CONFLICT-FREE COLORINGS OF UNIFORM HYPERGRAPHS WITH FEW EDGES

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Abstract. A coloring of the vertices of a hypergraph $\mathcal{H}$ is called *conflict-free* if each edge e of $\mathcal{H}$ contains a vertex whose color does not repeat in e. The smallest number of colors required for such a coloring is called the conflict-free chromatic number of $\mathcal{H}$, and is denoted by $\chi_{CF}(\mathcal{H})$. Pach and Tardos proved that for an $(2r-1)$-uniform hypergraph $\mathcal{H}$ with m edges, $\chi_{CF}(\mathcal{H})$ is at most of the order of $rm^{1/r}\log m$, for fixed r and large m. They also raised the question whether a similar upper bound holds for r-uniform hypergraphs. In this paper we show that this is not necessarily the case. Furthermore, we provide lower and upper bounds on the minimum number of edges of an r-uniform simple hypergraph that is not conflict-free k-colorable.

1. Introduction

Let $\mathcal{H}$ be a hypergraph with vertex set $V(\mathcal{H})$ and edge set $E(\mathcal{H})$. A coloring $c : V(\mathcal{H}) \to \{1, 2, 3, ...\}$ of $V(\mathcal{H})$ is a *proper coloring of* $\mathcal{H}$ if no edge of size at least 2 is monochromatic. The minimum number of colors required for such a coloring is called the *chromatic number* of $\mathcal{H}$, and is denoted by $\chi(\mathcal{H})$. A *rainbow coloring* of $\mathcal{H}$ is a proper coloring of $\mathcal{H}$ such that for every edge e, the colors of all vertices of e are distinct. The minimum number of colors required for a rainbow coloring is called the *rainbow chromatic number* of $\mathcal{H}$, and is denoted by $\chi_R(\mathcal{H})$.

In connection with some frequency assignment problems for cellular networks, Even *et al.* [8] introduced (in a geometric setting) the following intermediate coloring. A proper coloring of $\mathcal{H}$ is *conflict-free* if for each edge e of $\mathcal{H}$, some color occurs on exactly one vertex of e. The minimum number of colors required for a conflict-free coloring is called the *conflict-free chromatic number* of $\mathcal{H}$, and is denoted by $\chi_{CF}(\mathcal{H})$. Since edges of size one are always conflict free in any coloring, we may assume that all edges have size at least two in the rest of the paper. Because of applications and some interesting features, this parameter attracted considerable attention (see, e.g. [2, 3, 4, 6, 8, 9, 13, 12]). In particular, Pach and Tardos [12] discussed this notion for general hypergraphs and proved several interesting results. Clearly, $\chi(\mathcal{H}) \le \chi_{CF}(\mathcal{H}) \le \chi_R(\mathcal{H})$ for every $\mathcal{H}$ with equalities when $\mathcal{H}$ is an ordinary graph. However, for general hypergraphs, the behavior of χ_{CF} may differ significantly

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from that of χ and of χ_R. For example, if we truncate an edge of a hypergraph, then χ cannot decrease, χ_R cannot increase, while χ_{CF} may increase, decrease, or stay the same. As yet another example we mention that if $\mathcal{H}$ is a 10^6-uniform hypergraph with 10 edges, then $\chi(\mathcal{H}) = 2$ and $\chi_{CF}(\mathcal{H})$ can be 2, 3, or 4.

Pach and Tardos [12] analyzed the conflict-free colorings for graphs and hypergraphs. They proved that $\chi_{CF}(\mathcal{H}) \le 1/2 + \sqrt{2m + 1/4}$ for every hypergraph with m edges, and that this bound is tight. They also showed the following result.

Theorem 1 ([12]). *Let r be fixed. Let $\mathcal{H}$ be a hypergraph with m edges such that the size of every edge is at least $2r-1$. Then $\chi_{CF}(\mathcal{H}) \le C_r m^{1/r} \log m$, where C_r is a positive constant depending only on r.*

In fact, they proved a stronger result. Let us define the *edge degree* of an edge e in a hypergraph $\mathcal{H}$ as the number of other edges intersecting e. The maximum edge degree $D(\mathcal{H})$ is the maximum of the edge degrees over all the edges of $\mathcal{H}$. This parameter has been used to bound the chromatic number by a number of authors (e.g., [1, 5, 7, 11, 14, 15]). As for the confict-free chromatic number Pach and Tardos [12] showed that $\chi_{CF}(\mathcal{H}) \le C_r m^{1/r} \log m$ for all hypergraphs $\mathcal{H}$ in which the size of every edge is at least $2r-1$ and $D(\mathcal{H}) \le m$. They also posed the question whether the same upper bound holds also when every edge has size at least r and intersects at most m others. In this paper we show that this is not the case.

