Face 2-colourable triangular embeddings of complete graphs

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Abstract

A face 2-colourable triangulation of an orientable surface by a complete graph K_n exists if and only if $n \equiv 3$ or 7 $(mod\ 12)$. The existence of such triangulations follows from current graph constructions used in the proof of the Heawood conjecture. In this paper we give an alternative construction for half of the residue class $n \equiv 7$ $(mod\ 12)$ which lifts a face 2-colourable triangulation by K_m to one by K_{3m-2}. A nonorientable version of this result is discussed as well which enables us to produce non-isomorphic nonorientable triangular embeddings of K_n for half of the residue class $n \equiv 1$ $(mod\ 6)$. We also note the existence of non-isomorphic orientable triangular embeddings of K_n for $n \equiv 7$ $(mod\ 12)$ and $n \neq 7$.

1 Introduction

The search for triangular embeddings of complete graphs in closed surfaces has a long and exciting history (see [7]), culminating in the solution of the famous surface colouring problem of Heawood and giving birth to modern topological graph theory as treated in [5]. Despite the explosion of results in numerous new directions concerning embeddings, there still remain challenging unsolved problems which bring us "back to the roots", i.e., to triangulations of complete graphs.

It is well known [7] that a complete graph triangulates some orientable surface if and only if $n \equiv 0, 3, 4$ or 7 $(mod\ 12)$. Examining the triangular embeddings given in Ringel's book [7] as part of the solution of the Heawood problem, one finds that the known triangulations for $n \equiv 3$ $(mod\ 12)$ are *face 2-colourable*, that is, the triangular faces of the embeddings can be properly 2-coloured (say, black and white) such that no two monochromatic triangles share an edge. This fact has interesting consequences for design theory. Obviously, in each colour, the monochromatic triangles on the surface induce a Steiner triple system (STS) of order n; we thus have two STSs (the "black" and the "white" one) simultaneously "embedded" in an orientable surface. This aspect of face 2-colourable triangulations is studied in much more detail in [4]; see also [1, 9] for further information.

A face 2-colourable triangular embedding of K_n in an orientable surface can exist only if $n \equiv 3$ or 7 $(mod\ 12)$ since the vertex degrees must be even. As mentioned, their existence for $n \equiv 3$ $(mod\ 12)$ has been established. Their existence for $n \equiv 7$ $(mod\ 12)$ generally seems to have been overlooked but follows from work of Youngs [11] where triangular embeddings of K_n are produced by means of current assignments on ladder graphs. Amongst the variety of ladder graphs used in [11] it is possible to find, for each $n \equiv 7$ $(mod\ 12)$, one which is bipartite. Anderson [2] points out the significance of a bipartition; for our purposes it ensures that the corresponding triangular embedding is face 2-colourable. For further details see [4]; we are grateful to the referee who drew Youngs' paper to our attention. An additional face 2-colourable embedding of K_{19} is given in [6] and [9].

The aim of this paper is to provide an alternative recursive construction which takes as input an arbitrary face 2-colourable triangular embedding of K_n and produces the required embedding of K_{3n-2}. The proof extensively uses surface surgery and voltage assignment constructions, and is accompanied by a lot of details in order to facilitate its verification. For this purpose we also include a brief description of the technique of lifting embedded graphs by means of voltage assignments (Section 2).

Much of the above applies also to triangular embeddings of K_n in nonorientable surfaces; they exist if and only if $n \equiv 0$ or 1 $(mod\, 3)$, $n \geq 6$ and $n \neq 7$. But among the nonorientable triangulations of K_n collected in [7], one can find face 2-colourable ones only for $n \equiv 3$ $(mod\ 6)$, $n \geq 9$. Our main result implies their existence also for half of the residue class $n \equiv 1$ $(mod\ 6)$. This enables us to produce non-isomorphic nonorientable triangular embeddings of K_n for such values of n. We also note in connection with [6] that the apparently overlooked face 2-colourable orientable triangular embeddings of [11] establish that there are non-isomorphic orientable triangular embeddings of K_n for all $n \equiv 7$ $(mod\ 12)$ apart from the case $n = 7$.

