\omega_{k+1}}\right)(\omega_k + b - m_{k+1}).$$

Together with (3) we get the following recursive equation

$$\frac{\omega_{k-1}e^{-\lambda\omega_{k-1}} - \omega_{k+1}e^{-\lambda\omega_{k+1}}}{e^{-\lambda\omega_{k-1}} - e^{-\lambda\omega_{k+1}}} = \omega_k + b - \frac{1}{\lambda},$$

where $\omega_0 = 0$ and ω_k is an increasing sequence. Whenever $b \geq \frac{1}{\lambda}$, there is no solution to the recursion. In this case, the unique equilibrium is a babbling equilibrium.

Suppose $b < \frac{1}{\lambda}$. Denoting by $x_k \equiv \omega_k - \omega_{k-1}$, we can rewrite the recursion in terms of x_{k+1} and x_k as follows

$$x_{k+1} + x_k = \left(\frac{1}{\lambda} - b - x_k\right)\left(e^{\lambda(x_k + x_{k+1})} - 1\right). \tag{5}$$

The requirement that ω_k is increasing translates into x_k being positive.

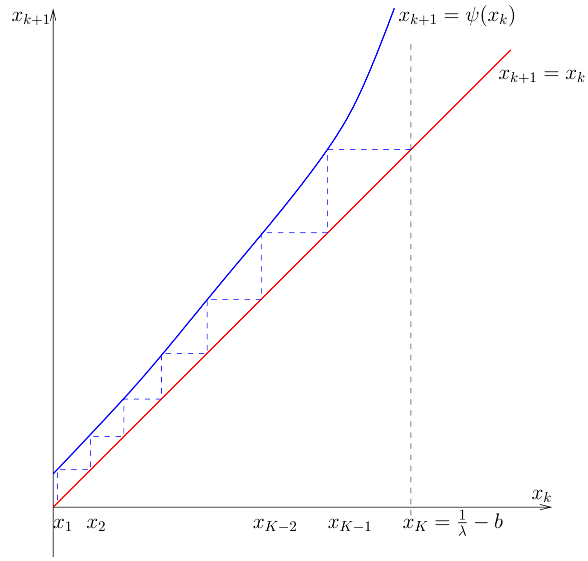


Figure 2: Recursion $x_{k+1} = \psi(x_k)$ for $\frac{1}{\lambda} > b > 0$.

Let ψ be the function that for any x_k gives a value of $x_{k+1} = \psi(x_k)$ such that x_k and x_{k+1} satisfy recursion (5). In Appendix, we show that function ψ is well defined, and verify that $f'(x) \geq 1$ for $x \in \left(0, \frac{1}{\lambda} - b\right]$, $\lim_{x \to 0} \psi(x) > 0$, and $\lim_{x \to \frac{1}{\lambda} - b} \psi(x) = \infty$. Therefore, $\psi(\cdot)$ is strictly above the diagonal line for $b > 0$. The graph of f for the case $\frac{1}{\lambda} > b > 0$ is depicted in Figure 2.

It is necessary that $x_{K+1} = \infty$ for some K (otherwise, eventually x_k becomes negative) and we can construct any equilibrium working backwards: $x_K = \frac{1}{\lambda} - b$, $x_{K-1} = \psi^{-1}(x_K)$ and so on until we reach x_1. Then the most informative equilibrium corresponds to K such that $\psi^{-1}(x_1) \leq 0$. Since $\lim_{x \to 0} \psi(x) > 0$, K is finite. From sequence $(x_k)_{k=1}^{K}$ we reconstruct threshold types $\omega_k = \omega_{k-1} + x_k$ and $\omega_0 = 0$, which completes the derivation.

□

In the next section, we show that all equilibria have the partition form and in fact recursion (5) allows us to construct any such equilibrium in our example.

We next characterize equilibria of the English auction and show that communication is perfect when $0 < b \leq \frac{1}{\lambda}$.

Proposition 2. *Suppose* $0 < b \neq \frac{1}{\lambda}$. *There the unique stationary equilibrium of the English auction satisfying NITS condition is characterized as follows:*

- *equilibrium is fully informative when* $b \in \left[0, \frac{1}{\lambda}\right)$ *and the advisor of type* v *sends message "quit" when* $p = v + b$ *and the bidder follows the recommendation of the advisor;*
- *babbling when* $b \in \left(\frac{1}{\lambda}, \infty\right)$ *and the bidder ignores messages from the advisor and quits when the running price reaches* $\frac{1}{\lambda}$.

When $b = \frac{1}{\lambda}$, *there is a continuum of equilibria indexed by* $v^* \in [0, \infty]$ *in which the advisor of type* v *sends message "quit" when* $p = v + b$ *and the bidder follows the recommendation of the advisor until the running price reaches* $v^* + b$.

Proof. If the bidder follows the recommendation, then the strategy to quit when $p = v + b$ is optimal for the advisor, as it is a weakly dominant strategy in the English auction where the advisor decides when to quit. When the bidder gets message "quit", $p > v$. Since p is increasing over time, it is optimal for the bidder to quit immediately. To finish the proof, we show that the bidder does not want to quit earlier. Let $v_p \equiv p - b$ for all p. The expected utility of the bidder i at time t from following the recommendation of the advisor is

$$\begin{aligned} V(v_p) &= \mathbb{E}\left[(v - \hat{v} - b)1\{v > \hat{v}\}|v, \hat{v} > v_p\right] \\ &= \frac{1}{2}\left(\mathbb{E}\left[\max\{v, \hat{v}\}|v, \hat{v} > v_p\right] - \mathbb{E}\left[\min\{v, \hat{v}\}|v, \hat{v} > v_p\right] - b\right) \\ &= \frac{1}{2}\left(\mathbb{E}\left[\max\{v, \hat{v}\}\right] - \mathbb{E}\left[\min\{v, \hat{v}\}\right] - b\right) \\ &= \frac{1}{2}\left(\frac{1}{\lambda} - b\right), \end{aligned}$$

where the first equality is by the symmetry of the auction, the second equality is by the memoryless property of the exponential distribution, and the last equality is by $\mathbb{E}\left[\max\{v_i, v_j\}\right] = \frac{3}{2\lambda}$ and $\mathbb{E}\left[\min\{v_i, v_j\}\right] = \frac{1}{2\lambda}$. Hence, when $0 < b \leq \frac{1}{\lambda}$, the bidder prefers to wait for a recommendation from the advisor rather than quit earlier. □

The dynamic communication strategy of the advisor attains perfect information transmission in the English auction. When the bidder learns v, she prefers a lower bid than the advisor. However, in the English auction there is a lower bound on bids equal to the running price that increases over time. This way if the advisor reveals v late in the auction, then she can ensure that the bidder will submit advisor's optimal bid. This is not possible in the second-price auction where the advisor optimally adds noise to her message.

When the support of values is finite, there is necessarily a pooling of types at the top. At some stage, the uncertainty about the value is sufficiently reduced and the bidder learns that she will likely overpay for the asset if she wins. Hence, the bidder prefers to quit immediately and does not give the advisor the opportunity to perfectly reveal her type in subsequent stages. This possibility is not present in our example because of the memory-less property of the exponential distribution.[10]

One can immediately see that the equilibrium in Proposition 2 satisfies NITS. At any stage p, the lowest type of the advisor sends the recommendation to quit. This is also an optimal action of the bidder who has the lowest value at stage p.

In general, the equilibrium of the second-price auction also constitutes an equilibrium of the English auction. We can specify that all communication happens at the initial bidding stage where bidders learn their values m_k. However, this equilibrium does not satisfy the NITS condition in the English auction. Indeed, consider stage $p = m_{k+1}$. The lowest remaining type ω_k satisfies (4) and so, gets a negative utility from remaining in the auction, while she gets zero if she persuades the bidder that her type is ω_k and the bidder optimally quits immediately. As we will show later this is a general phenomenon. All equilibria of static auctions have a partition structure, while all equilibria of the English auction have at least partial perfect information transmission.

We next show that the constructed equilibrium of the English auction dominates the equilibrium of the second-price auction in terms of efficiency and revenue. First, the English auction is efficient for $b \in \left(0, \frac{1}{\lambda}\right]$. In the second-price auction, since the communication is imperfect, ties arise in the auction with positive probability and lead to inefficient allocation. Observe that for $b = \frac{1}{\lambda}$, there is an equilibrium of the English auction with perfect communication, but all equilibria are babbling equilibria in the second-price auction. In Section 4, we show that this is a general phenomenon and informative communication can be sustained for a larger set of b in the

[10]Later, we will show that this is also the case for Pareto distribution. In general, one needs a fat tail of the distribution to guarantee that at every stage the probability of high values relative to the current running price is high.

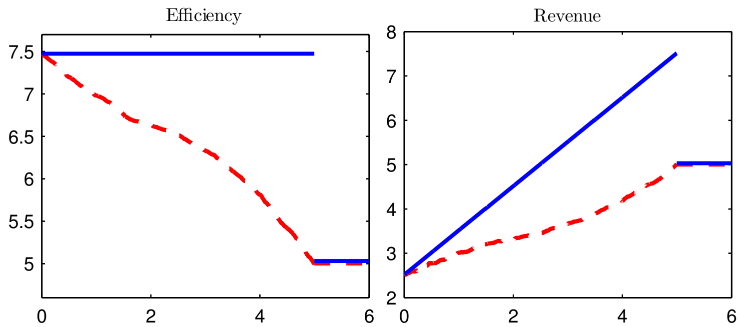


Figure 3: Efficiency and revenue comparison for $b > 0$: the English auction (solid line) and the second-price auction (dashed line). Bias b is plotted on the horizontal axis.

English auction. We also give an example of Pareto distribution, for which for a non-degenerate interval of b's all equilibria of the second-price auction are babbling, while there is an equilibrium with perfect communication in the English auction.

To explore these effect quantitatively let $\lambda = \frac{1}{5}$. The left panel of Figure 3 depicts the expected value of the winner in the auction which reflects the efficiency of different auction formats. Because of the perfect information transmission, the English auction is efficient for $b \leq \frac{1}{\lambda}$ and so its efficiency does not vary with b on this interval. The gap in efficiency between the English auction and the second-price auction increases as b increases up to $\frac{1}{\lambda}$. This happens because communication in the second-price auction becomes less and less informative as the bias increases.

To compare the revenue, we can use Myerson (1981), to write the revenue from different auction formats as follows

$$2\left(\mathbb{E}[\varphi(v)(1\{m(v) > m(\hat{v})\} + \frac{1}{2}1\{m(v) = m(\hat{v})\})] - U_A(0)\right) \qquad (6)$$

where $\varphi(v) = v + b - \frac{1}{\lambda}$ is the virtual valuation of advisor with value v, $U_A(0)$ is the expected utility of the advisor with type 0 and the suppress the notation for history in strategy m. We claim that (6) is higher in the English auctions. Since φ is increasing and the English auction is more efficient than the second-price auction, the first term in (6) is higher for the English auction (and strictly higher for $b \leq \frac{1}{\lambda}$). Hence, we need to show that $U_A(0)$ is lower in the English auction.

In the English auction, $U_A(0) = 0$ as the type 0 is the first to quit the auction and so, she wins with probability 0. In general, it is not true that $U_A(0)$ is non-negative in the second-price auction. As a simple example, consider the babbling equilibrium. The type 0 wins the auction with probability $\frac{1}{2}$, as both bidders ignore messages from their advisors and bid their expected value $\frac{1}{\lambda}$. Hence, in the babbling equilibrium, $U_A(0) = \frac{1}{2}\left(b - \frac{1}{\lambda}\right) < 0$ for $0 < b < \frac{1}{\lambda}$. The key observation is that the babbling equilibrium fails the NITS condition. At the same time, the most informative equilibrium of the second-price auction satisfies the NITS condition which implies that $U_A(0) \geq 0$ in this equilibrium and so, the revenue is higher in the English auction.

In the right panel of Figure 3, we depict the revenue of the seller for different auction formats for $\lambda = \frac{1}{5}$. The gap in efficiency and revenue between two auctions increases with the size of the bias for $b < \frac{1}{\lambda}$. Notice the discontinuity with respect to b. If b is greater than $\frac{1}{\lambda}$, then all equilibria are babbling and the seller gets revenue $\frac{1}{\lambda}$. That is, the seller benefits from having the bias only when this bias is not too large.

3 Static Auctions

When the interests of bidders and advisors are aligned ($b = 0$) and communication is perfect, the revenue equivalence (Myerson (1981), and Riley and Samuelson (1981)) states that the expected revenue does not depend on the auction format as long as the equilibrium allocation is efficient and the lowest type gets zero surplus. This section shows that in static auctions a similar result obtains: any equilibrium of any static auction, in which the asset is allocated to the bidder with the highest expected value, brings the same expected revenue and generates the same communication as some equilibrium of the second price auction. We characterize information transmission in any efficient equilibrium of any static auction and show that there is necessarily an efficiency loss due to imperfect communication. In the next sections, we show that dynamic auctions are quite different from the second-price auction both in terms of information transmission and generated revenue.

In any static auction, after bidders get messages, they update their information about their values. A *type* $\theta_i \equiv \mathbb{E}[v_i|\tilde{m}_i] \in [\underline{v}, \overline{v}]$ *of bidder* i is her expected value conditional on message $\tilde{m}_i$. Denote by F_θ the distribution of types of each bidder generated through communication in equilibrium. Notice that if the communication is imperfect, it can that the support of F_θ is a subset of $[\underline{v}, \overline{v}]$. In fact, in static auctions that we consider below, the support of F_θ is always finite. We extend the strategy a to types that are assigned probability zero under F_θ by simply specifying that they best respond to the the strategies of opponents.[11] Given equilibrium bidding strategy a, let $q_i(\theta_1, \ldots, \theta_N)$ be the expected probability of winning for bidder i given types of all bidders. In the analysis of static auctions, we focus on auctions with *efficient* equilibria, in which type $\overline{v}$ gets utility zero and the following holds:

$$q_i(\theta_1, \ldots, \theta_N) = \begin{cases} \frac{1}{n}, & \text{if } \theta_i \in \max\{\theta_1, \ldots, \theta_N\} \text{ and } n \equiv |\{j : \theta_j = \max\{\theta_1, \ldots, \theta_N\}| , \\ 0, & \text{otherwise.} \end{cases}$$

for all $i = 1, \ldots, N$ and all $(\theta_1, \ldots, \theta_N) \in [\underline{v}, \overline{v}]^N$. Such equilibria are efficient conditional on the information of bidders, but may fail to be efficient conditional on the information of advisors. An example of an efficient equilibrium is the truthful equilibrium of the second price auction. The equilibrium of the first price auction is not an efficient equilibrium if after the communication, only a discrete set of types of bidder is possible. Indeed, Riley (1989) shows that bidders use mixed strategies, and one can check that types of bidders that are not realized (are probability zero under F_θ) do not have a well-defined best-response and equilibruim cannot be extended to those types. However, we can slightly adjust the rules of the first price auction to both guarantee the existence of the extended equilibrium as well as that such equilibrium is an efficient equilibrium. Specify, that if there are ties, then tied bidders play the second price auction. This parallels the idea of Maskin and Riley (2000) to guarantee existence of equilibria in first price auctions by breaking ties via the second price auction. One can check that his guarantees that the highest type wins the auction and the best response of types that are not realized is well defined.

The next lemma states a version of the revenue-equivalence for static auctions with communication.

Lemma 2 (Revenue Equivalence). *Consider an efficient equilibrium of some static auction. There exists an equilibrium of the second price auction which generates the same expected revenue and the same distribution of bidders' types F_θ.*

In this paper, we are interested in whether the auction format affects information transmission and through it the revenue and efficiency. Lemma 2 tells us that it does not if one restricts attention to static auctions with the

[11] Since these types have probability zero under F_θ, the extended strategy still constitutes an equilibrium.

same allocation rule and rent to the lowest type. For example, one cannot expect to generate a better information transmission or higher revenue by switching from the second-price auction to the first-price auction or all-pay auction.

