Initial semantics in logics with constructors

Daniel Găină[1] and Kokichi Futatsugi[1]

[1]Japan Advanced Institute of Science and Technology
{daniel,futatsugi}@jaist.ac.jp

November 6, 2012

Abstract

The constructor-based logics constitute the logical foundation of the so-called OTS/CafeOBJ method, a modeling, specification and verification method of the observational transition systems. It is well known the important role played in algebraic specifications by the initial algebras semantics. Free models along presentation morphisms provide semantics for the modules with initial denotation in structured specification languages. Following Goguen and Burstall, the notion of logical system over which we build specifications is formalized as an institution. The present work is an institution-independent study of the existence of free models along sufficient complete presentation morphisms in logics with constructors in the signatures.

1 Introduction

Algebraic specification represents one of the most important classes of formal methods assisting the development of more reliable and efficient software systems. The fundamental mathematical structure underlying the theory of algebraic specifications is that of institution [20], a formal concept of logical system which arose within computing science with the ambition of covering the population of logics used in practice and working as much as possible at the abstract level, independent of any particular institution. In the present paper we study the existence of initial models for theories and free models along theory morphisms in arbitrary constructor-based institutions. The logics with constructors constitute the foundation of the OTS/CafeOBJ method, a modeling, specification and verification method of the observational transition systems, which has been previously explored in many case studies [27, 17, 26, 16].

Constructor-based institutions [3, 4, 19] are obtained from a base institution basically by enhancing the syntax with a sub-signature of constructor operators and by restricting the semantics to reachable models which consist of constructor-generated elements. The sentences and the satisfaction condition are preserved from the base institution and the signature morphisms are restricted such that the reducts of models which are reachable in the target signature are again reachable in the source signature. The result sorts of the constructors are called constrained and a sort that is not constrained it is called loose.

By introducing the constructor operators in the signatures we gain more expressivity for the specifications but some of the basic important properties are lost. For example, the constructor-based variants of Horn institutions [1] are not complete. However, in [19] we obtained a ω-completeness [2] result that depends on *sufficient completeness*. Intuitively, a presentation (Σ, Γ), where Σ is a signature and Γ a set of formulas, is sufficient complete if every term can be reduced to a term formed with constructors and operators of loose sorts using the equations in Γ. We argued that a complete and compact system of proof rules cannot be provided due to the Gödel's famous incompleteness result.

Firstly, we study the existence of pushouts in a concrete category of constructor-based signature morphisms (see Section 3). The method of constructing the pushouts can be used in other cases as well. As a main result it is shown that the existence can be guaranteed by imposing certain conditions on the signature morphisms. This result is important as the pushout construction is one of the most used approaches to combine specifications. A property closely related to combining specifications coherently with respect to the semantics is *semi-exactness*. The institution theoretic concept of semi-exactness is a basic property of institutions that has been intensively used by various works in algebraic specification [29, 15, 8, 30]. In the present work we conduct an institution dependent study of the semi-exactness property that is used later on to prove the existence of free models.

Secondly, we investigate the existence of initial and free models in arbitrary constructor-based institutions.

1. 'Reachable' institutions are obtained from constructor-based institutions by restricting the category of signatures such that all operators of constrained sorts are constructors. In Section 4, we prove in arbitrary 'reachable' institutions the existence of initial models of any set of *Horn sentences* of the form $(\forall X) \bigwedge H \Rightarrow C$, where H is a set of atoms, and C is an atom. Our hypothesis are natural and easy to check in concrete examples of institutions. Our results are much general than the ones obtained within the framework of factorization systems [2, 33, 1] or inclusion systems [10]. In 'reachable' institutions the class of models of a set of Horn sentences do not form a quasi-variety, and therefore the initiality results derived from preservation (i.e. the model class of a set of Horn sentences forms a quasi-variety, and quasi-varieties have initial objects) such as [2, 33, 1] or [10] cannot be applied. Initiality is then easily extended to the constructor-based institutions via sufficient completeness.

2. The existence of left adjoints of the forgetful functors gives the free models along the presentation morphisms which constitute the semantic of the modules with initial denotation of the structured algebraic specification languages. This was studied in the literature under the name of liberality and it has played a central role in institution theory from its beginning [20] (see also [23, 31]). In Section 5 we apply the results in [10] to extend the existence of initial models of sufficient complete presentations to the existence of free models along sufficient complete presentation morphisms in arbitrary constructor-based institutions.

[1] Horn institutions are obtained from a base institution, for example first-order logic, by restricting the sentences to the so-called Horn sentences of the form $(\forall X) \bigwedge H \Rightarrow C$, where H is a set of atoms in the base institution, and C is an atom.

[2] Some proof rules contain infinite premises which can only be checked with induction schemes. As a consequence, the resulting entailment system is not compact.

We assume that the reader is familiar with the basic notions of category theory. See [25] for the standard definitions of category, functor, pushout, etc., which are omitted here.

2 Institutions

An *institution* $I = (\mathbb{S}ig^I, \mathbb{S}en^I, \mathbb{M}od^I, \models^I)$ consists of

1. a category $\mathbb{S}ig^I$, whose objects are called *signatures*,
2. a functor $\mathbb{S}en^I : \mathbb{S}ig^I \to \mathbb{S}et$, providing for each signature a set whose elements are called *(Σ-)sentences*,
3. a functor $\mathbb{M}od^I : \mathbb{S}ig^I \to \mathbb{C}at^{op}$, providing for each signature Σ a category whose objects are called *(Σ-)models* and whose arrows are called *(Σ-)morphisms*,
4. a relation $\models^I_\Sigma \subseteq |\mathbb{M}od^I(\Sigma)| \times \mathbb{S}en^I(\Sigma)$ for each signature $\Sigma \in |\mathbb{S}ig^I|$, called *(Σ-)satisfaction*, such that for each morphism $\varphi : \Sigma \to \Sigma'$ in $\mathbb{S}ig^I$, the following *satisfaction condition* holds:
$$M' \models^I_{\Sigma'} \mathbb{S}en^I(\varphi)(e) \text{ iff } \mathbb{M}od^I(\varphi)(M') \models^I_\Sigma e$$
for all $M' \in |\mathbb{M}od^I(\Sigma')|$ and $e \in \mathbb{S}en^I(\Sigma)$.

We denote the *reduct* functor $\mathbb{M}od^I(\varphi)$ by $\_ \upharpoonright_\varphi$ and the sentence translation $\mathbb{S}en^I(\varphi)$ by $\varphi(\_)$. When $M = M' \upharpoonright_\varphi$ we say that M is the φ-reduct of M' and M' is a φ-expansion of M. When there is no danger of confusion, we omit the superscript from the notations of the institution components; for example $\mathbb{S}ig^I$ may be simply denoted by $\mathbb{S}ig$.

Example 1 (First Order Logic (FOL) [20]) The signatures are triplets (S,F,P), where S is the set of sorts, $F = \{F_{w\to s}\}_{(w,s)\in S^*\times S}$ is the ($S^*\times S$ -indexed) set of operation symbols, and $P = \{P_w\}_{w\in S^*}$ is the (S^*-indexed) set of relation symbols. If $w = \lambda$, an element of $F_{w\to s}$ is called a *constant symbol*, or a *constant*. By a slight notational abuse, we let F and P also denote $\bigcup_{(w,s)\in S^*\times S} F_{w\to s}$ and $\bigcup_{w\in S^*} P_w$ respectively. A signature morphism between (S,F,P) and (S',F',P') is a triplet $\varphi = (\varphi^{st}, \varphi^{op}, \varphi^{rl})$, where $\varphi^{st} : S \to S'$, $\varphi^{op} : F \to F'$, $\varphi^{rl} : P \to P'$ such that $\varphi^{op}(F_{w\to s}) \subseteq F'_{\varphi^{st}(w)\to\varphi^{st}(s)}$ and $\varphi^{rl}(P_w) \subseteq P'_{\varphi^{st}(w)}$ for all $(w,s) \in S^* \times S$. When there is no danger of confusion, we may let φ denote each of φ^{st}, φ^{rl} and φ^{op}. Given a signature $\Sigma = (S,F,P)$, a Σ-model A is a triplet $A = (\{A_s\}_{s\in S}, \{A_{w,s}(\sigma)\}_{(w,s)\in S^*\times S, \sigma\in F_{w\to s}}, \{A_w(R)\}_{w\in S^*, R\in P_w})$ interpreting each sort s as a set A_s, each operation symbol $\sigma \in F_{w\to s}$ as a function $A_{w,s}(\sigma) : A^w \to A_s$ (where A^w stands for $A_{s_1} \times \ldots \times A_{s_n}$ if $w = s_1 \ldots s_n$), and each relation symbol $R \in P_w$ as a relation $A_w(R) \subseteq A^w$. When there is no danger of confusion we may let A_σ and A_R denote $A_{w,s}(\sigma)$ and $A_w(R)$ respectively. Morphisms between models are the usual Σ-morphisms, i.e., S-sorted functions that preserve the structure. The Σ-sentences are obtained from

- equality atoms $t_1 = t_2$, where $t_1, t_2 \in (T_{(S,F)})_s$ and $T_{(S,F)}$ is the (S,F)-algebra of ground terms, or
- relational atoms $R(t_1, \ldots, t_n)$, where $R \in P_{s_1 \ldots s_n}$ and $t_i \in (T_{(S,F)})_{s_i}$ for all $i \in \{1, \ldots, n\}$,

by applying for a finite number of times:

- negation, conjunction, disjunction,
- universal or existential quantification over finite sets of constants (variables).