For the sake of completeness we note that recursive constructions are quite frequent in topological graph theory, and they even appear in connection with the Heawood problem. In particular, a recursive $n \rightarrow 3n-2$ construction for triangular embeddings of complete graphs is described in [7]. However, that construction does not produce *face 2-colourable* triangulations; this subtle point makes things much more difficult.

As an interesting aside, let us mention that there is a well known construction which produces an STS of order $3n-2$ from any given STS of order n (the so-called $3n-2$ construction, see for example Theorem 15 p200 of [8] with $v_3 = 3$.). Thus, our main result may be viewed as a "surface version" of the $3n-2$ construction for STSs. Actually, this was the way our main result was discovered - namely, by means of constructing pairs of simultaneously embedded STSs. This explains also the slight preference given here to orientable surfaces; the corresponding STSs may then be viewed as being depicted locally 2-dimensionally in a 3-dimensional space. For a recent study of surface representations of other types of block designs we refer to [10].

2 Graph embeddings (a brief account)

We assume that the reader is familiar with fundamentals of the classification of compact orientable surfaces (=2-manifolds). For the sake of completeness we review some basic notions about graph embeddings as well as the technique of *lifting* of embedded graphs. Let $\nu: G \rightarrow S$ be an embedding of a graph G in an orientable surface S. Connected components of the set $S \setminus \nu(G)$ are called *regions* or *(open) faces* of ν; the embedding ν is called

2-cell or *cellular* if all its regions are homeomorphic to an open disc. For the sake of brevity, we usually do not distinguish between G and its embedded copy $\nu(G)$.

Let now $\nu: G \to S$ be a cellular embedding of a *connected* graph G in our orientable surface S. At this stage we do not make any further assumptions on G – in particular, G may have loops and/or multiple edges. A *directed edge* is an edge endowed with one of the two possible directions. Fix an orientation of S; this orientation induces, for each vertex v of G, a cyclic permutation P_v of the directed edges that emanate from v. Since every $e \in E(G)$ gives rise to one pair of oppositely directed edges, every directed edge appears in precisely one cyclic permutation P_v. It is a well known fact that the product $P = \prod_{v \in V(G)} P_v$, called a *rotation system* in [5], carries the complete information about the cellular embedding $\nu: G \to S$.

Let Γ be a finite group and let $\alpha: D(G) \to \Gamma$ be a mapping, where $D(G)$ is the set of directed edges of G. If for each pair of oppositely directed edges x and x^- of G we have $\alpha(x^-) = (\alpha(x))^{-1}$, then α is called a *voltage assignment*, and the pair (G, α) is a *voltage graph*. The *lifted graph* G^α (called also *derived graph* in [5]) has vertex set $V(G) \times \Gamma$, and the set of directed edges of G^α is $D(G) \times \Gamma$. The incidence in G^α is defined as follows. If x is a directed edge of G "emanating" from the vertex v and "terminating" at u, then for each $g \in \Gamma$ the directed edge (x, g) of G^α emanates from the vertex (v, g) and terminates at the vertex $(u, g\alpha(x))$. If $\nu: G \to S$ is a 2-cell embedding in an orientable surface S, the *lifted embedding* $\nu^\alpha: G^\alpha \to S^\alpha$ is determined via the rotation system P^α such that, for each vertex (v, g) of G^α and each directed edge x of G emanating from v, $P^\alpha_{(v,g)}(x, g) = (P_v(x), g)$.

As mentioned earlier, the orientable surface S^α (as well as the embedding ν^α) is completely determined by the new rotation system P^α. To see this, we indicate how face boundaries may be recovered from rotation systems. Let $I: D(G) \to D(G)$ be the involutory permutation that sends every directed edge to its reverse, i.e., $I(x) = x^-$. Then, if P is the rotation system for ν, orbits of the composition PI correspond to face boundaries of the embedding $\nu: G \to S$. Similarly, the involutory permutation I^α for the lifted graph G^α is given by $I^\alpha(x, g) = (x^-, g\alpha(x))$, and face boundaries of the lifted embedding $\nu^\alpha: G^\alpha \to S^\alpha$ are determined by orbits of the permutation $P^\alpha I^\alpha$. In fact, a k-gon in G which is bounded by a closed walk, the composition of whose group elements under α has order s in Γ, lifts to $|\Gamma|/s$ ks-gons in G^α.