Lemma 2 provides a useful analytic tool. The second-price auctions are easier to analyze as they allow for a simple bidding equilibrium in weakly dominant strategies. At the same time, the equilibrium of the first-price auction with discrete types of bidders requires mixing by bidders. As we will see next, discrete types naturally arise in the communication between the bidder and the advisor.

Lemma 2 states that to characterize equilibria of a rich class of static auctions and compare their efficiency and revenue to dynamic auctions, one can simply analyze equilibria of the second-price auction. The next theorem uses this approach to characterize the information transmission in all efficient direct mechanisms

Theorem 1. *Suppose $\overline{v} < \infty$. The communication strategy in any efficient equilibrium of any static auction is characterized as follows. There exists a positive integer $\overline{K}$ such that for all $1 \leq K \leq \overline{K}$, there exists an equilibrium in which types of advisor $v \in [\omega_{k-1}, \omega_k)$ induce the same action of the bidder and signal to the bidder that bidder's value is equal to $m_k = \mathbb{E}[v|v \in [\omega_{k-1}, \omega_k)]$. Thresholds $(\omega_k)_{k=1}^{K}$ are determined as follows:*[12]

$$G(\omega_{k-1}, \omega_k)(1-\Lambda_k)(\omega_k+b-m_k)+G(\omega_k, \omega_{k+1})\Lambda_{k+1}(\omega_k+b-m_{k+1}) = 0. \quad (7)$$

where

$$\Lambda_k = \frac{1}{G(\omega_{k-1}, \omega_k)} \sum_{n=1}^{N-1} \binom{N-1}{n} F(\omega_{k-1}, \omega_k)^n F(\omega_{k-1})^{N-1-n} \frac{1}{n+1}.$$

Theorem 1 shows that in static auctions, the misalignment of interests on the bidder's side results into a coarsening of the information transmitted from the advisor to the bidder. In particular, this implies that with positive probability the object is allocated inefficiently when $b \neq 0$. Theorem 1 is a counter-part of Theorem 1 in Crawford and Sobel (1982). However, our result does not follow from their result. In our game, the cheap-talk game is endogenous. Each bidder and advisor play a cheap-talk game in which actions are bids. The attractiveness of each bid for the bidder and advisor is endogenous and depends on how opponents bid in the auction. The bidding

[12]Here and further, when a random variable v is distributed according to F, we use a short-hand notation $F(a, b)$ for $\mathbb{P}(v \in [a, b]) = F(b) - F(a)$.

behavior of opponents depends on the information communicated between opponent bidders and their advisors. Theorem 1 shows that main insights from the cheap-talk literature are still true even when the cheap-talk game is endogenously determined in equilibrium by the communication between opponents and their advisors.

Equation (7) reflects the incentive of threshold types ω_k in the second-price auction. Notice that Λ_k is the expected probability of winning a tie when the bidder submits bid m_k. Type ω_k is indifferent between sending message m_k and m_{k+1}. In the second-price auction, the bidder pays the second highest bid. Therefore, strategies bring different payoffs only when the bidder faces a highest opponent of type in the interval $[\omega_{k-1}, \omega_k)$ or in the interval $[\omega_k, \omega_{k+1})$. The first term in equation (7) represents the benefit from submitting a higher bid. A higher bid m_{k+1} increases the probability of winning a tie from Λ_k to 1. The second term in equation (7) is the cost associated with a higher bid. Sending message m_{k+1}, the advisor risks winning the auction at price m_{k+1}. Since the costs and the benefits, should be equalized for threshold types, the advisor with type ω_k prefers not to buy at a higher price m_{k+1}.

It will be useful to derive the necessary condition for informative communication in static auctions.

Corollary 1. *A necessary condition for a non-babbling equilibrium is*

$$b \leq \mathbb{E}v - \underline{v}.$$

It is also sufficient when $N = 2$.

Chen, Kartik, and Sobel (2008) shows that in the standard cheap-talk model, NITS always exists and selects equilibria that are sufficiently informative (induce a high number of actions). In particular, under some conditions, NITS selects the most informative equilibrium of the cheap-talk. We next verify that this result is also true in our model.

Theorem 2. *The most informative equilibrium of the second-price auction satisfies the NITS condition.*

The proof of Lemma 2 adapts the argument in Chen et al. (2008) showing that if there is an equilibrium in the cheap-talk game with K actions induced in equilibrium that fails to satisfy NITS, then there is also an equilibrium with $K + 1$ induced actions. Again we cannot apply their result directly, as the cheap-talk game between the bidder and the advisor is endogenous. Their result implies that in our model for a fixed equilibrium, we can construct a different cheap-talk equilibrium for one bidder and her advisor that is more informative. However, this need not be an equilibrium of the model, as once we change the cheap-talk equilibrium of bidders, this changes the cheap-talk game played and hence, this will not be an equilibrium of the model.

4 English Auction

In this section, we characterize stationary equilibria of the English auction satisfying the NITS condition when advisors are biased toward overbidding. Equilibirum communication in the English auction takes the following form: types below some v^* completely separate over time, while types above v^* pool and induce the same bid. As a result, the English auction induces better information transmission than any static auction. We show that the English auction is preferred to any static auction both in terms of efficiency and revenue.

4.1 Characterization

This subsection shows that all stationary equilibria of the English auction satisfying NITS are in delegation strategies defined as follows.

Definition 4. *Strategies of players are* delegation strategies *if for some v^*:*

- *the advisor sends message "quit" when the running price equals $v + b$;*
- *the bidder quits if either the running price is above v^*+b or she receives message "quit".*

If the advisor is in control of bids, then it is a weakly dominant strategy for her to quit when $p = v + b$. Hence, in delegation strategies, the bidder essentially delegates bidding to the advisor with the restriction that the advisor quits before the running price exceeds $v^* + b$.

An equilibrium in delegation strategies always exists. The advisor induces the bidder to quit either at her optimal price $v + b$ if $v \leq v^*$ or at price $v^* + b$ if $v > v^*$, which is still better than quitting at any price below $v^* + b$. Hence, the communication strategy is optimal. On the other hand, message "quit" at price p implies that the bidder's value is $p - b < p$ and the bidder prefers to quit immediately and get utility zero, rather than wait longer and face the risk of winning the auction at a price that exceeds her value. Finally, the cutoff v^* can be chosen so that at stage $p^* = v^* + b$

the option value to the bidder of staying in the auction and waiting for the advisor's recommendation hits zero for the first time.

When players use delegation strategies, the full revelation below v^* is possible because of the dynamic nature of the English auction. The advisor reveals the value to the bidder only when the running price equals her optimal quitting price. Because of the overbidding bias, the bidder gets negative utility if she wins at the current or any future running price. Therefore, the bidder prefers to quit immediately after getting the recommendation to quit. This simple mechanism ensures perfect communication for types below v^*.

However, there are also other stationary equilibria in the English auction. In particular, for any equilibrium of the second price auction, there exists an outcome-equivalent stationary equilibrium of the English auction. To construct such equilibrium, specify that types in $[\omega_{k-1}, \omega_k)$ that send message m_k in the second-price auction, in the English auction, wait until the running price reaches m_k and send message "quit" then and the bidder follows their recommendations. The next theorem is the main characterization result and it shows that the NITS condition rules out these equilibria.

Theorem 3. *Suppose $b > 0$ and $\overline{v} \leq \infty$. Any stationary equilibrium in the English auction that satisfies the NITS condition is in delegation strategies with cutoff v^* characterized as follows:*

1. *v^* satisfies*
$$v^* + b = \mathbb{E}[v|v \geq v^*] \tag{8}$$
when $\underline{v} < v^* < \infty$, and $v^* + b \geq \mathbb{E}[v|v \geq v^*]$ when $v^* = \underline{v}$.

2. *Let $v_0^* = \underline{v}$, $v_{K+1}^* = \overline{v}$ and $v_1^* < \dots v_k^* < \dots < v_K^*$ be all solutions to equation (8). Then v^* equals to some $v_k^* \in \{v_0^*, \dots, v_{K+1}^*\}$ such that for all v_j^*, $j < k$, and all $n = 1, \dots, N-1$:*
$$\int_{v_j^*}^{v_k^*} (1 - F(s))(\mathbb{E}[v|v > s] - s - b)dG_n(s) \geq 0, \tag{9}$$
where G_n is the distribution of the maximum of n random variables independently, identically distributed according to F.

Theorem 3 shows that equilibria of the English auction are quite different from equilibria in static auctions: types at the bottom perfectly reveals themselves over time, while types at the top pool with each other. The NITS condition effective rules out partition equilibria corresponding to equilibria of the second price auction. To see this, consider an equilibrium described in

Theorem 1. In the beginning of the game, type ω_k is indifferent between m_k and m_{k+1}. By the recursion (7), $\omega_k+b-m_{k+1}<0$ and so, when price exceeds m_k, type ω_k strictly prefers to separate from other types contradicting NITS. The intuition is that in the beginning of the game type ω_k is willing to submit price m_{k+1} as it increases her probability of winning against types in $[\omega_{k-1},\omega_k)$, despite the risk of winning at a higher price m_{k+1}. However, after the running price exceeds m_k, the benefits of submitting higher bid disappear, and only costs remain. Hence, at this stage, type ω_k would prefer to reveal herself and this way induce the bidder to quit immediately and avoid winning the auction which contradicts the NITS condition.

Conditions (8) and (9) on v^* reflect the option value to the bidder of following the advisor's recommendation. The bidder waits for the recommendation as long as this option value is positive. This option value can be calculated as follows. Suppose that the current running price is p, the lowest remaining type is v_p and there are n other bidders remaining in the auction. The bidder wins the auction if for some running price $s+b>p$, it holds $v>s=\hat{v}$. Then her expected payoff is $\mathbb{E}[v|v>s]-s-b$. Integrating over all s, we get that the option value to the bidder is equal to

$$\frac{1}{(1-F(v_p))^n F(v_p)^{N-1-n}}\int_{v_p}^{v^*}(1-F(s))(\mathbb{E}[v|v>s]-s-b)dG_n(s)\geq 0. \quad (10)$$

Condition (8) ensures that the bidder does not want to stop listening to the advisor slightly earlier or later. If $v^*+b<\mathbb{E}[v|v\geq v^*]$, then the bidder would prefer to quit slightly later, while she would prefer to quit slightly earlier if $v^*+b<\mathbb{E}[v|v\geq v^*]$. Condition (9) ensures that the option value stays positive up until $p=v^*+b$.

For many commonly-used parametric families of distributions this conditions on v^* can be further simplified. Introduce the *mean residual lifetime* function

$$MRL(s)=\mathbb{E}[v|v\geq s]-s,$$

which is well studied in statistics (see Bagnoli and Bergstrom (2005)). Many commonly-used distributions have monotone MRL. Function MRL is decreasing for such distribution as normal, logistic, extreme value, Weibull, gamma, power distribution with power greater than one, as well as their truncations from above or below.[13] For Pareto and log-normal distribution

[13] Bagnoli and Bergstrom (2005) shows that log-concavity of density f or log-concavity of reliability function $1-F$, which are preserved by truncations, are sufficient for a weakly decreasing MRL.

truncated from below at 1, MRL increasing.[14] For exponential distribution, MRL is constant.

We next characterize v^* for distributions with monotone MRL that cover most of the commonly-used distributions. After that we will return to the general characterization in Theorem 3 to discuss how equilibria look like for general distributions. We have already considered the exponential distribution in Section 2 which is a knife-edge case between increasing and decreasing MRL. The next corollary covers distributions with decreasing MRL.

Corollary 2. *Suppose that* $\overline{v} \leq \infty$ *and* MRL *is decreasing. Then for all* $b > 0$ *except* $b = \mathbb{E}v - \underline{v}$*, the unique stationary equilibrium of the English auction satisfying the NITS condition is in delegation strategies. The equilibrium is informative if and only if* $\mathbb{E}v \geq \underline{v} + b$*.*

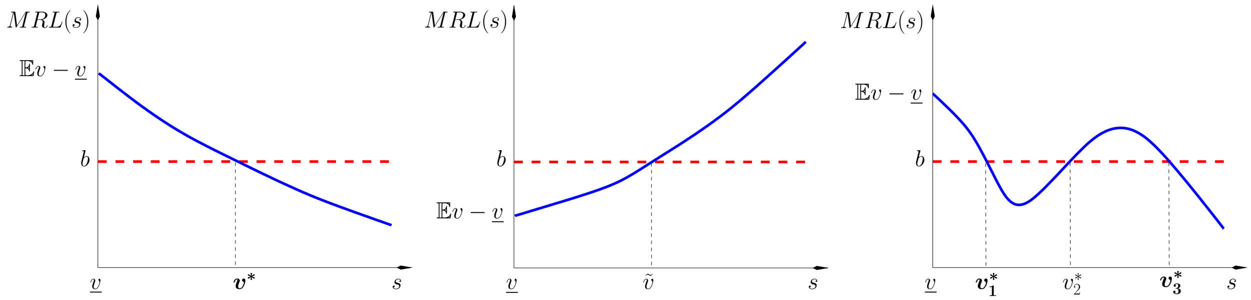
Figure 4a illustrates Corollary 2. When MRL is decreasing, there is v^* solving (8) if and only if $\mathbb{E}v - \underline{v}$, and it is unique whenever it exist. Since MRL crosses b from above, $\mathbb{E}[v|v \geq s] - s - b$ is positive for all $s < v^*$ and so, the option value (10) is positive for all $v_p < v^*$.

A new feature that was not present in the exponential example is that there is always pooling at the top, i.e. $v^* < \overline{v}$.[15] The bidder does not wait until all types of the advisor reveal themselves and at some point quits the auction before learning perfectly her value. Over the course of the auction, the bidder learns information about her value even if the advisor does not send any messages. The fact that there was no message so far indicates that her value cannot be lower than the running price minus bias b. When $\overline{v}$ is finite, after a certain time, the bidder knows that the value is close to $\overline{v}$. If she wins the auction, she will pay a price close to $\overline{v} + b$ and hence, it is very likely that she will overpay for the good. As a result, the bidder prefers to quit earlier and there is an interval of values at the top that she never learns.

Condition in Corollary 2 for informative equilibria is the same as the necessary condition for informative equilibria for static auctions (see Corollary 1). In particular, when $N = 2$, there is an informative equilibrium in the English auction if and only if there is an informative equilibrium in

[14]An (1998) shows that log-convexity of the density is sufficient for a weakly increasing MRL.

[15]Hence, if we consider a model with an exponential distribution truncated at the top at some $\overline{v}$ and let $\overline{v}$ go to infinity, then in equilibria of the English auction, there will be pooling at the top for each finite $\overline{v}$, but v^* will go to infinity as $\overline{v} \to \infty$ and in the limit, pooling will be degenerate and happen with probability zero.



(a) *decreasing MRL* (b) *increasing MRL* (c) *non-monotone MRL*

Figure 4: Graph of function MRL.

the static auction. Moreover, as b increases, the set of types that pool in equilibrium increases and equilibria become less informative. This comparative statics is different for distributions with increasing MRL as the next corollary shows.