Satisfaction is the usual first-order satisfaction and is defined using the natural interpretations of ground terms t as elements A_t in models A. The definitions of functors $\mathbb{S}en$ and $\mathbb{M}od$ on morphisms are the natural ones: for any signature morphism $\varphi : \Sigma \to \Sigma'$, $\mathbb{S}en(\varphi) : \mathbb{S}en(\Sigma) \to \mathbb{S}en(\Sigma')$ translates sentences symbol-wise, and $\mathbb{M}od(\varphi) : \mathbb{M}od(\Sigma') \to \mathbb{M}od(\Sigma)$ is the forgetful functor.

Example 2 (Constructor-based First Order Logic (CFOL)) The signatures are of the form (S,F,F^c,P), where (S,F,P) is a first-order signature, and $F^c \subseteq F$ (for all $(w,s) \in S^* \times S$ we have $F^c_{w\to s} \subseteq F_{w\to s}$) is a distinguished subfamily of sets of operation symbols called *constructors*. The constructors determine the set of *constrained* sorts $S^c \subseteq S$: $s \in S^c$ iff there exists a constructor $\sigma \in F^c_{w\to s}$. We call the sorts in $S^l = S - S^c$ *loose*. The (S,F,F^c,P)-sentences are the **FOL** sentences for the signature (S,F,P).

The (S,F,F^c,P)-models are the usual first-order structures M with the carrier sets for the constrained sorts consisting of interpretations of terms formed with constructors and elements of loose sorts, i.e. there exists a S-sorted set $Y = \{Y_s\}_{s\in S}$ of variables of loose sorts (i.e. for all $s \in S^c$ we have $Y_s = \emptyset$) and a S-sorted function $f = \{f_s : Y_s \to M_s\}_{s\in S}$ such that for every constrained sort $s \in S^c$, the function $f^\#_s : (T_{(S,F^c)}(Y))_s \to M_s$ is a surjection, where $T_{(S,F^c)}(Y)$ is the (S,F^c)-algebra of terms with variables from Y and $f^\# : T_{(S,F^c)}(Y) \to M \upharpoonright_{(S,F^c)}$ is the unique extension of f to a (S,F^c)-morphism.

A signature morphism $\varphi : (S,F,F^c,P) \to (S',F',F'^c,P')$ in **CFOL** is a first-order signature morphism $\varphi : (S,F,P) \to (S',F',P')$ such that the constructors are preserved along the signature morphisms (i.e. if $\sigma \in F^c$ then $\varphi(\sigma) \in F'^c$) and no 'new' constructors are introduced for 'old' constrained sorts (i.e. if $s \in S^c$ and $\sigma' \in (F'^c)_{w'\to\varphi(s)}$ then there exists $\sigma \in F^c_{w\to s}$ such that $\varphi(\sigma) = \sigma'$).

Lemma 3 [18] For every **CFOL** signature morphism $\varphi : (S,F,F^c,P) \to (S',F',F'^c,P')$, and any (S',F',F'^c,P')-model M', we have $M' \upharpoonright_{(S,F,P)} \in \mathbb{M}od(S,F,F^c,P)$.

Variants of constructor-based first-order logic were studied in[4] and [3].

Example 4 (Horn Clause Logic (HCL)) *A Horn sentence* for a **FOL** signature (S,F,P) is a sentence of the form $(\forall X)(\bigwedge H) \Rightarrow C$, where X is a finite set of variables, H is a finite set of (relational or equational) atoms, and C is a (relational or equational) atom. Classically, Horn clauses are Horn sentences in first-order logic without equality. Here, we call Horn clauses all Horn sentences of **FOL**. Thus **HCL** has the same signatures and models as **FOL** but only Horn clauses as sentences. One can define the constructor-based variant of **HCL** (i.e. Constructor-based Horn Clause Logic (**CHCL**)) as the restriction of **CFOL** to Horn sentences.

Example 5 (Preorder algebra (POA) [14, 13]) The **POA** signatures are just the ordinary algebraic signatures. The **POA** models are *preordered algebras* which are interpretations of the signatures into the category of preorders $\mathbb{P}re$ rather than the category of sets $\mathbb{S}et$. This means that each sort gets interpreted as a preorder, and each operation as a preorder functor, which means a preorder-preserving (i.e. monotonic) function. A *preordered algebra morphism* is just a family of preorder functors (preorder-preserving functions) which is also an algebra morphism.

The sentences have two kinds of atoms: equations and *preorder atoms*. A preorder atom $t \leq t'$ is satisfied by a preorder algebra M when the interpretations of the terms are in the preorder relation of the carrier, i.e. $M_t \leq M_{t'}$. Full sentences are constructed from equational and preorder atoms by using Boolean connectives and first-order quantification.

Horn preorder algebra (**HPOA**) and its constructor-based variant (**CHPOA**) are obtained as the restrictions of **POA** and **CPOA**, respectively, to Horn sentences.

Below we introduce a less refined class of constructor-based institutions that is used to import initiality and liberality to constructor-based institutions.

Example 6 (Reachable First Order Logic(RFOL)) This institution is obtained from **CFOL** by restricting the signatures such that all operation symbols of constrained sorts are constructors, i.e. for each (S,F,F^c,P) we have $F^c = F^{S^c}$ where

$$F^{S^c}_{w\to s} = \begin{cases} F_{w\to s} & : \quad s \in S^c \\ \emptyset & : \quad s \notin S^c \end{cases}$$

By restricting the sentences to Horn sentences we obtain **RHCL**. A similar construction can also be done for preorder algebra.

2.1 Substitutions

In **CFOL**, consider $\Sigma \overset{\chi_1}{\hookrightarrow} \Sigma(X_1)$ and $\Sigma \overset{\chi_2}{\hookrightarrow} \Sigma(X_2)$ two inclusion signature morphisms, where $\Sigma = (S,F,F^c,P)$ is a **CFOL** signature, X_i is a set of non-constructor constant symbols disjoint from the constants of F, and $\Sigma(X_i) = (S, F \cup X_i, F^c, P)$. A substitution between χ_1 and χ_2 in **CFOL** can be represented by a function $\theta : X_1 \to T_F(X_2)$. One can easily notice that θ can be extended to a function

$$\mathbb{S}en(\theta) : \mathbb{S}en(\Sigma(X_1)) \to \mathbb{S}en(\Sigma(X_2))$$

that replaces all the symbols in X_1 by the corresponding $F \cup X_2$-terms according to θ. On the semantics side, θ determines a functor

$$\mathbb{M}od(\theta) : \mathbb{M}od(\Sigma(X_2)) \to \mathbb{M}od(\Sigma(X_1))$$

such that for all $\Sigma(X_2)$-models M we have

- $\mathbb{M}od(\theta)(M)_z = M_z$, for each sort $z \in S$, or operation symbol $z \in F$, or relation symbol $z \in P$, and
- $\mathbb{M}od(\theta)(M)_z = M_{\theta(z)}$ for each $z \in X_1$.

Proposition 7 For every **CFOL** signature Σ and each substitution $\theta : X_1 \to \Sigma(X_2)$

$$\mathbb{M}od(\theta)(M) \models \rho \text{ iff } M \models \mathbb{S}en(\theta)(\rho)$$

for all $\Sigma(X_2)$-models M and all $\Sigma(X_1)$-sentences ρ.

The proof is the same as the one for **FOL**, which can be found in [11].

Assumption 8 Throughout this paper, for all institutions above, we assume that the signature morphisms allow mappings of constants to ground terms.

This makes it possible to treat first-order substitutions in the comma category [3] of signature morphisms.

Definition 9 Consider two signature morphisms $\Sigma \xrightarrow{\chi_1} \Sigma_1$ and $\Sigma \xrightarrow{\chi_2} \Sigma_2$ of an institution. A signature morphisms $\theta : \Sigma_1 \to \Sigma_2$ such that $\chi_1;\theta = \chi_2$ is called a Σ-*substitution* between χ_1 and χ_2.

2.2 Presentations

Let $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ be an institution. A presentation (Σ, E) consists of a signature Σ and a set E of Σ-sentences. A presentation morphism $\varphi : (\Sigma, E) \to (\Sigma', E')$ is a signature morphism $\varphi : \Sigma \to \Sigma'$ such that $E' \models \phi(E)$. The presentation morphisms form a category denoted $\mathbb{S}ig^{pres}$ with the composition inherited from the category of signatures. The model functor $\mathbb{M}od$ can be extended from the category of signatures $\mathbb{S}ig$ to the category of presentations $\mathbb{S}ig^{pres}$, by mapping a presentation (Σ, E) to the full subcategory [4] $\mathbb{M}od(\Sigma, E)$ of $\mathbb{M}od(\Sigma)$ consisting of models that satisfy E. The correctness of the definition of the overloaded model functor $\mathbb{M}od : \mathbb{S}ig^{pres} \to \mathbb{C}at^{op}$ is guaranteed by the satisfaction condition of the base institution. This leads to the *institution of presentations* $I^{pres} = (\mathbb{S}ig^{pres}, \mathbb{S}en, \mathbb{M}od, \models)$ over the base institution I, where the sentence functor $\mathbb{S}en$ and the satisfaction condition $\models$ of the base institution I are overloaded such that

- $\mathbb{S}en(\Sigma, E) = \mathbb{S}en(\Sigma)$, and
- for all $M \in \mathbb{M}od(\Sigma, E)$ and sentences $\rho \in \mathbb{S}en(\Sigma, E)$, $M \models_{(\Sigma,E)} \rho$ iff $M \models_\Sigma \rho$.