The goal of the paper is to give reasonable upper bounds on $\chi_{CF}(\mathcal{H})$ for r-uniform hypergraphs $\mathcal{H}$ with given number of edges or maximum edge degree. It will turn out that for a given m, the nature of the bounds for r-uniform hypergraphs with m edges significantly depends on whether r is small or large with respect to m. We also derive similar bound for *simple* r-uniform hypergraphs, i.e. for hypergraphs in which any two distinct edges share at most one vertex. It turns out that for positive integers r, k with $r \le k/8$, both upper and lower bounds on the minimum number of edges in an r-uniform simple hypergraph that have no conflict-free colorings with k colors are roughly squares of the corresponding bounds for hypergraphs without the restriction of being simple.

For a warm-up, in Section 2 we bound the number of edges in r-uniform hypergraphs with χ_{CF} equal to 3 or 4. In particular, for arbitrarily large even r, there is an r-uniform hypergraph $\mathcal{H}$ with just 7 edges and $\chi_{CF}(\mathcal{H}) = 4$. In Section 3, we find upper bounds on $\chi_{CF}(\mathcal{H})$ in terms of the size/maximum edge degree of $\mathcal{H}$ and present some constructions showing that our bounds are reasonable. In Section 4, we do the same for simple r-uniform hypergraphs.

2. Conflict-free coloring of hypergraphs with very few edges

We define the *s-blow up* of a graph G to be the hypergraph formed by replacing every vertex v of G with an s element set B_v. The set B_v is called a *blob*. If uv is an edge in G, then $B_u \cup B_v$ is an edge in the blow-up.

Observation 2. *For a hypergraph $\mathcal{H}$, if either the degree of every vertex of H is at most 1, or if there is a vertex contained in every edge of H, then $\chi_{CF}(\mathcal{H}) = 2$.* □

Observation 3. *Let $r \geq 2$. If $\mathcal{H}$ is an r-uniform hypergraph which is not conflict-free 2-colorable, then it has at least 3 edges and the only such graph with 3 edges is the $(r/2)$-blow up of K_3.*

Proof. By Observation 2, every hypergraph with 2 edges is conflict-free 2-colorable. Moreover, a blow-up of K_3 is not. Now assume that $\mathcal{H}$ is an r-uniform hypergraph with 3 edges e_1, e_2, e_3 which is not conflict-free 2-colorable. If every vertex has degree at most 1 or if there is a vertex of degree 3, then by Observation 2 it is conflict-free 2-colorable. So assume that the maximum degree is 2. Without loss of generality assume that $v \in e_1 \cap e_2$. If there exists $u \in e_3 - e_1 - e_2$, then we color v and u with color 1 and all the remaining vertices with color 2. This would give a conflict-free 2-coloring of $\mathcal{H}$, a contradiction. Hence $e_3 \subseteq \{e_1 - e_2\} \cup \{e_2 - e_1\}$. Since $\mathcal{H}$ is r-uniform, we have that $e_3 \nsubseteq e_1$ and $e_3 \nsubseteq e_2$. Thus, $e_1 \cap e_3 \neq \emptyset$ and the above argument holds if v is replaced by a vertex $w \in e_3$. Consequently, $e_1 \subseteq \{e_2 - e_3\} \cup \{e_3 - e_2\}$ and similarly $e_2 \subseteq \{e_1 - e_3\} \cup \{e_3 - e_1\}$. Moreover, since $\mathcal{H}$ is r-uniform, it must be the $(r/2)$- blow up of K_3. In particular, r is even. □

Lemma 4. *Let $r \geq 3$. If $\mathcal{H}$ is an r-uniform hypergraph with at most 6 edges, then it is always conflict-free 3-colorable. Moreover, if $r \geq 4$ and r is divisible by 4, then there exists an r-uniform hypergraph with 7 edges which is not conflict-free 3-colorable.*

Proof. We first show that if $\mathcal{H}$ has at most 6 edges then $\chi_{CF}(\mathcal{H}) \leq 3$. Let $\Delta(\mathcal{H})$ be the maximum degree of $\mathcal{H}$.

Case 1. $\Delta(\mathcal{H}) \geq 4$. Let v be a vertex of degree at least 4. We color v with color 1. By Observation 3, there is a conflict-free coloring of $\mathcal{H} - v$ with colors 2 and 3. This gives a conflict-free 3-coloring of $\mathcal{H}$.

Case 2. $\Delta(\mathcal{H}) \leq 2$. Since $\chi_{CF}(\mathcal{G}) \leq \Delta(\mathcal{G}) + 1$ for every hypergraph $\mathcal{G}$ we can conflict-free 3-color $\mathcal{H}$ (see [12]).

