Stable Higher Order Triangular Finite Elements with Mass Lumping for the Wave Equation

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Abstract

Solving the wave equation by a C^0 finite element method requires to mass-lump the term in time of the variational formulation in order to avoid the inversion of a n-diagonal symmetric matrix at each time-step of the algorithm. One can easily get this mass-lumping on quadrilateral meshes by using a h-version of the spectral elements, based on Gauss-Lobatto quadrature formulae but the equivalent method is not obvious for triangular meshes. In this paper we construct and analyze new families of triangular finite elements which fulfill the same requirements as spectral quadratic and cubic finite elements.

Key words: wave equations, finite elements, mass-lumping.

AMS subject classifications: 35L05, 65L20, 65N30.

1 Introduction

Solving the wave equation in time domain by finite element methods is challenging but fundamental in order to model problems closer to the needs of industry. However, the use of such techniques rises some difficulties due the presence of a mass-matrix which grows with the order of the method and the dimension of the problem and must be inverted at each time-step. For this reason, finite difference methods were preferred to FEM for a long time.

Recent developments of FEM with mass-lumping, such as spectral finite elements enable to overcome this difficulty by using quadrilateral or hexahedral finite elements mass-lumped with Gauss-Lobatto quadrature rules [8]. Such elements were used and analyzed in their h-version and some properties of superconvergence were pointed out, in particular for the error committed on the velocity [4].

However, the use of quadrilateral finite elements is not so easy (in particular because non-regular quadrilaterals lead to isoparametric elements) and the use of triangles remains more popular in the industrial community. For that reason, we construct and analyze, in this paper, higher order triangular finite elements fitted to the numerical resolution of the wave equation. This purpose implies an adequate mass-lumping using a quadrature rule with positive weights in order to ensure the positivity of the discrete harmonic operator appearing in the scheme. Moreover, the accuracy of the method without mass-lumping must be kept.

In order to get all these properties, we construct a class of H^1-conform triangular finite elements which correspond to P_2 and P_3 spectral finite elements. However, to get the positivity of the discrete operator, we must modify the classical spaces of polynomials P_k.

The P_2 standard space must be replaced by $\widetilde{P}_2 = P_2 \oplus \{b\}$ where $b = \lambda_1\lambda_2\lambda_3$ in barycentric coordinates is the "bubble" function equal to 1 at the center of the triangle and 0 on its three edges. The new element is an element with 7 degrees of freedom which are those of the classical P_2 triangle plus its center. Then, the corresponding quadrature rule is the well known Simpson's rule the weights of which are positive [11].

For P_3, the process is more complex. The new element has 12 degrees of freedom and the space of polynomials is $\widetilde{P}_3 = P_3 \oplus \{b_1, b_2, b_3\}$ where b_1, b_2, b_3 are polynomials equal to 1 at three points symmetrically located on the three medians of the triangle respectively and 0 on the three edges of the triangle. We show that the problem has a solution for a unique set of points.

An other approach can be found in [6].

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2 A non-stable approach

We shall consider the following model problem:

$$(1)\quad \begin{cases} \text{Find } u : \mathbb{R}^2\times]0,T[\to \mathbb{R} \quad \text{so that:} \\ \dfrac{\partial^2 u}{\partial t^2}(x,t) - \Delta u(x,t) = 0 \quad \text{in } \mathbb{R}^2 \text{ x }]0,T[\\ u(x,0) = u_0(x),\ \dfrac{\partial u}{\partial t}(x,0) = u_1(x) \quad \text{in } \mathbb{R}^2 \end{cases}$$

Its variational formulation is:

$$\frac{d^2}{dt^2}\int_{\mathbb{R}^2} u\ v\ dx \ + \int_{\mathbb{R}^2} \nabla u\ \nabla v\ dx = 0 \quad \forall v \in H^1(\mathbb{R}^2)$$
(2)

In this section, we shall only deal with the space approximation which is the main point of this study. Let $V_h^k(\mathbb{R}^2) = \{v \in H^1(\mathbb{R}^2) \ \ /\forall i \in N \quad v_{/T_i} \in P_k\}$, be the Lagrange finite element space of kth order associated to a triangular mesh $\{\mathcal{T}_h\}$ of $\mathbb{R}^2$, $T_i \in \mathcal{T}_h$. The semi-discretized formulation of the problem can be written as follows:

$$(3)\quad \begin{cases} \text{Find } u_h(.,t) \ \in V_h^k(\mathbb{R}^2),\ t \in \ ]0,T[\quad \text{so that:} \\ \dfrac{d^2}{dt^2}\displaystyle\int_{\mathbb{R}^2} u_h(x,t) v_h(x) dx \\ + \displaystyle\int_{\mathbb{R}^2} \nabla u_h(x,t) \nabla d v_h(x) dx = 0 \\ \forall v_h \in V_h^k(\mathbb{R}^2) \\ u_h(x,0) = u_{0,h}(x),\ \dfrac{\partial u_h}{\partial t}(x,0) = u_{1,h}(x) \quad \text{in } \mathbb{R}^2 \end{cases}$$

