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On a family of quintic Thue equations

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We consider the parametrized quintic family of Thue equations

$$X(X^2 - Y^2)(X^2 - a^2Y^2) - Y^5 = \pm 1 \qquad a \in \mathbb{Z}$$

and prove that it only has trivial solutions for $|a| \geq 3.6 \cdot 10^{19}$ using a recent estimate for linear forms in three logarithms of algebraic numbers by P. Voutier.

1. Introduction

Let $F \in \mathbb{Z}[X, Y]$ be an irreducible form of degree ≥ 3 and m a nonzero integer. The diophantine equation

$$F(X, Y) = m \tag{1.1}$$

is called *Thue equation* in honour of A. Thue (1909), who proved that (1.1) only has a finite number of solutions $(x, y) \in \mathbb{Z}^2$. His result was a consequence of his theorem on diophantine approximation, which is not effective. A. Baker (1968) gave an effective upper bound for $\max(|x|, |y|)$ based on his studies on linear forms in logarithms of algebraic numbers; applying a numerical reduction method due to Baker and Davenport (1969), it is possible to calculate all solutions of a single Thue equation with a computer, see Pethő and Schulenberg (1987), Tzanakis and de Weger (1989), Bilu and Hanrot (1996).

In the last years, several parametrized families of Thue equations have been investigated; Thomas (1990) — he was the first considering parametric families —, Mignotte *et al.* (1996b), Lee (1992), Mignotte and Tzanakis (1991), and Thomas (1993) considered cubic families. Quartic families have been solved by Pethő (1991), Mignotte *et al.* (1996a), Lettl and Pethő (1995), Chen and Voutier (1997), and Pethő and Tichy (1997); in Lettl *et al.* (1997b, 1997a) a sextic family has been completely solved. Halter-Koch *et al.* (1997) considered a very general family of arbitrary degree, assuming a very deep conjecture of Lang and Waldschmidt (see Lang (1978)).

In this paper we consider the family of quintic Thue equations

$$F_a(X, Y) := X(X^2 - Y^2)(X^2 - a^2Y^2) - Y^5 = u = \pm 1. \tag{1.2}$$

We will prove:

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Theorem 1.1. *Let* $|a| \geq 3.6 \cdot 10^{19}$. *Then the only solutions of (1.2) are* $F_a(\pm 1, 0) = \pm 1$ *and* $F_a(0, \pm 1) = F_a(1, \pm 1) = F_a(-1, \pm 1) = F_a(a, \pm 1) = F_a(-a, \pm 1) = \mp 1$.

Using Kash (see Daberkow *et al.* (1997)), the Theorem has been verified for $|a| \leq 100$, which took about one week on a DEC Alpha with 275 MHz, we conjecture that the Theorem is valid for all $a \in \mathbb{Z}$.

For some comments on the size of the constant in the Theorem see the remark at the end of the paper.

Put

$$f_a(X) := F_a(X, 1) = X(X^2 - 1)(X^2 - a^2) - 1$$

and denote its roots by $\alpha = \alpha^{(1)}, \ldots, \alpha^{(5)}$. For a solution (x, y) of (1.2) we have

$$F_a(x, y) = \prod_{\nu=1}^{5} (x - \alpha^{(\nu)} y) = \mathrm{N}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(x - \alpha y) = \pm 1. \tag{1.3}$$

This means that $x - \alpha y$ is a unit in the order $\mathcal{O} := \mathbb{Z}[\alpha]$.

To prove Theorem 1.1, we will first investigate the structure of $\mathcal{O}^\times$, the unit group of $\mathcal{O}$, then we will define linear forms in logarithms, derive upper bounds for them contradicting to the lower bounds, given by a Theorem of Voutier (1997), for large values of a.

Most of the calculations involve heavy manipulations with asymptotic approximations, they have been carried out by some kind of symbolic interval arithmetic using the computer algebra system Maple.

2. Elementary observations

Since $F_{-a} = F_a$, it suffices to consider nonnegative values of a. By $F_a(-X, -Y) = -F_a(X, Y)$, we only have to consider solutions (x, y) with $y \geq 0$.

If $y = 0$, then $|x| = 1$; if $y = 1$ we have either

$$x(x^2 - 1)(x^2 - a^2) = 0,$$

which has the solutions indicated in Theorem 1.1, or

$$x(x^2 - 1)(x^2 - a^2) = 2,$$

which leads to $3|(x-1)x(x+1)|2$, a contradiction. Therefore, we can suppose $y \geq 2$.

Throughout the paper, we will need estimates for the roots $\alpha^{(\nu)}$, $\nu = 1, \ldots, 5$.

Lemma 2.1. *Let* $a \geq 3$. *Then* f_a *has five real roots* $\alpha^{(1)} > \alpha^{(2)} > \alpha^{(3)} > \alpha^{(4)} > \alpha^{(5)}$ *satisfying the following estimates for* $a \geq 13$:

$$\begin{aligned}
a + \frac{1}{2}\frac{1}{a^4} + \frac{49}{100}\frac{1}{a^6} &\leq \alpha^{(1)} \leq a + \frac{1}{2}\frac{1}{a^4} + \frac{51}{100}\frac{1}{a^6}\\
1 - \frac{1}{2}\frac{1}{a^2} - \frac{7}{8}\frac{1}{a^4} - \frac{63}{50}\frac{1}{a^6} &\leq \alpha^{(2)} \leq 1 - \frac{1}{2}\frac{1}{a^2} - \frac{7}{8}\frac{1}{a^4} - \frac{31}{25}\frac{1}{a^6}\\
\frac{1}{a^2} + \frac{99}{100}\frac{1}{a^6} &\leq \alpha^{(3)} \leq \frac{1}{a^2} + \frac{101}{100}\frac{1}{a^6}\\
-1 - \frac{1}{2}\frac{1}{a^2} - \frac{1}{8}\frac{1}{a^4} - \frac{19}{25}\frac{1}{a^6} &\leq \alpha^{(4)} \leq -1 - \frac{1}{2}\frac{1}{a^2} - \frac{1}{8}\frac{1}{a^4} - \frac{37}{50}\frac{1}{a^6}\\
-a + \frac{1}{2}\frac{1}{a^4} + \frac{49}{100}\frac{1}{a^6} &\leq \alpha^{(5)} \leq -a + \frac{1}{2}\frac{1}{a^4} + \frac{51}{100}\frac{1}{a^6}.
\end{aligned}$$

Proof. These inequalities can easily be verified by considering the sign of f_a at the points $a + \frac{1}{2}a^{-4} + \frac{49}{100}a^{-6}, \dots$ □

Sometimes, approximations of higher order will be needed; they can be obtained performing two or three symbolic Newton steps starting at $a, 1, 0, -1, -a$ respectively, calculating an asymptotic expansion by Maple and verifying as in the proof above.