Corollary 3. *Suppose $b > 0$, MRL is increasing on $[\underline{v}, \infty)$, and let $\bar{b}$ be the largest b for which*

$$\int_{\underline{v}}^{\infty} (1 - F(s))(\mathbb{E}[v|v > s] - s - b)dG_n(s) \geq 0, \tag{11}$$

for all $n = 1, \ldots, N-1$. Then all stationary equilibria of the English auction satisfying the NITS condition are in delegation strategies with v^* characterized as follows:

- *$v^* = \infty$ when $b \in [0, \mathbb{E}v - \underline{v})$;*
- *$v^* = \underline{v}$ or $v^* = \infty$ when $b \in [\mathbb{E}v - \underline{v}, \bar{b}]$;*
- *$v^* = \underline{v}$ when $b \in (\bar{b}, \infty)$.*

Figure 4*b* illustrates Corollary 2. When MRL is increasing, there is at most one solution $\tilde{v}$ to (8), however, it cannot be an equilibrium cutoff v^*. The reason is that since $\mathbb{E}[v|v \geq s] - s$ crosses b from below at $\tilde{v}$, if $v^* = \tilde{v}$ the option value to the bidder of waiting for advisor's recommendation is negative for $v < \tilde{v}$. Therefore, the equilibrium is either fully separating ($v^* = \infty$) or babbling ($v^* = 0$).[16] Intuitively, in the beginning of the auction,

[16] Observe that it is necessary for MRL to be increasing that $\bar{v} = \infty$. Indeed, if $\bar{v} < \infty$, then $\lim_{s \to \bar{v}} \mathbb{E}[v|v > s] - s = 0 < \mathbb{E}v - \underline{v}$.

winning is a bad news, as the bidder gets negative utility if she wins. As the auction continues, eventually, the bidder gets positive utility from winning, as $\mathbb{E}[v|v \geq v_p]$ increases faster than the price $v_p + b$ that the bidder pays in case she wins. The bidder is willing to follow the advice of the bidder if the benefits of winning later in the auction outweigh the risk of winning early in the auction. Hence, the condition on b: when b is sufficiently low, there is a fully informative equilibrium. As b increases, at some point, the babbling equilibrium is possible, and a for sufficiently high b, babbling equilibrium is the only equilibrium. Notice that the babbling equilibrium is an equilibrium only for sufficiently large b, for which the lowest type $\underline{v}$ gets positive utility $\frac{1}{N}(\underline{v} + b - \mathbb{E}v)$ from winning and so, the NITS condition is satisfied.

Corollary 3 allows the existence of non-babbling equilibria even when $b > \mathbb{E}v - \underline{v}$ and all equilibia of the static auctions are babbling as long as (11) holds. To give a concrete example, suppose $N = 2$ and $F(v) = 1 - \left(\frac{1}{v}\right)^2$ is a Pareto distribution on $[1, \infty)$. We have $\mathbb{E}[v] - \underline{v} = 1$ and so all equilibria in static auctions are babbling whenever $b > 1$. We can compute (11) as follows[17]

$$\int_1^\infty \frac{2\left(\hat{v} - b\right)}{\hat{v}^5} d\hat{v} = 2\int_1^\infty d\left(\frac{1}{\hat{v}^4} - \frac{b}{\hat{v}^5}\right) = 2\int_1^\infty d\left(\frac{b}{4\hat{v}^4} - \frac{1}{3\hat{v}^3}\right) = \frac{4 - 3b}{6},$$

which is positive whenever $b < \frac{4}{3}$. Hence, for $b \in \left(1, \frac{4}{3}\right)$ there exists an informative equilibrium, even though all equilibria of static auctions are babbling. The reason is that the term $\mathbb{E}[v|v > v^*] = 2v^*$ grows faster than $v^* + b$ so that for $b < \frac{4}{3}$, (11) holds.

Let us now return to the equilibria of the English auction for general distributions characterized in Theorem 3. For decreasing MRL, there is a unique candidate for v^* corresponding to the unique solution to (8), while for increasing MRL, there are two candidates $v^* = \underline{v}$ and $v^* = \overline{v}$. For general distributions, there can be multiple solutions to equation (8) which are, together with $\underline{v}$ and $\overline{v}$ (when $\overline{v} = \infty$), are candidates for v^*.

Condition (9) ensures that the option value to the bidder of following advisor's recommendation given by (10) is positive for all $v_p < v^*$. The integral (10) can be split into several integrals with limits of integration given by $(v_k^*)_{k=0}^{K+1}$. Since (10) should hold for every v_p up to v^*, only solutions to (8), in which MRL crosses b from above are possible candidates for the equilibrium cutoff. Moreover, the option value (10) is the smallest at the solutions to (8) where MRL crosses b from below.

As an illustration, consider general function MRL depicted in Figure 4c. There are three solutions v_1^*, v_2^*, and v_3^* to equation (8). By Theorem 3,

[17] Observe that for Pareto distribution $f(v) = \frac{2}{v^3}$ and $\mathbb{E}[v|v > \hat{v}] = \hat{v}^2 \int_{\hat{v}}^\infty \frac{2}{v^2} dv = 2\hat{v}$.

there can be at most four equilibria satisfying NITS in this situation. First, v_0^* is not an equilibrium cutoff, as it fails the NITS ($\mathbb{E}v > \underline{v} + b$), and $v_4^* = \overline{v}$ is not an equilibrium cutoff, as MRL is below b at $\overline{v}$. Neither is v_2^*, as MRL crosses b from below at v_2^*. Hence, only candidates for the equilibrium cutoff are v_1^* and v_3^*. There is an equilibrium with cutoff v_1^*, as for any $v_p \leq v_1^*$, the integrand in (10) is positive. There is an equilibrium with cutoff v_3^* if and only if the integral (10) for $v_p = v_2^*$ is positive.

It is interesting to observe how the number of bidders affects communication and efficiency of the English. First, when MRL is decreasing the number of bidders does not affect the communication in any way. Perhaps surprisingly, in a general case, increased competition reduces the scope of information transmission. Equation (8) for v^* does not depend on N, but condition (9) becomes stringent when N increases, as it needs to hold for a larger set of n. The reduction in the communication happens because the value of the advice is reduced can depend on how competitive the auction is. Moreover, this dependence can be non-monotone. When n is larger, the value of the highest opponent bidder is higher in the sense of first-order stochastic dominance of distribution G_n. However, the integrand in (9) need not be a monotone function, and so, the option can both increase and decrease as n increases.

Discussion Before proceeding with the comparison of auction formats, we discuss the generality of our results and underlying assumptions.

The dependence of the option value of advice on the competitiveness of the auction is also linked to whether the focus on equilibria in stationary strategies is restrictive. When MRL is decreasing, one can show that stationary strategies are without loss of generality in a sense that if we were to consider a fine grid for p and analyze limits of equilibria as the grid becomes arbitrary fine (one way of circumventing the indeterminacy of outcome in the game with continuous p), then in all of the limits, strategies would not depend on the number of remaining bidders. This is quite intuitive, as winning is always good news for the bidder when MRL is decreasing. However, if MRL is increasing or non-monotone, then it can be optimal for the bidder to condition on the number of remaining bidders, adso other non-stationary equilibria can exist.

The characterization in Theorem 3 can be extended in several directions. First, it is more realistic to assume that the seller instead of knowing bias b, has some prior beliefs F_b about b supported on $\mathbb{R}_+$. That is the seller knows that there is a conflict of interest on the bidder's side, but does not know the scope of this conflict. Our characterization of the English auction still obtain in this environment, as long as bidders and advisors know b. Importantly, the efficiency and revenue comparison that we carry out in the next subsections also hold in this case. Hence, the seller benefits from switching to the English auction from a static auction, and this does not require knowledge of fine details about the conflict of interest and only the direction of the bias. Second, the characterization can be immediately generalized to the case of heterogeneous bidders. In particular, we can allow for different distributions F_i of values and different biases $b_i > 0$ and Theorem 3 would still hold with obvious changes in notation. This is in contrast with the characterization of equilibria in static auctions in Theorem 1 which becomes more complicated for asymmetric bidders, as threshold types are different for different bidders and pinned down by a system of recursive equations.

Finally, the NITS condition has lots of bite in the English auction, often leading to the unique equilibrium. Chen, Kartik and Sobel (2008) provide several justification for the NITS condition in a general cheap-talk model. We next provide an additional justification for the partially separating equilibria of the English auction obtained in Theorem 3.

One can think of our model as the model of information acquisition in which the information acquisition technology is endogenous. The seller affects how much information can be transmitted between the advisor and the bidder through her choice of the auction format. However, we can also make the information acquisition itself endogenous via following *auction with delegation*. Suppose that each bidder can commit to a rule that maps advisor's type announcements into bids in the English auction. That is, the bidder can offer the advisor incentives to reveal information and hence, can choose how much information is transmitted. In the beginning of the auction, they offer their advisors a contract specifying such mappings and advisors reveal their private information. The following proposition shows that in fact, the equilibria in partially separating strategies in Theorem 3 are also equilibria in the auction with delegation.

Proposition 3. *Suppose $b > 0$, MRL is decreasing and* $\inf_{v\in[\underline{v},\overline{v}]}(\ln f(v))' \geq -b$*. Strategies described in Theorem* 3 *also constitute an equilibrium of the auction with delegation.*

Proposition 3 states that if all bidders in the English auction follow the advisor's recommendation up to threshold v^*, then even if each bidder had a commitment power to offer the optimal contract to the advisor in the beginning of the game, she would not be able to do better. In particular, committing to a coarse information transmission is not optimal and the bidder prefers to use a finer information of the advisor even though it comes at cost of implementing an optimal bid of the advisor.

Proposition 3 follows from a general analysis of the delegation problem in Amador and Bagwell (2013). In the literature on optimal delegation, strategies in Theorem 3 are called *interval delegation*. One intricacy in applying the analysis of Amador and Bagwell (2013) is that they make concavity assumptions on payoff functions which are not satisfied in the English auction. Indeed, choosing to quit at price higher than $v^* + b$ leads to probability one of winning the auction, but does not change the expected payment conditional on winning compared to quitting at price $v^* + b$. We augment the argument of Amador and Bagwell (2013) to account for this possibility.

Theorem 3 shows that in the English auction, the bidder does not need commitment to implement the optimal contract. However, we show this under an important restriction. First, other bidders follow strategies in Theorem 3. It is interesting question whether when bidders follow partial pooling strategies as in equilibria of the second price auction, the bidder still benefits from the interval delegation. This requires a more general analysis of the delegation problem which is beyond the scope of this paper but is an interesting direction of research.

4.2 Efficiency Comparison

We next compare the efficiency of auction formats. In the exponential example, the efficiency comparison was clear, as the English auction always allocated the asset to the bidder with the highest value. Theorem 3 shows that in general, there can be pooling at the top which distorts the efficiency. This leads to a loss of efficiency and depending on the size of this pooling region, it is possible that the equilibrium of the static auction is more efficient if it generates a more efficient outcomes for types above v^*. The next theorem shows that this is not the case. More strongly, the superior efficiency stems from a superior information transmission. We say, that an equilibrium of an auction is *more informative* than an equilibrium of potentially different auction, if the partition of advisor types generated by the former is finer than the partition generated by the latter.

Theorem 4. *Suppose $b > 0$. Then any equilibrium of the English auction satisfying NITS is more informative and more efficient than any efficient equilibrium of any static auction.*

Theorem 4 shows that there is no partition generated by a static auction that is finer than the partition generated by the English auction. That is, there is no $\omega_k > v^*$ where ω_k and v^* are as in Theorems 1 and 3. This implies that the English auction generates a finer information partition than any static auction and hence, is more efficient.

The argument for Theorem 4 can be sketched as follows. Suppose that there exists an equilibrium of the static auction such that $\omega_{k-1} = v^* < \omega_k < \omega_{k+1} = \overline{v}$. For simplicity, also assume that $N = 2$. Then equation (7) implies that

$$\frac{1}{2}F(\omega_{k-1}, \omega_k)(\omega_k + b - m_k) = -\frac{1}{2}F(\omega_k, \omega_{k+1})(\omega_k + b - m_{k+1}),$$

or

$$\omega_k + b = \frac{F(\omega_{k-1}, \omega_k)}{F(\omega_{k-1}, \omega_{k+1})}m_k + \frac{F(\omega_k, \omega_{k+1})}{F(\omega_{k-1}, \omega_{k+1})}m_{k+1} = \mathbb{E}[v|v \geq v^*].$$

However, this contradicts the fact that $v^* < \omega_k$ solves (8). Intuitively, if there were a variation in bids among types above v^*, any type above v^* would strictly prefer to submit a higher bid and increase her chances of winning against types below. This happens because v^* is already sufficiently close to $\overline{v}$ and price m_{k+1} does not vary much from price m_k.

Theorem 4 also sheds light on the communication in static auctions. In static auctions, the dependence of the communication on N is more convoluted. The number of bidders N enters recursion (5) in a complicated way and from it, it is not clear how N affects the communication partition. However, from Theorem 4, the communication partition in the English auction is finer than the partition generated by any static auction. This implies that the communication in static auctions does not become perfect as we increase the competitiveness of the auction, which is a priori not obvious from recursion (5).

Finally, for distributions with increasing MRL we can go beyond the comparison with static auctions.

Corollary 4. *Suppose* $0 < b < \mathbb{E}v - \underline{v}$ *and* MRL *is increasing on* $[\underline{v}, \infty)$. *Then the unique stationary equilibrium of the English auction satisfying NITS is fully efficient.*

Corollary 4 solves the problem of efficient mechanism design for distributions with increasing MRL and moderate bias b. It shows that despite the conflict of interest, an efficient outcome is implementable as a unique outcome via the English auction. We conjecture that this holds more generally for any distribution. In order to show this one needs to show that no other auction format either static or dynamic can attain a finer communication for types above v^*. Theorem 4 guarantees that this is the case when the comparison is with efficient equilbiria of static auctions.

4.3 Revenue Comparison

We next compare the revenue from different auction formats. Denote by $\varphi(v) \equiv v + b - \frac{1-F(v)}{f(v)}$ the virtual valuation of advisor.

Theorem 5. *Suppose $b > 0$ and φ is increasing. Then any equilibrium of the English auction satisfying NITS brings higher revenue than any efficient equilibrium of any static auction satisfying NITS.*

Proof. We can view the problem that the seller faces as an optimal mechanism design problem from informed advisors. The fact that bids are submitted not directly by advisors, but by bidders implies that there is a restriction on the set of mechanisms that the seller can implement. However, we can still use Lemma 3 in Myerson (1981) to write the expected revenue of the seller as follows:

$$N\left(\mathbb{E}[\varphi(v)p(v)] - U_A(0)\right), \tag{12}$$

where $p(v)$ is the probability that type v wins the auction and $U_A(0)$ is the expected utility of type 0 from the auction. In equation (12), only $p(\cdot)$ and $U_A(0)$ depend on the format of the auction. By Lemma 2, it is sufficient to compare the English auction with the second-price auction. By NITS, $U_A(0) \geq 0$ for the second-price auction, while $U_A(0) = 0$ for the English auction. To prove the comparison, we need to show that the first term in (12) is larger for the English auction. This is indeed the case, as φ is increasing and the English auction is more efficient by Theorem 2. □

The key insight of Theorem 5 is that we can view the problem of the seller of extracting maximal revenue as the a problem of designing a mechanism that extracts rents from informed advisors. In this case, the fact that there is a communication puts a restriction on the set of mechanisms that are the seller can implement. However, one can still use the envelope formula in Myerson (1981) to write the revenue in the form (12). The higher efficiency of the English auction implies that the first term in (12) is higher than in any static auction, while the NITS guarantees that the rent of the lowest type is positive in static auctions, while it is zero in the English auction.

While superior efficiency of the English auction because of the better information transmission is intuitive, it is a priori not clear if the English auction should also bring higher expected revenue. If types in some interval pool and induce the same bid in the second-price auction, then it can potentially increase the revenue of the seller. As we have already seen, at least some types above ω_k get negative utility from winning at price m_{k+1}, but with positive probability they end up winning the asset at this price. We show that despite this occasional overpaying, it does not occur often enough to reduce significantly the information rents of advisors. The key in ensuring this is the NITS condition. To see this, let us return to our exponential example and consider a babbling equilibrium in which all types pool and bidders submit bids $\frac{1}{\lambda}$. The revenue from such equilibrium of the second-price auction is $\mathbb{E}[v] = \frac{1}{\lambda}$. However, the equilibrium of the English auction brings revenue $\mathbb{E}[\min\{v_1, v_2\}] = \frac{1}{2\lambda} + b$ and so for $b < \frac{1}{2\lambda}$, the babbling equilibrium of the second-price auction brings higher revenue. In this case, a significant amount of low types make a bid that exceeds their value, as they cannot credibly transmit their value to the bidder. This way the seller extracts an extra revenue. However, for $b < \frac{1}{2\lambda}$, babbling equilibrium fails to satisfy the NITS conditions.