Let $(\xi_i)_{i\in N}$ be a basis of $V_h^k(\mathbb{R}^2)$. Then (3) is equivalent to the following (infinite) ordinary differential equations system:

$$(4)\quad \begin{cases} M_{2,h}\dfrac{\partial^2 u_h}{\partial t^2}(t) \ + \ K_{2,h} u_h(t) \ = 0 \\ \text{with} \begin{cases} (M_{2,h})_{l,i} \ = \displaystyle\int_{\mathbb{R}^2} \xi_l(x)\ \xi_i(x)\ dx \\ (K_{2,h})_{l,i} \ = \displaystyle\int_{\mathbb{R}^2} \nabla\xi_l(x)\ \nabla\xi_i(x)\ dx\ , \\ (l,i) \in N^2 \end{cases} \end{cases}$$

So, we get a matrix $(M_{2,h})$ which is n-diagonal symmetric and must be inverted at each time-step for any explicit scheme, and we wish to lump this mass-matrix in order to avoid this inversion. This mass-lumping implies to find quadrature rules the points of which coincide with the degrees of freedom of the elements. Moreover, in order to keep the same accuracy as that of the scheme without mass-lumping, the quadrature rules must be exact for polynomials of degree $2k-2$ [2].

For P_1, the degrees of freedom are the vertices of the triangle and the suitable rule is the trapezoidal rule.

For P_2, the degrees of freedom are the vertices of the triangle and the middles of the edges, and the suitable rule is so that the weights are equal to 0 at the vertices and 1/3 at the edges [11].

For these two first kinds of finite elements, the nodes of the element coincide with the points of the quadrature rule in a natural way. For cubic elements, the location of the nodes on the edges of the elements must be changed so that these nodes coincide with the points of the quadrature rule. Actually, the same fact occurs for quadrilaterals [4].

Without mass-lumping, the degrees of freedom of cubic elements are the vertices of the triangle, two points located at one third and two thirds of the edge and the center of the triangle. In order to mass-lump, we must define a new element with the same degrees of freedom but in which the distance of the points of the edges to the nearest vertex is $(3-\sqrt{3})/6$. So, the weights of the suitable quadrature rule are -1/120 at the vertices, 1/20 on the edges and 9/40 at the the center of the element [7]. As one can see, for P_2 and P_3, the weights at the vertices of the quadrature rules are null or negative. So, these rules will not provide a proper approximation of $-\Delta$: for P_2, we get an ill-posed discrete problem and for P_3, the stability is not ensured.

Some other quadrature rules which do not fit to our problem can be found in [5].

3 New finite element spaces

Since the quadrature formulae above defined are unique, P_2 and P_3 seem to have no chance to fulfill the requirements of mass-lumping for the wave equation. So, in order to overcome the difficulty risen by the non-positivity of the weights, we are going to construct new spaces, namely $\widetilde{P}_2$ and $\widetilde{P}_3$ which will be slightly larger than P_2 and P_3. This boils down to add some interior nodes to the previous P_2 and P_3 triangles with the hope that we will be able to find quadrature rules which will be suitable for achieving mass lumping while keeping the order of the method.

In fact, one can find also in [2] that if a space of polynomials $\widetilde{P}$ satisfies $P_k \subset \widetilde{P} \subset P_{k'}$, $k \leq k'$, one gets the "P_k accuracy" as soon as the quadrature formula is exact for $P_{k+k'-2}$. Considering the symmetries of the triangle, one can show that it is sufficient to have three degrees of freedom (not fixed by symmetry) to integrate P_3 and five degrees of freedom to integrate P_5 [9].

3.1 The case of P_2 elements

In order to have the same accuracy as quadratic elements, one would like to have a space $\widetilde{P}_2$ satisfying $P_2 \subset \widetilde{P}_2 \subset P_3$. Moreover, the quadrature formula should integrate exactly P_3 (since $3 = 3 + 2 - 2$).