Clearly, there is no linear factor dividing f_a, indeed, it is elementary to show that f_a is irreducible for all values $a \in \mathbb{Z}$.

3. Algebraic properties of the number field

Since a solution of (1.3) corresponds to a unit in $\mathcal{O}$, we have to investigate the unit group of this order.

Theorem 3.1. *Let $a \geq 3$, α be a root of f_a and $\mathcal{O} = \mathbb{Z}[\alpha]$. Then*

$$\mathcal{O}^\times = \langle -1, \alpha, \alpha - 1, \alpha + 1, \alpha - a \rangle .$$

Proof. Since $f_a(\alpha) = 0$, we see that $\alpha, \alpha - 1, \alpha + 1, \alpha - a$ are units in $\mathcal{O}$.

We will derive an upper bound for the index of $\langle -1, \alpha, \alpha - 1, \alpha + 1, \alpha - a \rangle$ in the unit group of $\mathcal{O}$ by estimating the regulators of the two groups. The discriminant $D_\mathcal{O} = D(f_a)$ is given by

$$D(f_a) = 16a^{14} - 64a^{12} - 12a^{10} + 296a^8 - 364a^6 - 380a^4 + 360a^2 + 3017 > 15.99a^{14}$$

for $a \geq 81$. By Pohst and Zassenhaus (1989), chapter V, (6.22), the regulator of $\mathcal{O}$ can be estimated as follows:

$$R_\mathcal{O} \geq \left[\left(\frac{3 \log^2(D_\mathcal{O}/16)}{4 \cdot 5 \cdot 6} \right)^4 \frac{2^0}{5 \cdot 4} \right]^{\frac{1}{2}} \geq \frac{\log^4(D_\mathcal{O}/16)}{2^7 \cdot 5^2 \sqrt{5}} \geq \frac{\log^4\left(\frac{15.99}{16}a^{14}\right)}{2^7 \cdot 5^2 \sqrt{5}} .$$

Let R_α be the regulator of $\langle -1, \alpha, \alpha - 1, \alpha + 1, \alpha - a \rangle$:

$$R_\alpha = \begin{vmatrix} \log|\alpha^{(1)}| & \log|\alpha^{(1)} - 1| & \log|\alpha^{(1)} + 1| & \log|\alpha^{(1)} - a| \\ \log|\alpha^{(2)}| & \log|\alpha^{(2)} - 1| & \log|\alpha^{(2)} + 1| & \log|\alpha^{(2)} - a| \\ \log|\alpha^{(3)}| & \log|\alpha^{(3)} - 1| & \log|\alpha^{(3)} + 1| & \log|\alpha^{(3)} - a| \\ \log|\alpha^{(4)}| & \log|\alpha^{(4)} - 1| & \log|\alpha^{(4)} + 1| & \log|\alpha^{(4)} - a| \end{vmatrix} . \tag{3.1}$$

Application of Lemma 2.1 yields for $a \geq 101$:

$$20 \log^4 a < R_\alpha < 20 \log^4 a + 19.659 \log^3 a + 4 \log^2 a + 0.05 \log a + 0.01.$$

Hence $R_\alpha > 0$ and $\alpha, \alpha - 1, \alpha + 1, \alpha - a$ are independent units.

Now, we can bound the index $I := [\mathcal{O}^\times :< -1, \alpha, \alpha - 1, \alpha + 1, \alpha - a >]$ by

$$I = \frac{R_\alpha}{R_\mathcal{O}} < \frac{20 \log^4 a + 19.659 \log^3 a + 4 \log^2 a + 0.05 \log a + 0.01}{5.368 \log^4 a - .001 \log^3 a + .6 \log^2 a - 2 \cdot 10^{-12} \log a} < 4 \tag{3.2}$$

for $a \geq 686\,312$. Hence $I \in \{1, 2, 3\}$ in this case.

In order to exclude the cases $I = 2$ and $I = 3$, we need the following result of Mahler (1964):

Proposition 3.2. *Let γ be an algebraic integer of degree $d \geq 2$ with conjugates $\gamma = \gamma^{(1)}, \dots, \gamma^{(d)}$ and*

$$M(\gamma) := \prod_{k=1}^{d} \max\left\{1, \left|\gamma^{(k)}\right|\right\}.$$

Then

$$\left|D_{\mathbb{Z}[\gamma]}\right| \leq d^d \cdot M(\gamma)^{2(d-1)}.$$

In our case, this yields for $a \geq 81$ and $\gamma \in \mathcal{O} \setminus \mathbb{Z}$

$$M(\gamma) \geq \left(\frac{15.99}{5^5}\right)^{1/8} \cdot a^{7/4}. \tag{3.3}$$

Assume now $I = 3$. Then there exists an $1 \neq \varepsilon \in \mathcal{O}^\times$ satisfying

$$\varepsilon^3 = \alpha^i (\alpha-1)^j (\alpha+1)^k (\alpha-a)^l \tag{3.4}$$

with integers i, j, k, l, because we can assume ε to be positive. Furthermore we can assume $-1 \leq i, j, k \leq 1$ and $0 \leq l \leq 1$.

As $K = \mathbb{Q}(\alpha)$ is a quintic number field, it is primitive, and ε is a quintic algebraic integer. Let

$$h(X) = \prod_{i=1}^{5} (X - \varepsilon^{(i)}) = \sum_{j=0}^{5} d_j X^j$$

be its minimal polynomial with $d_j \in \mathbb{Z}$.

Application of Lemma 2.1 yields

$$M(\alpha) < 1.01\, a^2, \quad M(\alpha-1) < 2.0\, a^2, \quad M(\alpha+1) < 2.0\, a^2.$$

First assume that $l = 0$ and that at most two of the three numbers i, j, k are nonzero. By (3.3), (3.4), $M(\gamma^n) = M(\gamma)^{|n|}$ and $M(\gamma_1 \cdot \gamma_2) \leq M(\gamma_1) M(\gamma_2)$ we have

$$\left(\frac{15.99}{5^5}\right)^{1/8} a^{7/4} \leq M(\varepsilon) \leq (2a^2)^{2/3},$$

which is a contradiction for $a \geq 8$.