Case 3. $\Delta(\mathcal{H}) = 3$. Let v be a vertex of degree 3 contained in the edges e_1, e_2 and e_3. If $\mathcal{H} - \{e_1, e_2, e_3\} = \{e_4, e_5, e_6\}$ is conflict-free 2-colorable, then we color them conflict-free with colors 2 and 3, color v with color 1 and arbitrarily color the remaining vertices with colors 2 and 3. This gives a conflict-free 3-coloring of $\mathcal{H}$. If not, then by Observation 3, $\{e_4, e_5, e_6\}$ forms the $(r/2)$-blow up of K_3. We may assume that $e_4 \cup e_5 \cup e_6 = B_4 \cup B_5 \cup B_6$, where B_4, B_5 and B_6 are the blobs $e_5 \cap e_6$, $e_4 \cap e_6$ and $e_4 \cap e_5$, respectively. Now, suppose that there is a vertex $u \in (e_4 \cup e_5 \cup e_6) - (e_1 \cup e_2 \cup e_3)$. Without loss of generality assume that $u \in B_6$. Let w be a vertex in B_5. We now color v and u with color 1, w with color 2 and the rest of the vertices with color 3. This gives a conflict-free 3-coloring of $\mathcal{H}$. Hence $\{e_4 \cup e_5 \cup e_6\} \subseteq \{e_1 \cup e_2 \cup e_3\}$. Thus every vertex in $\{e_4 \cup e_5 \cup e_6\}$ has degree 3.
The above argument holds for each vertex $u \in e_4 \cup e_5 \cup e_6$ by replacing v with u and e_1, e_2, e_3 with the three edges containing u. Hence by symmetry, the degree of every vertex of $\mathcal{H}$ is 3. We also know that deleting any vertex, leaves a copy of the $(r/2)$-blow up of K_3. Moreover, since $\mathcal{H}$ is r-uniform, $\mathcal{H}$ must be the $(r/2)$-blow up of K_4. A blow-up of K_4 can be conflict-free 3-colored as follows. In the first blob we color a vertex with color 1 and another with color 2 and the rest with 3. In the second blob

we color one vertex with 2 and the rest with 3. In the third blob we color one vertex with 1 and the rest with 3 and in the fourth blob we color everything with color 3.

Now to show that there exists a hypergraph $\mathcal{H}$ with 7 edges which is not 3-conflict-free colorable, we consider the $(r/4)$-blow up of the Fano plane and take the complement of every edge. The resulting hypergraph $\mathcal{H}$ has seven blobs $B_1, B_2, ...B_7$ and the following edges: $e_1 = B_1 \cup B_2 \cup B_6 \cup B_7$, $e_2 = B_2 \cup B_3 \cup B_4 \cup B_7$, $e_3 = B_4 \cup B_5 \cup B_6 \cup B_7$, $e_4 = B_1 \cup B_2 \cup B_4 \cup B_5$, $e_5 = B_1 \cup B_3 \cup B_4 \cup B_6$, $e_6 = B_2 \cup B_3 \cup B_5 \cup B_6$, and $e_7 = B_1 \cup B_3 \cup B_5 \cup B_7$. Suppose that there is a conflict-free 3-coloring f of $\mathcal{H}$ with colors $1, 2$ and 3.

Claim 1: No color can appear in exactly one blob.
Proof: Assume that a color, say 1, appears in exactly one blob. Consider the three edges e_2, e_3, e_6 not containing B_1. They must be conflict-free 2-colorable with colors $2, 3$. But they form the $(r/2)$-blow up of K_3 which is not conflict-free 2-colorable, a contradiction.

Claim 2: No color can appear in exactly two blobs.
Proof: Suppose that color 1 appears in exactly two blobs. Let B_1, B_2 be the blobs containing vertices of color 1. Consider the two edges e_1, e_4 containing both B_1 and B_2 and the edge e_3 containing neither B_1 nor B_2. These three edges form the $(r/2)$-blow up of K_3 with at least two vertices of color 1 present in a single blob. All other vertices gets color 2 or 3. With these restrictions there exists no conflict-free 3-coloring of the blow up of K_3.

Hence by the above claims, every color appears in at least three blobs.

Since f is a conflict-free 3-coloring of $\mathcal{H}$ which has seven edges, some color is unique for at least three edges. Assume that this color is 1.

Claim 3: A vertex with color 1 cannot be unique for more than one edge.
Proof: If not, then without loss of generality, assume that a vertex with color 1 belonging to B_1 is unique for edges e_4 and e_5. Hence the blobs B_2, B_3, B_4, B_5, B_6 do not have any vertices of color 1. So color 1 appears only in at most two blobs. This contradicts Claims 1 and 2.