2.3 Basic set of sentences

A set of sentences $B \subseteq Sen(\Sigma)$ is called *basic* [9] if there exists a Σ-model M_B such that, for all Σ-models M, $M \models B$ iff there exists a morphism $M_B \to M$. We say that M_B is a *basic model* of B. If in addition the morphisms $M_B \to M$ is unique then the set B is called *epi basic*.

Remark 10 Any set of epi basic sentences has an initial model.

It is well-known that any set of atoms in **FOL** and **POA** is epi basic (see for example [9] or [12]). In the followings we lift this result to reachable institutions.

Lemma 11 Any set of atoms in **RFOL** and **RPOA** is epi basic.

PROOF. Let (S, F, F^c, P) be a a **RFOL**-signature. Since all operators of constrained sorts are constructors we have $T_{(S,F,P)} \in \mathbb{M}od(S, F, F^c, P)$, where $T_{(S,F,P)}$ is the term model interpreting each relation symbol as empty set. Moreover, for any congruence relation $\equiv \subseteq T_{(S,F,P)} \times T_{(S,F,P)}$ we have that $(T_{(S,F,P)})_\equiv \in \mathbb{M}od(S, F, F^c, P)$.

Let B be a set of atomic (S, F, F^c, P)-sentences. The basic model M_B it is the initial model of B and it is constructed as follows: on the quotient $(T_{(S,F,P)})_{\equiv_B}$ of the term model $T_{(S,F,P)}$

[3] Given a category $\mathcal{C}$ and an object $A \in |\mathcal{C}|$, the comma category $A/\mathcal{C}$ has arrows $A \xrightarrow{f} B \in \mathcal{C}$ as objects, and $h \in \mathcal{C}(B, B')$ with $f, h = f'$ as arrows.

[4] $\mathcal{C}'$ is a full subcategory of $\mathcal{C}$ if $\mathcal{C}'$ is a subcategory of $\mathcal{C}$ such that $\mathcal{C}(A, B) = \mathcal{C}'(A, B)$ for all objects $A, B \in |\mathcal{C}'|$.

by the congruence generated by the equational atoms of B, we interpret each relation symbol $\pi \in P$ by $(M_B)_\pi = \{(t_1/_{\equiv_B}, \ldots, t_n/_{\equiv_B}) \mid \pi(t_1, \ldots, t_n) \in B\}$.

By defining a notion of congruence compatible with the preorder (see [13] or [7]) one may obtain the same result for **RPOA**. ∎

A direct consequence of Lemma 11 is the following corollary.

Corollary 12 In **RFOL** and **RPOA**, any set of atoms has an initial model.

Given a **CFOL** signature (S, F, F^c, P), the model (S, F, P)-model $T_{(S,F,P)}$ is not reachable by the constructors in F^c, i.e. $T_{(S,F,P)} \in |\mathbb{M}od(S, F, F^c, P)|$ does not hold, in general. It follows that in **CFOL** not all sets of sentences are epi basic, and hence, not all sets of atoms have an initial model.

2.4 Reachable models

Below, we give an institution-independent characterization of the models with elements that consist of interpretations of terms.

Definition 13 Let $\mathcal{D}$ be a broad subcategory [5] of signature morphisms of an institution $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$. We say that a Σ-model M is *$\mathcal{D}$-reachable* if for each span of signature morphisms $\Sigma_1 \xleftarrow{\chi_1} \Sigma_0 \xrightarrow{\chi} \Sigma$ in $\mathcal{D}$, each χ_1-expansion M_1 of $M \upharpoonright_\chi$ determines a substitution $\theta : \chi_1 \to \chi$ such that $M \upharpoonright_\theta = M_1$.

The proof of the following proposition can be found in [18].

Proposition 14 In **FOL** and **POA** assume that $\mathcal{D}$ is the class of signature extensions with a finite number of constants. A model M is $\mathcal{D}$-reachable iff its elements are exactly the interpretations of ground terms.

In concrete institutions underlying the algebraic specification languages, $\mathcal{D}$ consists of signature extensions with a finite number of constants. Since $\mathcal{D}$-reachable models have elements which consist of interpretations of ground terms, we may refer to $\mathcal{D}$-reachable models as ground reachable models.

Remark 15 In **FOL** and **POA**, the models defining a set of atoms as basic set of sentences are ground reachable.

For each **RFOL** Σ-model M there exists a signature extension $\Sigma \hookrightarrow \Sigma'$ with constants of loose sorts, and a ground reachable Σ'-model M' such that the reduct of M' to the signature Σ is M. Note that Σ' can be the extension of Σ with the elements of loose sorts of the model M. In this case M' is just like M but interpreting each element of loose sort by itself. The **RFOL** models are called reachable. The **CFOL** models are reachable in the sub-signature of constructors. Actually, there is an abstract characterization of reachable models (see [19, 18]) which may be applied to a base institution in order to obtain the constructor-based variant.

[5] $\mathcal{C}'$ is a broad subcategory of $\mathcal{C}$ if $\mathcal{C}'$ is a subcategory of $\mathcal{C}$ and $|\mathcal{C}'| = |\mathcal{C}|$.

2.5 Internal logic

The following institutional notions dealing with logical connectives and quantifiers were defined in [32].

Let $\Sigma \in |\mathbb{S}ig|$ be a signature of an institution $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$.

- A Σ-sentence $\neg e$ is a *(semantic) negation* of the Σ-sentence e when for every Σ-model M we have $M \models_\Sigma \neg e$ iff $M \not\models_\Sigma e$.
- A Σ-sentence $e_1 \wedge e_2$ is a *(semantic) conjunction* of the Σ-sentences e_1 and e_2 when for every Σ-model M we have $M \models_\Sigma e_1 \wedge e_2$ iff $M \models_\Sigma e_1$ and $M \models_\Sigma e_2$.
- A Σ-sentence $(\forall\chi)e'$, where $\Sigma \xrightarrow{\chi} \Sigma' \in \mathbb{S}ig$ and $e' \in \mathbb{S}en(\Sigma')$, is a *(semantic) universal χ-quantification* of e' when for every Σ-model M we have $M \models_\Sigma (\forall\chi)e'$ iff $M' \models_{\Sigma'} e'$ for all χ-expansions M' of M.

We will use the symbol $\bigwedge$ to denote the conjunction of a set of sentences even if it is infinite. Very often quantification is considered only for a restricted class of signature morphisms. For example, quantification in **FOL** considers only the signature extensions with a finite number of constants.

3 Pushouts of constructor-based signatures

There is a difference between the signature morphisms in [3] and the signature morphisms in **CFOL**. The signature morphisms in [3] do not allow 'new' constructors of 'old' sorts, and they are too restrictive for applications as one can see below. We allow constructors of 'old' sorts that are loose, but the signature morphisms in **CFOL** do not have pushouts, in general. We will study here the conditions when the pushouts do exist.

3.1 Pushouts

Let $\varphi : (S, F, F^c, P) \to (S', F', F'^c, P')$ be a signature morphism in **CFOL**. We say that φ^{op} is *injective* if for all arities $w \in S^*$ and sorts $s \in S$, $\varphi^{op}_{w\to s}$ is injective. The same applies to φ^{ct}, the constructor symbols component, and φ^{rl}, the relation symbols component. φ^{op} is encapsulated means that no 'new' operation symbol, i.e. outside the image of φ, is allowed to have the sort in the image of φ. More precise, if $\sigma' \in F'_{w'\to s'}$ then for all $s \in S$ such that $s' = \varphi^{st}(s)$ there exists $\sigma \in F_{w\to s}$ such that $\varphi^{op}(\sigma) = \sigma'$. Same applies to φ^{ct}.

Definition 16 ($(xyzt)$-signature morphisms) A **CFOL** signature morphism

$$\varphi : (S, F, F^c, P) \to (S', F', F'^c, P')$$

is a $(xyzt)$-morphism, with $x, t \in \{i, *\}$ and $y, z \in \{i, e, *\}$, where i stands for 'injective', e for 'encapsulated', and $*$ for 'all', when

1. it does not map constants to terms different from constants,
2. the sort component $\varphi^{st} : S \to S'$ has the property x,
3. the operation component $\varphi^{op} = (\varphi^{op}_{w\to s} : F_{w\to s} \to F'_{\varphi^{st}(w)\to\varphi^{st}(s)})_{\substack{w\in S^* \\ s\in S}}$ has the property y,

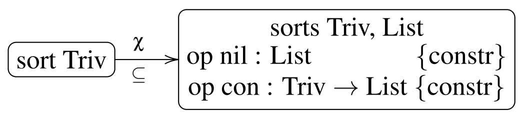
4. the constructor component $\varphi^{ct} = (\varphi^{ct}_{w\to s} : F^c_{w\to s} \to F'^c_{\varphi^{st}(w)\to\varphi^{st}(s)})_{w\in S^*, s\in S}$ has the property z, and
5. the relation component $\varphi^{rl} = (\varphi^{rl}_w : P_w \to P'_{\varphi^{st}(w)})_{w\in S^*}$ has the property t.

This notational convention can be extended to other institutions too, such as, for example, **FOL** or **CPOA**. In case of **FOL**, because there are no constructor symbols, the third component is missing. In case of **CPOA**, because there are no relation symbols, the last component is missing.