For much more detail about this construction, which has become a standard tool in topological graph theory, we refer to [5].

3 The main results

Theorem 1 *Let $n \equiv 3$ (mod 12). If K_n has a face 2-colourable orientable triangular embedding, then K_{3n-2} also has a face 2-colourable orientable triangular embedding.*

Proof. Let η be a face 2-colourable triangular embedding of K_n in an orientable surface. Let the triangular faces of the embedding be properly coloured black and white, and let a fixed orientation of the surface be chosen (say, clockwise). When working with embedded graphs, we shall use the same notation for vertices (edges) of the abstract graph as well as for the *embedded* graph; no confusion will be likely.

Fix a vertex $z \in V(K_n)$ and consider the restricted embedding of the vertex-deleted graph $G = K_n - z \simeq K_{n-1}$ which will be obtained in the following way. Remove from the original embedding η of K_n the vertex z, together with all open arcs that correspond on the surface to edges incident with z; also, remove all (open) triangular faces of the embedding that correspond to triangles originally incident with the point z. Cutting out the point z together with a surrounding topological disk (formed by the union of all open triangular faces incident with z and the open arcs incident with this point) leaves a *hole* in our surface. Thus, as the result, we obtain a face 2-coloured triangular embedding $\phi:\ G \to S$ in a *surface with boundary* (or *bordered surface*); note that the boundary of the hole in S has the form $\phi(D)$ where D is some Hamiltonian cycle in G.

We now take three disjoint copies of the embedding ϕ (including the proper 2-colouring of triangular faces inherited from η). More precisely, for each $i = 0, 1, 2$ we take a complete graph G^i of order $n-1$ on the vertex set $V^i = \{u^i; u \in V(G)\}$ and a face 2-colourable triangular embedding $\phi^i:\ G^i \to S^i$ of G^i in a bordered clockwise oriented surface S^i such that the natural mapping $f^i:\ G \to G^i$ which assigns the superscript i to each vertex of G is a colour-preserving and orientation-preserving isomorphism of the triangular embeddings ϕ and ϕ^i (we assume that the surfaces S^i are mutually disjoint).

An easy counting argument shows that in the embedding ϕ we have $t = (n-1)(n-3)/6$ white triangular faces; let $\mathcal{T}$ be the set of these faces and let $\mathcal{T}^i = f^i(\mathcal{T})$ be the corresponding set of all white triangular faces in the embedding ϕ^i for $i = 0, 1, 2$. In what follows we describe a procedure which, when carried out successively for each $T \in \mathcal{T}$, will "put together" the bordered surfaces S^i in a way suitable for our purposes. The idea itself is not new and was probably first used in [3].

Let u, v, w be vertices of G such that u, v, w are corners of a fixed white triangular face T of ϕ; we may without loss of generality assume that the chosen clockwise orientation of S induces the cyclic permutation (uvw) of vertices on the boundary cycle C of the face T.

For this particular T, consider the auxiliary face-2-coloured embedding ψ_T of the complete tripartite graph $K_{3,3,3}$ in a torus with three holes cut in its surface, as depicted in Fig. 1 (the holes are shown as dotted regions). The three vertex-parts of our $K_{3,3,3}$ in Fig. 1 consist of vertices u^i_T, v^i_T and w^i_T, $i = 0, 1, 2$. The three boundary curves of holes in the torus are the three 3-cycles $C^i_T = (u^i_T v^i_T w^i_T)$ $(i = 0, 1, 2)$. We assume that the torus is disjoint from all surfaces S^i and that its orientation induces the clockwise cyclic permutations $(u^i_T w^i_T v^i_T)$ of vertices on the boundary curves C^i_T, respectively. (Notice the important difference between the cyclic permutations (uvw) on S and $(u^i_T w^i_T v^i_T)$ on the torus.)

Now, for each $i = 0, 1, 2$, remove from the embedding ϕ^i the open triangular face $T^i = f^i(T)$. We thus create in each S^i a new hole with boundary curve C^i where $C^i = f^i(C)$ is the 3-cycle $(u^i v^i w^i)$ in ϕ^i. Finally, for $i = 0, 1, 2$ we identify the closed curve C^i in the embedding ϕ^i with the curve C^i_T in the embedding ψ_T in such a way that $u^i \equiv u^i_T$, $v^i \equiv v^i_T$, and $w^i \equiv w^i_T$.