Consider now the case $i = j = k = 1$ and $l = 0$. Then Mahler's estimate does not help us, but using approximations of $\alpha^{(\nu)}$ of order 10 and asymptotic expansions of third roots, we obtain for $a \geq 227$

$$-a^2 + \frac{2}{3} + \frac{3}{a^{4/3}} < d_3 < -a^2 + \frac{2}{3} + \frac{3.1}{a^{4/3}}.$$

Therefore d_3 cannot be an integer, a contradiction.

As a third case, consider $(i, j, k, l) = (2, 2, 0, 1)$, which is equivalent to $(i, j, k, l) = (-1, -1, 0, 1)$. In this case, we obtain

$$\begin{aligned} 3\,a^4 + 6\,a^2 + 2\,a - 1.26 + \frac{3}{a^{1/3}} \\ < \left(-3\,a^3 - 4\,a - 1\right) d_1 + \left(3\,a^2 + 2\right) d_2 - 3\,d_3 a + 3\,d_4 < \\ < 3\,a^4 + 6\,a^2 + 2\,a - 1.25 + \frac{4}{a^{1/3}}. \end{aligned}$$

The expression in the middle should be an integer, but this is impossible for $a \geq 4097$.

Actually, all possible cases occurring for the exponents i, j, k, l can be dealt with by using one of the above three types of arguments, if $a \geq 765\,432$; for a complete list see Heuberger (1997). The case $I = 2$ can be treated in the same way, here we have to be careful of the signs of the $\varepsilon^{(\nu)}$, a complete list can be received from the author.

For $3 \leq a \leq 800\,000$ the index bound (3.2) has been computed explicitely and the equation

$$\varepsilon^n = \alpha^i(\alpha-1)^j(\alpha+1)^k(\alpha-a)^l$$

has been checked for all possible tuplets (n,i,j,k,l) using the computational number theory system PARI. In all cases the only solution was $i = j = k = l = 0$. This verification took about one week on a Pentium 100 computer. □

We will also work in the splitting field of f_a, so we investigate its Galois group.

Theorem 3.3. *For $a \in \mathbb{Z}$, we have* $\mathrm{Gal}(f_a) = S_5$.

Proof. According to Cohen (1996), Algorithm 6.3.9., we have to check that

$$R = \prod_{\sigma\in H}\left(X - F(\alpha^{(\sigma(1))}, \dots \alpha^{(\sigma(5))})\right)$$

with

$$\begin{aligned} F(x_1,x_2,x_3,x_4,x_5) \quad = \quad & {x_1}^2\left(x_2x_5+x_3x_4\right)+{x_2}^2\left(x_1x_3+x_4x_5\right)+{x_3}^2\left(x_1x_5+x_2x_4\right) \\ & +{x_4}^2\left(x_1x_2+x_3x_5\right)+{x_5}^2\left(x_1x_4+x_2x_3\right) \end{aligned}$$

and

$$H = \{\mathrm{id}, (12), (13), (14), (15), (25)\}$$

does not have an integral root. But this can immediately be checked by calculating bounds for $\varepsilon_\sigma := F(\alpha^{(\sigma(1))}, \dots \alpha^{(\sigma(5))})$ with $\sigma \in H$ applying Lemma 2.1. □

4. Approximation properties of the solutions

Let $(x,y) \in \mathbb{Z}^2$ be a solution of (1.2) with $y \geq 2$. For $\nu = 1,\dots,5$, we define $\beta^{(\nu)} := x - \alpha^{(\nu)}y$, which are units by (1.3). Writing

$$\eta_1^{(\nu)} := \alpha^{(\nu)}, \quad \eta_2^{(\nu)} := \alpha^{(\nu)} - 1, \quad \eta_3^{(\nu)} := \alpha^{(\nu)} + 1, \quad \eta_4^{(\nu)} := \alpha^{(\nu)} - a$$

and applying Theorem 3.1 we have

$$\beta^{(\nu)} = \pm\left(\eta_1^{(\nu)}\right)^{u_1}\left(\eta_2^{(\nu)}\right)^{u_2}\left(\eta_3^{(\nu)}\right)^{u_3}\left(\eta_4^{(\nu)}\right)^{u_4} \tag{4.1}$$

with $u_1,\dots,u_4 \in \mathbb{Z}$. We define

$$U := \max\{|u_1|,\dots,|u_4|\}.$$

We will now use standard material — cf. Bilu and Hanrot (1996) — to derive asymptotic expressions for the $\beta^{(\nu)}$. By (1.3) we see that

$$\prod_{\nu=1}^{5}\left|\frac{x}{y} - \alpha^{(\nu)}\right| = \frac{1}{y^5},$$

hence x/y is an approximation to some $\alpha^{(\nu)}$. To record this, we define the index j by

$$\left|\beta^{(j)}\right| = \min_{\nu\in\{1,\dots,5\}} \left|\beta^{(\nu)}\right|$$

and say that (x, y) is a solution of type j. Then the following lemma holds:

LEMMA 4.1. *Let $a \geq 14$ and (x, y) be a solution of (1.2) of type j with $y \geq 2$. Then we have*

$$\left|x - \alpha^{(j)}y\right| \leq \frac{16}{y^4\left|f_a'(\alpha^{(j)})\right|} \leq \frac{1}{2y}, \tag{4.2}$$

hence x/y is a principal convergent of $\alpha^{(j)}$, and for $\nu \neq j$

$$\left|x - \alpha^{(\nu)}y\right| \geq \frac{\left|\alpha^{(\nu)} - \alpha^{(j)}\right|}{2}y. \tag{4.3}$$

PROOF. We have

$$y \cdot \left|\alpha^{(\nu)} - \alpha^{(j)}\right| \leq \left|x - \alpha^{(\nu)}y\right| + \left|x - \alpha^{(j)}y\right| \leq 2\left|x - \alpha^{(\nu)}y\right|,$$

and (4.3) is proved.