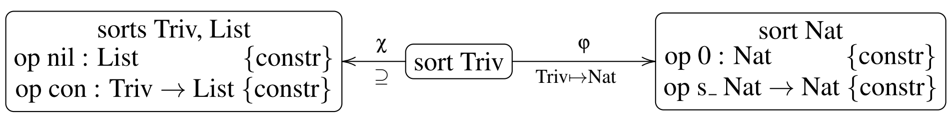
Proposition 17 [3] The subcategory of **CFOL** signature $(**e*)$-morphisms has pushouts.

Consider the following example of parameterized specification.

Example 18 Consider the following parameterized specification of lists of arbitrary elements:



If we want to obtain lists of natural numbers then we need to construct the pushout of the following span of signature morphisms:

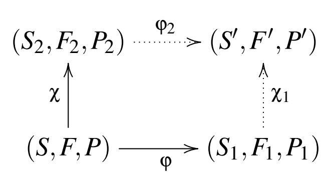


Note that χ is a signature morphism in the sense of [3] but φ is not.

We will investigate the existence of pushouts of constructor-based signature morphisms in order to cover the example above.

Proposition 19 The category of **CFOL** signature morphisms has $((****),(i*e*))$-pushouts [6]. Moreover, if $\{\Sigma_2 \stackrel{\chi}{\leftarrow} \Sigma \stackrel{\varphi}{\rightarrow} \Sigma_1, \Sigma_1 \stackrel{\chi_1}{\rightarrow} \Sigma' \stackrel{\varphi_2}{\leftarrow} \Sigma_2\}$ is a pushout of **CFOL** signature morphisms such that φ is a $(****)$-morphism and χ is a $(i*e*)$-morphism then φ_2 is $(****)$-morphism and χ_1 is a $(i*e*)$-morphism.

PROOF. Consider the following pushout of **FOL** signature morphisms.



Remark 20 By the pushout construction in the category of sets, the above pushout has the following properties:

[6] We say that a category C has $(\mathcal{L},\mathcal{R})$-pushouts for two subcategories $\mathcal{L},\mathcal{R}\subseteq C$, if for each span of morphisms $A_2 \stackrel{v}{\leftarrow} A_0 \stackrel{u}{\rightarrow} A_1$ such that $u\in\mathcal{L}$ and $v\in\mathcal{R}$ there exists a pushout $\{A_2 \stackrel{v}{\leftarrow} A_0 \stackrel{u}{\rightarrow} A_1, A_1 \stackrel{v_1}{\rightarrow} A \stackrel{u_2}{\leftarrow} A_2\}$.

1. χ_1 is injective on sorts,

2. for all $s_1 \in S_1$ and $s_2 \in S_2$ such that $\chi_1(s_1) = \varphi_2(s_2)$ there exists $s \in S$ such that $\varphi(s) = s_1$ and $\chi(s) = s_2$, and

3. φ_2 is injective on $S_2 - \chi(S)$.

Also note that χ_1 and φ_2 map constants to constants only.

Let $F'^c = \chi_1(F_1^c) \cup \varphi_2(F_2^c)$, where

- $F'^c = \{F'^c_{w' \to s'}\}_{\substack{w' \in S'^* \\ s' \in S'}}$, and
- $F'^c_{w' \to s'} = (\cup_{\chi_1(w_1,s_1)=(w',s')} \chi_1((F_1^c)_{w_1 \to s_1})) \cup (\cup_{\varphi_2(w_2,s_2)=(w',s')} \varphi_2((F_2^c)_{w_2 \to s_2}))$.

We prove that

$$\begin{array}{ccc}
(S_2, F_2, F_2^c, P_2) & \overset{\varphi_2}{\dashrightarrow} & (S', F', F'^c, P') \\
\chi \uparrow & & \uparrow \chi_1 \\
(S, F, F^c, P) & \xrightarrow[\varphi]{} & (S_1, F_1, F_1^c, P_1)
\end{array}$$

is a pushout of **CFOL** signature morphisms.

Firstly, we show that χ_1 is a $(i * e *)$-morphism. By the definition of F'^c all the constructors in F_1^c are preserved by χ_1. Now let $\sigma' \in F'^c_{w' \to \chi_1(s_1)}$, where $s_1 \in S_1$. There are two cases:

1. *There exists $\sigma_1 \in (F_1^c)_{w_1 \to st_1}$ such that $\chi_1(\sigma_1) = \sigma'$.* Note that $\chi_1(s_1) = \chi_1(st_1)$ and since χ_1 is injective on sorts, $s_1 = st_1$. Therefore $\chi_1(\sigma_1 : w_1 \to s_1) = \sigma' : w' \to \chi_1(s_1)$.

2. *There exists $\sigma_2 \in (F_2^c)_{w_2 \to st_2}$ such that $\varphi_2(\sigma_2) = \sigma'$.* We have $\varphi_2(st_2) = \chi_1(s_1)$. By Remark 20 there exists $s \in S$ such that $\varphi(s) = s_1$ and $\chi(s) = st_2$. Since χ is a $(i * e *)$-morphism there exists $\sigma \in F^c_{w \to s}$ such that $\chi(\sigma) = \sigma_2$. Now take $\sigma_1 : w_1 \to st_1 = \varphi(\sigma : w \to s)$. We have $\chi_1(st_1) = \chi_1(s_1)$ and by the injectivity of χ_1 we get $s_1 = st_1$. Therefore $\chi_1(\sigma_1 : w_1 \to s_1) = \sigma' : w' \to \chi_1(s_1)$.

Secondly, we show that φ_2 is a **CFOL** signature morphism. Again by the definition of F'^c all the constructors in F_2^c are preserved by φ_2. Let $\sigma' \in F'^c_{w' \to \varphi_2(s_2)}$, where $s_2 \in S_2^c$. We have two cases:

1. $s_2 \in \chi(S^c)$. Let $s \in S^c$ such that $\chi(s) = s_2$. Since χ_1 is a $(i * e *)$-morphism and $\varphi(s) \in S_1^c$, there exists $\sigma_1 \in (F_1^c)_{w_1 \to \varphi(s)}$ such that $\chi_1(\sigma_1) = \sigma'$. Because $s \in S^c$, there exists $\sigma \in F^c_{w \to s}$ such that $\varphi(\sigma) = \sigma_1$. Take $\sigma_2 = \chi(\sigma)$ and we have $\varphi_2(\sigma_2 : \chi(w) \to s_2) = \sigma' : w' \to \varphi_2(s_2)$.

2. $s_2 \in S_2^c - \chi(S^c)$. Since χ is a $(i * e *)$-morphism we have $s_2 \in S_2 - \chi(S)$. By the definition of F'^c we have two subcases:

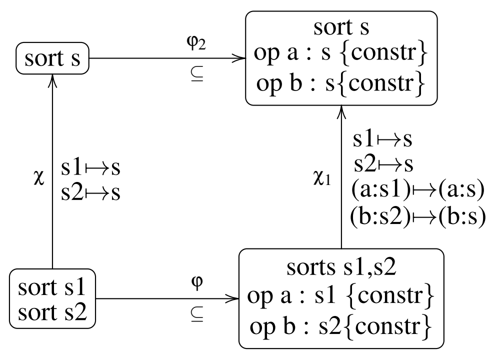
   (a) *There exists $\sigma_1 \in (F_1^c)_{w_1 \to st_1}$ such that $\chi_1(\sigma_1) = \sigma'$.* Because $\chi_1(st_1) = \varphi_2(s_2)$ there is $s \in S$ such that $\chi(s) = s_2$ and $\varphi(s) = st_1$ which is a contradiction with $s_2 \in S_2 - \chi(S)$.

(b) *There exists* $\sigma_2 \in (F_2^c)_{w_2 \to st_2}$ *such that* $\varphi_2(\sigma_2) = \sigma'$. We have $s_2 \in S_2 - \chi(S)$ and $\varphi_2(s_2) = \varphi_2(st_2)$ which by Remark 20 implies $s_2 = st_2$. Therefore $\varphi_2(\sigma_2 : w_2 \to s_2) = \sigma' : w' \to \varphi_2(s_2)$.

Now we show that for all $v_i : (S_i, F_i, F_i^c, P_i) \to (S'', F'', F''^c, P'')$, where $i \in \{1,2\}$, such that $\varphi; v_1 = \chi; v_2$ there exists a unique $v : (S', F', F'^c, P') \to (S'', F'', F''^c, P'')$ such that $\chi_1; v = v_1$ and $\varphi_2; v = v_2$. A unique v exists as a **FOL** signature morphism. We need to prove that v is a **CFOL** signature morphism. Let $\sigma'' \in F''^c_{w'' \to v(s')}$, where $s' \in S'^c$. By the definition of F'^c, either there is $s_1 \in S_1^c$ such that $\chi(s_1) = s'$ or there is $s_2 \in S_2^c$ such that $\varphi(s_2) = s'$. Assume that there exists $s_1 \in S_1^c$ such that $\chi_1(s_1) = s'$ (the other case is similar). Since v_1 is a **CFOL** signature morphism there is $\sigma_1 \in (F_1^c)_{w_1 \to s_1}$ such that $v_1(\sigma_1) = \sigma''$. Now take $\sigma' = \chi_1(\sigma_1)$ and we have $v(\sigma' : \chi_1(w_1) \to s') = \sigma'' : w'' \to v(s')$. ■

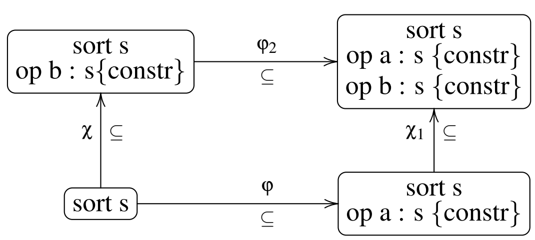
The condition χ is a $(i * e*)$-morphism is necessary for Proposition 19 as one can see in the examples below.