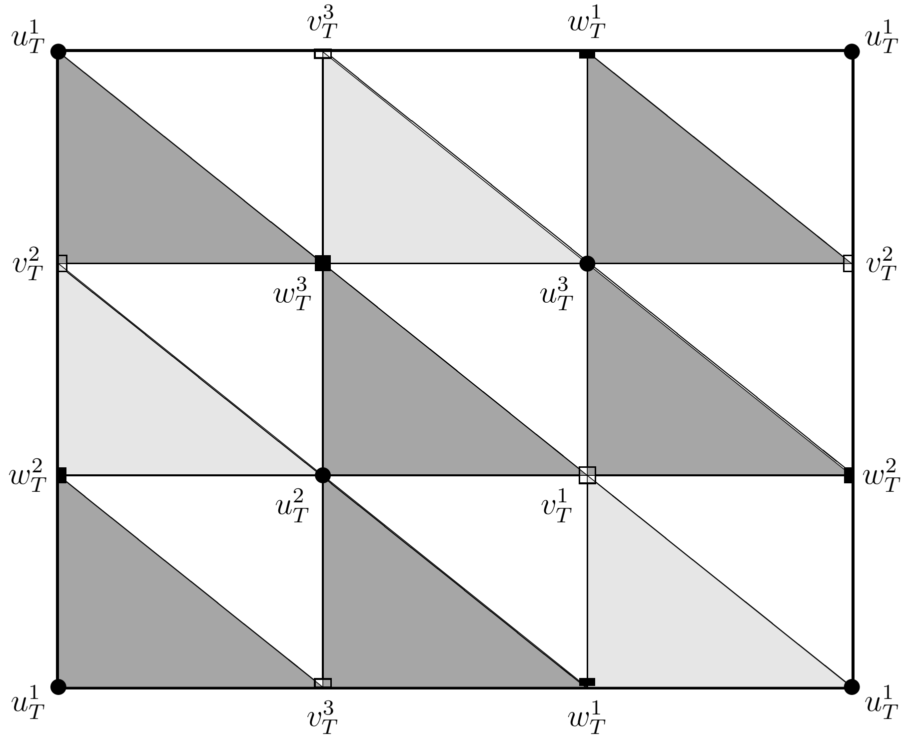

Fig. 1. The toroidal embedding ψ_T of $K_{3,3,3}$.

Applying the above procedure successively to *each* white triangular face $T \in \mathcal{T}$ (and assuming that the corresponding auxiliary toroidal embeddings ψ_T are mutually disjoint), we obtain from S^0, S^1 and S^2 a new connected triangulated surface with boundary, which we denote by $\hat{S}$. Roughly speaking, $\hat{S}$ is obtained from S^0, S^1 and S^2 by adding $|\mathcal{T}|$ toroidal 'bridges' raised (for each $T \in \mathcal{T}$) above the white triangular faces $T^i = f^i(T)$, $i = 0, 1, 2$. Clearly, $\hat{S}$ has three holes, and their (disjoint) boundary curves correspond to the three Hamiltonian cycles $D^i = f^i(D)$ in the graphs G^i. Also, it is easy to see that the chosen orientations of ϕ^i and ψ_T guarantee that the bordered surface $\hat{S}$ is orientable. In fact, $\hat{S}$ inherits the clockwise orientation from the embeddings ϕ^i, $i = 0, 1, 2$, and ψ_T, $T \in \mathcal{T}$. Note that $\hat{S}$ also inherits the proper 2-colouring of triangular faces from these embeddings. Since we have $t = (n-1)(n-3)/6$ black triangles in S (and hence in each S^i) and for each of the t white triangles T in S we added, in the auxiliary toroidal embedding

ψ_T, another 15 triangles, the total number of triangular faces on $\hat{S}$ is equal to $3t + 15t = 3(n-1)(n-3)$. For each collection of 15 triangles added, 9 are white and 6 are black; hence it is easy to check that exactly half of the triangles on $\hat{S}$ are black, as expected.