For this purpose, we shall define $\widetilde{P}_2$ by :

$$\widetilde{P}_2 = P_2 \oplus \{b\} \tag{5}$$

where b denotes the "bubble" function expressed in barycentric coordinates $\{\lambda_1, \lambda_2, \lambda_3\}$ as :

$$b = \lambda_1 \lambda_2 \lambda_3 \tag{6}$$

The triangle corresponding to $\widetilde{P}_2$ is the classical P_2 triangle (the nodes of which are the vertices S_j, $j = 1,3$ and the middles of the edges E_j, $j = 1,3$) to which we add its center of gravity G. The new finite element has seven degrees of freedom and it is immediate to check that we do get the $\widetilde{P}_2$-unisolvence. Moreover, as b vanishes on the edges of K, the degree of any element of $\widetilde{P}_2$ on any edge of K remains equal to 2.

Now, if we can consider the space :

$$V_h = \{v \in C^0(\bar{\Omega}) / \ \forall K \in \mathcal{T}_h, \ v/_K \in \widetilde{P}_2\} \tag{7}$$

as a space of approximation of $H^1(\Omega)$. V_h clearly admits three types of basis functions :

- functions associated to the vertices of the triangles the support of which is equal to the number of triangles admitting a given node as a common vertex
- functions associated to the edges of the mesh the support of which is made of two triangles
- "bubble"functions supported by one triangle.

In this case, the Simpson's rule mass-lumps properly. Its points are the nodes of the degrees of freedom and its weights are :

$$\omega_S = \frac{1}{20} \qquad \omega_E = \frac{2}{15} \qquad \omega_G = \frac{9}{20} \tag{8}$$

which are all positive.

3.2 The case of P_3 elements

Now, we would like to construct a triangular finite element which will have the same properties as P_3 and will provide a positive mass-lumping. For that purpose, we shall define $\widetilde{P}_3$ such that $P_3 \subset \widetilde{P}_3 \subset P_4$. The corresponding quadrature rule must be exact for P_5 (since $3 + 4 - 2 = 5$). Of course, we wish to have a set of points as small as possible in order to keep a reasonable computational cost but large enough to lead to the five free parameters required for P_5, as mentioned above. Moreover, our choice is that the traces of the functions of $\widetilde{P}_3$ on the edges of the triangle should be of third order.

So, all these required properties lead us to define a set of quadrature points classified as follows :

- the three vertices $\{S_1, S_2, S_3\}$
- 6 boundary points $\{M_{12}(\alpha), M_{21}(\alpha), M_{13}(\alpha), M_{31}(\alpha), M_{23}(\alpha), M_{32}(\alpha)\}$
- 3 interior points $\{G_1(\beta), G_2(\beta), G_3(\beta)\}$

In what precedes, (α, β) denotes two real parameter between 0 and 1, $G_1(\beta)$ has barycentric coordinates $(\beta, \frac{1-\beta}{2}, \frac{1-\beta}{2})$, $G_2(\beta)$ $(\frac{1-\beta}{2}, \beta, \frac{1-\beta}{2})$ and $G_3(\beta)$ $(\frac{1-\beta}{2}, \frac{1-\beta}{2}, \beta)$, while $M_{ij}(\alpha)$ denotes the barycentre of S_i and S_j with respective weights α and $(1-\alpha)$ (see figure 1).

Note that, with respect to the quadrature points considered in section 2, we gained one parameter (namely β)) by splitting the center of gravity G into three interior points $G_1(\beta)$, $G_2(\beta)$ and $G_3(\beta)$.

On the other hand, the set $\widetilde{P}_3$ must satisfy :

(i) $P_3 \subset \widetilde{P}_3 \subset P_4$.

(ii) The previous quadrature points are $\widetilde{P}_3$-unisolvent.

(iii) Traces on ∂K of functions of $\widetilde{P}_3$ have degree 3.

All this is obtained by choosing :

$$\widetilde{P}_3 = P_3 \oplus bP_1 \tag{9}$$

the dimension of which is equal to 12, which coincides with the number of quadrature points we have considered. It is easy to construct the three basis functions b_1, b_2 and b_3 associated to $G_1(\beta)$, $G_2(\beta)$ and $G_3(\beta)$ as bubble type functions defined by :

$$b_j = b(\lambda_j - \frac{1-\beta}{2}) \tag{10}$$

To see that the quadrature points are $\widetilde{P}_3$-unisolvent it suffices to remark that any $\widetilde{p}$ of $\widetilde{P}_3$ has a unique decomposition