Example 21 Consider the following pushout in **FOL**:



Note that χ is not injective on sorts, and since there is no constructor $(c : s1)$ such that $\chi_1(c : s1) = (b : s)$, χ_1 is not a **CFOL** signature morphisms.

Example 22 Consider the following pushout in **FOL**:



Note that the condition “no constructors of ‘old’ sorts” is not fulfilled by χ and φ, and since χ_1 adds the constructor b of the ‘old’ constrained sort s, χ_1 is not a **CFOL** signature morphism.

Proposition 19 provides a method of constructing pushouts of signatures in other constructor-based institutions such as **CPOA**. Initially, the construction is done in the base institution, for example in **POA**, and then it is extended to the constructor-based variant, such as **CPOA**.

3.2 Pushouts of presentation morphisms

The pushouts of presentation morphisms have been playing a very important role in algebraic specifications [20, 15] as it constitutes the basis of constructing large specifications out of smaller ones.

Proposition 23 [20] Let $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ be an institution, and $\mathcal{L}, \mathcal{R} \subseteq \mathbb{S}ig$ two subcategories of signature morphisms. If $\mathbb{S}ig$ has $(\mathcal{L}, \mathcal{R})$-pushouts then the presentation morphisms have $(\mathcal{L}^{pres}, \mathcal{R}^{pres})$-pushouts, where

- $\mathcal{L}^{pres}$ consists of presentation morphisms $(\Sigma, E) \stackrel{\varphi}{\to} (\Sigma', E')$ such that $\Sigma \stackrel{\varphi}{\to} \Sigma' \in \mathcal{L}$,
- $\mathcal{R}^{pres}$ consists of presentation morphisms $(\Sigma, E) \stackrel{\chi}{\to} (\Sigma', E')$ such that $\Sigma \stackrel{\chi}{\to} \Sigma' \in \mathcal{R}$.

Parameterization is one of the most important technique used in structuring formal specifications. A parameterized presentation is a presentation morphism $\chi : (P, E_P) \to (B[P], E_B[E_P])$, where (P, E_P) is the parameter and $(B[P], E_B[E_P])$ the body, such that $\chi : P \to B[P]$ is

1. an inclusive signature morphism, and
2. a $(**e*)$-morphism (it does not add new constructors to the parameter).

Parameterization allows to abstract away the elements of a system that are not part of the essence, and can be obtained at a later time by instantiation. The pushout construction constitutes the basis of the instantiation mechanism. To instantiate (P, E_P) with (T, E_T) requires a parameter mapping $\varphi : (P, E_P) \to (T, E_T)$, and the result of the instantiation is the vertex of the pushout $\{(B[P], E_B[E_P]) \stackrel{\chi}{\leftarrow} (P, E_P) \stackrel{\varphi}{\to} (T, E_T), (B[P], E_B[E_P]) \stackrel{\varphi_2}{\to} (B[T], E_B[E_T] \stackrel{\chi_1}{\leftarrow} (T, E_T)\}$, where $E_B[E_T] = \varphi_2(E_B[E_P]) \cup \chi_1(E_T)$.

3.3 Semi-exactness

A basic institutional property that is necessary for combining specifications coherently with respect to the semantics is the *semi-exactness* property.

Definition 24 An institution $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ is $(\mathcal{L}, \mathcal{R})$-*semi-exact*, where $\mathcal{L}, \mathcal{R} \subseteq \mathbb{S}ig$ are two subcategories of signature morphisms, if for each $(\mathcal{L}, \mathcal{R})$-pushout of signature morphisms

$$\begin{array}{ccc} \Sigma_2 & \xrightarrow{\varphi_2} & \Sigma' \\ {\scriptstyle \chi}\uparrow & & \uparrow{\scriptstyle \chi_1} \\ \Sigma & \xrightarrow[\varphi]{} & \Sigma_1 \end{array}$$

the following diagram

$$\begin{array}{ccc} \mathbb{M}od(\Sigma_2) & \xleftarrow{\mathbb{M}od(\varphi_2)} & \mathbb{M}od(\Sigma') \\ {\scriptstyle \mathbb{M}od(\chi)}\downarrow & & \downarrow{\scriptstyle \mathbb{M}od(\chi_1)} \\ \mathbb{M}od(\Sigma) & \xleftarrow[\mathbb{M}od(\varphi)]{} & \mathbb{M}od(\Sigma_1) \end{array}$$

is a pullback in $\mathbb{C}at$. The institution I is semi-exact if $\mathcal{L} = \mathcal{R} = \mathbb{S}ig$.

Proposition 25 [30] **FOL** is semi-exact.

We lift up the semi-exactness property from the base institution **FOL** to its constructor-based variant **CFOL**.

Proposition 26 **CFOL** is $((****),(i*e*))$-semi-exact.

PROOF. Assume a pushout of **CFOL** signature morphisms

$$\begin{array}{ccc} (S_2,F_2,F_2^c,P_2) & \overset{\varphi_2}{\dashrightarrow} & (S',F',F'^c,P') \\ \chi \uparrow & & \uparrow \chi_1 \\ (S,F,F^c,P) & \underset{\varphi}{\longrightarrow} & (S_1,F_1,F_1^c,P_1) \end{array}$$

such that χ is a $(i*e*)$-morphism and φ is a $(****)$-morphism. Since **FOL** is semi-exact and for all **CFOL** signatures (S_i,F_i,F_i^c,P_i), $\mathbb{M}od(S_i,F_i,F_i^c,P_i)$ is the full subcategory of $\mathbb{M}od(S_i,F_i,P_i)$, it suffices to prove that for all models $M_1 \in \mathbb{M}od(S_1,F_1,F_1^c,P_1)$ and $M_2 \in \mathbb{M}od(S_2,F_2,F_2^c,P_2)$ such that $M_1 \upharpoonright_{\varphi} = M_2 \upharpoonright_{\chi}$ there exists $M' \in \mathbb{M}od(S',F',F'^c,P')$ such that $M' \upharpoonright_{\chi_1} = M_1$ and $M' \upharpoonright_{\varphi_2} = M_2$.

By Proposition 25, there exists a model $M' \in \mathbb{M}od(S',F',P')$ such that $M' \upharpoonright_{\chi_1} = M_1$ and $M' \upharpoonright_{\varphi_2} = M_2$. Let M'^l be the S'-sorted set such that for all $s' \in S'$, $M'^l_{s'} = \begin{cases} M'_{s'} & \text{if } s' \in S'^l \\ \emptyset & \text{if } s' \notin S'^l \end{cases}$. Let $con_{M'} : T_{(S',F'^c)}(M'^l) \to M' \upharpoonright_{(S',F'^c)}$ be the unique extension of the inclusion $\{M'^l_{s'} \hookrightarrow M'_{s'}\}_{s' \in S'}$ to a model (S',F'^c)-morphism. If we prove that $con_{M'}$ is surjective on the sorts in $\chi_1(S_1)$ and $\varphi_2(S_2)$, since $S' = \chi_1(S_1) \cup \varphi_2(S_2)$, it follows that $con_{M'}$ is surjective, meaning that $M' \in \mathbb{M}od(S',F',F'^c,P')$.

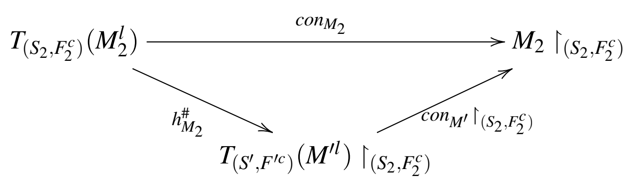
1. We define the S_1-sorted set M_1^l such that for all sorts $s_1 \in S_1$, we have $(M_1^l)_{s_1} = \begin{cases} (M_1)_{s_1} & \text{if } s_1 \in S_1^l \\ \emptyset & \text{if } s_1 \notin S_1^l \end{cases}$. Let $con_{M_1} : T_{(S_1,F_1^c)}(M_1^l) \to M_1 \upharpoonright_{(S_1,F_1^c)}$ be the unique extension of the inclusion $\{(M_1^l)_{s_1} \hookrightarrow (M_1)_{s_1}\}_{s_1 \in S_1}$ to a (S_1,F_1^c)-morphism. By Proposition 19, χ_1 is a $(i*e*)$-morphism, which implies that $\chi_1(S_1^l) \subseteq S'^l$. It follows that the following inclusion exists $\{(M_1^l)_{s_1} \hookrightarrow (T_{(S',F'^c)}(M'^l))_{\chi_1(s_1)}\}_{s_1 \in S_1}$.