Furthermore, (1.3) gives

$$\left|f_a'(\alpha^{(j)})\right| = \prod_{\nu\neq j}\left|\alpha^{(\nu)} - \alpha^{(j)}\right| \leq \frac{16}{y^4}\prod_{\nu\neq j}\left|x - \alpha^{(\nu)}y\right| = \frac{16}{y^4\left|x - \alpha^{(j)}y\right|}.$$

Application of Lemma 2.1 yields for $a \geq 14$

$$2a^4 - 2a^2 + \frac{6}{a} \leq f'(\alpha^{(1)}) \leq 2a^4 - 2a^2 + \frac{8}{a} \tag{4.4}$$

$$-2a^2 + 4 \leq f'(\alpha^{(2)}) \leq -2a^2 + 6 \tag{4.5}$$

$$a^2 - \frac{4}{a^2} \leq f'(\alpha^{(3)}) \leq a^2 - \frac{3}{a^2} \tag{4.6}$$

$$-2a^2 - 2 \leq f'(\alpha^{(4)}) \leq -2a^2 \tag{4.7}$$

$$2a^4 - 2a^2 - \frac{8}{a} \leq f'(\alpha^{(5)}) \leq 2a^4 - 2a^2 - \frac{6}{a}, \tag{4.8}$$

hence $\left|f_a'(\alpha^{(j)})\right| \geq a^2/2 \geq 4$, and the lemma is proved, since we assume $y \geq 2$. □

5. A linear form in logarithms of algebraic numbers

For pairwise distinct $l, p, q \in \{1, \dots, 5\}$ Siegel's identity holds:

$$\frac{x - \alpha^{(p)}y}{x - \alpha^{(q)}y} \cdot \frac{\alpha^{(q)} - \alpha^{(l)}}{\alpha^{(p)} - \alpha^{(l)}} - 1 = \frac{x - \alpha^{(l)}y}{x - \alpha^{(q)}y} \cdot \frac{\alpha^{(q)} - \alpha^{(p)}}{\alpha^{(p)} - \alpha^{(l)}} \tag{5.1}$$

If we choose $l = j$, the right hand side will become small, and so by (4.1) and Lemma 4.1 the absolute value of the linear form

$$\Lambda_{p,q,j} := u_1 \log\left|\frac{\eta_1^{(p)}}{\eta_1^{(q)}}\right| + u_2 \log\left|\frac{\eta_2^{(p)}}{\eta_2^{(q)}}\right| + u_3 \log\left|\frac{\eta_3^{(p)}}{\eta_3^{(q)}}\right| + u_4 \log\left|\frac{\eta_4^{(p)}}{\eta_4^{(q)}}\right| + \log\left|\frac{\alpha^{(q)} - \alpha^{(j)}}{\alpha^{(p)} - \alpha^{(j)}}\right|$$

will be very small. According to the type of (x, y), we will now choose p and q, give an upper bound for this linear form and investigate relations between the u_i.

LEMMA 5.1. *Let $a \geq 12\,089$ and (x, y) be a solution of (1.2) of type j with $y \geq 2$. The following estimates hold, according to the value of j:*

$j \in \{1, 5\}$: *$U > c_j a^2 \log a$, $|u_1 - u_3| < U/(2a \log a)$ and $|2u_1 - u_2 - u_3| < U/(c_j a^2 \log a)$, where $c_1 = 1$ and $c_5 = 2$. We have $\log |\Lambda_{2,4,j}| < -H_j U \log a$ with $H_1 = 5$ and $H_5 = 500/103$.*

$j \in \{2, 4\}$: *$U > 2a \log a$, $|u_4| < U/(2a \log a)$ and $|3u_1 + u_2 + u_3| < U/(c_j a \log a)$, where $c_2 = 2/3$ and $c_4 = 1$. We have $\log |\Lambda_{1,5,j}| < -(5/2)U \log a$.*

$j = 3$: *$U > 4a^3 \log^2 a$, $|u_4| < U/(4a^3 \log^2 a)$ and $|u_2 - u_3| < (3U)/(4a^2 \log a)$. We have $\log |\Lambda_{1,5,3}| < -(500/203)U \log a$*

The symmetric nature of our equation (1.2) enabled us to collect similar cases, as it can be seen in this lemma.

PROOF. We prove this lemma only for $j = 2$, because the proofs of the other cases are analogous.

Lemma 2.1 yields $\lfloor \alpha^{(2)} \rfloor = 0$ and $\lfloor \frac{1}{\alpha^{(2)}} \rfloor = 1$, hence the continued fraction expansion of $\alpha^{(2)}$ starts with $\left[0, 1, \alpha_2^{(2)}\right]$, where

$$2a^2 - 4.5 + \frac{1}{a^2} < \alpha_2^{(2)} = \frac{\alpha^{(2)}}{1 - \alpha^{(2)}} < 2a^2 - 4.5 + \frac{2}{a^2}.$$

Therefore we have $\left\lfloor \alpha_2^{(2)} \right\rfloor = 2a^2 - 5$ and hence by Lemma 4.1 we get

$$y \geq 2a^2 - 5 > a^2. \tag{5.2}$$

By (4.2), (4.5) and (5.2) we have

$$-\frac{16}{a^{10}} \leq -\frac{16}{y^4 a^2} < \beta^{(2)} = x - \alpha^{(2)} y < \frac{16}{y^4 a^2} \leq \frac{16}{a^{10}},$$

hence

$$\alpha^{(2)} - \frac{16}{a^{12}} < \frac{x}{y} < \alpha^{(2)} + \frac{16}{a^{12}},$$

and for $\nu \neq 2$

$$y\left(\alpha^{(2)} - \alpha^{(\nu)} - \frac{16}{a^{12}}\right) < \beta^{(\nu)} = x - \alpha^{(\nu)} y < y\left(\alpha^{(2)} - \alpha^{(\nu)} + \frac{16}{a^{12}}\right). \tag{5.3}$$