Assume that a vertex of color 1 in B_1 is unique for e_1. So the blobs B_2, B_6, B_7 do not have vertices of color 1. Again without loss of generality assume that a vertex of color 1 in B_3 is unique for the edge e_2. So the blob B_4 does not have any vertex of color 1. Now there must be a vertex of color 1 in B_5 which is unique for e_3. We now consider the edges e_4, e_5, e_6. Each of these edges contains exactly two vertices of color 1. We delete these vertices and consider the new edges e'_4, e'_5, e'_6. The hypergraph formed by these edges must be conflict-free 2-colorable with colors $2, 3$. The edges e'_4, e'_5, e'_6 form the $((r/2) - 1)$-blow up of K_3 which is not conflict-free 2-colorable, a contradiction. $\square$

3. Conflict-free coloring of hypergraphs with few edges

Having dealt with small cases, now we study the bounds for the conflict-free chromatic number of hypergraphs with few edges. We start with a simple probabilistic fact we shall use later on.

Lemma 5. *Color a set T of t points, randomly, with s colors, so that each of s^t colorings is equally likely. Let $p_{t,s}$ be the probability that no color appears exactly once on T and let $\hat{p}_{t,s}$ be the probability that at most one color appears exactly once on T. Then*

$$p_{t,s} \le \left(\frac{2t}{s}\right)^{\lceil t/2 \rceil} \tag{1}$$

and

$$\hat{p}_{t,s} \le \left(\frac{8t}{s}\right)^{\lceil (t-1)/2 \rceil}. \tag{2}$$

Proof. To prove (1), let us randomly color all elements of T, one by one. Note that if no color appears exactly once we shall use at most $\lfloor t/2 \rfloor$ of them, and the set T' of the elements that are colored with a color which we have already used has at least $\lceil t/2 \rceil$ elements. Thus, since the number of ways to choose T' is at most 2^t, we get

$$p_{t,s} < 2^t \left(\frac{t}{2s}\right)^{\lceil t/2 \rceil} \le \left(\frac{2t}{s}\right)^{\lceil t/2 \rceil}.$$

In order to show (2), we again randomly color all elements of T one by one. Note that we shall use at most t colors. Furthermore, in this case the set T' of the elements that are colored with a color which we have already used has at least $\lceil (t-1)/2 \rceil$ elements and the number of ways to choose T' is at most 2^t. Hence

$$\hat{p}_{t,s} < 2^t \left(\frac{t}{s}\right)^{\lceil (t-1)/2 \rceil} \le \left(\frac{8t}{s}\right)^{\lceil (t-1)/2 \rceil}.$$

□

Now we can bound $\chi_{CF}(\mathcal{H})$ for an r-uniform hypergraph $\mathcal{H}$ with m edges.

Theorem 6. *Let $\mathcal{H}$ be a r-uniform hypergraph with m edges and maximum edge degree $D(\mathcal{H})$.*

(i) If $D(\mathcal{H}) \le 2^{r/2}$, and $D(\mathcal{H})$ (and thus r) is large enough, then there exists a vertex coloring of $\mathcal{H}$ with $120 \ln D(\mathcal{H})$ colors such that each edge has at least one color appearing exactly once. In particular,

$$\chi_{CF}(\mathcal{H}) \le 120 \ln D(\mathcal{H}) \le 120 \ln m.$$

(ii) If $m \ge 2^{r/2}$, then $\chi_{CF}(\mathcal{H}) \le 4r(16m)^{2/(r+2)}$.

Proof. In order to show (i) we set $p = 1.34 \ln D(\mathcal{H})/r$, choose a subset $\hat{T}$ of vertices of $\mathcal{H}$ independently with probability p, and then color each vertex of $\hat{T}$ independently with one of $s = 120 \ln D(\mathcal{H})$ colors. Let A_e be the event that no color appears exactly once in the edge e. Then, by Lemma 5,

$$\begin{aligned}\mathbb{P}(A_e) &\le \sum_{i=0}^{i_0} \binom{r}{i} p^i (1-p)^{r-i} \Big(\frac{2i}{s}\Big)^{i/2} + \sum_{i=i_0+1}^{r} \binom{r}{i} p^i (1-p)^{r-i} \\ &\le \sum_{i=0}^{i_0} \binom{r}{i} p^i (1-p)^{r-i} \Big(\frac{2i_0}{s}\Big)^{i/2} + \sum_{i=i_0+1}^{r} \binom{r}{i} p^i (1-p)^{r-i},\end{aligned}$$

where here and below $i_0 = \lfloor 2.5 \cdot 1.34 \ln D(\mathcal{H}) \rfloor$.