   Let $h_{M_1} : T_{(S_1,F_1^c)}(M_1^l) \to T_{(S',F'^c)}(M'^l) \upharpoonright_{(S_1,F_1^c)}$ be the unique extension of the inclusion $\{(M_1^l)_{s_1} \hookrightarrow (T_{(S',F'^c)}(M'^l))_{\chi_1(s_1)}\}_{s_1 \in S_1}$ to a model (S_1,F_1^c)-morphism. Note that for all $m \in M_1^l$, $con_{M_1}(m) = (h_{M_1};con_{M'} \upharpoonright_{(S_1,F_1^c)})(m)$, which implies that the following diagram is commutative.

$$\begin{array}{ccc} T_{(S_1,F_1^c)}(M_1^l) & \xrightarrow{con_{M_1}} & M_1 \upharpoonright_{(S_1,F_1^c)} \\ & {\scriptstyle h_{M_1}} \searrow \quad \nearrow {\scriptstyle con_{M'} \upharpoonright_{(S_1,F_1^c)}} & \\ & T_{(S',F'^c)}(M'^l) \upharpoonright_{(S_1,F_1^c)} & \end{array}$$

   Since con_{M_1} is surjective we obtain that $con_{M'} \upharpoonright_{(S_1,F_1^c)}$ is surjective, meaning that $con_{M'}$ is surjective on the sorts in $\chi_1(S_1)$.

2. Let M_2^l be the S_2-sorted set such that for all $s_2 \in S_2$, $(M_2^l)_{s_2} = \begin{cases} (M_2)_{s_2} & \text{if } s_2 \in S_2^l \\ \emptyset & \text{if } s_2 \notin S_2^l \end{cases}$.
Let $con_{M_2} : T_{(S_2,F_2^c)}(M_2^l) \to M_2 \upharpoonright_{(S_2,F_2^c)}$ be the unique extension of following inclusion $\{(M_2^l)_{s_2} \hookrightarrow (M_2)_{s_2}\}_{s_2 \in S_2}$ to a (S_2,F_2^c)-morphism. We define the S_2-sorted function $h_{M_2} = \{(h_{M_2})_{s_2} : (M_2^l)_{s_2} \to (T_{(S',F'^c)}(M'^l))_{\varphi_2(s_2)}\}_{s_2 \in S_2}$. For all $s_2 \in S_2^l$ such that $\varphi_2(s_2) \in S'^l$, and $m \in (M_2^l)_{s_2}$, we define $h_{M_2}(m) = m$. Let $s_2 \in S_2^l$ such that $\varphi_2(s_2) \in S'^c$, and $m \in (M_2^l)_{s_2}$. By the pushout construction, $S'^c = \chi_1(S_1^c) \cup \varphi_2(S_2^c)$. There exists $s_1 \in S_1^c$ such that $\chi_1(s_1) = \varphi_2(s_2)$. Since $con_{M'}$ is surjective on the sorts in $\chi_1(S_1)$, there exists $t' \in (T_{(S',F'^c)}(M'^l))_{\varphi_2(s_2)}$ such that $con_{M'}(t') = m$. We define $h_{M_2}(m) = t'$. Note that for all sorts $s_2 \in S_2$, we have $(h_{M_2})_{s_2}; (con_{M'})_{\varphi_2(s_2)} = (con_{M_2})_{s_2}$.
Let $h_{M_2}^{\#} : T_{(S_2,F_2^c)}(M_2^l) \to T_{(S',F'^c)}(M'^l) \upharpoonright_{(S_2,F_2^c)}$ be the unique extension of h_{M_2} to a model (S_2,F_2)-morphism, and note that the following diagram is commutative.



Since con_{M_2} is surjective, $con_{M'} \upharpoonright_{(S_2,F_2^c)}$ is surjective, which implies that $con_{M'}$ is surjective on the sorts in $\varphi_2(S_2)$.

■

The ideas of lifting semi-exactness from the base institution to the constructor-based variant provided by Proposition 26 can be applied in other cases such as preorder algebra.

4 Initial models

In this section we provide sufficient institution-independent conditions for a set of Horn sentences to have an initial model.

Definition 27 Let $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ be an institution with a sub-functor $\mathbb{S}en_0 : \mathbb{S}ig \to \mathbb{S}et$ of $\mathbb{S}en$, and a subcategory $\mathcal{D} \subseteq \mathbb{S}ig$ of signature morphisms. A *$\mathcal{D}$-Horn sentence* over $\mathbb{S}en_0(\Sigma)$ is a sentence semantically equivalent to $(\forall\chi)\bigwedge H \Rightarrow C$, where $\Sigma \xrightarrow{\chi} \Sigma' \in \mathcal{D}$, $C \in \mathbb{S}en_0(\Sigma')$ and $H \subseteq \mathbb{S}en_0(\Sigma')$.

When the sub-functor $\mathbb{S}en_0$ and the subcategory $\mathcal{D}$ are fixed, we call $(\forall\chi)\bigwedge H \Rightarrow C$, simply, Horn sentence.

An example of institution $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ as in Definition 27 is **CHCL**, where the institution $I_0 = (\mathbb{S}ig, \mathbb{S}en_0, \mathbb{M}od, \models)$ is the restriction **CHCL**$_0$ of **CHCL** to atomic sentences, and $\mathcal{D}$ consists of signature extensions with a finite number of constants.

Note that not all sets of sentences in **CHCL** have an initial model. If we restrict the signatures such that all operators of constrained sorts are constructors then we obtain initiality for Horn sentences (i.e. any set of sentences in **RHCL** has an initial model). Abstractly we assume

1. an institution I as in Definition 27, and
2. a subcategory $\mathbb{S}ig^r \subseteq \mathbb{S}ig$ of signature morphisms.

Let $\mathbb{S}en^r : \mathbb{S}ig^r \to \mathbb{S}et$, $\mathbb{S}en_0^r : \mathbb{S}ig^r \to \mathbb{S}et$ and $\mathbb{M}od^r : (\mathbb{S}ig^r) \to \mathbb{C}at^{op}$ be the functors define as the restrictions of $\mathbb{S}en : \mathbb{S}ig \to \mathbb{S}et$, $\mathbb{S}en_0 : \mathbb{S}ig \to \mathbb{S}et$ and $\mathbb{M}od : \mathbb{S}ig \to \mathbb{C}at^{op}$, respectively, to $\mathbb{S}ig^r$. We also define $\models^r \stackrel{def}{=} \{\models_\Sigma\}_{\Sigma \in |\mathbb{S}ig^r|}$.

Fact 28 $I^r = (\mathbb{S}ig^r, \mathbb{S}en^r, \mathbb{M}od^r, \models^r)$ and $I_0^r = (\mathbb{S}ig^r, \mathbb{S}en_0^r, \mathbb{M}od^r, \models^r)$ are institutions.

We provide 'easy-to-check' institution-independent conditions for any set of sentences in I^r to have an initial model. Our methodology of proving initiality follows exactly the structure of the sentences:

1. we assume that all sentences of I_0^r are epi basic;
2. based on this assumption, which is checked in concrete examples of institutions, we prove that any set of quantifier-free Horn sentences in I^r has an initial model;
3. we extend the initiality result to the quantified Horn sentences of I^r.

One may wonder what is the role played by I in the abstract setting. The answer is simple: I provides the subcategory $\mathcal{D}$ of signature morphisms and the satisfaction relation for quantified sentences. If I is **CHCL** and I^r is **RHCL** then it is easy to notice that a signature extension with constants of constrained sorts is not a signature morphism in **RHCL**. Therefore in concrete examples we have $\mathcal{D} \not\subseteq \mathbb{S}ig^r$.

4.1 Quantifier-free layer

Assume an institution $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ and a sub-functor $\mathbb{S}en_0 : \mathbb{S}ig \to \mathbb{S}et$ of $\mathbb{S}en$. We denote by I_0 the institution $(\mathbb{S}ig, \mathbb{S}en_0, \mathbb{M}od, \models)$. A concrete example of such institution is **RHCL** such that I_0 is the restriction of **RHCL** to atomic sentences.

Proposition 29 If all sets of sentences in I_0 are epi basic then every set Γ of Σ-sentences semantically equivalent to $\bigwedge H \Rightarrow C$, where $H \subseteq \mathbb{S}en_0(\Sigma)$ and $C \in \mathbb{S}en_0(\Sigma)$, has an initial model .

PROOF. Let $\Sigma \in |\mathbb{S}ig|$ and $\Gamma \subseteq \mathbb{S}en(\Sigma)$. We define $\Gamma_0 = \{e \in \mathbb{S}en_0(\Sigma) \mid \Gamma \models e\}$. Let M_{Γ_0} be the basic model of Γ_0. We prove that M_{Γ_0} is the initial model of Γ. If $M \models \Gamma$ then $M \models \Gamma_0$, and since Γ_0 is epi basic, there exists a unique morphism $M_{\Gamma_0} \to M$. We only need to show $M_{\Gamma_0} \models \Gamma$. Let $\bigwedge H \Rightarrow C \in \Gamma$ and assume $M_{\Gamma_0} \models H$. Since H is basic, there exists a model morphism $M_H \to M_{\Gamma_0}$, which implies $\Gamma_0 \models H$, and we obtain $\Gamma \models H$. Since $\bigwedge H \Rightarrow C \in \Gamma$ and $\Gamma \models H$, we get $\Gamma \models C$. It follows that $C \in \Gamma_0$ and hence $M_{\Gamma_0} \models C$. ∎

Corollary 30 In **RHCL** and **RHPOA**, any set of quantifier-free sentences has an initial model that is ground reachable.