A natural next question is whether the revenue can be further improved. We know from Myerson (1981) that introducing the reservation price increases the revenue whenever the virtual valuation φ is negative for some types. Then introducing a reserve price $r = v_r + b$, where v_r is given by the solution to $\varphi(v_r) = 0$, increases further the revenue. By setting the reservation price at $v_r + b$, the seller does not allocate to types below v_r which contribute negatively into the expected profit (12). In our exponential example, $v_r = \frac{1}{\lambda} - b$ for $b < \frac{1}{\lambda}$. The knowledge of b is important to set the reservation price optimally as φ depends on b. Interestingly, in the family of distributions with increasing MRL we can go even further and find an optimal mechanism.

Corollary 5. *Suppose $0 < b < \mathbb{E}v - \underline{v}$ and MRL is increasing on $[\underline{v}, \infty)$. Then there exists a reservation price r such that the unique stationary equilibrium satisfying NITS of the English auction with a reservation price r is optimal.*

Corollary 5 follows from our characterization in Corollary 3. Indeed, Myerson (1981) shows that generally a second price auction with a reservation price is an optimal mechanism. In particular, it is an optimal mechanism from extracting rents from informed advisors and the question is whether the constraints imposed by the communication between bidders and advisors prevent us from implementing this outcome. Corollary 3 can be easily modified to allow for a reservation price by simply assuming that the seller start increasing price starting from r. Hence, for distributions with increasing MRL, the fact that bids are submitted by bidders does not prevent us from implementing an optimal outcome. We conjecture that this is true more generally which is related to the general efficiency of the English auction. If one shows that no mechanism induces finer partition of types above v^*, then this would imply that the English auction with the reservation price is an optimal mechanism for general distributions.

While the analysis of the English auction is does not change with the introduction of the reserve price and one can easily compute an optimal reserve price, this is not the case in static auctions. Indeed, if the seller restricts bids in the second price auction to be above some r, then this affects the equilibrium communication. Essentially, after the introduction of the reserve price, the distribution of values is $F(\cdot|v \in [r, \overline{v}])$ and generally the partition of types generated in equilibrium changes, which in turn changes which types tie with each other in equilibrium. Hence, determining the revenue price is less straightforward in the static auction and requires more knowledge of the strategic environment from the seller, while only the knowledge of the distribution and b is necessary in the English auction.

5 M&A Contests

This section applies our model to qualitatively study M&A contests. We first introduce an M&A auction which closely resembles the actual sale procedure in M&A contests and is a generalization of the English auction. Using the estimates from the literature of the advisors' bias, we compare the efficiency and profitability of different auction formats with and without the conflict of interest.

5.1 M&A Auction

M&A contests are conducted as follows. The target company approaches potential buyers. Interested buyers sign confidentiality agreement and the seller reveals non-public information about the company. The seller elicits preliminary non-binding bids indicating the interest of bidders and narrows down the circle of potential bidders. After that, the formal bidding starts and a smaller circle of bidders submits binding bids. At this stage, the seller decides whether any bidder wins the auction or the bidding continues, in which case the seller continues negotiation with bidders and further increases the price.

We capture key features of M&A contests with the following auction format which we call an *M&A auction*. First, bidders submit initial bids coming from some discrete set $\{\beta_1, \dots, \beta_K\}$. The bidder with the highest bid wins the auction and pays her bid. If there is tie, then to bidders who tied proceed to the next stage of bidding which follows the rules of the English auction. That is, if several bidders tied at bid β_k, the seller continuously increases the price starting from β_k and bidders choose when to quit the auction until only one bidder remains who gets the asset and pays the price at which the last of her competitors quit. The communication in the M&A auction happens as follows. In the first stage of bidding, advisors send recommendations β_k. In the formal round, the communication happens continuously over time and each advisor determines when to send her bidder a recommendation to "quit" the auction.

The interpretation is that in the first stage of bidding, the seller does not distinguish between close bids. For example, if the seller gets bids 1m, 2m, 10m, 10.5m, the seller thinks that two highest bids are sufficiently close and interprets them as a tie and continues the auction with bidders submitting 10 and 10.5 to extract additional revenue. The following theorem constructs equilibria of the M&A auction.

Theorem 6. *Suppose that $b > 0$. The following strategies constitute an equilibrium of the M&A auction.*

- *In the first stage, advisors of type in $[\omega_{k-1}, \omega_k)$ send message β_k to their bidders indicating that the true value v belongs to some interval $[\omega_{k-1}, \omega_k)$ and bidders submit β_k.*
- *In the formal round, advisors send message "quit" when the running price is equal to $v+b$ and bidders quit either when they receive message "quit" from their advisors or when running price equals v_k^*.*

Cutoffs v_k^* are the smallest solutions to

$$\mathbb{E}[v | v \in [v_k^*, \omega_k)] = v_k^* + b, \tag{13}$$

belonging to $[\omega_{k-1}, \omega_k]$ and thresholds $(\omega_k)_{k=0}^{K+1}$ with $\omega_0 = \underline{v}$ and $\omega_{K+1} = \overline{v}$ satisfy

$$G(\omega_{k-1})(\omega_k + b - \beta_k) + \int_{\omega_{k-1}}^{v_k^*} (\omega_k - \hat{v}) dG(\hat{v}) + L_k(\omega_k - v_k^*) = G(\omega_k)(\omega_k + b - \beta_{k+1}), \tag{14}$$

and

$$G(\omega_k)(m_{k+1} - \beta_{k+1}) + U_B(\omega_k, \omega_{k+1}, m_{k+1}) \geq G(\omega_{k-1})(m_{k+1} - \beta_k) + U_B(\omega_{k-1}, \omega_k, m_{k+1}) \tag{15}$$

$$G(\omega_k)(m_{k+1} - \beta_{k+1}) + U_B(\omega_k, \omega_{k+1}, m_{k+1}) \geq G(\omega_{k+j})(m_{k+1} - \beta_{k+j+1}) \tag{16}$$

where

$$\begin{aligned} L_k &= \sum_{n=1}^{N-1} \binom{N-1}{n} F(v_k^*, \omega_k)^n F(v_k^*)^{N-1-n} \frac{1}{n+1}, \\ U_B(\omega_{k-1}, \omega_k, m) &= \int_{\omega_{k-1}}^{v_k^*} (m - \hat{v} - b) dG(\hat{v}) + L_k(m - v_k^* - b). \end{aligned}$$

First, notice that the English auction is an M&A auction in which advisors do not transmit any information in the first bidding stage and hence, all bidders submit the same bid. In particular, it immediately follows from Theorem 2, there exists an equilibrium of the M&A auction and it satisfies the NITS condition. On the other extreme, when in equilibrium, bidders ignore the information from advisors after the initial bidding stage, the equilibrium in the M&A auction is the equilibrium of the first price auction. Therefore, M&A auction is the auction format that is in between the English auction and the first price auction. However, one can construct an equilibrium in the M&A auction that differs from the equilibrium in the English auction and the first price auction. In the Online Appendix we provide details of such a construction.

5.2 Cost of Conflict of Interest

It is commonly believed that strategic bidders overpay in M&A contests for the target because of the agency problems. Our theoretical analysis confirms this point of view. In this subsection, we use our model to give a qualitative estimate of the amount that shareholders of bidders overpay for the target because of the conflicting interests of the management.

6 Preference for Underbidding

In this section, we consider advisors biased toward underbidding. This bias can be explained by the "quiet life" model, incorporating additional business requires additional effort from managers and managers prefer not to increase the size of the firm. Bertrand and Mullainathan (2003) document that in manufacturing, under the weakening of the threat of takeover, the management reduces the creation of new plants with no net effect on firm size.

In this case, if the bidder knew the value, then she would submit a higher bid than the advisor. Then the English auction does not have an advantage over static auctions, as it only restricts the bidder to submit bids lower than the running price. However, the Dutch auction can allow for a better information transmission because it restricts the bidder from submitting bids higher than the running price. In this section, we construct an equilibrium of the Dutch auction satisfying NITS that is more efficient than any static auction, however, it can be worse in terms of revenue. As before, the approach is to illustrate results with an exponential example and then proceed to more general results.

6.1 Example

We start with an equilibrium of the second-price auction described in the following proposition.

Proposition 4. *Under F exponential and $b < 0$, the following strategies constitute an equilibrium of the second-price auction. The bidder submits a bid m. The advisor with type $v \in I_k$ sends message $m_k = \mathbb{E}[v|[\omega_{k-1}, \omega_k)]$ for $k = 1, 2, \ldots$ where $\omega_k = kx$ and $x \in \left[-b, \frac{1}{\lambda} - b\right]$ is given by the solution to equation*

$$x\frac{e^{2\lambda x} + 1}{e^{2\lambda x} - 1} = \frac{1}{\lambda} - b.$$

We can use the analysis of the case $b > 0$ in Proposition 1 to prove Proposition 4. Indeed, the derivation of the recursion (5) does not depend on the value of b. Recall the function ψ which is defined implicitly as a value of $x_{k+1} = \psi(x_k)$ that satisfies (5) for given x_k. When $b < 0$, there is a fixed point of ψ that gives the stationary equilibrium described in Proposition 4 (see Figure 5).

Since $\overline{v}_w = \infty$, it is not clear in what sense the stationary equilibrium in Proposition 4 satisfies the NITS condition. However, we can construct a sequence of equilibria with $\overline{v} < \infty$ that satisfies NITS and converges to the equilibrium in Proposition 4 as $\overline{v} \to \infty$. Indeed, fix an integer K. Let $x_{K+1} = 0$ and recursively define $x_k = \psi^{-1}(x_{k+1})$. It is easy to verify that these strategies constitute an equilibrium when F is the exponential distribution with parameter λ truncated at $\overline{v} = \sum_{k=1}^{K} x_k \to \infty$ as $K \to \infty$.

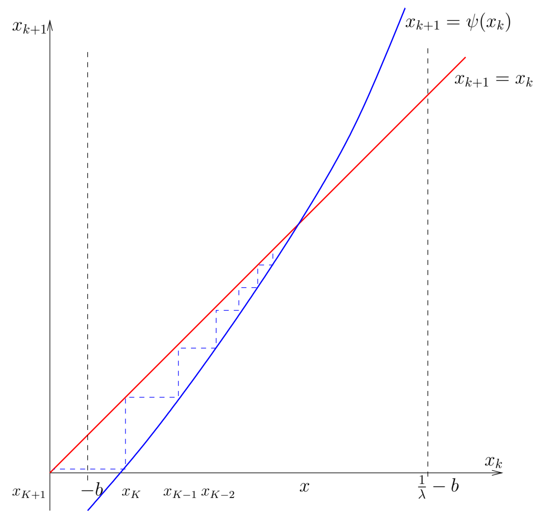


Figure 5: Recursion $x_{k+1} = \psi(x_k)$ for $b < 0$.

Since $x_{K+1} = 0$, type $\overline{v}$ perfectly reveals herself and so, this equilibrium satisfies NITS. Moreover, for any $\varepsilon > 0$ there exists $\overline{K}$ such that for any K, $x > x_k > x - \varepsilon$ for all but $\overline{K}$ indexes k. This way, even though the equilibrium in Proposition 4 cannot be verified to satisfy the NITS condition, it is a limit of equilibria satisfying NITS.

Now, we show that a partial separation is possible in the Dutch auction similarly to the English auction.

Proposition 5. *Suppose $b < 0$. There exists an equilibrium of the Dutch auction described by a tuple $\{v^*, \sigma(\cdot)\}$ as follows. Types of advisor $v \leq v^*$ send message "stop" when $p = v^* + b$. Any type of advisor $v \geq v^*$ sends message "stop" at time t when $p = \sigma(v)$. The bidder follows the recommendation of advisor when the running price is above $v^* + b$ and stops the auction if the running price is $v^* + b$. Threshold v^* is the solution to*

$$\frac{v^*}{1 - e^{-\lambda v^*}} = \frac{1}{\lambda} - b \tag{17}$$

and bidding strategy $\sigma(\cdot)$ is given by

$$\sigma(v) = \mathbb{E}[\max\{v^*, \hat{v}\} + b | \hat{v} < v], \text{ for } v \geq v^*. \tag{18}$$

In the equilibrium constructed in Proposition 5, the advisor perfectly reveals her type to the bidder at a price $\sigma(v)$ that is optimal for her. Because of the underbidding bias of the advisor, the optimal price of stopping

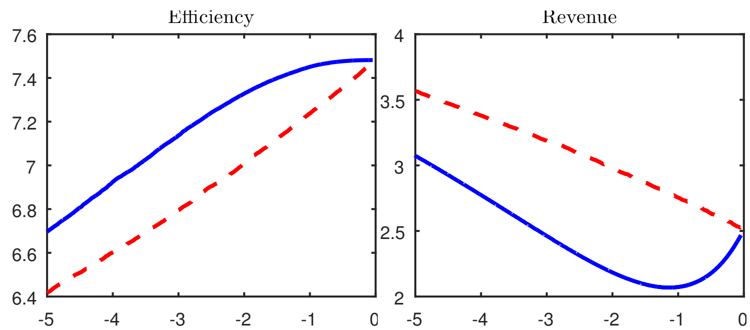


Figure 6: Efficiency and revenue comparison for $b < 0$: the Dutch auction (solid line) and the second-price auction (dashed line). Bias b is plotted on the horizontal axis.

the auction for the bidder is higher. Hence, it is optimal for her to stop immediately after she gets a recommendation from the advisor. However, information transmission is perfect only up to some cutoff v^* and all types below v^* pool with each other. The reason for this is that at a certain stage, the uncertainty of the bidder about her value is sufficiently reduced. Then the bidder prefers stopping the auction immediately to guarantee the victory, rather than trying to win at a lower price, but facing the risk of loosing the auction.

We now compare the efficiency and revenue of the Dutch auction and the second-price auction. In the Dutch auction for $b < 0$, one can show that[18] $v^* < x$ and so, the equilibrium partition of the Dutch auction is finer than the equilibrium partition of the second-price auction. This implies that the equilibrium in Proposition 5 is more efficient than the stationary equilibrium of the second-price auction.

The revenue comparison is ambiguous for $b < 0$. We can use again the argument in Myerson (1981) to derive the expression (6) for the revenue. Because of the higher efficiency the first term in (6) is higher in the Dutch auction. On the other hand, the second term is higher in the second-price auction. Indeed, $U_A(0) = \frac{1}{2}\left(1 - e^{-\lambda x}\right)(b - m_1)$ for the second-price auction and $U_A(0) = \frac{1}{2}\left(1 - e^{-\lambda v^*}\right)(-v^*)$ for the Dutch auction. By $v^* < x$, $1 - e^{-\lambda x} > 1 - e^{-\lambda v^*}$ and so the probability of winning is smaller for type 0

[18]One needs to show that

$$v\frac{e^{\lambda v}}{e^{\lambda v} - 1} > v\frac{e^{2\lambda v} + 1}{e^{2\lambda v} - 1},$$

which clearly holds.

in the Dutch auction. At the same time, $b - m_1 < -v^* < 0$.[19] Therefore, the second-price auction may bring higher revenue than the Dutch auction if the second term in (6) dominates the first term.

To explore the difference between auctions quantitatively take $\lambda = \frac{1}{5}$. The left panel of Figure 6 depict the expected value of the winner in the auction and it shows that the Dutch auction is more efficient than the second-price auction and the gap in the efficiency increases as the size of the bias increases. In the right panel of Figure 6, we depict the revenue of the seller for different auction formats. For $b < 0$, the Dutch auction brings lower revenue than the second-price auction and so the effect of a lower second term auction in (6) dominates. This implies that in our example, there is a trade-off between efficiency and revenue when the advisor is biased toward underbidding.