Taking logarithms of the conjugates of (4.1), we have the following system of linear equations in the u_i:

$$\begin{aligned}
\log\left|\beta^{(5)}\right| &= u_1 \log\left|\eta_1^{(5)}\right| + u_2 \log\left|\eta_2^{(5)}\right| + u_3 \left|\eta_3^{(5)}\right| + u_4 \log\left|\eta_4^{(5)}\right| \\
\log\left|\frac{\beta^{(1)}}{\beta^{(5)}}\right| &= u_1 \log\left|\frac{\eta_1^{(1)}}{\eta_1^{(5)}}\right| + u_2 \log\left|\frac{\eta_2^{(1)}}{\eta_2^{(5)}}\right| + u_3 \log\left|\frac{\eta_3^{(1)}}{\eta_3^{(5)}}\right| + u_4 \log\left|\frac{\eta_4^{(1)}}{\eta_4^{(5)}}\right| \\
\log\left|\frac{\beta^{(3)}}{\beta^{(5)}}\right| &= u_1 \log\left|\frac{\eta_1^{(3)}}{\eta_1^{(5)}}\right| + u_2 \log\left|\frac{\eta_2^{(3)}}{\eta_2^{(5)}}\right| + u_3 \log\left|\frac{\eta_3^{(3)}}{\eta_3^{(5)}}\right| + u_4 \log\left|\frac{\eta_4^{(3)}}{\eta_4^{(5)}}\right| \\
\log\left|\frac{\beta^{(4)}}{\beta^{(5)}}\right| &= u_1 \log\left|\frac{\eta_1^{(4)}}{\eta_1^{(5)}}\right| + u_2 \log\left|\frac{\eta_2^{(4)}}{\eta_2^{(5)}}\right| + u_3 \log\left|\frac{\eta_3^{(4)}}{\eta_3^{(5)}}\right| + u_4 \log\left|\frac{\eta_4^{(4)}}{\eta_4^{(5)}}\right|
\end{aligned}$$

By (5.3) and Lemma 2.1, we have good estimates for $\log\left|\beta^{(\nu)}/\beta^{(5)}\right|$ in terms of a. Solving the system by Cramer's rule, we obtain

$$\begin{aligned}
Ru_1 &= \left(-10\log^3 a - 14\log 2\log^2 a - 4\log^2 2\log a + \vartheta_{11}\frac{\log^2 a}{a}\right)\log\left|\beta^{(5)}\right| \\
&\quad +10\log^4 a + 14\log 2\log^3 a + 4\log^2 2\log^2 a + \vartheta_{12}\frac{\log^3 a}{a} \\
Ru_2 &= \left(40\log^3 a + 31\log 2\log^2 a + 6\log^2 2\log a + \vartheta_{21}\frac{\log^2 a}{a}\right)\log\left|\beta^{(5)}\right| \\
&\quad -20\log^4 a - 3\log 2\log^3 a + 2\log^2 2\log^2 a + \vartheta_{22}\frac{\log^3 a}{a} \\
Ru_3 &= \left(-10\log^3 a + 11\log 2\log^2 a + 6\log^2 2\log a + \vartheta_{31}\frac{\log^2 a}{a}\right)\log\left|\beta^{(5)}\right| \\
&\quad +10\log^4 a - 11\log 2\log^3 a - 6\log^2 2\log^2 a + \vartheta_{32}\frac{\log^3 a}{a} \\
Ru_4 &= \left(-20\frac{\log^2 a}{a} + \vartheta_{41}\frac{1}{a^2}\right)\log\left|\beta^{(5)}\right| + 20\frac{\log^3 a}{a} + \vartheta_{42}\frac{\log^2 a}{a^2},
\end{aligned}$$

where

$$R = R_\alpha > 20\log^4 a + 28\log 2\log^3 a + 8\log^2 2\log^2 a - 3 \tag{5.4}$$

is the regulator of $\mathbb{Z}[\alpha^{(1)}]$ — see (3.1) — and the ϑ_{ik} lie in the following intervals:

ϑ_{11}	ϑ_{12}	ϑ_{21}	ϑ_{22}	ϑ_{31}	ϑ_{32}	ϑ_{41}	ϑ_{42}
$[-11,-9]$	$[20,22]$	$[-11,-9]$	$[-34,-30]$	$[-11,-9]$	$[18,20]$	$[-1,1]$	$[19,21]$

By (5.3) and (5.2), we have $\left|\beta^{(5)}\right| \geq a^3$, hence $\log\left|\beta^{(5)}\right| \geq 3\log a$. This yields $Ru_2 > 0$, $Ru_4 < 0$ and $U = |u_2|$. Furthermore we have

$$Ru_2 + Ru_4 2a\log a \geq (21\log^2 a + 2\log a - 3)\log\left|\beta^{(5)}\right| + 20\log^4 a - 3\log^3 a - 5 > 0.$$

Since u_4 is an integer, this yields

$$U = |u_2| > 2a\log a\,|u_4| \geq 2a\log a. \tag{5.5}$$

From (5.4) we conclude

$$\begin{aligned}
U(20\log^4 a + 28\log 2\log^3 a + 8\log^2 2\log^2 a - 3) &\leq |Ru_2| = Ru_2 \\
&\leq (40\log^3 a + 31\log 2\log^2 a + 6\log^2 2\log a)\log\left|\beta^{(5)}\right| \\
&\qquad -20\log^4 a - 3\log 2\log^3 a + 2\log^2 2\log^2 a,
\end{aligned}$$

and which together with (5.5) implies

$$\log\left|\beta^{(5)}\right| \geq \frac{1}{2}U\log a + \frac{9}{20}\log a > a\log^2 a. \tag{5.6}$$

Moreover, we have

$$3Ru_1 + Ru_2 + Ru_3 \le \left(-50\frac{\log^2 a}{a} + \frac{0.8}{a}\right)\log\left|\beta^{(5)}\right| \\ +20\log^4 a + 20\log^3 a + 4\log^2 a + 50 < 0$$

and

$$3Ru_2 + 2a\log a(3Ru_1 + Ru_2 + Ru_3) > (20\log^3 a + 64\log^2 a + 8\log a - 76)\log\left|\beta^{(5)}\right| \\ +40\log^5 a + 38\log^4 a + 7\log^3 a - 7 > 0,$$

which yields $|3u_1 + u_2 + u_3| \le U/((2/3)a\log a)$.