Since $p \le 1.34(r/2)(\ln 2)/r \le 0.47$, for $i \ge i_0 + 1$ we have

$$\frac{\binom{r}{i+1} p^{i+1} (1-p)^{r-i-1}}{\binom{r}{i} p^i (1-p)^{r-i}} \le \frac{r}{i} \frac{p}{1-p} \le \frac{1}{2.5(1-p)} < \frac{1}{1.325}.$$

so the second sum can be bounded from above by a geometric series and consequently

$$\sum_{i=i_0+1}^{r} \binom{r}{i} p^i (1-p)^{r-i} \le 4.08 \binom{r}{i_0+1} p^{i_0+1} (1-p)^{r-i_0-1}.$$

Since $\binom{r}{j} \le (\frac{er}{j})^j$ and $(1-p)^{r-j} \le (1-p)^r \le (e^{-pr/j})^j$ we have

$$\begin{aligned}\mathbb{P}(A_e) &\le \Big(1 + \Big(\sqrt{\frac{2i_0}{s}} - 1\Big)p\Big)^r + 4.08 \Big(\frac{erp}{i_0+1} \cdot e^{-pr/(i_0+1)}\Big)^{i_0+1} \\ &\le \exp(-0.76pr) + 4.08 \exp\big(-0.79 \cdot 1.34 \ln D(\mathcal{H})\big) \\ &\le D(\mathcal{H})^{-1.01} + 4.08 D(\mathcal{H})^{-1.05} \le 1/(4D(\mathcal{H}))\end{aligned}$$

for sufficiently large $D(\mathcal{H})$. Consequently, $D(\mathcal{H})\mathbb{P}(A_e) < 1/4$ and by Lovász Local Lemma, there exists a conflict-free coloring of $\mathcal{H}$ with $s = 120 \ln D(\mathcal{H})$ colors, so (i) follows.

Now let $s = 2r(16m)^{2/(r+2)}$ and $k = 2s$. We shall show that $\mathcal{H}$ has a conflict-free coloring with at most k colors. Let v be a vertex of maximum degree in $\mathcal{H}$. Reserve a color c for v and delete v along with all the edges containing it. Repeat this procedure and reserve a different color every time we delete a vertex of maximum degree in the remaining hypergraph. This procedure is repeated $k/2$ times. Let $\mathcal{H}_1$ denote hypergraph obtained by $k/2$ repetitions of this procedure. We consider the following two cases.

Case 1. $D(\mathcal{H}_1) < m^{r/(r+2)}$. Color each vertex of $\mathcal{H}_1$ by a color chosen randomly among s colors. Let A_e be the event that no color appears exactly once in the edge e. By Lemma 5, $\mathbb{P}(A_e) < (2r/s)^{r/2}$. Thus for $r \ge 2$,

$$4 \cdot D(\mathcal{H}_1) \cdot \mathbb{P}(A_e) < 4 \cdot m^{r/(r+2)} \cdot (2r/s)^{r/2} = 4 \cdot m^{r/(r+2)} \cdot (2r/2r(16m)^{2/(r+2)})^{r/2} \le 1.$$

Hence by Lovász Local Lemma, there exists a conflict-free coloring of $\mathcal{H}_1$ with $k/2$ colors. Together with the other $k/2$ colors, we have a conflict-free coloring of $\mathcal{H}$ with $k = 2s = 4r(16m)^{2/(r+2)}$ colors.

Case 2. $D(\mathcal{H}_1) \geq m^{r/(r+2)}$. Note that since each time we have deleted a vertex of maximum degree in the remaining hypergraph, we have removed at least $\Delta(\mathcal{H}_1) \geq \frac{D(\mathcal{H}_1)}{r} \geq \frac{m^{r/(r+2)}}{r}$ edges $k/2$ times. Thus, $m \geq km^{r/(r+2)}/(2r)$ which implies $k \leq 2rm^{2/(r+2)}$, a contradiction. This completes the proof of (ii). □

It is not hard to see that the bound given by Theorem 6(i) is tight up to a constant factor. Indeed, the following holds.

Proposition 7. *For all $m \geq 1$ and for all even $r \geq 2$, there exists an r-uniform hypergraph $\mathcal{H}$ with m edges such that $\chi_{CF}(\mathcal{H}) > \frac{1}{2}\log_2 m$.*

Proof. If $1 \leq m \leq 4$, then $\frac{1}{2}\log_2 m \leq 1$, and the statement follows. Let $m \geq 5$ and let n be the largest integer such that $\binom{n}{2} \leq m$. Let $\mathcal{H}'$ be the $(r/2)$-blow up of K_n, where the blobs are $B_1, \dots, B_n$. Consider the hypergraph $\mathcal{H}$ obtained from $\mathcal{H}'$ by adding $m - \binom{n}{2}$ isolated edges. By construction, $\mathcal{H}$ has m edges.