Let H be the graph that triangulates the bordered surface $\hat{S}$; we need a precise description of H. Let $D = (u_1 u_2 \dots u_{n-1})$ be our Hamiltonian cycle in $G = K_n - z$ (thus, $V(G) = \{u_j;\ 1 \le j \le n-1\}$). Since n is odd, every other edge of D is incident to a white triangle on $\hat{S}$; let these edges be u_2u_3, u_4u_5,...,$u_{n-1}u_1$. From the above construction it is easy to see that the graph H is obtained as follows. For $1 \le j \ne j' \le n-1$, each vertex u_j of G gives rise to three vertices u_j^0, u_j^1, u_j^2 of H, and each edge $u_ju_{j'}$ of G incident to a white triangle gives rise to 9 edges $u_j^i u_{j'}^{i'}$, $i, i' = 0, 1, 2$, of H. Since each edge of G except the $(n-1)/2$ edges u_1u_2, u_3u_4,..., $u_{n-2}u_{n-1}$ is incident to exactly one white triangle, the total number of edges of the graph H is $9(|E(G)| - (n-1)/2) + 3(n-1)/2 = 3(n-1)(3n-8)/2$. To see its structure, note that for each edge $u_ju_{j'}$ of $G \simeq K_{n-1}$ (except when $\{u_j, u_{j'}\} = \{u_l u_{l+1}\}$, $l = 1, 3, 5, \dots, n-2$), H contains all edges of the form $u_j^i u_{j'}^{i'}$, $i, i' = 0, 1, 2$. However, if $\{u_j, u_{j'}\} = \{u_l u_{l+1}\}$ for some $l = 1, 3, \dots, n-2$ then H contains no edge $u_j^i u_{j'}^{i'}$ with $i \ne i'$. Also, H contains no edge of the form $u_j^i u_j^{i'}$, $i, i' = 0, 1, 2$. We see that, abstractly, H is isomorphic to K_{3n-3} minus $(n-1)$ pairwise disjoint 3-cycles (the ones of the form $(u_j^0 u_j^1 u_j^2)$) and minus $(n-1)/2$ pairwise disjoint 6-cycles (the cycles $(u_l^0 u_{l+1}^1 u_l^2 u_{l+1}^0 u_l^1 u_{l+1}^2)$ for $l = 1, 3, 5, \dots, n-2$).

Let $\omega :\ H \to \hat{S}$ be the embedding just constructed. We recall that the boundary curves of the three holes in $\hat{S}$ are D_i, the images of our Hamiltonian cycle D under the isomorphisms f^i, $i = 0, 1, 2$. In order to complete the construction to obtain a face 2-colourable triangular embedding of K_{3n-2} we need one more modification of the bordered surface $\hat{S}$: We build one more auxiliary triangulated bordered surface $\bar{S}$ and paste it to $\hat{S}$ so that all three holes of $\hat{S}$ will be capped. The construction itself will use voltage assignments.

Let ν be the plane embedding of the (multi)graph L with faces of length 1 and 3 coloured black and white, as depicted in Fig. 2.