Putting $l = j = 2, p = 1$ and $q = 5$ in Siegel's identity (5.1) and using (4.1), Lemma 2.1 and Lemma 4.1, we get

$$\left|\left(\frac{\eta_1^{(1)}}{\eta_1^{(5)}}\right)^{u_1}\left(\frac{\eta_2^{(1)}}{\eta_2^{(5)}}\right)^{u_2}\left(\frac{\eta_3^{(1)}}{\eta_3^{(5)}}\right)^{u_3}\left(\frac{\eta_4^{(1)}}{\eta_4^{(5)}}\right)^{u_4}\frac{\alpha^{(5)}-\alpha^{(2)}}{\alpha^{(1)}-\alpha^{(2)}} - 1\right| = \\ \left|\frac{x-\alpha^{(2)}y}{x-\alpha^{(5)}y}\cdot\frac{\alpha^{(5)}-\alpha^{(1)}}{\alpha^{(2)}-\alpha^{(1)}}\right| \le 3.1\left|\frac{\beta^{(2)}}{\beta^{(5)}}\right| \le 3.1\frac{16}{y^4a^2}\frac{2}{\left|\alpha^{(5)}-\alpha^{(2)}\right|y} \le \frac{100}{a^{13}},$$

hence we have

$$|\Lambda_{1,5,2}| < 3.5\left|\frac{\beta^{(2)}}{\beta^{(5)}}\right| = 3.5\left|\frac{\left(\beta^{(5)}\right)^3}{\beta^{(1)}\beta^{(3)}\beta^{(4)}}\right|\frac{1}{\left|\beta^{(5)}\right|^5} < \frac{1.8a^2}{\left|\beta^{(5)}\right|^5}.$$

Together with (5.6) this means

$$\log|\Lambda_{1,5,2}| < \log(1.8a^2) - \frac{5}{2}U\log a - \frac{9}{4}\log a < -\frac{5}{2}U\log a.$$

□

6. A lower bound for a linear form in logarithms

We rewrite the linear form $\Lambda_{p,q,j}$ defined in Lemma 5.1 in the following way as a linear form in three logarithms: For $j \in \{1,5\}$ we write

$$\Lambda_{2,4,j} = u_1\log\left|\frac{\eta_1^{(2)}}{\eta_1^{(4)}}\frac{\eta_2^{(2)}}{\eta_2^{(4)}}\frac{\eta_3^{(2)}}{\eta_3^{(4)}}\right| + u_4\log\left|\frac{\eta_4^{(2)}}{\eta_4^{(4)}}\right| \\ +\log\left|\frac{\alpha^{(4)}-\alpha^{(j)}}{\alpha^{(2)}-\alpha^{(j)}}\right|\left(\left|\frac{\eta_3^{(2)}}{\eta_3^{(4)}}\right|\left|\frac{\eta_2^{(2)}}{\eta_2^{(4)}}\right|^{-1}\right)^{u_3-u_1}\left|\frac{\eta_2^{(2)}}{\eta_2^{(4)}}\right|^{u_2+u_3-2u_1},$$

for $j \in \{2,4\}$

$$\Lambda_{1,5,j} = u_1\log\left|\frac{\eta_1^{(1)}}{\eta_1^{(5)}}\right|\left|\frac{\eta_3^{(1)}}{\eta_3^{(5)}}\right|^{-3} + u_2\log\left|\frac{\eta_2^{(1)}}{\eta_2^{(5)}}\right|\left|\frac{\eta_3^{(1)}}{\eta_3^{(5)}}\right|^{-1} \\ +\log\left|\frac{\alpha^{(5)}-\alpha^{(j)}}{\alpha^{(1)}-\alpha^{(j)}}\right|\left|\frac{\eta_3^{(1)}}{\eta_3^{(5)}}\right|^{3u_1+u_2+u_3}\left|\frac{\eta_4^{(1)}}{\eta_4^{(5)}}\right|^{u_4}$$

and for $j = 3$

$$\Lambda_{1,5,3} = u_1 \log \left| \frac{\eta_1^{(1)}}{\eta_1^{(5)}} \right| + u_2 \log \left| \frac{\eta_2^{(1)}}{\eta_2^{(5)}} \frac{\eta_3^{(1)}}{\eta_3^{(5)}} \right| + \log \left| \frac{\alpha^{(5)} - \alpha^{(3)}}{\alpha^{(1)} - \alpha^{(3)}} \right| \left| \frac{\eta_3^{(1)}}{\eta_3^{(5)}} \right|^{u_3 - u_2} \left| \frac{\eta_4^{(1)}}{\eta_4^{(5)}} \right|^{u_4}.$$

We know an upper bound for this linear form in logarithms by Lemma 5.1, we will now derive a lower bound using a recent result of Voutier (1997). For an algebraic number γ with minimal polynomial $\sum_{i=0}^{d} a_i X^i$ and conjugates $\gamma = \gamma^{(1)}, \dots, \gamma^{(d)}$ the absolute logarithmic Weil height of γ is defined as

$$h(\gamma) := \frac{1}{d} \log \left[a_d \prod_{i=1}^{d} \max \left(1, \left|\gamma^{(i)}\right|\right) \right].$$

Proposition 6.1. (Voutier) *Let γ_1, γ_2 and γ_3 be positive algebraic numbers and put $D := [\mathbb{Q}(\gamma_1, \gamma_2, \gamma_3) : \mathbb{Q}]$. Let b_1, b_2 and b_3 be integers with $b_3 \neq 0$ and let h_1, h_2, h_3, B and $E > 1$ be real numbers which satisfy*

$$h_i \geq \max \left(\frac{\log E}{D}, h(\gamma_i), \frac{E \left|\log \gamma_i\right|}{D} \right) \qquad 1 \leq i \leq 3, \tag{6.1}$$

$$B \geq \max \left\{ 2, E^{1/D}, \frac{\log^2 E}{D^2} \left(\frac{|b_1|}{h_3} + \frac{|b_3|}{h_1} \right) \left(\frac{|b_2|}{h_3} + \frac{|b_3|}{h_2} \right) \right\} \tag{6.2}$$

and $E < 4.6^D$. If $\log \gamma_1$, $\log \gamma_2$, and $\log \gamma_3$ are linearly independent over $\mathbb{Q}$, then

$$\log \left| b_1 \log \gamma_1 + b_2 \log \gamma_2 + b_3 \log \gamma_3 \right| > -\frac{2.4 \cdot 10^6 \cdot D^5 \log^2 B}{\log^4 E} \cdot h_1 \cdot h_2 \cdot h_3.$$

Applying this proposition, we get

Lemma 6.2. *Let $a \geq 12\,089$ and (x, y) be a solution of (1.2) of type j with $y \geq 2$. We define h_1, h_2, h_{32}, and B according to the value of j:*

$j \in \{1, 5\}$: *Define*

$$h_1 := \frac{3}{2} \log a, \qquad h_2 := \frac{1}{20} \log(267 a^{20}), \qquad h_{32} := \frac{\log(65\,580\ a^{16})}{40 a \log a} + \frac{\log(16.1 a^{14})}{20 a^2 \log a},$$

and $B := (0.11a)^2$.