Let $k = \lfloor \log_2 n \rfloor$. Suppose that $\mathcal{H}$ has a conflict-free coloring f with k colors. For $i = 1, \dots, n$, let S_i be the set of colors that appear in the blob B_i. Since there are $2^k - 1$ nonempty distinct subsets of the set $\{1, \dots, k\}$ and $n > 2^k - 1$, there are some $1 \leq i < j \leq n$ with $S_i = S_j$. Then each color occurs in the edge $B_i \cup B_j$ an even number of times, a contradiction. So, $\chi_{CF}(\mathcal{H}) \geq 1 + k$.

Since $m \leq \binom{n+1}{2} - 1 = \frac{n^2+n-2}{2} < n^2$, we have $\log_2 m < 2\log_2 n < 2(1+k) \leq 2\chi_{CF}(\mathcal{H})$. □

To construct a matching bound for Theorem 6(ii), when m is much larger than r, is a harder task. Pach and Tardos [12] showed that if $\mathcal{H}$ is a r-uniform hypergraph with m edges, then $\chi_{CF}(\mathcal{H}) \leq rm^{2/(r+1)}\log m$, and they ask whether $\chi_{CF}(\mathcal{H}) \leq rm^{1/r}\log m$. We answer their question in the negative. More precisely, we show that if r is much smaller than m, then there exists r-uniform hypergraph $\mathcal{H}$ such that $\chi_{CF}(\mathcal{H}) \geq C_r m^{2/(r+2)}/\log m$. Let us start with a simple observation.

Observation 8. *Given any coloring f of an n-element set with k colors, we can choose a family $\mathcal{A}_f$ of k disjoint sets such that each set in $\mathcal{A}_f$ has size $\lfloor n/2k \rfloor$ and is monochromatic.*

Proof. Consider the color classes $A_1, A_2, ...A_k$. For each color class A_i we partition it into subclasses B_{i_j} of size equal to $\lfloor n/2k \rfloor$ until we cannot anymore. The last subclass say $B_{i_{j'}}$ for each i will have size less than $\lfloor n/2k \rfloor$. Summing the sizes of these $B_{i_{j'}}s$ we get at most $n/2$ vertices. The remaining at least $n/2$ vertices give us a family of k sets such that each set in $\mathcal{A}_f$ has size $\lfloor n/2k \rfloor$ and is monochromatic. □

Theorem 9. *For each positive even fixed r, there exists a constant $c_r \leq 4(8e^2/r)^{r/2}$ such that for every integer $k \geq r/2$, there exists an r-uniform hypergraph $\mathcal{H}$ with less than $1 + c_r k^{(r+2)/2}\log k$ edges such that $\chi_{CF}(\mathcal{H}) > k$.*

Proof. Consider a vertex set V of size n, a multiple of $4k$. Let

$$m = \left\lceil 4(8e^2/r)^{r/2}k^{(r+2)/2}\log k \right\rceil . \tag{3}$$

We form a random r-uniform hypergraph $\mathcal{H}$ with m edges by choosing m subsets $F_1, F_2, ..., F_m$ of V of size r randomly with equal probability and repetitions allowed.

We will prove that with a positive probability the conflict-free chromatic number of $\mathcal{H}$ is larger than k.

Let f be any fixed k-coloring of V. By Observation 8, there exists a family $\mathcal{A}_f$ of k sets $\{A_1, A_2, ..., A_k\}$ such that each of these sets has size $\lfloor n/2k \rfloor$ and is monochromatic. So the probability that edge F_i has a conflict is bounded from below by the probability that it has exactly 2 or 0 vertices from each of the sets in $\mathcal{A}_f$ and no vertices outside $\mathcal{A}_f$, which, in turn, is equal to $\binom{k}{r/2}\binom{\lfloor n/(2k) \rfloor}{2}^{r/2}\binom{n}{r}^{-1}$. Since

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k,$$

we get

$$\mathbb{P}(\text{edge } F_i \text{ has a conflict}) \geq \left(\frac{k}{r/2}\right)^{r/2}\left(\frac{n^2}{16k^2}\right)^{r/2}\left(\frac{r^2}{e^2n^2}\right)^{r/2} = \left(\frac{r}{8e^2k}\right)^{r/2}.$$