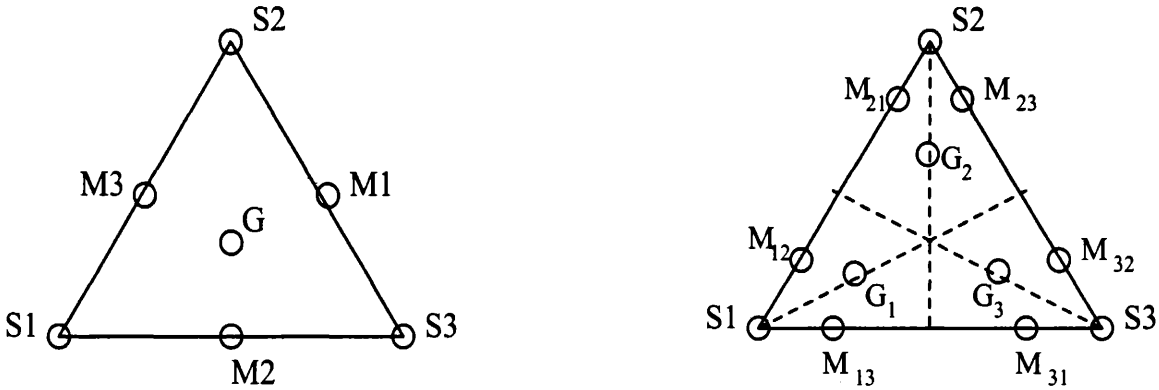

Figure 1: The degrees of freedom for $\widetilde{P}_2$ (*left*) and $\widetilde{P}_3$ (*right*).

in the form :

$$
(11) \qquad \begin{cases} \widetilde{p} = p + bq \\ p \in P_3, \ p(G) = 0 \\ q \in P_1 \end{cases}
$$

Then if $\widetilde{p}$ vanishes in all quadrature points, p vanishes at all boundary points and also at G. These points being P_3-unisolvent, we deduce that $p = 0$. Now the fact that $\widetilde{p}(G_j(\beta)) = 0$ yields $q(G_j(\beta))$ which yields $q = 0$ since $q \in P_1$ and since the three points $G_1(\beta)$, $G_2(\beta)$ and $G_3(\beta)$ are not aligned as soon as $\beta \neq \frac{1}{3}$.

Taking into account the different kinds of points, the quadrature formula can be written :

$$
(12) \qquad \begin{cases} E(f) = & mesK\{\omega_s \sum_{j=1}^{3} f(S_j) \\ & +\omega_\alpha \sum_{\substack{i,j=1 \\ i \neq j}}^{3} f(M_{ij}(\alpha)) \\ & +\omega_\beta \sum_{j=1}^{3} f(G_j(\beta))\} \end{cases}
$$

which is a formula with five parameters (ω_s, ω_α, ω_β, α and β) while we have 5 classes of equivalence in P_5.

One shows there exists a unique set for the parameters in order that formula (12) integrates exactly P_5. This set is :

$$
(13) \qquad \begin{cases} \beta & = \frac{1}{3} + \frac{2}{21}\sqrt{7} \simeq 0.5853 \\ \alpha & = -\frac{-42 - 21\sqrt{7} + \sqrt{35 + 16\sqrt{7}}\sqrt{3}\sqrt{7}}{84 + 42\sqrt{7}} \\ & \simeq 0.2935 \\ \omega_s & = 2\frac{919\sqrt{7} + 2471}{124080\sqrt{7} + 330960} \simeq 0.0148 \\ \omega_\alpha & = 2\frac{\sqrt{7}\,(2+\sqrt{7})^4}{25280 + 9520\sqrt{7}} \simeq 0.0488 \\ \omega_\beta & = 2\frac{147 + 42\sqrt{7}}{400\sqrt{7} + 1280} \simeq 0.2208 \end{cases}
$$

Of course, we now construct a space of approximation of $H^1(\Omega)$ as :

$$
(14) \qquad V_h = \{v \in C^0(\bar{\Omega}) /\ \forall K \in \mathcal{T}_h,\ v/_K \in \widetilde{P}_3\}
$$

Once again there are three types of basis functions. The difference with the $\widetilde{P}_2$ space lies in the fact that there is still one basis functions by vertex (as for $\widetilde{P}_2$) but two basis functions by edge and three basis functions by triangle.

The approximation of the term in time leads to a diagonal mass-matrix only when one uses the appropriate quadrature formula to compute the integrals appearing in the variational formulation but the computation of the integrals coming from the harmonic operator can be made in two ways : either exactly or by using the same quadrature formula (which will not provide an exact value of the integrals). We shall present here the first point of view.