$j \in \{2, 4\}$: *Define*

$$h_1 := \frac{1}{20} \log(65 a^{36}), \qquad h_2 := \frac{1}{20} \log(65\,580\ a^{16}), \qquad h_{32} := \frac{1}{20 a \log a} \log(264 a^{24}),$$

and $B := (0.03a)^2$ or $(0.04a)^2$ if $j = 2$ or $j = 4$, respectively.

$j = 3$: *Define*

$$h_1 := \frac{1}{20} \log(1.0001 a^{14}), \qquad h_2 := \frac{6}{5} \log a, \qquad h_{32} := \frac{3 \log(16.1 a^{14})}{80 a^2 \log a} + \frac{\log(267 a^{20})}{80 a^3 \log^2 a}$$

and $B := (0.09a^2)^2$.

Furthermore define

$$h_{31} := \frac{1}{60} \log(2^{85} a^{292}), \qquad h_3 := h_{31} + U h_{32}, \qquad C := 2.4 \cdot 10^6 \cdot 60^5 \cdot \log^{-4} 12.$$

Then we have

$$\log|\Lambda_{p,q,j}| > -C\log^2 Bh_1h_2h_3,$$

where p and q are defined in Lemma 5.1.

PROOF. We prove the lemma for $j=2$, all remaining cases are analogous.

First, we have to check the linear independance of the logarithms. Assume that they are linearly dependent. Since $\operatorname{Gal} f_a = S_5$, we can apply the Galois automorphisms induced by $\sigma_1 = (23)$ and $\sigma_2 = (24)$, which yields

$$\left|\frac{\alpha^{(5)}-\alpha^{(2)}}{\alpha^{(1)}-\alpha^{(2)}}\right| = \left|\frac{\alpha^{(5)}-\alpha^{(3)}}{\alpha^{(1)}-\alpha^{(3)}}\right| = \left|\frac{\alpha^{(5)}-\alpha^{(4)}}{\alpha^{(1)}-\alpha^{(4)}}\right|,$$

because $\mathbb{Q}(\alpha^{(1)},\dots,\alpha^{(5)})$ is a totally real number field. But this leads to a contradiction.

In our case, $D = \left[\mathbb{Q}(\alpha^{(1)},\alpha^{(2)},\alpha^{(5)}):\mathbb{Q}\right] = 60$; we choose $E := 12$ and estimate the heights using approximations of the $\alpha^{(\nu)}$ of order 12:

$$\begin{aligned}
h\left(\left|\frac{\eta_1^{(1)}}{\eta_1^{(5)}}\right|\left|\frac{\eta_3^{(5)}}{\eta_3^{(1)}}\right|^3\right) &\leq \frac{1}{20}\log(65a^{36})\\
\left|\log\left(\left|\frac{\eta_1^{(1)}}{\eta_1^{(5)}}\right|\left|\frac{\eta_3^{(5)}}{\eta_3^{(1)}}\right|^3\right)\right| &\leq \frac{12}{a},\\
h\left(\left|\frac{\eta_2^{(1)}}{\eta_2^{(5)}}\right|\left|\frac{\eta_3^{(5)}}{\eta_3^{(1)}}\right|\right) &\leq \frac{1}{20}\log(65\,580\;a^{16})\\
\left|\log\left(\left|\frac{\eta_2^{(1)}}{\eta_2^{(5)}}\right|\left|\frac{\eta_3^{(5)}}{\eta_3^{(1)}}\right|\right)\right| &\leq \frac{4.1}{a},
\end{aligned}$$

and h_1 and h_2 satisfy (6.1).

To estimate the height of $\frac{\alpha^{(5)}-\alpha^{(2)}}{\alpha^{(1)}-\alpha^{(2)}}$ which — in general — is not an algebraic integer, we have to estimate the leading coefficient a_{60} of its minimal polynomial. Since

$$\operatorname{resultant}_Y(f_a(X+Y), f_a(Y)) = -X^5(X^{20}+\ldots+D(f_a))$$

and $\left[\mathbb{Q}(\alpha^{(p)}-\alpha^{(q)}):\mathbb{Q}\right] = 20$ for $p \neq q$ by Theorem 3.3, the minimal polynomial $m(X)$ of $\alpha^{(p)}-\alpha^{(q)}$ is

$$m(X) = X^{20}+\ldots+D(f_a)$$

and the minimal polynomial $\widehat{m}(X)$ of $(\alpha^{(p)}-\alpha^{(q)})^{-1}$ is

$$\widehat{m}(X) = D(f_a)X^{20}+\ldots+1.$$

Now $\frac{\alpha^{(5)}-\alpha^{(2)}}{\alpha^{(1)}-\alpha^{(2)}}$ is a root of

$$\operatorname{resultant}_Y\left(Y^{20}m\left(\frac{X}{Y}\right),\widehat{m}(Y)\right) = D(f_a)^{20}\prod_{\substack{p\neq q\\ r\neq s}}\left(X-\frac{\alpha^{(p)}-\alpha^{(q)}}{\alpha^{(r)}-\alpha^{(s)}}\right) \in \mathbb{Z}[X],$$

and therefore the leading coefficient $a_{60}|D(f_a)^{20}$.

Hence we get

$$\begin{aligned} h\left(\left|\frac{\alpha^{(5)}-\alpha^{(2)}}{\alpha^{(1)}-\alpha^{(2)}}\right|\right) &\leq \frac{1}{60}\log(2^{85}a^{292}) \\ \left|\log\left|\frac{\alpha^{(5)}-\alpha^{(2)}}{\alpha^{(1)}-\alpha^{(2)}}\right|\right| &\leq \frac{2.1}{a} \\ h\left(\left|\frac{\eta_3^{(1)}}{\eta_3^{(5)}}\right|\right) &\leq \frac{1}{20}\log(16.1a^{14}) \\ \left|\log\left|\frac{\eta_3^{(1)}}{\eta_3^{(5)}}\right|\right| &\leq \frac{2}{a} \\ h\left(\left|\frac{\eta_4^{(1)}}{\eta_4^{(5)}}\right|\right) &\leq \frac{1}{20}\log(267a^{20}) \\ \left|\log\left|\frac{\eta_4^{(1)}}{\eta_4^{(5)}}\right|\right| &\leq \log(4a^5), \end{aligned}$$

where the last two lines determined the choice of E, hence by Lemma 5.1 and because of

$$h_{32} = \frac{1}{20a\log a}\log(1056a^{31}) \geq \frac{1}{\frac{2}{3}a\log a}\frac{1}{20}\log(16.1a^{14}) + \frac{1}{2a\log a}\frac{1}{20}\log(267a^{20})$$

h_3 satisfies (6.1).