6.2 General Results

This subsection generalizes the insights from the analysis of the exponential example. Theorems 2 and 1 do not depend on the sign of b and so, in static auctions, we still obtain revenue equivalence and crude information transmission in case of preference for underbidding. The next theorem generalizes the equilibrium constructed in Proposition 5.

Theorem 7. *Suppose $b < 0$ and let v^* be the largest solution to*

$$\mathbb{E}[v|v < v^*] = v^* + b, \tag{19}$$

where $v^* = \overline{v}$ if equation (19) does not have a solution. There exists an equilibrium of the Dutch auction satisfying the NITS condition characterized by $\{\sigma(\cdot), v^*\}$ as follows. The advisor of type $v > v^*$ sends message "stop" when running price p reaches $\sigma(v) \equiv \mathbb{E}[\max\{\hat{v}, v^*\} + b|\hat{v} < v]$. and the advisor of type $v < v^*$ sends "stop" when price p reaches $\sigma(v^*)$. The bidder immediately stops the auction after she receives the message "stop" or when price p reaches $\sigma(v^*)$.

The equilibrium of the Dutch auction in Theorem 7 is similar to equilibria in the English auction. There is an interval of types at the bottom that pool with each other, and types at the top perfectly reveal themselves over time.

[19] Indeed,

$$b - m_1 = b - \frac{1}{\lambda} + x\frac{e^{-\lambda x}}{1 - e^{-\lambda x}} = b - \frac{1}{\lambda} + x\frac{e^{\lambda x}}{e^{\lambda x} - 1} - x > b - \frac{1}{\lambda} + x\frac{e^{2\lambda x} +}{e^{2\lambda x} - 1} - x = -x < -v^* < 0.$$

Function $\sigma(v)$ in Theorem 7 is the equilibrium bidding strategy in the Dutch auction if bids were submitted directly by advisors. The following theorem shows that the equilibrium of the Dutch auction that we constructed is more efficient than any equilibrium of any static auction.

Theorem 8. *Suppose $b < 0$. The equilibrium of the Dutch auction in Theorem 7 is more efficient than any efficient equilibrium of any static auction.*

As we have seen in the example, the revenue comparison does not carry over to the case of bias toward underbidding.

7 Conclusion

This paper studies the interaction between the information transmission and bidding in auctions. In static auctions, the revenue-equivalence result holds giving in particular equivalence of the first- and second- price auctions. However, dynamic auctions, such as the English and the Dutch auctions, are generally more efficient than static auctions. This happens because in dynamic auctions the set of bids available to the bidder shrinks. Therefore, by sending the information later in the game, the advisor can induce the bidder to choose a more favorable action and hence, would provide a more refined information to the bidder. Moreover, the English auction also dominates static auctions in terms of revenue when advisors are biased toward overbidding, the case most relevant empirically. This paper characterize equilibria in different auction formats and show the efficiency/revenue comparison.

8 Appendix

Proofs for Section 2

Proof of Lemma 1. Specify new online strategies m' and a' as follows. Let $m'(v,h) = a(h, m(v,h))$ and $a'(h,\tilde{a}) = \tilde{a}$ for all $h \in \mathcal{H}$ and all $\tilde{a}$ in the image of $m'(\cdot,h)$. For any h, fix an action $\tilde{a}(h)$ in the image of $m'(\cdot,h)$. For any recommendation that does not belong to the image of $m'(\cdot,h)$, the bidder interprets this deviation as a recommendation of action $\tilde{a}(h)$. Hence, it is sufficient to guarantee that advisors do not deviate to recommendations that happens with positive probability on-path. Clearly, strategy profiles m' and a' generate the same outcome. The proof that they constitute an equilibrium is provided in the text. □

Details of Proof of Proposition 1. First, we show that f is well-defined. Since $x_k > 0$, we car rewrite (5) as

$$\frac{x_{k+1}+x_k}{e^{\lambda(x_{k+1}+x_k)}-1} = \frac{1}{\lambda} - b - x_k.$$

Let $h(x)$ be the function implicitly defined by the solution to the equation

$$\frac{h}{e^{\lambda h}-1} = \frac{1}{\lambda} - b - x. \tag{20}$$

Left-hand side of (20) is decreasing in h and it takes values in $\left(0, \frac{1}{\lambda}\right]$ for $h \geq 0$. Therefore, the solution $h(x)$ always exists whenever $x \in \left(0, \frac{1}{\lambda} - b\right]$.

Now, we show that $\psi'(x) \geq 1$ for $x \in \left(0, \frac{1}{\lambda} - b\right]$. The derivative of h

$$h'(x) = \frac{(e^{\lambda h(x)}-1)^2}{1-e^{\lambda h(x)}+\lambda h(x)e^{\lambda h(x)}} \geq 0.$$

Then $\psi(x) = h(x) - x$. Moreover,

$$\psi'(x) = \frac{(e^{\lambda h(x)}-1)^2}{1-e^{\lambda h(x)}+\lambda h(x)e^{\lambda h(x)}} - 1 = \frac{e^{\lambda h(x)}-1-\lambda h(x)}{e^{-\lambda h(x)}-1+\lambda h(x)} \geq 1,$$

where to show the inequality we need to show that $e^{\lambda h} - e^{-\lambda h} - 2\lambda h \geq 0$ for $h \geq 0$. This is implied by the fact that $e^{\lambda h} - e^{-\lambda h} - 2\lambda h$ is increasing in h and equals zero at 0.[20]

Finally, from (5) it follows that

$$\lim_{x\to 0}(1-\lambda b)\,\frac{e^{\lambda\psi(x)}-1}{\lambda\psi(x)} = 1$$

and so, $\lim_{x\to 0}\psi(x) > 0$. Also from (5) it follows that $\psi(x) \to \infty$ as $x \to \frac{1}{\lambda} - b$. □

Proofs for Section 3

Proof of Lemma 2. Denote by $q_i(\theta_i) = \mathbb{E}[q_i|\theta_i]$ the expected probability of allocation for type θ_i and by $t_i(\theta_i) = \mathbb{E}[t_i|\theta_i]$ the expected transfer from type θ_i given that other bidders use their equilibrium strategies. Necessary conditions for q_i and t_i to be part of equilibrium are the following for all $i = 1, \ldots, N$:

$$q_i(\theta_i)\theta_i - t_i(\theta_i) \geq q_i(\theta_i')\theta_i - t_i(\theta_i') \quad \text{for all } \theta_i' \in [\underline{v}, \overline{v}], \tag{21}$$

$$q_i(\theta_i)\theta_i - t_i(\theta_i) \geq 0 \quad \text{for all } \theta_i \in [\underline{v}, \overline{v}]. \tag{22}$$

[20]Its derivative $\lambda e^{\lambda h} + \lambda e^{-\lambda h} - 2\lambda = \lambda e^{-\lambda h}(e^{\lambda h}-1)^2 \geq 0$.

Denote $U_i(\theta_i) \equiv q(\theta_i)\theta_i - t_i(\theta_i)$. Lemma 2 in Myerson (1981) gives the following integral formula for U_i.

Lemma 3 (**Myerson (1981)**). *Conditions* (21) *and* (22) *imply that for all* $i = 1, \ldots, N$ *and all* $\theta_i \in [\underline{v}, \overline{v}]$*:*

$$U_i(\theta_i) = U_i(\underline{v}) + \int_{\underline{v}}^{\theta_i} \left(\int_{[\underline{v},\overline{v}]^{N-1}} q_i(\theta, \theta_{-i}) dF_\theta(\theta_{-i}) \right) d\theta.$$

Consider a strategy m_i of the advisor i in the efficient equilibrium of a static auction $\mathcal{A}$ and corresponding probability distribution F_θ generated by strategy m_i. The expected probability of allocation from following the equilibrium strategy for type θ_i is $\int_{[\underline{v},\overline{v}]^{N-1}} q_i(\theta_i, \theta_{-i}) dF_\theta(\theta_{-i})$, and by Lemma 3, the expected transfer from reporting type θ_i is $P(\theta_i)\theta_i - U_i(\mathcal{M}, \theta_i)$. Both quantities depend only on function q_i and $U_i(\underline{v})$ and hence are the same for the auction $\mathcal{A}$ and the second price auction. This implies that the strategy m_i is also constitutes equilibrium in the second-price auction. □

Proof of Theorem 1. Clearly, the second-price auction is an efficient mechanism. By Theorem 2 it is sufficient to analyze equilibria of the second-price auction.

To any profile of bids $\bar{a} = (a_i)_{i \in N}$ corresponds an allocation $(q_1(\bar{a}), \ldots, q_N(\bar{a}))$ such that $\sum_{i=1}^N q_i(\bar{a}) = 1$ and transfers $(t_1(\bar{a}), \ldots, t_N(\bar{a}))$. Denote by $q(a_i) \equiv \mathbb{E}[q_i(a_i, a_{-i})]$ and $t(a_i) \equiv \mathbb{E}[t_i(a_i, a_{-i})]$ the expected probability of allocation and transfer, respectively, from action a_i, where expectations are taken fixing strategies of other bidders and advisors m_{-i} and a_{-i}. Bidder i chooses a bid from A given that her expected value is $\theta_i = \mathbb{E}[v_i|a_i]$. Let where $Q = \{q(a_i), a_i \in A\}$ and $t(q) = \min_{a_i: q = q(a_i)} t(a_i)$. Then the bidder and the advisor play the cheap-talk game with payoffs given by

$$\text{Bidder} \quad : \quad qv - t(q), \tag{23}$$

$$\text{Advisor} \quad : \quad q(v+b) - t(q). \tag{24}$$

Since the mixed derivatives of (23) and (24) are positive, the set of types of the advisor that induce the same probability of allocation q is an interval. Therefore, to characterize equilibria of the second-price auction, we need to determine incentives of threshold types of the advisor ω_k. Consider any such type ω_k. In the second-price auction, a message is simply an expected value of the bidder m_k. Let $\hat{m}$ be the message of the highest bidder among $N-1$ opponents of the bider. From submitting a message m_k, type ω_k gets utility

$$\mathbb{E}[\omega_k + b - \hat{m} | \hat{v} < \omega_{k-1}] + G(\omega_{k-1}, \omega_k)\Lambda_k(\omega_k + b - m_k).$$

From submitting a message m_{k+1}, type ω_k gets utility

$$\mathbb{E}[\omega_k+b-\hat{m}|\hat{v} < \omega_{k-1}]+G(\omega_{k-1},\omega_k)(\omega_k+b-m_k)+G(\omega_k,\omega_{k+1})\Lambda_{k+1}(\omega_k+b-m_{k+1}).$$

Type ω_k should be indifferent between the two which gives equation (7).

Claim 1. If $\omega_{k+1} = \omega_k$, then either $k = 0$ or $k = K$.

Proof: Suppose to contradiction that for some $0 < k < K$, $\omega_{k+1} = \omega_k$. This implies that $H(\omega_k, \omega_{k+1}) = 0$ and so, from (7), $H(\omega_{k-1}, \omega_k)(1 - \Lambda_{k-1})(\omega_k + b - m_k) = 0$ and $H(\omega_{k+1}, \omega_{k+2})\Lambda_{k+2}(\omega_{k+1} + b - m_{k+2}) = 0$. This implies that $\omega_k + b = m_k$ and $\omega_{k+1} + b = m_{k+2}$. But only the first equality can hold if $b < 0$ and only the second equality can hold if $b > 0$, contradiction. **q.e.d.**

Claim 2. There exists $\varepsilon > 0$ such that for all k, either $\omega_{k+1} - \omega_k > \varepsilon$ for $0 < k < K$.

Proof: It follows from (7) that whenever $\omega_{k-1} < \omega_k < \omega_{k+1}$, we have

$$\omega_k + b > \mathbb{E}[v|v \in [\omega_{k-1}, \omega_k)] \tag{25}$$

and

$$\omega_k + b < \mathbb{E}[v|v \in [\omega_k, \omega_{k+1})]. \tag{26}$$

First, consider $b > 0$. If for any $\varepsilon > 0$, there exists and equilibrium such that $\omega_{k+1} - \omega_k < \varepsilon$, then for such equilibrium $\mathbb{E}[v|v \in [\omega_k, \omega_{k+1})] \leq \omega_k + \varepsilon$ which contradicts (26) for sufficiently small ε. Now, consider $b < 0$. If for any $\varepsilon > 0$, there exists and equilibrium such that $\omega_k - \omega_{k-1} < \varepsilon$, then for such equilibrium $\mathbb{E}[v|v \in [\omega_{k-1}, \omega_k)] \geq \omega_k - \varepsilon$ which contradicts (25) for sufficiently small ε. **q.e.d.**

The fact that there exists $\overline{K}$ such that there is an equilibrium with K segments for any $1 \leq K \leq \overline{K}$, but not for $K > \overline{K}$ can be proven by the same argument as in the proof Theorem 1 in Crawford and Sobel (1982). □

Proof of Corollary 1. Sine there is an equilibrium partition for any $K \leq \overline{K}$, it is sufficient to show that there is an equilibrium with two intervals in the partition. For $K = 2$, we can use the same argument as in the proof of Theorem 4 to show that (7) implies

$$\omega_1 + b - \mathbb{E}[v|v \in [\omega_0, \omega_2)] \leq 0. \tag{27}$$

Since in the equilibrium with $K = 2$, $\omega_0 = \underline{v}$ and $\omega_2 = \overline{v}$, and $\omega_1 \geq \underline{v}$, we get the desired conclusion. It is easy to check that for $N = 2$, the inequality in (27) is an equality and an equilibrium with two segments exists whenever equation $\omega_1 + b - \mathbb{E}[v] = 0$ has a solution. Since $b > 0 \geq \mathbb{E}[v] - \underline{v}$, whenever $b \leq \mathbb{E}[v] - \underline{v}$, such solution exists by continuity which proves the sufficiency of condition in the corollary. □

Proof of Theorem 2. In this proof, it is useful to introduce the following notations:

$$\begin{aligned}
\Phi(\omega_{k-1}, \omega_k) &\equiv \sum_{n=1}^{N-1} \binom{N-1}{n} F(\omega_{k-1}, \omega_k)^n F(\omega_{k-1})^{N-1-n} \frac{1}{n+1}, \\
\Psi(\omega_{k-1}, \omega_k) &\equiv \sum_{n=1}^{N-1} \binom{N-1}{n} F(\omega_{k-1}, \omega_k)^n F(\omega_{k-1})^{N-1-n} \frac{n}{n+1}, \\
m(\omega_{k-1}, \omega_k) &\equiv \mathbb{E}[v | v \in (\omega_{k-1}, \omega_k)].
\end{aligned}$$

Define function

$$H(\omega_{k-1}, \omega_k, \omega_{k+1}) \equiv \Psi(\omega_{k-1}, \omega_k)(\omega_k + b - m(\omega_{k-1}, \omega_k)) + \Phi(\omega_k, \omega_{k+1})(\omega_k + b - m(\omega_k, \omega_{k+1})). \tag{28}$$

It is easy to check that function H coincides with the left-hand side of equation (7). By Theorem 1, any equilibrium of the second-price auction is outcome-equivalent to an equilibrium having the partition structure with thresholds $(\tilde{\omega}_k)_{k=1}^K$ solving the recursion

$$H(\tilde{\omega}_{k-1}, \tilde{\omega}_k, \tilde{\omega}_{k+1}) = 0 \tag{29}$$

with $\tilde{\omega}_0 = \underline{v}$ and $\tilde{\omega}_{K+1} = \overline{v}$. We show that if the NITS condition fails, then for any such solution, there exists a different solution to the recursion (29) with $K+1$ partition cells. Since there are at most $\overline{K}$ partition cells, this implies that there exists an equilibrium satisfying NITS, and in particular, the most informative equilibrium satisfies NITS. We consider separately cases $b > 0$ and $b < 0$.