Consequently,

$$\begin{aligned}\mathbb{P}(f \text{ is a conflict-free coloring of } \mathcal{H}) &\leq \left(1 - \left(\frac{r}{8e^2k}\right)^{r/2}\right)^m \\ &< \exp\left(-m\left(\frac{r}{8e^2k}\right)^{r/2}\right).\end{aligned}$$

There are k^n distinct colorings of $V(\mathcal{H})$, so

$$\begin{aligned}\mathbb{P}(\mathcal{H} \text{ is conflict-free colorable with } k \text{ colors}) &< k^n \exp\left(-m\left(\frac{r}{8e^2k}\right)^{r/2}\right) \\ &\leq \exp\left(-m\left(\frac{r}{8e^2k}\right)^{r/2} + n\log k\right).\end{aligned}$$

If $n = 4k$, then by (3) the probability that $\mathcal{H}$ is conflict-free colorable is strictly smaller than 1. Hence there exists an r-uniform hypergraph $\mathcal{G}$ with m edges such that $\chi_{CF}(\mathcal{G}) > k$. □

Remark: Solving (3) for k, we get $k \sim C_r m^{2/(r+2)}/\log m$, where C_r is a function of r. Thus, Theorem 9 shows that for a given m and $r \leq C_r m^{2/(r+2)}/\log m$, there exists an r-uniform hypergraph $\mathcal{H}$ with m edges such that $\chi_{CF}(\mathcal{H}) > C_r m^{2/(r+2)}/\log m$.

4. Conflict-free coloring of simple hypergraphs

Although one can show that there exist simple r-uniform hypergraphs $\mathcal{H}$ with $m = C^r$ such that $\chi(\mathcal{H}) = \Theta(r)$, the second part of Theorem 6(ii) can be improved in the case of simple hypergraphs. Let us start with the following simple consequence of Lemma 5.

Lemma 10. *Let $r \leq k/8$ and let $\mathcal{H}$ be an r-uniform hypergraph. If $D(\mathcal{H}) < \frac{1}{4}(\frac{k}{8r})^{\lceil (r-1)/2 \rceil}$, then there exists a vertex coloring of $\mathcal{H}$ with k colors such that each edge has at least two colors appearing exactly once.*

Proof. Consider a random k-coloring of $\mathcal{H}$ and let A_e be the event that the edge e has at most one color appearing exactly once. By Lemma 5, the probability of A_e, $\mathbb{P}(A_e) \leq (\frac{8r}{k})^{\lceil (r-1)/2 \rceil}$. Now note that for a given edge e, the event A_e is independent of all but at most $D(\mathcal{H})$ other events $A_{e'}$. Thus, for $D(\mathcal{H}) < \frac{1}{4}(\frac{k}{8r})^{\lceil (r-1)/2 \rceil}$, we have $4 \cdot \mathbb{P}(A_e) \cdot D(\mathcal{H}) < 1$, and so by Lovász Local Lemma there exists a coloring where none of the events A_e occur. Consequently, there exists a coloring of $\mathcal{H}$ with k colors such that every edge has at least two colors appearing exactly once. □

Remark. By Lemma 7, for a given m, even if r is arbitrarily large (but even), there is an r-uniform hypergraph $\mathcal{H}$ with m edges and $\chi_{CF}(\mathcal{H}) > 0.5 \log_2 m$. There is no similar statement for simple hypergraphs. Indeed, if the maximum edge degree of a simple r-uniform hypergraph $\mathcal{H}$ is less than r, then we can choose in each edge e a vertex v_e that belongs only to e. Then we color each v_e with 1, and every other vertex with 2. So, such a hypergraph has a conflict-free coloring with just 2 colors.

Theorem 11. *Let $r \leq k/8$ and let $\mathcal{H}$ be an r-uniform simple hypergraph with m edges. If $m \leq \frac{1}{16r(r-1)^2}(\frac{k}{8(r-1)})^{r-2}$, then $\chi_{CF}(\mathcal{H}) \leq k$.*

Proof. Assume that $\chi_{CF}(\mathcal{H}) > k$. Let $\mathcal{H}_1$ be the hypergraph obtained from $\mathcal{H}$ by truncating each edge e by a vertex v_e of maximum degree. Observe that $\mathcal{H}_1$ is an $(r-1)$-uniform simple hypergraph and if f is a k-coloring of $\mathcal{H}_1$, then there exists an edge of $\mathcal{H}_1$ which has at most one color appearing exactly once, otherwise $\mathcal{H}$ would be conflict-free k-colorable. Now by Lemma 10, $D(\mathcal{H}_1) \geq \frac{1}{4}(\frac{k}{8(r-1)})^{\lceil (r-2)/2 \rceil}$. Furthermore, $\mathcal{H}_1$ has a vertex of degree at least $D(\mathcal{H}_1)/(r-1)$. If $\mathcal{H}_1$ has a vertex v of degree at least d, then every edge e in $\mathcal{H}_1$ containing v must have a vertex v_e whose degree in $\mathcal{H}$ is at least d. Moreover, since $\mathcal{H}$ is simple, all these d vertices are distinct. Hence $\mathcal{H}$ has at least $D(\mathcal{H}_1)/(r-1)$ vertices of degree at least $D(\mathcal{H}_1)/(r-1)$. So by the degree-sum formula,