Of course, this kind of result could be extended to higher order triangular finite elements but, even for $\widetilde{P}_3$, the computations, made with the help of MAPLE, which led to this result were not immediate and it is obvious that such computations will rapidly reach the bounds of any software of this kind for higher order elements. So, although conceptually possible, the extension to higher order does not seem easy in practice. Moreover, we don't have any theoretical result ensuring that it is possible to construct an adequate quadrature formula with positive weights at any order.

Immediate generalization of such elements to 3D provided non-positive quadrature rules until now.

3.3 Discretization in time

The higher order character of the approximation in space suggests to use a higher order approximation in time in order not to sully the accuracy of the global approximation. Of course, the most natural way to get a fourth order time discretization would be to discretize the time derivative by using a centered fourth order finite difference scheme. Unfortunately, such schemes are unconditionally unstable. So two solutions remain: either use a standard second order finite difference scheme :

$$\text{(15)} \qquad \frac{\partial^2 u_h(t^n)}{\partial t^2} \simeq \frac{u_h^{n+1} - 2u_h^n + u_h^{n-1}}{\Delta t^2}$$

which is stable but reduces the convergence of the method to second order or apply a modified equation approach described, for instance, in [3] but in a slightly different way, as described below:

By writing down the Taylor expansion of (15) we get:

$$\text{(16)} \qquad \left\{ \begin{array}{rl} \dfrac{\partial^2 u_h(t^n)}{\partial t^2} = & \dfrac{u_h^{n+1} - 2u_h^n + u_h^{n-1}}{\Delta t^2} \\ & -\dfrac{\Delta t^2}{12} \dfrac{\partial^4 u_h(t_n)}{\partial t^4} + O(h^6) \end{array} \right.$$

At this step, we replace $\dfrac{\partial^4 u_h}{\partial t^4}$ by $N_h^2 u_h$ ($N_h = M_{2,h}^{-1} K_{2,h}$) (these matrices were defined in (4)).

The new formulation can then be written:

$$\text{(17)} \quad u_h^{n+1} = 2u_h^n - u_h^{n-1} - \frac{\Delta t^2}{h^2} N_h [u_h^n - \frac{\Delta t^2}{12} N_h u_h^n]$$

This new algorithm involves two computations of the discrete Laplace operator but this additional cost will be balanced by the increase of the stability condition, as we shall see below.

A plane wave (or, equivalently, Fourier) analysis of the method leads to an eigenvalue problem in ω_h^2 (ω_h is the pulsation of the discrete plane wave). The eigenvectors of this problem are in $\mathbb{R}^6$ for $\widetilde{P}_2$ and $\mathbb{R}^{13}$ for $\widetilde{P}_3$. Its solution (computed numerically) provides the following stability conditions :

$$\text{(18)} \quad \frac{\Delta t}{h} \leq 0.2187 \text{ for } \widetilde{P}_2 \text{ and } \frac{\Delta t}{h} \leq 0.1244 \text{ for } \widetilde{P}_3$$

for the second order scheme in time,

$$\text{(19)} \quad \frac{\Delta t}{h} \leq 0.3788 \text{ for } \widetilde{P}_2 \text{ and } \frac{\Delta t}{h} \leq 0.2155 \text{ for } \widetilde{P}_3$$

for the fourth order scheme in time which allows to use time-steps almost twice larger than a second order scheme. This balances the increase of computation introduced by the method.

Higher order approximations in time do not have the same properties and provide too expensive algorithms.

Moreover, the ratio $q_h = \omega_h/k$ gives the error committed on the velocity. The study of q_h versus the inverse of the number of elements per wavelength and for different values of the angle of the direction of propagation θ and the ratio $\alpha = \Delta t/h$ gives the dispersion curves [10]. On the other hand, log-like curves of q_h shows that the error is of 4th order for P_2 and 6th order for P_3, which points out the same superconvergence phenomenon as for quadrilaterals.

Remark : A classical finite element analysis gives an error in h^4 for the L^∞-norm and in h^3 for the H^1-norm [1], [9].

4 Numerical results

We solved the following test problem for regular meshes in P_1, $\widetilde{P}_2$ and $\widetilde{P}_3$:

$$\text{(20)} \qquad \left\{ \begin{array}{l} \dfrac{\partial^2 u}{\partial t^2}(x,y,t) - \Delta u(x,y,t) = g(x,y) f(t) \\ \text{in }]0,12[^2 \times]0,50[\\ u(x,y,0) = \dfrac{\partial u}{\partial t}(x,y,0) = 0 \quad \text{in }]0,12[^2 \\ u(0,y,t) = u(12,y,t) = u(x,0,t) = \\ u(x,12,t) = 0 \quad \text{in }]0,12[^2 \text{ x }]0,50[\end{array} \right.$$

where $g(x,y)$ is a Gaussian function in polar coordinates and $f(t)$ is the second derivative of a Gaussian function (Ricker function).