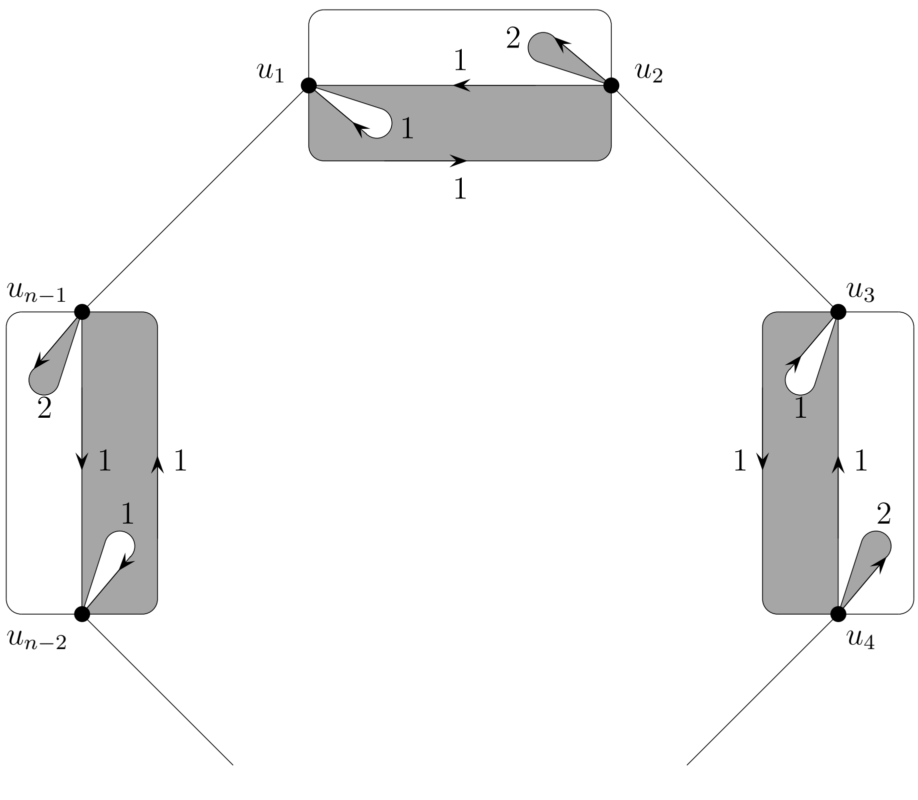

Fig. 2. The plane embedding of the graph L.

Our Fig. 2 also shows voltages α on directed edges of L, taken in the group $\mathcal{Z}_3 = \{0, 1, 2\}$. (It is assumed that edges with no direction assigned carry the zero voltage.)

The lifted graph L^α has the vertex set $\{u_j^i;\ 1 \le j \le n-1,\ i \in \mathcal{Z}_3\}$. (We are deliberately using the same letters for vertices of L as for vertices of the graphs G_i, but assume that these graphs are disjoint; such notation will be of advantage later.) The edge set of L^α can be described as follows. For each fixed $l = 1, 3, 5, \ldots, n-2$, the 6 vertices u_l^i, u_{l+1}^i ($i \in \mathcal{Z}_3$) induce a complete graph $J_l \simeq K_6$ in L^α. Moreover, two successive complete subgraphs J_l and

J_{l+2} (indices mod $(n-1)$) are joined by three edges $u^i_{l+1}u^i_{l+2}$, $i \in \mathcal{Z}_3$. (Thus we have a total of $15(n-1)/2+3(n-1)/2 = 9(n-1)$ edges in L^α, and there are neither loops nor multiple edges there.)

The lifted embedding $\nu^\alpha : L^\alpha \to \bar{S}$ has $4(n-1)$ triangular faces: the white ones are bounded by the triangles $(u^0_l u^1_l u^2_l)$ and $(u^0_l u^0_{l+1} u^2_{l+1})$, $(u^1_l u^1_{l+1} u^0_{l+1})$, $(u^2_l u^2_{l+1} u^1_{l+1})$, $l = 1, 3, 5, \ldots, n-2$, and the black ones are bounded by $(u^0_l u^2_l u^1_l)$ and $(u^0_l u^1_{l-1} u^2_{l-1})$, $(u^1_l u^2_{l-1} u^0_{l-1})$, $(u^2_l u^0_{l-1} u^1_{l-1})$, where $l = 2, 4, \ldots, n-1$. In addition, there are 4 more faces in the embedding ν^α; three faces, which we denote by F^i, bounded by $(n-1)$-gons of the form $(u^i_1 u^i_2 \ldots u^i_{n-1})$, $i \in \mathcal{Z}_3$, and one face F' bounded by the $(3n-3)$-gon $(u^0_1 u^1_2 u^1_3 u^2_4 \ldots u^2_{n-2} u^0_{n-1})$; here we use the fact that $n-1$ is coprime with 3. Observe that, in fact, the boundary of the latter face is a Hamiltonian cycle (say, B) in L^α whose $(3n-3)$ vertices (in the above cyclic order) can be encoded as $u^{\chi(j)}_j$ where $\chi(j) = \lfloor j/2 \rfloor$ $(mod\ 3)$.