PROOF. We set the parameters of Proposition 29 for **RHCL**. The other case is similar. $I =$ **RHCL** and $I_0 =$ **RHCL**$_0$ is the restriction **RHCL** to atomic sentences. By Lemma 11, the **RHCL**$_0$ sentences are epi basic, and by Proposition 29, any set of quantifier-free **RHCL** sentences has an initial model. Since the initial models are basic models, by Remark 15, they are also ground reachable. ∎

4.2 Quantification layer

Assume an institution $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ with a broad subcategory $\mathcal{D} \subseteq \mathbb{S}ig$ of signature morphisms, and a sub-functor $\mathbb{S}en_1$ of $\mathbb{S}en$ such that all sentences of I are semantically equivalent to $(\forall\chi)\rho$, where $\Sigma \xrightarrow{\chi} \Sigma' \in \mathcal{D}$ and $\rho \in \mathbb{S}en_1(\Sigma')$. An example of such institution is **CHCL** such that $\mathcal{D}$ is the subcategory of signature extensions with a finite number of constants, $I_1 = (\mathbb{S}ig, \mathbb{S}en_1, \mathbb{M}od, \models)$ is the restriction $\mathbf{CHCL}_1$ of **CHCL** to quantifier-free sentences.

Any subcategory of signature morphisms $\mathbb{S}ig^r \subseteq \mathbb{S}ig$ determines two institutions $I^r = (\mathbb{S}ig^r, \mathbb{S}en^r, \mathbb{M}od^r, \models^r)$ and $I_1^r = (\mathbb{S}ig^r, \mathbb{S}en_1^r, \mathbb{M}od^r, \models^r)$ as in Fact 28. If $\mathbb{S}ig^r$ is the full subcategory of **CHCL** signature morphisms such that all operators of constrained sorts are constructors then $I^r = \mathbf{RHCL}$, and $I_1^r = \mathbf{RHCL}_1$, the restriction of **RHCL** to quantifier-free sentences.

Theorem 31 All sets of sentences in I^r have an initial model when all sets of sentences in I_1^r have an initial model, which is $\mathcal{D}$-reachable in I.

PROOF. Let $\Sigma \in |\mathbb{S}ig^r|$ and $\Gamma \subseteq \mathbb{S}en(\Sigma)$. We define $\Gamma_1 = \{e \in \mathbb{S}en_1(\Sigma) \mid \Gamma \models e\}$. The set Γ_1 has an initial model M_{Γ_1} which is $\mathcal{D}$-reachable. If $M \models \Gamma$ then $M \models \Gamma_1$. Since M_{Γ_1} is the initial model of Γ_1, there exists a unique morphism $M_{\Gamma_1} \to M$. We only need to prove $M_{\Gamma_1} \models \Gamma$. Let $(\forall\chi)\rho \in \Gamma$, where $\Sigma \xrightarrow{\chi} \Sigma' \in \mathcal{D}$ and $\rho \in \mathbb{S}en_1(\Sigma')$, and N be a χ-expansion of M_{Γ_1}. Since M_{Γ_1} is $\mathcal{D}$-reachable, there exists a substitution $\theta : \chi \to 1_\Sigma$ such that $M_{\Gamma_1} \upharpoonright_\theta = N$. Since $(\forall\chi)\rho \models \theta(\rho)$, we obtain $\theta(\rho) \in \Gamma_1$. It follows $M_{\Gamma_1} \models \theta(\rho)$, and by the satisfaction condition $N \models \rho$. ■

Corollary 32 In **RHCL** and **RHPOA**, any set of sentences has an initial model.

PROOF. We set the parameters of Theorem 31 for **RHCL**. The other case is similar. I is **CHCL**, I_1 is the restriction of **CHCL** to the quantifier-free sentences, I^r is **RHCL** and I_1^r is the restriction of **RHCL** to the quantifier-free sentences. By Corollary 30, every set of quantifier-free sentences in **RHCL** has an initial model which is ground reachable. By Theorem 31, every set of sentences in **RHCL** has an initial model. ■

4.3 Sufficient completeness

For any **CHCL** signature (S, F, F^c, P), recall that $F^{S^c}_{w\to s} = \begin{cases} F_{w\to s} & : \ s \in S^c \\ \emptyset & : \ s \notin S^c \end{cases}$. A **CHCL** presentation $((S, F, F^c, P), E)$ is called *sufficient complete* if for all (S, F, F^c, P)-models M that satisfies E we have $M \in \mathbb{M}od(S, F, F^c, P)$. Let $\mathbb{S}ig^{\mathbf{CHCL}^{sc}} \subseteq \mathbb{S}ig^{\mathbf{CHCL}^{pres}}$ be the full subcategory of sufficient complete presentations: $|\mathbb{S}ig^{\mathbf{CHCL}^{sc}}|$ consists of **CHCL** presentations $((S, F, F^c, P), E)$ that are sufficient complete. We define the institution $\mathbf{CHCL}^{sc}$ of sufficient complete presentations as the the restriction of $\mathbf{CHCL}^{pres}$ to the sufficient complete presentations:

- $\mathbb{S}en^{\mathbf{CHCL}^{sc}} : \mathbb{S}ig^{\mathbf{CHCL}^{sc}} \to \mathbb{S}et$ and $\mathbb{M}od^{\mathbf{CHCL}^{sc}} : \mathbb{S}ig^{\mathbf{CHCL}^{sc}} \to \mathbb{C}at^{op}$ are the restrictions of the $\mathbb{S}en^{\mathbf{CHCL}^{pres}} : \mathbb{S}ig^{\mathbf{CHCL}^{pres}} \to \mathbb{S}et$ and $\mathbb{M}od^{\mathbf{CHCL}^{pres}} : \mathbb{S}ig^{\mathbf{CHCL}^{pres}} \to \mathbb{C}at^{op}$, respectively, to $\mathbb{S}ig^{\mathbf{CHCL}^{sc}}$;
- $\models^{\mathbf{CHCL}^{sc}} = \{\models^{\mathbf{CFOL}^{pres}}_{(\Sigma,E)}\}_{(\Sigma,E)\in|\mathbb{S}ig^{\mathbf{CHCL}^{sc}}|}$.

Similarly, one can define the institution $\mathbf{CHPOA}^{sc}$ of sufficient complete **CHPOA** presentations.

Proposition 33 In $\mathbf{CHCL}^{sc}$ and $\mathbf{CHPOA}^{sc}$, any set of sentences has an initial model.

PROOF. Let $((S,F,F^c,P),E)$ be a sufficient complete **CHCL** presentation, and Γ a set of (S,F,F^c,P)-sentences. By Corollary 32, $\mathbb{M}od((S,F,F^{S^c},P),E\cup\Gamma)$ has an initial model $O_{E\cup\Gamma}$. Since $((S,F,F^c,P),E)$ is sufficient complete, $O_{E\cup\Gamma}\in\mathbb{M}od(S,F,F^c,P)$. Since $O_{E\cup\Gamma}$ is the initial model of $Mod((S,F,F^{S^c},P),E\cup\Gamma)$ and $\mathbb{M}od((S,F,F^c,P),E\cup\Gamma)$ is the full subcategory of $\mathbb{M}od((S,F,F^{S^c},P),E\cup\Gamma)$, we obtain that $O_{E\cup\Gamma}$ is the initial model of $\mathbb{M}od((S,F,F^c,P),E\cup\Gamma)$.

The case of $\mathbf{CHPOA}^{sc}$ is similar. ■

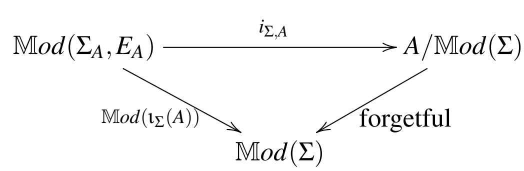
5 Free models

In this section we give sufficient institution independent conditions for proving the existence of free models along sufficient complete presentation morphisms in constructor-based institutions. Our ideas are based upon [31] and especially [10]. More concrete, we notice that one of the conditions of the result in [10] is too restrictive and we replace it with a somewhat less restrictive variant, such that we can apply it in more cases. We also define the concept of *weak diagrams* for **CFOL** which is essential for proving the existence of free models and provides ideas to define it for other constructor-based institutions.

5.1 Institution-independent diagrams

In this paper we present a simplified version of institutional diagrams of [10], which omits the compatibility of this notion with the signature morphisms.

Definition 34 An institution $I=(\mathbb{S}ig,\mathbb{S}en,\mathbb{M}od,\models)$ has *weak diagrams* ι if for each signature Σ, and Σ-model A there exists a signature morphism $\iota_\Sigma(A):\Sigma\to\Sigma_A$ called *the extension of* Σ *via* A, and a set E_A of Σ_A-sentences called *the diagram of the model* A such that $\mathbb{M}od(\Sigma_A,E_A)$ and the comma category $A/\mathbb{M}od(\Sigma)$ are naturally isomorphic, i.e. the following diagram commutes by the isomorphism $i_{\Sigma,A}$ natural in Σ and A.



Remark 35 Since the comma category $A/\mathbb{M}od(\Sigma)$ has an initial model $A\xrightarrow{1_A}A$ and $i_{\Sigma,A}$ is an isomorphism of categories, the followings hold:

1. $\mathbb{M}od(\Sigma_A,E_A)$ has an initial model A_A such that $A_A\upharpoonright_{\iota_\Sigma(A)}=A$, and
2. for all $M\in\mathbb{M}od(\Sigma_A,E_A)$, $i_{\Sigma,A}(M)=(A_A\to M)\upharpoonright_{\iota_\Sigma(A)}$, where $A_A\to M$ is the unique morphism from A_A to M.

We define the diagrams for **CFOL**. Let A be a Σ-model, where $\Sigma=(S,F,F^c,P)$.