Case $b > 0$. If type $\underline{v}$ reveals herself to the bidder, then the bidder prefers to submit a loosing bid. Suppose to contradiction that NITS fails and $\underline{v} + b < m(\underline{v}, \tilde{\omega}_1)$. We show by induction that for any $k \leq K+1$, there exists another solution $(\omega_j^k)_{j=1}^{K_j}$ to (29) such that $\omega_0^k = \underline{v}$, $\omega_k^k > \tilde{\omega}_{k-1}$, and $\omega_{k+1}^k = \tilde{\omega}_k$. Theorem 2 follows from the claim applied to $k = K+1$.

For $k = 1$, the failure of NITS implies that $H(\underline{v}, \underline{v}, \tilde{\omega}_1) = \Phi(\underline{v}, \tilde{\omega}_1)(\underline{v} + b - m(\underline{v}, \tilde{\omega}_1)) < 0$. At the same time, $H(\underline{v}, \tilde{\omega}_1, \tilde{\omega}_1) = \Psi(\underline{v}, \tilde{\omega}_1)(\tilde{\omega}_1 + b - m(\underline{v}, \tilde{\omega}_1)) > 0$, as $\tilde{\omega}_1 = m(\tilde{\omega}_1, \tilde{\omega}_1) \geq m(\underline{v}, \tilde{\omega}_1)$. By continuity, there exists $x \in (\underline{v}, \tilde{\omega}_1)$ such that $H(\underline{v}, x, \tilde{\omega}_1) = 0$ proving the claim for $k = 1$.

Suppose the statement is true for k and we next prove it for $k+1$. Since $\tilde{\omega}_k$ solves (29), $H(\tilde{\omega}_{k-1}, \tilde{\omega}_k, \tilde{\omega}_{k+1}) = 0$ or

$$\Psi(\tilde{\omega}_{k-1}, \tilde{\omega}_k)(\tilde{\omega}_k + b - m(\tilde{\omega}_{k-1}, \tilde{\omega}_k)) + \Phi(\tilde{\omega}_k, \tilde{\omega}_{k+1})(\tilde{\omega}_k + b - m(\tilde{\omega}_k, \tilde{\omega}_{k+1})) = 0. \tag{30}$$

Let $\omega_k^k > \tilde{\omega}_{k-1}$ and $\omega_{k+1}^k = \tilde{\omega}_k$ as in the inductive hypothesis and consider $H(\omega_k^k, \tilde{\omega}_k, \tilde{\omega}_{k+1})$:

$$\Psi(\omega_k^k, \tilde{\omega}_k)(\tilde{\omega}_k + b - m(\omega_k^k, \tilde{\omega}_k)) + \Phi(\tilde{\omega}_k, \tilde{\omega}_{k+1})(\tilde{\omega}_k + b - m(\tilde{\omega}_k, \tilde{\omega}_{k+1})),$$

which differs from (30) only in the first term. Since $\omega_k^k > \tilde{\omega}_{k-1}$, $m(\omega_k^k, \tilde{\omega}_k) > m(\tilde{\omega}_{k-1}, \tilde{\omega}_k)$. Moreover, the binomial distribution with probability of success $\frac{F(\tilde{\omega}_{k-1}, \tilde{\omega}_k)}{F(\tilde{\omega}_k)}$ first-order stochastic dominance the binomial distribution with probability of success $\frac{F(\omega_k^k, \tilde{\omega}_k)}{F(\tilde{\omega}_k)}$. Hence,

$$\begin{aligned}
\frac{\Psi(\omega_k^k, \tilde{\omega}_k)}{F(\tilde{\omega}_k)^N} &= \sum_{n=1}^{N-1} \binom{N-1}{n} \left(\frac{F(\omega_k^k, \tilde{\omega}_k)}{F(\tilde{\omega}_k)}\right)^n \left(\frac{F(\omega_k^k)}{F(\tilde{\omega}_k)}\right)^{N-1-n} \frac{n}{n+1} \\
&< \sum_{n=1}^{N-1} \binom{N-1}{n} \left(\frac{F(\tilde{\omega}_{k-1}, \tilde{\omega}_k)}{F(\tilde{\omega}_k)}\right)^n \left(\frac{F(\tilde{\omega}_{k-1})}{F(\tilde{\omega}_k)}\right)^{N-1-n} \frac{n}{n+1} = \frac{\Psi(\tilde{\omega}_{k-1}, \tilde{\omega}_k)}{F(\tilde{\omega}_k)^N},
\end{aligned}$$

as and $\frac{n}{n+1}$ is increasing in n. Therefore, $H(\omega_k^k, \tilde{\omega}_k, \tilde{\omega}_{k+1}) < 0$.

On the other hand, since $\omega_{k+1}^k = \tilde{\omega}_k$,

$$H(\omega_k^k, \tilde{\omega}_k, \omega_{k+1}^k) = \Psi(\omega_k^k, \tilde{\omega}_k)(\tilde{\omega}_k + b - m(\omega_k^k, \tilde{\omega}_k)) > 0.$$

By continuity, there exists $x \in (\tilde{\omega}_k, \tilde{\omega}_{k+1})$ such that $H(\omega_k^k, \omega_{k+1}^k, x) = 0$. By continuity, we can find solution $(\omega_j^{k+1})_{j=1}^{K_{k+1}}$ to (29) with $\omega_{k+1}^{k+1} > \tilde{\omega}_k$ and $\omega_{k+2}^{k+1} = \tilde{\omega}_{k+1}$, which completes the proof of the inductive step.

Case $b < 0$. If type $\overline{v}$ reveals herself to the bidder, then the bidder prefers to submit a bid that is guaranteed to win. Type $\overline{v}$ does not want to reveal herself if and only if $\overline{v} + b - m(\tilde{\omega}_K, \overline{v}) \leq 0$. Suppose to contradiction that NITS fails and $\overline{v} + b > m(\tilde{\omega}_K, \overline{v})$. We show by induction that for any $k \leq K+1$, there exists another solution $(\omega_j^k)_{j=1}^{K_j}$ to (29) such that $\omega_{K_j+1}^k = \overline{v}$, $\omega_{K_j-k}^k < \tilde{\omega}_{K-k+1}$, and $\omega_{K_j-k-1}^k = \tilde{\omega}_{K-k}$. Theorem 2 follows from the claim applied to $k = K$.

For $k = 1$, the failure of NITS implies that $H(\tilde{\omega}_K, \overline{v}, \overline{v}) = \Psi(\tilde{\omega}_K, \overline{v})(\overline{v} + b - m(\tilde{\omega}_K, \overline{v})) > 0$. At the same time, $H(\tilde{\omega}_K, \tilde{\omega}_K, \overline{v}) = \Phi(\tilde{\omega}_K, \overline{v})(\tilde{\omega}_K + b -$

$m(\tilde{\omega}_K, \overline{v})) > 0$, as $\tilde{\omega}_K$ satisfies (29). By continuity, there exists $x \in (\tilde{\omega}_K, \overline{v})$ such that $H(\tilde{\omega}_K, x, \overline{v}) = 0$ proving the claim for $k = 1$.

Suppose the statement is true for k and we next prove it for $k+1$. Since $\tilde{\omega}_k$ solves (29), $H(\tilde{\omega}_{K-k-1}, \tilde{\omega}_{K-k}, \tilde{\omega}_{K-k+1}) = 0$ or

$$\Psi(\tilde{\omega}_{K-k-1}, \tilde{\omega}_{K-k})(\tilde{\omega}_{K-k} + b - m(\tilde{\omega}_{K-k-1}, \tilde{\omega}_{K-k}))+ \\ \Phi(\tilde{\omega}_{K-k}, \tilde{\omega}_{K-k+1})(\tilde{\omega}_{K-k} + b - m(\tilde{\omega}_{K-k}, \tilde{\omega}_{K-k+1})) = 0. \quad (31)$$

Let $\omega^k_{K_j-k} < \tilde{\omega}_{K-k+1}$ and $\omega^k_{K_j-k-1} = \tilde{\omega}_{K-k}$ as in the inductive hypothesis and consider $H(\tilde{\omega}_{K-k-1}, \tilde{\omega}_{K-k}, \omega^k_{K_j-k})$:

$$\Psi(\tilde{\omega}_{K-k-1}, \tilde{\omega}_{K-k})(\tilde{\omega}_{K-k} + b - m(\tilde{\omega}_{K-k-1}, \tilde{\omega}_{K-k}))+ \\ \Phi(\tilde{\omega}_{K-k}, \omega^k_{K_j-k})(\tilde{\omega}_{K-k} + b - m(\tilde{\omega}_{K-k}, \omega^k_{K_j-k})),$$

which differs from (31) only in the second term. Since $\omega^k_{K_j-k} < \tilde{\omega}_{K-k+1}$, $m(\tilde{\omega}_{K-k}, \omega^k_{K_j-k}) < m(\tilde{\omega}_{K-k}, \tilde{\omega}_{K-k+1})$. Moreover,

$$\begin{aligned}\Phi(\tilde{\omega}_{K-k}, \omega^k_{K_j-k}) &= \sum_{n=1}^{N-1} \binom{N-1}{n} F(\tilde{\omega}_{K-k}, \omega^k_{K_j-k})^n F(\tilde{\omega}_{K-k})^{N-1-n} \frac{1}{n+1} + G(\omega^k_{K_j-k}, \tilde{\omega}_{K-k}) \cdot 0 \\ &< \sum_{n=1}^{N-1} \binom{N-1}{n} F(\tilde{\omega}_{K-k}, \tilde{\omega}_{K-k+1})^n F(\tilde{\omega}_{K-k})^{N-1-n} \frac{1}{n+1} \\ &= \Phi(\tilde{\omega}_{K-k}, \tilde{\omega}_{K-k+1}).\end{aligned}$$

Hence, $H(\tilde{\omega}_{K-k-1}, \tilde{\omega}_{K-k}, \omega^k_{K_j-k+1}) > 0$. On the other hand, since $\omega^k_{K_j-k-1} = \tilde{\omega}_{K-k}$,

$$H(\tilde{\omega}_{K-k}, \tilde{\omega}_{K-k}, \omega^k_{K_j-k+1}) = \Phi(\tilde{\omega}_{K-k}, \omega^k_{K_j-k+1})(\tilde{\omega}_{K-k} + b - m(\tilde{\omega}_{K-k}, \omega^k_{K_j-k+1})) < 0.$$

Therefore, there exists $x \in (\tilde{\omega}_k, \tilde{\omega}_{k+1})$ such that $H(x, \tilde{\omega}_{K-k}, \omega^k_{K_j-k+1}) = 0$. By continuity, we can find solution $(\omega^{k+1}_j)^{K_{k+1}}_{j=1}$ to (29) with $\omega^k_{K_j-k-1} < \tilde{\omega}_{K-k}$ and $\omega^k_{K_j-k-2} = \tilde{\omega}_{K-k-1}$, which completes the proof of the inductive step. □

Proofs for Section 4

By Lemma 1, the strategy of the advisor can be described by a function $m(v)$ which specifies at what price the advisor sends message "quit" to the bidder. The following lemma shows that in the English auction, types of advisor either perfectly reveal themselves to the bidder or pool with neighboring types.

Lemma 4. *Function $m(v)$ is increasing on a subset of $[\underline{v}, \overline{v}]$ of Lebesgue measure $\overline{v} - \underline{v}$.*

Proof of Lemma 4. Suppose to contradiction that $m(v)$ is strictly decreasing on a set of positive measure I. Let $v = \inf I$ and $v' = \sup I$. Then $m(v) > m(v')$ and $G(v') - G(v) > 0$. Let q and t be the probability of winning and expected price paid conditional on using strategy $m(v)$ and q' and t' be the probability of winning and expected price paid conditional on using strategy $m(v')$. Then

$$\begin{aligned} qv - t &\geq q'v - t', \\ q'v' - t' &\geq qv' - t, \end{aligned}$$

implies that $q' \geq q$. By quitting at price $m(v)$ instead of $m(v')$, the advisor increases the probability of winning by at least $G(v') - G(v) > 0$ and so, $q > q'$ which is a contradiction. □

Lemma 5. *m is strictly increasing on $[\underline{v}, v^*)$ and is constant almost everywhere on $[v^*, \overline{v}]$ where v^* satisfies* (8) *when $v^* > \underline{v}$ and $v^* + b \geq \mathbb{E}[v|v \geq v^*]$ when $v^* = \underline{v}$.*

Proof of Lemma 5. Any equilibrium generates a partition Π of $[\underline{v}, \overline{v}]$ satisfying for any $\pi \in \Pi$, $v, v' \in \pi \iff m(v) = m(v')$. We say that types in $\pi \in \Pi$ pool if $m(v)$ is constant on π, i.e. these types send message "quit" at the same price. We say that types in $[v', v'']$ separate, if $m(v)$ is strictly increasing on $[v', v'']$, i.e. all these types send message "quit" at different prices. Define by Π^P the closure of the set of all types that pool with some other type. Then $\Pi^S = [\underline{v}, \overline{v}] \backslash \Pi^P$ is the set of all types that separate and denote by $\partial \Pi^P$ the boundary of Π^P.

Notice that the babbling equilibrium is an equilibrium of the English auction and it satisfies NITS if and only if $\mathbb{E}[v] \leq \underline{v} + b$. So we focus on the case when there is a non-trivial information transmission in equilibrium, i.e. $\Pi^S \neq \emptyset$.

We first show that whenever an interval of types perfectly reveals their value to the bidder in the auction, then these types quit at the optimal time.

Claim 3. *If m is strictly increasing on a subset S of (v', v'') of (Lebesgue) measure $|v'' - v'|$, then $m(v) = v + b$ on (v', v'').*

Proof: There exists at most countable number of discontinuities of m on S. Consider a type v at which m is continuous, i.e. there exist sequences $v_j^- \to v - 0$ and $v_j^+ \to v + 0$ such that $m(v_j^-) \to m(v) - 0$ and $m(v_j^+) \to m(v) + 0$. We show that $m(v) = v + b$. Suppose to contradiction that $m(v) < v + b$.

Choose j large so that $m(v) < m(v_j^+) < v + b$. If type v sends "quit" at price $m(v_j^+)$ instead of $m(v)$, then she can additionally win against types in $[v, v_j^+)$ and pay at most $m(v_j^+) < v + b$. Therefore, her utility is higher contradicting the rationality of type v. Now, suppose to contradiction that $m(v) > v + b$. Choose j large so that $v + b < m(v_j^-) < m(v)$. If type v sends "quit" at price $m(v)$ instead of $m(v_j^-)$, then she additionally wins against types in $[v_j^-, v)$ and pays at least $m(v_j^-) > v+b$. Therefore, she strictly gains from sending "quit" at price $m(v_j^-)$ contradicting the rationality of type v. Therefore, $m(v) = v + b$ for all continuity points of S. Since $m(v) = v + b$ on a dense subset of S, it is also true on the whole S. Since S has measure $|v'' - v'|$, set S is dense in (v', v'') and so, $m(v) = v + b$ on (v', v''). **q.e.d.**

Claim 4. $\Pi^P = [v^*, \overline{v}]$ for some $v^* \geq \underline{v}$.

Proof: Consider $v \in \partial\Pi^P$. Type v is indifferent between pooling with some interval of types $\pi \ni v$ and separating. Indeed, since $\pi \in \partial\Pi^P$ and $\Pi^S \neq \emptyset$, there exists a sequence $v_j \to v$ such that $m(v_j) = v_j + b$ by Claim 1. Type v can mimic type v_j and for large j get utility arbitrarily close to her maximal utility. Therefore,

$$m(v) = v + b = \mathbb{E}[v | v \in \pi]. \tag{32}$$

Suppose to contradiction to Claim 2 there exists a sequence $v_j \to v$ such that $v_j \in \Pi^P$ and $v_j < v$. Then $\mathbb{E}[v|v \in \pi] < v + b$ which is a contradiction. **q.e.d.**

Next, we show that all types in Π^P send "quit" at the same price.