Let us now cut out from $\bar{S}$ the three (open) faces F^i, $i \in \mathcal{Z}_3$, bounded by the above three disjoint $(n-1)$-gons, obtaining thereby an (orientable) bordered surface S^*. Let L^* be the graph obtained from L^α by adding a new vertex u^* and joining it to each vertex of L^α (and keeping all edges in L^α unchanged). We construct an embedding $\nu^* : L^* \to S^*$ from ν^α in an obvious way: in the embedding ν^α (after the removal of the three open faces), we insert the vertex u^* in the centre of the face F' bounded by the $(3n-3)$-gon and join this point by open arcs (within F') to *every* vertex on the boundary of F' (that is, with every vertex of the Hamiltonian cycle B or, equivalently, with every vertex of L^α). Instead of F' we now have $(3n-3)$ new triangular faces on S^*; they are bounded by the 3-cycles $u^* u^{\chi(j)}_j u^{\chi(j+1)}_{j+1}$. We now colour the new triangular faces as follows: the face of ν^* bounded by the 3-cycle $u^* u^{\chi(j)}_j u^{\chi(j+1)}_{j+1}$ will be black (white) if the triangular face of the embedding ν^α containing the edge $u^{\chi(j)}_j u^{\chi(j+1)}_{j+1}$ is white (black). It is easy to check that this rule indeed well defines a 2-colouring of the triangular embedding $\nu^* : L^* \to S^*$. We thus have $4(n-1) + (3n-3) = 7(n-1)$ triangular faces on S^*, exactly half of which are black.

We are ready for the final step of the construction. Our method of constructing the orientable surface $\hat{S}$ guarantees that a chosen orientation of $\hat{S}$ induces *consistent* orientations of the boundary cycles of the three holes of $\hat{S}$; we may assume that the orientation induces the cyclic ordering of the cycles D^i in the form that was used before, namely, $D^i = f^i(D) = (u^i_1 u^i_2 \ldots u^i_{n-1})$, $i = 0, 1, 2$. The bordered surface S^* has three holes as well, and again, the method of construction implies that an orientation of S^* can be chosen so that the boundary cycles of the holes are oriented in the form $D^{*i} = (u^i_{n-1} \dots u^i_2 u^i_1)$, $i \in \mathcal{Z}_3$. It remains to do the obvious – namely, for $i = 0, 1, 2$ to paste together the boundary cycles D^i and D^{*i} so that corresponding vertices u^j_i get identified. As the result we obtain an orientable surface $\hat{S}\#S^*$, known as the *connected sum* of the bordered surfaces $\hat{S}$ and S^*, and a triangular embedding $\sigma: K \to \hat{S}\#S^*$ of some graph K. We claim that $K \simeq K_{3n-2}$ and that the triangulation is face 2-colourable.

Obviously, $|V(K)| = 3n - 2$. A straightforward edge count shows that $|E(K)| = |E(H)| + |E(L^*)| - 3|E(D)| = 3(n-1)(3n-8)/2 + 12(n-1) - 3(n-1) = (3n-2)(3n-3)/2$. It is easy to check that, except for edges incident with u^* and edges contained in the three $(n-1)$-cycles D^{*i}, the graph L^* contains *exactly those edges which are missing in* H. This shows that there are no repeated edges or loops in K, and thus $K \simeq K_{3n-2}$. As regards the face-2-colouring, we just have to see what happens along the identified $(n-1)$-cycles D^i and D^{*i} (since both triangulations of $\hat{S}$ and S^* are already known to be face 2-colourable). But according to the construction, if $l = 1, 3, 5, \dots, n-2$, a triangular face on $\hat{S}$ that contains the edge $u^i_l u^i_{l+1}$ is black, while the face on S^* bounded by the triangle $(u^i_l u^i_{l+1} u^{i-1}_{l+1})$ is white.