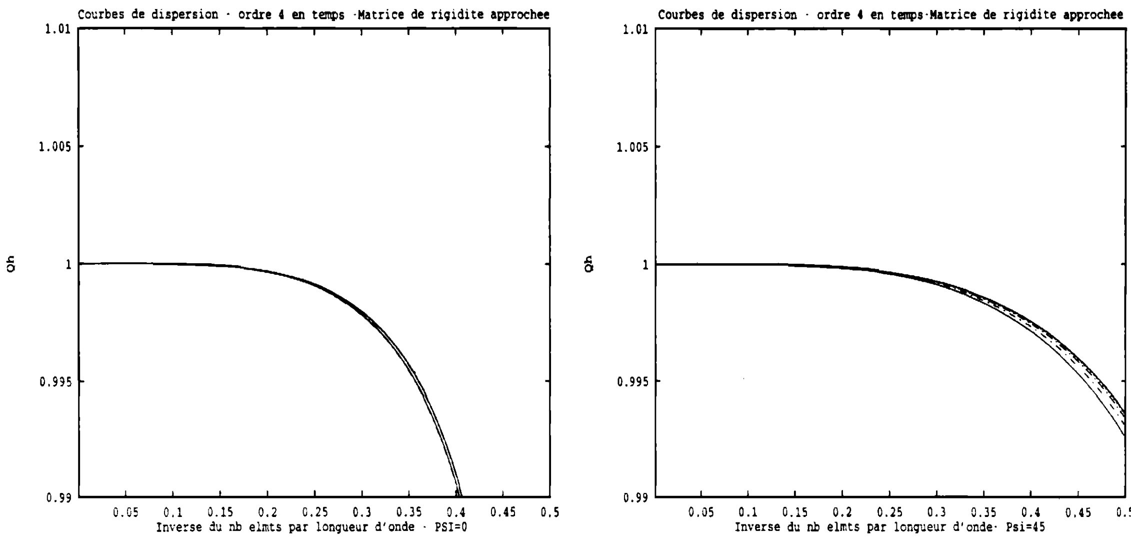

Figure 2: Dispersion curves for $\widetilde{P}_2$ and fourth order in time, α varying from 0.05 to 0.35, for $\theta = 0$ (left) and $\theta = \frac{\pi}{4}$ (right).

We give, in the figures below, the seismograms of the solutions on the interval in time $[25, 50]$ (i.e. after a trip of 100 wavelengths) at the point (9,3) on a regular meshes containing roughly the same number of degrees of freedom for $\widetilde{P}_2$, $\widetilde{P}_3$ and P_1. The "exact" solution is in dotted line and the numerical one in continuous line.

These figures show the gain of accuracy given by $\widetilde{P}_3$ and the importance of a good accuracy in time. In fact, in order to obtain an "exact" solution on a non-regular mesh, $\widetilde{P}_2$ will take twice more CPU time than $\widetilde{P}_3$.

5 Conclusion

We constructed and analyzed triangular finite elements with mass-lumping for the wave equation with an accuracy comparable to quadratic and cubic spectral finite elements. These new elements ensure a stable approximation of the wave equation. Higher order elements could be found but no automatic algorithm is known for the moment. Generalization to tetrahedra is being studied.