Claim 5. $\Pi^P = \pi$ for some $\pi \in \Pi$.

Proof: Suppose to contradiction that there are two adjacent intervals of types π and π' such that types in π send "quit" at price m and types in π' send "quit" at price $m' > m$. Consider type v that is at the boundary of π and π'. By continuity, type v is indifferent between sending "quit" at price m and m'. The benefit of quitting at m' rather than m is that type v wins against types in π, but there is a risk that she will tie with types in π'. The indifference of type v implies that $m' > v + b$. But then consider a time when the running price reaches m'. Type v is the lowest type. However, she gets a negative utility from pooling with types in π'. This contradicts the NITS condition. **q.e.d.**

Finally, equation (8) follows from (32) and Claim 3. □

Proof of Theorem 3. By Lemma 5 condition (8) is a necessary condition. For any v^* satisfying in addition (11), we construct an equilibrium in online strategies satisfying NITS for dynamic auctions. Then we show that if v^* fails (11), then it cannot be part of equilibrium.

Consider strategies described in the theorem. The optimality of the advisor and the bidder after she receives the message "quit" is verified in the text. Let us check the optimality of the bidder. Let N_p be the number of bidders remaining in the game at price p and v_p be the lowest type remaining in the game at price p. The utility of the bidder from following the recommendation of the advisor starting from running price p is equal to

$$V(N_p, v_p) = \frac{1}{(1-F(v_p))^{N_p} F(v_p)^{N-1-N_p}} \left(\int_{v_p}^{v^*} (1-F(s))(\mathbb{E}[v|v>s] - s - b) dG_{N_p}(s) \right) +$$
$$\frac{1}{(1-F(v_p))^{N_p} F(v_p)^{N-1-N_p}} \sum_{n=1}^{N_p-1} \binom{N_p-1}{n} (1-F(v^*))^{n+1} (F(v^*) - F(v_p))^{N_p-1-n} \frac{1}{n} (\mathbb{E}[v|v \geq v^*] - v^*$$

By the definition of v^*, the last term is zero and so,

$$V(N_p, v_p) = \frac{1}{(1-F(v_p))^{N_p} F(v_p)^{N-1-N_p}} \left(\int_{v_p}^{v^*} (1-F(s))(\mathbb{E}[v|v>s] - s - b) dG_{N_p}(s) \right). \tag{34}$$

The bidder prefers to quit immediately at the first time $V(N_p, v_p)$ becomes negative. Moreover, (11) implies for all $v_p \leq v^*$, $V(N_p, v_p) \geq 0$ which proves the optimality of the bidder's strategy. □

Proof of Corollary 2. To show that $v^* < \overline{v}$, notice that the left-hand side of (8) is greater than the right-hand side for v^* sufficiently close to $\overline{v}$. Therefore, $v_1^* < \overline{v}$ and for $\tilde{v} \in (v_1^*, v_0^*)$, $\mathbb{E}[v|v > \tilde{v}] - \tilde{v} - b < 0$. This implies that $v^* < \overline{v}$. □

Proof of Proposition 3. Given that all other bidders follow advisor's recommendations up to $p^* = v^* + b$, all bids above p^* lead to probability one of winning and the same expected payment $\mathbb{E}[\hat{v}]$. We can equivalently formulate the choice problem in terms of the expected probability of winning $q = G(p - b)$. Let $\overline{q} = \lim_{p \to p^* - 0} G(p - b)$ and q^* be the probability of winning from submitting bid p^*. Then q is chosen from the set $[0, \overline{q}) \cup \{q^*, 1\}$, and the preferences over q's of the bidder and the advisor are

$$\text{Bidder} \quad : \quad qv - t(q), \tag{35}$$
$$\text{Advisor} \quad : \quad q(v+b) - t(q), \tag{36}$$

where $t(q) = q\mathbb{E}[\hat{v}+b|\hat{v} \leq G^{-1}(q)]$. The bidder designs a contract $q : [\underline{v}, \overline{v}] \to [0, \overline{q}) \cup \{q^*, 1\}$ that solves the program A

$$\max_{q \in ([0,\overline{q}) \cup \{q^*,1\})^{[\underline{v},\overline{v}]}} \int_{\underline{v}}^{\overline{v}} (q(v)v - t(q(v)))\, dF(v) \text{ subject to:}$$
$$v \in \arg\max_{v'}\{(v+b)q(v') - t(q(v'))\} \text{ for all } v$$

Denote by q^A the solution to program A. The proof proceeds as follows. We extrapolation $t(q)$ to all q in $[0, 1]$ and solve program B where we maximize over contracts $q : [\underline{v}, \overline{v}] \to [0, 1]$. Then we show that the solution to program B is also a solution to the program A. As a preliminary step, we show that function t is strictly convex on $[0, \overline{q})$.

Claim 6. *$t(q)$ is strictly convex and twice differentiable on* $[0, \overline{q})$.

Proof: For any $q < \overline{q}$, in the English auction type $v = G^{-1}(q)$ of the advisor wins with probability q. Consider any $q, q' < \overline{q}$ and types v and v' that win in the English auction with probabilities q and q', respectively. Since in the English auction bidding $v+b$ is strictly optimal for advisor type v, $qv - t(q) > q'v - t(q')$. This implies that $v > \frac{t(q)-t(q')}{q-q'}$ whenever $q > q'$ and $v < \frac{t(q')-t(q)}{q'-q}$ whenever $q < q'$, which in turn, implies strict convexity of t on $[0, \overline{q})$. Differentiability of t is implied by $t(q) = q\mathbb{E}[\hat{v} + b|\hat{v} \leq G^{-1}(q)]$. **q.e.d.**

Notice that function t cannot be extrapolated to a strictly convex function to the whole interval $[0, 1]$ because of the following claim.

Claim 7. Points $(\overline{q}, \lim_{q \nearrow \overline{q}} t(q)), (q^*, t(q*))$, and $(1, t(1))$ lie on the same line with slope $\lim_{q \nearrow \overline{q}} t'(q) = v^* + b$.

Proof: In the English auction, the bidder with expected value $\mathbb{E}[v|v > v^*]$ gets expected profit 0 from winning at price p^*. Therefore, he is indifferent between $(\overline{q}, \lim_{q \nearrow \overline{q}} t(q)), (q^*, t(q*))$, and $(1, t(1))$ and so, they lie on the same line (her indifference curve) with slope $\mathbb{E}[v|v > v^*] = v^* + b$. Since $t(q)$ is differentiable at $q < \overline{q}$, for every $v < v^*$, $v+b = t'(q(v))$ which implies that $\lim_{q \nearrow \overline{q}} t(q) = v^* + b$. **q.e.d.**

Claim 7 implies that $t(q)$ cannot be extrapolated to a strictly convex function on the whole $[0, 1]$, and it has a linear piece above $\overline{q}$ (see Figure 7). In order to apply Amador and Bagwell (2013), we perturb function t on $[\overline{q}, 1]$ so that the perturbation t_ε is strictly convex and twice differentiable.[21]

[21]For example, we can consider perturbation $t_\varepsilon(q) = t(q) + \varepsilon \max\{0, (q - \overline{q})^3\}$.

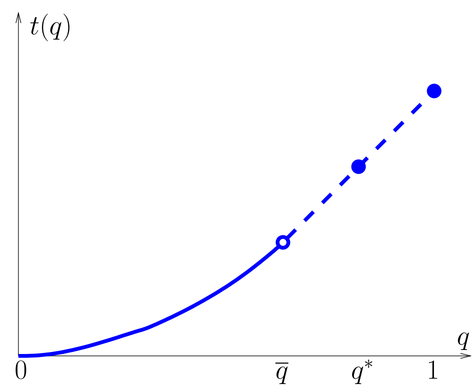


Figure 7: Function t

Then we solve program B:

$$\max_{q\in[0,1]^{[\underline{v},\overline{v}]}} \quad \int_{\underline{v}}^{\overline{v}} \left(q(v)v - t_\varepsilon(q(v))\right) dF(v) \text{ subject to:}$$
$$v \in \arg\max_{v'}\{(v+b)q(v') - t_\varepsilon(q(v'))\} \text{ for all } v,$$

and show that the solution of program A cannot be different from the limits (as $\varepsilon \to 0$) of solutions of programs B.

Claim 8. *An interval delegation with cutoff v^* is a solution to program B.*

Proof: We verify that conditions (c1), (c2), (c3') of Proposition 1 in Amador and Bagwell (2013) as well as convexity and differentiability assumptions are satisfied when we fix that other bidders follow their equilibrium strategies.

First, Claim 6 verifies the differentiability and convexity assumptions in in Amador and Bagwell (2013). Second, in program B, κ in Amador and Bagwell (2013) is equal to 1. Let $q_f(v)$ be the optimal choice of q of advisor of type v. Since $F(v) - (v - t'(q_f(v)))f(v) = F(v) - bf(v)$ which is nondecreasing if and only if $(\ln f(v))' \geq -\frac{1}{b}$ by the assumption of the proposition, which verifies (c1). Third, $\tilde{v} - v^* - \int_{\tilde{v}}^{\overline{v}}(v - v^* - b)\frac{f(v)}{1-F(\tilde{v})}dv = \tilde{v} + b - \mathbb{E}[v|v \geq \tilde{v}] \leq 0$ for $\tilde{v} > v^*$by the decreasing MRL, which verifies (c2). Finally, $\underline{v} - t'(q_f(\underline{v})) = -b < 0$ which verifies (c3'). Therefore, the interval delegation is optimal and v^* is the only candidate for cutoff. **q.e.d.**

We now use Claim 8 to show that the interval delegation is a solution to program A as well. Indeed, suppose that this is not the case and there exists a q^A that brings higher value to program A than q^B. For any $\epsilon > 0$, we can find ε small enough so that there exists q_ε^A that is ϵ-close to q^A in the sup-norm and satisfies constraints of program B and in particular the maximized function is at most ϵ away from the value of program A. But this contradicts optimality of the interval delegation in program B. □

Proof of Theorem 4. We will show that for any equilibrium of the second-price auction, there is no $\omega_k \in (v^*, \overline{v})$. This implies that the partition generated by the second-price auction is cruder, and so the English auction is more efficient. Suppose to contradiction that there is $\omega_k \in (v^*, \overline{v})$ such that $\omega_{k-1} \leq v^*$. Notice that in equation (7) $\Lambda_{k+1} \leq \frac{1}{2}$,$1-\Lambda_k \geq \frac{1}{2}$, $\omega_k+b-\mathbb{E}[v|v \in [\omega_{k-1}, \omega_k]] \geq 0$ and $\omega_k + b - \mathbb{E}[v|v \in [\omega_k, \omega_{k+1}]] \leq 0$. Therefore, equation (7) implies

$$G(\omega_{k-1}, \omega_k)(\omega_k+b-\mathbb{E}[v|v \in [\omega_{k-1}, \omega_k)])+G(\omega_k, \omega_{k+1})(\omega_k+b-\mathbb{E}[v|v \in [\omega_k, \omega_{k+1})]) \leq 0$$

or

$$\omega_k+b-\frac{G(\omega_{k-1}, \omega_k)}{G(\omega_{k-1}, \omega_{k+1})}\mathbb{E}[v|v \in [\omega_{k-1}, \omega_k)]-\frac{G(\omega_k, \omega_{k+1})}{G(\omega_{k-1}, \omega_{k+1})}\mathbb{E}[v|v \in [\omega_k, \omega_{k+1})] \leq 0.$$

Observe that[22]

$$\frac{G(\omega_k, \omega_{k+1})}{G(\omega_{k-1}, \omega_{k+1})} \geq \frac{F(\omega_k, \omega_{k+1})}{F(\omega_{k-1}, \omega_{k+1})}.$$

Since $\mathbb{E}[v|v \in [\omega_k, \omega_{k+1})] \geq \mathbb{E}[v|v \in [\omega_{k-1}, \omega_k)]$,

$$\omega_k+b-\frac{F(\omega_{k-1}, \omega_k)}{F(\omega_{k-1}, \omega_{k+1})}\mathbb{E}[v|v \in [\omega_{k-1}, \omega_k)]-\frac{F(\omega_k, \omega_{k+1})}{F(\omega_{k-1}, \omega_{k+1})}\mathbb{E}[v|v \in [\omega_k, \omega_{k+1})] \leq 0$$

or

$$\omega_k + b - \mathbb{E}[v|v \in [\omega_{k-1}, \omega_{k+1})] \leq 0.$$

Then

$$\begin{aligned} \omega_k - b - \mathbb{E}[v|v \in [\omega_{k-1}, \omega_{k+1}]] &\geq \omega_k - b - \mathbb{E}[v|v \in [v^*, \omega_{k+1}]] \\ &\geq \omega_k - b - \mathbb{E}[v|v \geq v^*] \\ &= \omega_k - v^* > 0, \end{aligned}$$

which is a contradiction. □

[22]Indeed

$$\frac{F^{N-1}(\omega_{k+1}) - F^{N-1}(\omega_k)}{F^{N-1}(\omega_{k+1}) - F^{N-1}(\omega_{k-1})} \geq \frac{F(\omega_{k+1}) - F(\omega_k)}{F(\omega_{k+1}) - F(\omega_{k-1})}$$

if and only if

$$\gamma F^{N-1}(\omega_{k+1}) + (1-\gamma)F^{N-1}(\omega_{k-1}) \geq F^{N-1}(\omega_k)$$

for γ satisfying $\gamma F(\omega_{k+1}) + (1-\gamma)F(\omega_{k-1}) = F(\omega_k)$ which holds by Jensen's inequality.

Proofs for Section 5

Proof of Theorem 6. Let us verify that no player wants to deviate from described strategies. In the formal round, the value of each bidder is distributed according to $F(\cdot|v \in [\omega_{k-1}, \omega_k))$. By Theorem 2 specified strategies constitute an equilibrium in the formal round.

In the formal round, it is a weakly dominant strategy of bidders to truthfully reveal the interval to which their values belong and announce m_k. Indeed, by misreporting bidders affect whether they win or lose against opponents announcing m, but do not affect price at which they win. If a bidder reports $m > m_k$, then she faces a risk of winning at a price that is higher than her expected value if she wins against an opponent bidding m_{k+j}. At the same time, if all opponents bid weakly below m_k and some opponents bid exactly m_k, then the bidder wins against them and pays m_k. In this case, her expected payoff is 0. Notice that if instead she bid m_k, then she would tie with opponents bidding m_k and her payoff would be at least 0, as in the formal round, there is an option to quit in the beginning of the round and get 0. If the bidder reports $m < m_k$, then she misses an opportunity to win and pay a price lower than her expected value which is profitable. To complete the verification, we need to show that advisors do not have incentives to deviate. This is ensure by an appropriate choice of thresholds ω_k specified in equations (14). In particular, threshold type ω_k is indifferent between messages m_{k-1} and m_k. $\square$

Proofs for Section 6

Proof of Proposition 5. First, observe that (17) is the equation $\mathbb{E}[v|v < v^*] = v^* + b$ for the exponential distribution. The left-hand side of (17) is a strictly increasing function[23] which is $\frac{1}{\lambda}$ at $v^* = 0$ and converges to infinity as $v^* \to \infty$, while the right-hand side is greater than $\frac{1}{\lambda}$. Hence, there is a unique solution to (17).