As an additional check we independently determine the genus of the surface $\hat{S}\#S^*$. Denoting the genus of a graph G (or surface S) by $\gamma(G)$ (or $\gamma(S)$), we have

$$\gamma(H) = 3\gamma(K_n) + 3|\mathcal{T}| - 2 = 3(n-3)(n-4)/12 + 3(n-1)(n-3)/6 - 2 \ .$$

We may determine $\gamma(\bar{S})$ from Euler's formula; this gives $\gamma(\bar{S}) = n-2$. Hence, finally,

$$\gamma(K) = \gamma(H) + \gamma(\bar{S}) + 2 = (3n^2 - 11n + 10)/4 = \gamma(K_{3n-2}) \ ,$$

as required. This completes the proof of Theorem 1. □

Theorem 2 *Let* $n \equiv 7$ *(mod 12). If* K_n *has a face 2-colourable orientable triangular embedding, then* K_{3n-2} *also has a face 2-colourable orientable triangular embedding.*

Proof. The proof is similar to that of the previous Theorem, differing mainly in the definition of the graph L. Referring to our earlier definition of L, we take one of the two-point subgraphs, say that containing u_1 and

u_2, and replace the voltage assignment 1 by 2 and vice versa, the remaining part of L being unaltered. The proof then proceeds on the same lines as before with the modified version of L. Note that this alteration ensures that the lifted embedding still has a $(3n-3)$-gon face even though $n-1$ is not coprime with 3. The order of the vertices around this face differs from that in the previous Theorem, but it is still possible to insert a new vertex u^* and to complete a 2-colouring of the resulting triangular embedding. □

4 Conclusion

Applying Theorem 1 to the existing face 2-colourable orientable triangulations of K_n for $n \equiv 3 \ (mod\ 12)$ described e.g. in [7], we can deduce the existence of a face 2-colourable orientable triangular embedding of the complete graph K_n for $n \equiv 7 \ (mod\ 36)$. Then applying Theorem 2 we obtain embeddings for $n \equiv 19 \ (mod\ 108)$. Proceeding in this fashion we obtain face 2-colourable triangular embeddings for $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots = \frac{1}{2}$ of the residue class $n \equiv 7 \ (mod\ 12)$.

We also observe that the "twist" given to the pair (u_1, u_2) in the proof of Theorem 2 may be applied to any individual pair. In fact, in the constructions of both Theorems 1 and 2 we may "twist" any number, say k, of the pairs (u_{2i-1}, u_{2i}) provided in the former case that $k \equiv 0$ or $1 \ (mod\ 3)$ and in the latter case that $k \equiv 1$ or $2 \ (mod\ 3)$ as these values are consistent with obtaining a $(3n-3)$-gon face in the lifted embedding of L^α. We can thereby produce a large number of different face 2-colourable triangular embeddings which may or may not be isomorphic.

An inspection of the proofs in the preceding section shows that, in fact, they apply also to the nonorientable case in the following sense. If one starts with a face 2-colourable embedding of K_n in a *nonorientable* surface (which can exist only if $n \equiv 1$ or $3 \ (mod\ 6)$), then the method of the proof of Theorems 1 and 2 produces a nonorientable face 2-colourable triangulation of K_{3n-2}. We state this as a separate result.

Theorem 3 *Let $n \equiv 1$ or $3 \ (mod\ 6)$. If K_n has a face 2-colourable triangular embedding in a nonorientable surface, then K_{3n-2} also has a face 2-colourable nonorientable triangular embedding.* □

Nonorientable face 2-colourable triangulations of K_n are known to exist for all $n \equiv 3 \ (mod\ 6)$, $n \geq 9$ (see also the discussion at the end of Chapter 9 in [7]). Combining this fact with Theorem 3 yields:

Theorem 4 *For half of the residue class $n \equiv 1$ (mod 6) there exists a face 2-colourable nonorientable triangular embedding of K_n.* □

Our final observations concern the problem of constructing non-isomorphic triangulations of complete graphs as stated in [6]. Essentially this is settled for $n \equiv 7$ (*mod* 12) by the orientable embeddings given in [7] and [11]. Those in [7] (for $n \neq 7$) are not face 2-colourable, while from [11] we can obtain face 2-colourable embeddings (see also [4]). Hence we may state:

Theorem 5 *For $n \equiv 7$ (mod 12) and $n \neq 7$ there exist at least two non-isomorphic orientable triangular embeddings of K_n.* □

Furthermore, for $n \equiv 1$ (*mod* 6) and $n \neq 1$ or 7, [7] gives a nonorientable triangular embedding of K_n which is not face 2-colourable. Combining this with Theorem 4 above we have:

Theorem 6 *For half of the residue class $n \equiv 1$ (mod 6) there exist at least two non-isomorphic nonorientable triangular embeddings of K_n.* □

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