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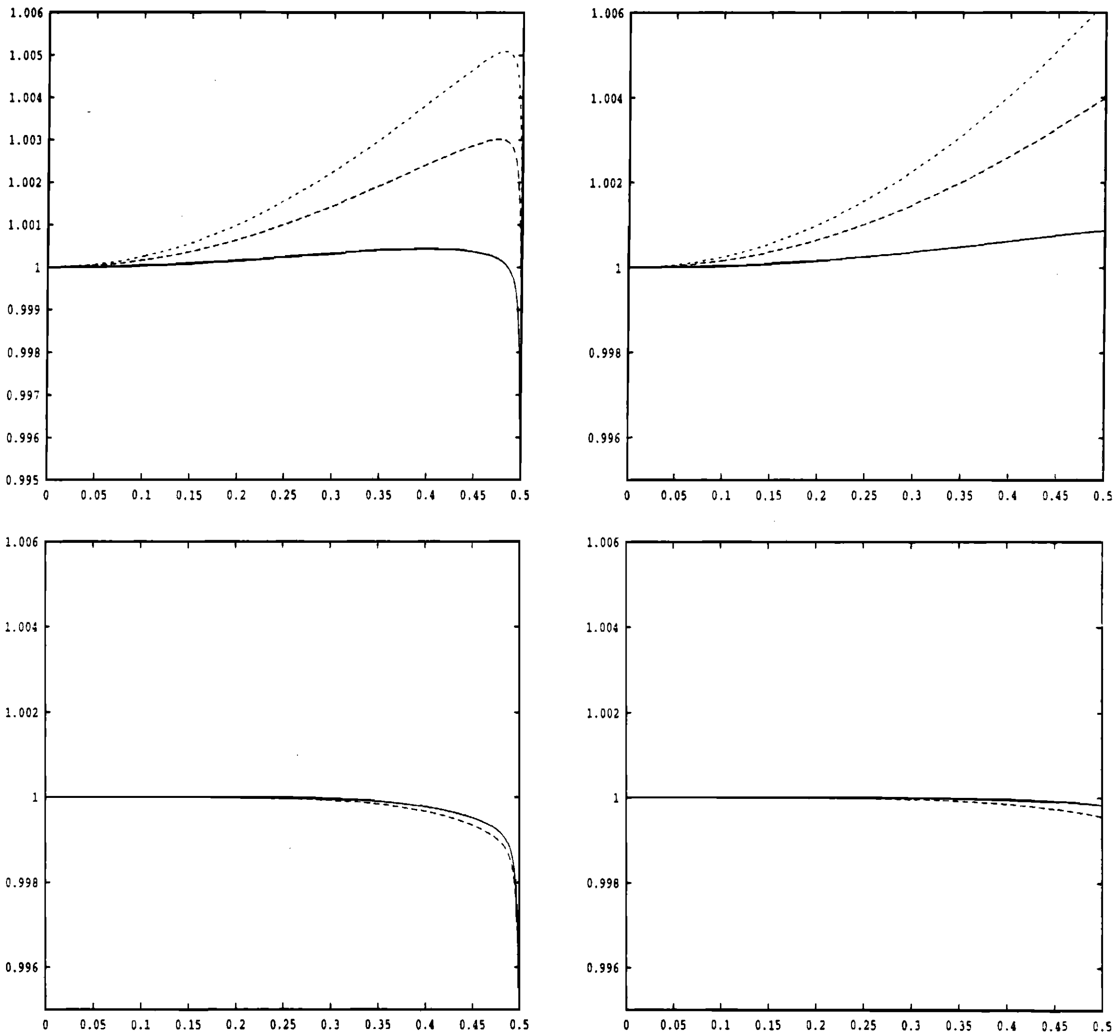

Figure 3: Dispersion curves for $\widetilde{P}_3$ and second order in time, α varying from 0.05 to 0.124, for $\theta = 0$ (left) and $\theta = \dfrac{\pi}{4}$ (right) (*top*) and fourth order in time, α varying from 0.1 to 0.21, for $\theta = 0$ (left) and $\theta = \dfrac{\pi}{4}$ (right) (*bottom*).

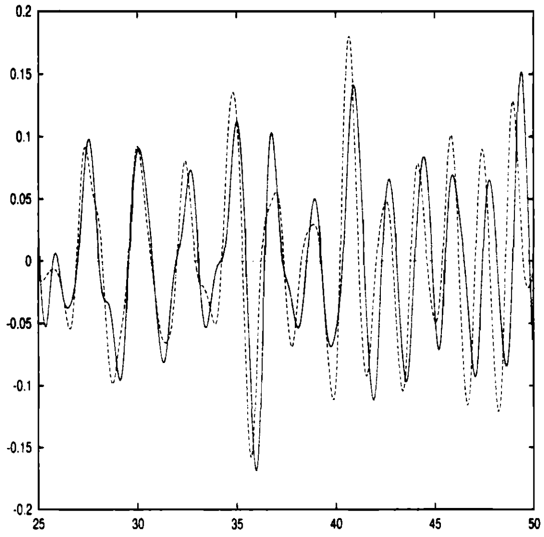

Figure 4: Seismogram for P_1, second order in time and space, 3.75 elements per wavelength, $\alpha = 0.3$.