We first verify that the advisor does not have incentives to deviate from her strategy. As a preliminary step, we derive the equilibrium of the first-price auction where bids are submitted directly by advisors and the lowest participating bidder has type v^* and simply bids her value $v^* + b$. The

[23] Indeed, its derivative is equal to

$$\frac{e^{-\lambda v}}{(1 - e^{-\lambda v})^2}(e^{\lambda v} - (1 + \lambda v)) > 0$$

advisor with type v solves the following problem

$$\max_{\sigma}(v+b-\sigma)F(\sigma^{-1}(\sigma)), \tag{37}$$

for which the first-order condition is

$$f(v)(v+b) = (F(v)\sigma(v))' \tag{38}$$

with the initial condition $\sigma(v^*) = \mathbb{E}[v|v<v^*] = v^* + b$. From (38),

$$\sigma(v) = \sigma(v^*)\frac{F(v^*)}{F(v)} + \frac{1}{F(v)}\int_{v^*}^{v} f(\hat{v})(\hat{v}+b)d\hat{v} = \mathbb{E}[\hat{v}|\hat{v}<v] + \frac{F(v)-F(v^*)}{F(v)}b, \tag{39}$$

which gives equation (18). Let $p^* = \sigma(v^*)$. The utility of the advisor from winning the auction is

$$v - \mathbb{E}[\hat{v}|\hat{v}<v] + b\frac{F(v^*)}{F(v)} \geq v + b - \mathbb{E}[\hat{v}|\hat{v}<v].$$

Since v^* solves (17), the advisor gets a positive utility from the auction for $v > v^*$.

If the bidder follows the recommendation of the advisor, then the strategy to stop when $p = \sigma(v)$ is optimal for the advisor when $v > v^*$, as it is an equilibrium strategy in the Dutch auction where the advisor decides when to stop. For $v \leq v^*$, the advisor gets utility $\frac{1}{N}(v - v^*) \leq 0$ if she follows the strategy and $v + b - \sigma(v_p)$ if she stop at a price above p^*. Since

$$v + b - \sigma(v_p) \leq v + b - \sigma(v^*) = v - v^* \leq \frac{1}{N}(v - v^*),$$

sending the message "stop" at price p^* is optimal for the advisor.

Notice that the mixed derivative in b and σ of the maximized function (37) is positive. Hence, if the bidder submits the bid, then she chooses a higher bid. Therefore, it is optimal for her to stop when she gets the message from the advisor with type $v > v^*$.

To finish the proof, we show that the bidder does not want to stop the auction earlier. Let $v_p \equiv \sigma^{-1}(p)$ for all $p > p^*$. Denote by $\hat{v}$ the value of the opponent bidder. The expected utility of the bidder at time t from following the recommendation of the advisor is

$$\mathbb{E}\left[(v-\sigma(v))1\{v>\hat{v}\}|v,\hat{v}<v_p\right] = \mathbb{E}\left[(v-\sigma(v))1\{v>\hat{v}\}|v^*<v<v_p;\hat{v}<v_p\right]\frac{F(v_p)-F(v^*)}{F(v_p)},$$

where we used the fact that at stage p^*, the bidder gets utility zero from winning. We need to compare this utility with the utility that the bidder gets if she quits before the advisor's message

$$\mathbb{E}[v|v < v_p] - \sigma(v_p) = -b\frac{F(v_p) - F(v^*)}{F(v_p)},$$

which boils down to showing that

$$\mathbb{E}\left[(v - \sigma(v))1\{v > \hat{v}\}|v^* < v < v_p; \hat{v} < v_p\right] + b \tag{40}$$

is non-negative. Using (39) and $\mathbb{E}[\hat{v}|\hat{v} < v] = \frac{1}{\lambda}\left(1 - \frac{vf(v)}{F(v)}\right)$ for the exponential distribution, we can re-write (40) as follows

$$\int_{v^*}^{v_p}\left(-b - \frac{1}{\lambda} + \frac{v + bF(v^*)}{F(v)}\right)\frac{F(v)}{F(v_p)}\frac{dF(v)}{F(v_p) - F(v^*)} + b$$

or rearranging terms

$$\int_{v^*}^{v_p}\left(F(v)\left(-b - \frac{1}{\lambda}\right) + v + bF(v^*)\right)dF(v) + bF(v_p)(F(v_p) - F(v^*)). \tag{41}$$

We will show that (41) is increasing in v_p. Since (41) is zero at $v_p = v^*$, this would imply that (41) is non-negative for all $v_p > v^*$. The derivative of (41) is equal to

$$f(v_p)\left(F(v_p)\left(-b - \frac{1}{\lambda}\right) + v_p + bF(v^*) + b(2F(v_p) - F(v^*))\right) = f(v_p)F(v_p)\left(b - \frac{1}{\lambda} + \frac{v_p}{F(v_p)}\right) > 0$$

where the inequality follows from the fact that v^* is the unique solution to (17). □

Proof of Theorem 7. We first show that strategies described in Theorem 7 constitute an equilibrium of the Dutch auction.

Indeed, the left-hand side of equation (19) is greater than $\underline{v}$ and bounded from above by $\mathbb{E}[v]$. The right-hand side of equation (19) is less than $\underline{v}$ for small v^* and is greater than $\mathbb{E}[v]$ for sufficiently large v^*. By continuity, equation (19) has a solution.

To prove that conjectured strategies constitute an equilibrium, we need to show that the advisor sends the message "stop" at the optimal time given that bidder follows her recommendation, and that the bidder prefers to follow recommendations of the advisor.

Optimality of the advisor First, we show that strategy $\sigma(\cdot)$ is optimal for the advisor. The advisor of type v solves the following problem

$$\max_{\sigma}(v + b - \sigma)G(\sigma^{-1}(\sigma)), \tag{42}$$

for which the first-order condition is

$$g(v)(v + b) = (G(v)\sigma(v))' \tag{43}$$

with the initial condition $\sigma(v^*) = v^* + b$. From (43),

$$\sigma(v) = \frac{G(v^*)}{G(v)}(v^* + b) + \frac{1}{G(v)}\int_{v^*}^{v} g(\hat{v})(\hat{v} + b)d\hat{v} =$$

$$\frac{G(v^*)}{G(v)}(\mathbb{E}[v|v < v^*]) + \frac{G(v) - G(v^*)}{G(v)}(\mathbb{E}[\hat{v}|\hat{v} \in [v^*, v]] + b) =$$

$$b\frac{G(v) - G(v^*)}{G(v)} + \mathbb{E}[\hat{v}|\hat{v} < v] - (\mathbb{E}[\hat{v}|\hat{v} < v^*] - \mathbb{E}[v|v < v^*])\,\frac{G(v^*)}{G(v)} =$$

$$b + \mathbb{E}[\hat{v}|\hat{v} < v] + (v^* - \mathbb{E}[\hat{v}|\hat{v} < v^*])\,\frac{G(v^*)}{G(v)}. \tag{44}$$

The equilibrium bid is equal to expectation of $\max\{v^*, \hat{v}\} + b$ conditional on $\hat{v} < v$. Given (44), the utility of the advisor from winning the auction equals

$$v - \mathbb{E}[\hat{v}|\hat{v} < v] - (v^* - \mathbb{E}[\hat{v}|\hat{v} < v^*])\frac{G(v^*)}{G(v)} =$$

$$\frac{1}{G(v)}\left(G(v)v - G(v^*)v^* - \mathbb{E}[\hat{v} : \hat{v} \in [v^*, v]]\right) =$$

$$\int_{v^*}^{v} \frac{G(\omega)}{G(v)}d\omega > 0. \tag{45}$$

Hence, if the bidder follows her strategy, then it is optimal for the advisor to follow her strategy.

Optimality of the bidder By the single-crossing property of payoffs, when the bidder knows v, the bidder prefers to stop the auction earlier. Hence, having received the message "stop" from the advisor, the bidder prefers to stop immediately. It remains to check that the bidder does not want to quit the auction before she gets a recommendation from the advisor. By (44), if the bidder quits at time t, then her payoff equals

$$\mathbb{E}[v|v<v_p]-\sigma(v_p)=\mathbb{E}[v|v<v_p]-b-\mathbb{E}[\hat{v}|\hat{v}<v_p]-(v^*-\mathbb{E}[\hat{v}|\hat{v}<v^*])\frac{G(v^*)}{G(v_p)}. \tag{46}$$

On the other hand, if the bidder follows her equilibrium strategy, then her expected utility is given by

$$\mathbb{E}\left[(v-\sigma(v))1\{v>\hat{v}\}|\hat{v},v<v_p\right]=$$

$$\mathbb{E}\left[(v-\sigma(v))1\{v>\hat{v}\}|v^*<v<v_p,\hat{v}<v_p\right]\frac{F(v_p)-F(v^*)}{F(v_p)}=$$

$$\frac{1}{F(v_p)^2}\int_{v^*}^{v_t}\left(v-b-\mathbb{E}[\hat{v}|\hat{v}<v]-(v^*-\mathbb{E}[\hat{v}|\hat{v}<v^*])\frac{G(v^*)}{G(v)}\right)F(v)dF(v), \tag{47}$$

where the first equality is by the fact that $\mathbb{E}\left[(v-\sigma(v))1\{v>\hat{v}\}|v<v^*,\hat{v}<v_p\right]=0$, the second equality is by (44). We need to show that (46) is less than(47). We evaluate the difference

$$\int_{v^*}^{v_p}\left(v-b-\mathbb{E}[\hat{v}|\hat{v}<v]-(v^*-\mathbb{E}[\hat{v}|\hat{v}<v^*])\frac{G(v^*)}{G(v)}\right)F(v)dF(v)-$$
$$F^2(v_p)\left(\mathbb{E}[v|v<v_p]-b-\mathbb{E}[\hat{v}|\hat{v}<v_p]-(v^*-\mathbb{E}[\hat{v}|\hat{v}<v^*])\frac{G(v^*)}{G(v_p)}\right). \tag{48}$$

The derivative of (48) divided by $f(v_p)F(v_p)$ is equal to

$$v_p-b-\mathbb{E}[\hat{v}|\hat{v}<v_t]-(v^*-\mathbb{E}[\hat{v}|\hat{v}<v^*])\frac{G(v^*)}{G(v_p)}-$$
$$2\left(\mathbb{E}[v|v<v_p]-b-\mathbb{E}[\hat{v}|\hat{v}<v_p]-(v^*-\mathbb{E}[\hat{v}|\hat{v}<v^*])\frac{G(v^*)}{G(v_p)}\right)-$$
$$F(v_p)\left(\frac{1}{F(v_p)}(v_t-\mathbb{E}[v|v<v_p])-\frac{(N-1)F^{N-2}(v_p)}{G(v_p)}\left(v_p-\mathbb{E}[\hat{v}|\hat{v}<v_p]-(v^*-\mathbb{E}[\hat{v}|\hat{v}<v^*])\frac{G(v^*)}{G(v_p)}\right)\right)=$$
$$v_p-2\mathbb{E}[v|v<v_p]+b+\mathbb{E}[\hat{v}|\hat{v}<v_p]+(v^*-\mathbb{E}[\hat{v}|\hat{v}<v^*])\frac{G(v^*)}{G(v_p)}-$$
$$\left(v_p-\mathbb{E}[v|v<v_p]-(N-1)(v_p-\mathbb{E}[\hat{v}|\hat{v}<v_p])+(v^*-\mathbb{E}[\hat{v}|\hat{v}<v^*])\frac{G(v^*)}{G(v_p)}(N-1)\right)=$$
$$(N-1)(v_p-\mathbb{E}[\hat{v}|\hat{v}<v_p])+(\mathbb{E}[\hat{v}|\hat{v}<v_p]-\mathbb{E}[v|v<v_p])+b-(v^*-\mathbb{E}[\hat{v}|\hat{v}<v^*])\frac{G(v^*)}{G(v_p)}(N-2)=$$

$$v_p - \mathbb{E}[v|v < v_p] + b + (N-2)\left(v_p - \mathbb{E}[\hat{v}|\hat{v} < v_p] - (v^* - \mathbb{E}[\hat{v}|\hat{v} < v^*])\,\frac{G(v^*)}{G(v_p)}\right) \tag{49}$$

In (49) $v_p - \mathbb{E}[v|v < v_p] + b > 0$ by the fact that v^* is the largest solution to (19). The remaining term in (49) is positive by (45). Hence, the derivative of (48) is positive. Therefore, since at $v_p = v^*$, the expression (48) is equal to zero by (19), for $\geq v^*$ the expression (48) is non-negative. This proves that the bidder prefers to follow recommendations of the advisor rather than stop the auction earlier. □

Proof of Theorem 8. We want to show that there is no partition $(\omega_k)_{k=1}^K$ induced by the equilibrium of the second-price auction such that $\omega_k \in [\underline{v}, v^*]$. Since v^* is the unique solution to (19) and $\underline{v} + b - \mathbb{E}[v|v \leq \underline{v}] = b < 0$, $\omega_k + b - \mathbb{E}[v|v \leq \omega_k] < 0$. Therefore,

$$\omega_k + b - \mathbb{E}[v|v \in [\omega_{k-1}, \omega_k)] \quad \leq \quad \omega_k + b - \mathbb{E}[v|v < \omega_k] < 0,$$

which contradicts the fact that the first term in (7) should be positive. □

9 Online Appendix (Not for Publication)

Analysis of the M&A Auction

We provide calculations that for the M&A auction under the log-normal specification of values. Let $y_k^* = \frac{\ln v_k^* - \mu}{\sqrt{2}\sigma}$ and $y_k = \frac{\ln \omega_k - \mu}{\sqrt{2}\sigma}$. Then

$$\begin{aligned}
\mathbb{E}[v|v \in [v_k^*, \omega_k)] &= \frac{\int_{v_k^*}^{\omega_k} \frac{v}{v\sigma\sqrt{2\pi}} e^{-\frac{(\ln v - \mu)^2}{2\sigma^2}} dv}{\frac{1}{2}\text{erfc}\left(\frac{\ln v_k^* - \mu}{\sqrt{2}\sigma}\right) - \frac{1}{2}\text{erfc}\left(\frac{\ln \omega_k - \mu}{\sqrt{2}\sigma}\right)} \\
&= \frac{\frac{2}{\sqrt{\pi}} \int_{y_k^*}^{y_k} e^{-y^2 + \sqrt{2}\sigma y + \mu} dy}{\text{erfc}\left(y_k^*\right) - \text{erfc}\left(y_k\right)} \\
&= e^{\mu + \frac{\sigma^2}{2}} \frac{\frac{2}{\sqrt{\pi}} \int_{y_k^*}^{y_k} e^{-\left(y - \frac{\sigma}{\sqrt{2}}\right)^2} dy}{\text{erfc}\left(y_k^*\right) - \text{erfc}\left(y_k\right)} \\
&= e^{\mu + \frac{\sigma^2}{2}} \frac{\text{erfc}\left(y_k^* - \frac{\sigma}{\sqrt{2}}\right) - \text{erfc}\left(y_k - \frac{\sigma}{\sqrt{2}}\right)}{\text{erfc}\left(y_k^*\right) - \text{erfc}\left(y_k\right)},
\end{aligned}$$

and equation (13) becomes

$$e^{\sqrt{2}\sigma y_k^* - \frac{\sigma^2}{2}} + be^{-\mu - \frac{\sigma^2}{2}} = \frac{\text{erfc}\left(y_k^* - \frac{\sigma}{\sqrt{2}}\right) - \text{erfc}\left(y_k - \frac{\sigma}{\sqrt{2}}\right)}{\text{erfc}\left(y_k^*\right) - \text{erfc}\left(y_k\right)}$$

Construction of an equilibrium To construct an equilibrium with communication in both stages of bidding, fix an equilibrium of the English auction characterized by cutoff v^*. Then we can start with some v_1^* and ω_1 that satisfy (13) and get β_2 from equation (14). Indeed, β_1 can be chosen arbitrary as no bidder wins and pays price β_1 and $\omega_{k-1} = \underline{v}$. Given β_2, we can try to choose ω_2 (and hence m_2) so that inequalities (15) and (16). Knowing ω_2 we determine v_2^* from equation (13) and proceed this way until we reach ω_{K+1} exceeding v^*. At this point, we need to check whether there is a price β_K that types of advisor in $[\omega_K, \overline{v}]$ recommend so that inequalities (15) and (16) are satisfied. If not, we check if they are satisfied for some β_{K-1} and types of advisor in $[\omega_{K-1}, \overline{v}]$ and proceed this way until we construct an equilibrium.

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