$$m \geq D(\mathcal{H}_1)^2/r(r-1)^2 > \frac{1}{16r(r-1)^2}\left(\frac{k}{8(r-1)}\right)^{r-2}.$$

□

Note that if we solve the equation $m = \frac{1}{16r(r-1)^2}(\frac{k}{8(r-1)})^{r-2}$ with respect to k, then we get $k \sim C'_r m^{1/(r-2)}$. So for large r, the upper bound for the conflict-free chromatic number for simple hypergraphs provided by Theorem 11 is roughly a square of the bound given by Theorem 6 for the general case. The following result shows that, at least for large r, this estimate is not very far from being optimal.

Lemma 12. *Let $r \leq k$. Then, there exists an r-uniform simple hypergraph $\mathcal{H}$ with $(1+o(1))(4k \ln k)^2(\frac{4e^2k}{r})^r$ edges such that $\chi_{CF}(\mathcal{H}) > k$.*

Proof. We first construct an auxiliary $4k$-uniform simple hypergraph $\mathcal{H}_1$ as follows. Let q be a prime which will be chosen later. The vertex set of $\mathcal{H}_1$ is $S = S_1 \cup \ldots \cup S_{4k}$

where all S_i are disjoint copies of $GF(q) = \{0, 1, ..., q-1\}$. The edges of $\mathcal{H}_1$ are $4k$-tuples $(x_1, ..., x_{4k}) \in S_1 \times ... \times S_{4k}$ that are solutions of the system of linear equations

$$\sum_{i=1}^{4k} i^j x_i = 0, \qquad j = 0, 1, ..., 4k-3, \tag{4}$$

over $GF(q)$.

For any fixed pair of variables in (4), we have a $(4k-2) \times (4k-2)$ system of linear equations with Vandermond's determinant which has a unique solution over $GF(q)$. This means that $\mathcal{H}_1$ is $4k$-uniform simple hypergraph with $4kq$ vertices in which each vertex is contained in q edges, so $|E(\mathcal{H}_1)| = q^2$.

Now from each edge e of $\mathcal{H}_1$ we choose an r-subset A_e randomly and independently. Let $\mathcal{H}$ be the r-uniform simple hypergraph obtained from $\mathcal{H}_1$ by taking the subsets A_e as its edges. Our goal is to show that with a positive probability the conflict-free chromatic number of $\mathcal{H}$ is large.

To this end, fix a coloring f. Let B_e denote the event that the edge e has a conflict in the coloring f, and $p = \mathbb{P}(B_e)$. Arguing as in the proof of Theorem 9, one can show that

$$p \geq \Big(\frac{r}{8e^2k}\Big)^{r/2}.$$

Since the edges of $\mathcal{H}$ were chosen independently, the probability that f is a conflict-free coloring of $\mathcal{H}$ is $(1-p)^{q^2}$. Moreover, the total number of colorings is k^{4kq}, so the probability that there exists a conflict-free coloring of $\mathcal{H}$ with k colors is at most $k^{4kq} \cdot (1-p)^{q^2}$. This probability is less than 1, provided

$$k^{4kq} \cdot e^{-pq^2} < 1\,,$$

which holds whenever

$$q > \frac{4k \ln k}{p}.$$

Now if we take the smallest prime q such that $q > q_0 = 4k \ln k (\frac{8e^2k}{r})^{r/2}$, then we have an r-uniform simple hypergraph with q^2 edges and $\chi_{CF}(\mathcal{H}) > k$. It is known (see, for instance, [10]) that one can take $q = (1 + o(1))q_0$. Hence

$$|E(\mathcal{H})| = (1 + o(1))(4k \ln k)^2 \Big(\frac{8e^2k}{r}\Big)^r.$$

□

Finally, let us remark that if we take $k = r$, then we get a simple r-uniform hypergraph $\mathcal{H}$ with $m = 2^{O(r)}$ edges such that $\chi(\mathcal{H}) > r = \Omega(\ln m)$. So Theorem 6(i) cannot be much improved in the case of simple hypergraphs, at least when m grows exponentialy with r.

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