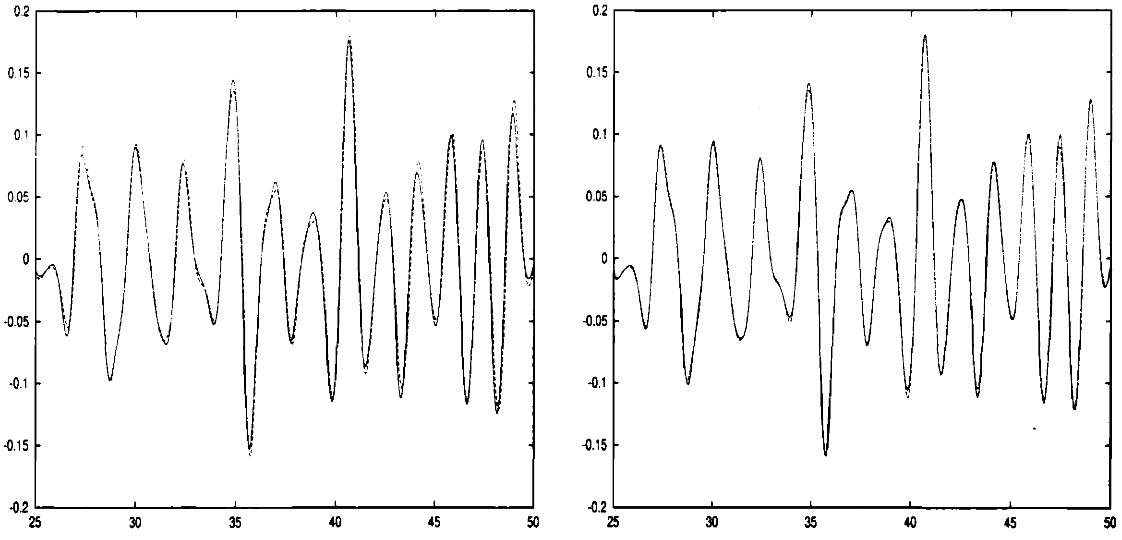

Figure 5: Seismogram for $\widetilde{P}_2$, fourth order in space and second order in time and $\alpha = 0.15$ (left) and fourth order in time and $\alpha = 0.3$ (right), 1.875 elements per wavelength (CPU time on a Dec station $\simeq$ 3.9mn).

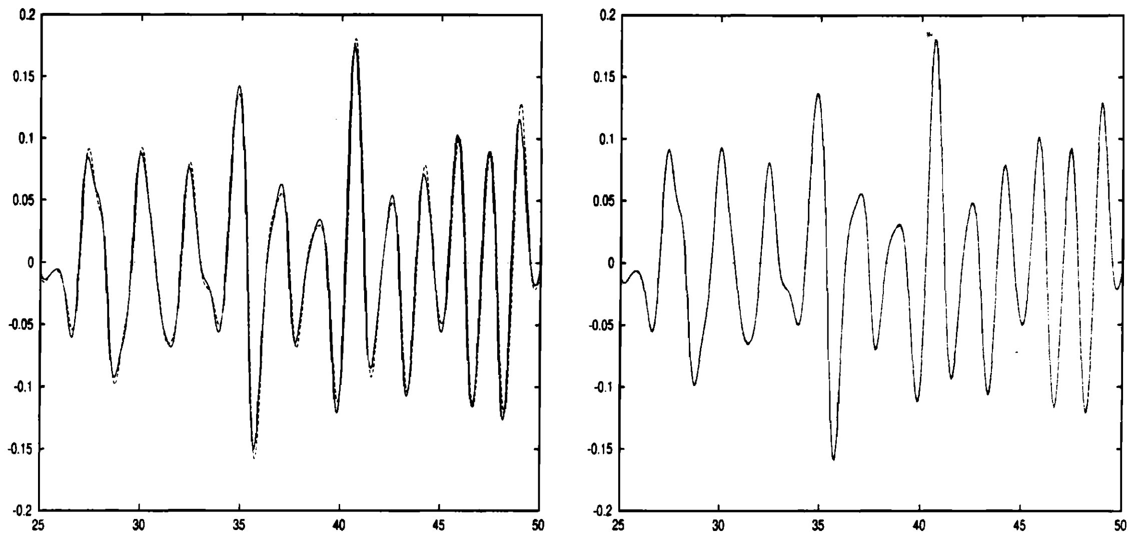

Figure 6: Seismogram for $\widetilde{P}_3$, fourth order in space and second order in time and $\alpha = 0.1$ (left) and fourth order in time and $\alpha = 0.2$ (right), 1.25 elements per wavelength (CPU time on a Dec station $\simeq$ 5.5mn).

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