For our application $|b_1|, |b_2| \leq U$ and $b_3 = 1$. Hence for $i = 1, 2$ we have

$$\frac{|b_i|}{h_3} + \frac{|b_3|}{h_i} \leq \frac{1}{h_{32}} + \frac{1}{h_i} \leq \frac{20a}{31} + \frac{20}{16\log a},$$

and (6.2) is satisfied with $B = (0.03a)^2$. Hence the lemma follows from Proposition 6.1.
□

7. Proof of Theorem 1.1

With the notations of the previous sections and putting

j	1	2	3	4	5
H_j	5	5/2	500/203	5/2	500/103
$\hat{a}_j$	$1.6\cdot 10^{18}$	$1.3\cdot 10^{19}$	$1.1\cdot 10^{9}$	10^{19}	$1.7\cdot 10^{18}$
a_j	$1.6\cdot 10^{18}$	$3.6\cdot 10^{19}$	$1.1\cdot 10^{9}$	$3.3\cdot 10^{19}$	$1.7\cdot 10^{18}$

,

Lemmata 5.1 and 6.2 yield

$$-H_jU\log a > \log|\Lambda_{p,q,j}| > -Ch_1h_2h_3\log^2 B.$$

Hence

$$Ch_1h_2h_{31}\log^2 B > U\left(H_j\log a - Ch_1h_2h_{32}\log^2 B\right) =: U\cdot g(a). \tag{7.1}$$

For $a \geq \hat{a}_j$, $g(a)$ is positive, and therefore we can insert the lower bound for U from Lemma 5.1, and (7.1) cannot be true for $a \geq a_j$, hence the assumption that a solution with $y > 1$ exists leads to a contradiction and Theorem 1.1 is proved.

Remark: The size of the constants a_j depends mainly on the constant C in Lemma 6.2, which comes from the factor D^5 in Proposition 6.1, which is very high because of the Galois group S_5.

Furthermore, we have to use linear forms in three logarithms, because in the cases $j = 1, 2, 4, 5$, there is no dominating exponent u_i, which would enable us to group the linear form into a linear form in two logarithms, where the constants would be far better.

References

Baker, A. (1968). Contribution to the theory of Diophantine equations I. On the representation of integers by binary forms. *Philos. Trans. Roy. Soc. London Ser. A*, **263**:173–191.

Baker, A., Davenport, H. (1969). The equations $3x^2 - 2 = y^2$ and $8x^2 - 7 = z^2$. *Quart. J. Math. Oxford*, **20**:129–137.

Bilu, Y., Hanrot, G. (1996). Solving Thue Equations of High Degree. *J. Number Theory*, **60**:373–392.

Chen, J. H., Voutier, P. M. (1997). Complete solution of the Diophantine Equation $X^2 + 1 = dY^4$ and a Related Family of Quartic Thue Equations. *J. Number Theory*, **62**:71–99.

Cohen, H. (1996). *A Course in Computational Algebraic Number Theory*, volume 138 of *Graduate Texts in Mathematics*. Springer, Berlin *etc.*, third edition.

Daberkow, M., Fieker, C., Klüners, J., Pohst, M. E., Roegner, K., Wildanger, K. (1997). KANT V4. To appear in J. Symbolic Comput.

Halter-Koch, F., Lettl, G., Pethő, A., Tichy, R. F. (1997). Thue equations associated with Ankeny-Brauer-Chowla Number Fields. To appear in J. London Math. Soc.

Heuberger, C. (1997). Algorithmische Lösung parametrisierter Thue-Gleichungen. Diplomarbeit, Technische Universität Graz.

Lang, S. (1978). *Elliptic Curves: Diophantine Analysis*, volume 23 of *Grundlehren der Mathematischen Wissenschaften*. Springer, Berlin – New York.

Lee, E. (1992). *Studies on Diophantine equations*. PhD thesis, Cambridge University.

Lettl, G., Pethő, A. (1995). Complete Solution of a Family of Quartic Thue Equations. *Abh. Math. Sem. Univ. Hamburg*, **65**:365–383.

Lettl, G., Pethő, A., Voutier, P. (1997a). On the arithmetic of simplest sextic fields and related Thue equations. In Győry, K., Pethő, A., Sós, V. T., editors, *Number Theory, Diophantine, Computational and Algebraic Aspects*. W. de Gruyter Publ. Co. To appear.

Lettl, G., Pethő, A., Voutier, P. (1997b). Simple families of Thue inequalities. To appear in Trans. Amer. Math. Soc.

Mahler, K. (1964). An inequality for the discriminant of a polynomial. *Michigan Math. J.*, **11**:257–262.

Mignotte, M., Pethő, A., Roth, R. (1996a). Complete solutions of quartic Thue and index form equations. *Math. Comp.*, **65**:341–354.

Mignotte, M., Pethő, A., Lemmermeyer, F. (1996b). On the family of Thue equations $x^3 - (n-1)x^2y - (n+2)xy^2 - y^3 = k$. *Acta Arith.*, **76**:245–269.

Mignotte, M., Tzanakis, N. (1991). On a family of cubics. *J. Number Theory*, **39**:41–49.

Pethő, A. (1991). Complete solutions to families of quartic Thue equations. *Math. Comp.*, **57**:777–798.

Pethő, A., Schulenberg, R. (1987). Effektives Lösen von Thue Gleichungen. *Publ. Math. Debrecen*, **34**:189–196.

Pethő, A., Tichy, R. F. (1997). On two-parametric quartic families of diophantine problems. To appear in J. Symbolic Comput.

Pohst, M., Zassenhaus, H. (1989). *Algorithmic algebraic number theory*. Cambridge University Press, Cambridge *etc.*.

Thomas, E. (1990). Complete Solutions to a Family of Cubic Diophantine Equations. *J. Number Theory*, **34**:235–250.

Thomas, E. (1993). Solutions to Certain Families of Thue Equations. *J. Number Theory*, **43**:319–369.

Thue, A. (1909). Über Annäherungswerte algebraischer Zahlen. *J. reine angew. Math.*, **135**:284–305.

Tzanakis, N., de Weger, B. M. M. (1989). On the practical solution of the Thue equation. *J. Number Theory*, **31**:99–132.

Voutier, P. (1997). Linear forms in three logarithms. Preprint.