- Σ_A consists of the signature Σ enriched with all elements of loose sorts of A, i.e. $\Sigma_A = (S, F_A, F^c, P)$, where $(F_A)_{w\to s} = \begin{cases} F_{\to s} \cup A_s & if\ w = \lambda\ and\ s \in S^l \\ F_{w\to s} & otherwise \end{cases}$

- $\iota_\Sigma(A) : (S,F,F^c,P) \hookrightarrow (S,F_A,F^c,P)$ is the inclusion of signatures,

- A_A is the $\iota_\Sigma(A)$-expansion of A interpreting each $a \in A$ as a, and

- E_A is the set of all Σ_A-atoms satisfied by A_A.

Proposition 36 In **CFOL**, for each Σ-model A, where $\Sigma = (S,F,F^c,P)$ there exists an isomorphisms of categories $i_{\Sigma,A} : \mathbb{M}od(\Sigma_A, E_A) \to A/\mathbb{M}od(\Sigma)$.

PROOF. Let $f_{A_A} : T_{(S,F_A,P)} \to A_A$ be the unique morphisms from $T_{(S,F_A,P)}$ to A_A. Since A is a reachable model, f_{A_A} is a surjection.

We show that A_A is the basic model of E_A. Assume a Σ_A-model M, and let $f_M : T_{(S,F_A,P)} \to M$ be the unique morphism from $T_{(S,F_A,P)}$ to M.

1. If $M \models E_A$ then $Ker(f_{A_A}) \subseteq Ker(f_M)$, and there exists a unique model morphism $h_M : A_A \to M$ such that $f_{A_A};h_M = f_M$.

2. If there exists $h_M : A_A \to M$, then by the initiality of $T_{(S,F_A,P)}$, we have $f_{A_A};h_M = f_M$. Since A_A is ground reachable, for all atomic Σ_A-sentences e, $A_A \models e$ implies $M \models e$. It follows that $M \models E_A$.

Since A_A is the basic model of E_A, $A_A/\mathbb{M}od(\Sigma_A) = A_A/\mathbb{M}od(\Sigma_A, E_A)$. Because A_A is the initial model of $\mathbb{M}od(\Sigma_A, E_A)$, the forgetful functor $F : A_A/\mathbb{M}od(\Sigma_A,E_A) \to \mathbb{M}od(\Sigma_A,E_A)$ is an isomorphisms of categories. Also, the functor $I_{\Sigma,A} : A_A/\mathbb{M}od(\Sigma_A) \to A/\mathbb{M}od(\Sigma)$ defined by $I_{\Sigma,A}(A_A \stackrel{h_M}{\to} M) = (A_A \stackrel{h_M}{\to} M)\upharpoonright_{\iota_\Sigma(A)}$ on models, and by $I_{\Sigma,A}(h_M \stackrel{h}{\to} h_N) = (h_M \stackrel{h}{\to} h_N)\upharpoonright_{\iota_\Sigma(A)}$ on morphisms, is an isomorphism of categories. Then $i_{\Sigma,A} = F^{-1};I_{\Sigma,A}$ is also an isomorphism of categories as a composition of two isomorphisms. ■

In institutions with no constructors, such as **FOL**, all sorts are loose, and the extension $\iota_\Sigma(A)$ of a signature Σ via a Σ-model A is obtained by adding all the elements of the model A to the signature Σ. This is the classical approach, which can be found, for example, in [10].

Proposition 37 [12] If I is an institution with weak diagrams ι then the institution of presentations I^{press} has also elementary diagrams.

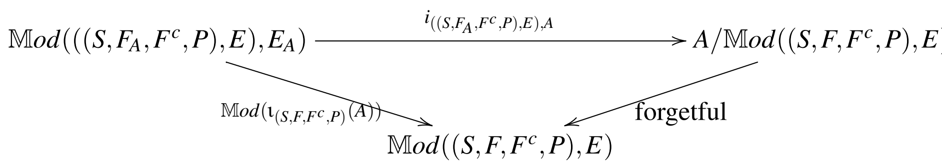
PROOF. [Sketch] Let (Σ,E) be a presentation, and A a Σ-model that satisfies E. The extension of (Σ,E) via A in I^{pres} is $\iota_\Sigma(A) : (\Sigma,E) \to (\Sigma_A, \iota_\Sigma(A)(E))$ and the diagram of the (Σ,E)-model A is E_A. ■

Corollary 38 **CHCL**^{sc} has weak diagrams.

PROOF. Assume a sufficient complete **CHCL** presentation $(S,F,F^c,P),E)$, and let A be a (S,F,F^c,P)-model that satisfies E. Note that $(S,F_A,F^c,P),E)$ is sufficient complete, where $\iota_{(S,F,F^c,P)(A)} : (S,F,F^c,P) \hookrightarrow (S,F_A,F^c,P)$ is the extension of (S,F,F^c,P) via A. Indeed, for any $M \in \mathbb{M}od((S,F_A,F^{S^c},P),E)$, $M\upharpoonright_{(S,F,P)} \in \mathbb{M}od((S,F,F^{S^c},P),E)$; since $(S,F,F^c,P),E)$ is sufficient complete, $M\upharpoonright_{(S,F,P)} \in \mathbb{M}od((S,F,F^c,P),E)$, which implies $M \in \mathbb{M}od((S,F_A,F^c,P),E)$. In **CHCL**^{sc}, let

- $\iota_{(S,F,F^c,P)(A)} : ((S,F,F^c,P),E) \hookrightarrow ((S,F_A,F^c,P),E)$ be the extension of the presentation $((S,F,F^c,P),E)$ via A, and
- the diagram E_A of A in **CHCL** be the diagram of A in $\mathbf{CHCL}^{sc}$.

By Proposition 37, the following diagram is commutative.



■

Note that the signature morphisms in **CFOL** do not preserve the loose sorts, in general. It follows that our notion of diagrams is not 'functorial' in the sense of [10], which implies that **CFOL** does not have elementary diagrams.

5.2 Liberality

An institution $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ is $\mathcal{L}$-*liberal* [20], where $\mathcal{L} \subseteq \mathbb{S}ig$, if for all presentation morphisms $(\Sigma,E) \xrightarrow{\varphi} (\Sigma',E')$ such that $\Sigma \xrightarrow{\varphi} \Sigma' \in \mathcal{L}$ the functor

$$\mathbb{M}od(\Sigma',E') \xrightarrow{\mathbb{M}od(\varphi)} \mathbb{M}od(\Sigma,E)$$

has a left adjoint.

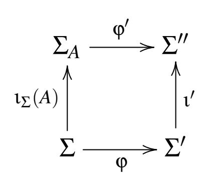
The following theorem can be proved as in [10].

Theorem 39 Let $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models, \iota)$ be an institution with weak diagrams. Assume two categories $\mathcal{L}, \mathcal{R} \subseteq \mathbb{S}ig$ of signature morphisms such that

1. $\iota_\Sigma(A) \in \mathcal{R}$, for all $\Sigma \in |\mathbb{S}ig|$ and $A \in |\mathbb{M}od(\Sigma)|$,
2. $\mathbb{S}ig$ has $(\mathcal{L},\mathcal{R})$-pushouts,
3. I is $(\mathcal{L},\mathcal{R})$-semi-exact, and
4. each presentation has an initial model.

Then I is $\mathcal{L}$-liberal.

PROOF. [sketch] Let $\varphi : (\Sigma,E) \to (\Sigma',E')$ be a presentation morphism such that $\Sigma \xrightarrow{\varphi} \Sigma' \in \mathcal{L}$, and let A be a Σ-model that satisfies E. Consider the following pushout of signature morphisms



We define the free model A^φ to be $(A_A^{\varphi'}) \upharpoonright_{\iota'}$ where $A_A^{\varphi'}$ is the initial model of $\varphi'(E_A) \cup \iota'(E')$, and the the universal arrow $\eta_A : A \to (A^\varphi) \upharpoonright_\varphi$ to be $(A_A \to (A_A^{\varphi'}) \upharpoonright_{\varphi'}) \upharpoonright_{\iota_\Sigma(A)}$. ■

Below is a corollary of the above theorem.

Corollary 40 $\mathbf{CHCL}^{sc}$ is $(**e*)$-liberal.

PROOF. We set the parameters of Theorem 39. The institution I is $\mathbf{CHCL}^{sc}$, $\mathcal{L}$ consists of sufficient complete presentation $(**e*)$-morphisms, and $\mathcal{R}$ consists of sufficient complete presentation morphisms that add non-constructor constants of loose sorts. By Proposition 33, each set of sentences in $\mathbf{CHCL}^{sc}$ has an initial model. For each span of $\mathbf{CHCL}^{sc}$ signature morphism $((S,F\cup C,F^c,P),E_C)\xleftarrow{\chi}((S,F,F^c,P),E)\xrightarrow{\varphi}((S_1,F_1,F_1^c,P_1),E_1)$ such that φ is a $(**e*)$-morphism, and C is set of constants of loose sorts, the following pushout

$$\begin{array}{ccc}
((S,F\cup C,F^c,P),E_C) & \xrightarrow{\varphi_C} & ((S_1,F_1\cup C',F_1^c,P_1),E') \\
\chi \uparrow & & \uparrow \chi_1 \\
((S,F,F^c,P),E) & \xrightarrow[\varphi]{} & ((S_1,F_1,F_1^c,P_1),E_1)
\end{array}$$

is constructed as follows: $\{(S,F\cup C,F^c,P)\xleftarrow{\chi}(S,F,F^c,P)\xrightarrow{\varphi}(S_1,F_1,F_1^c,P_1),(S,F\cup C,F^c,P)\xrightarrow{\varphi_C}(S_1,F_1\cup C',F_1^c,P_1)\xleftarrow{\chi_1}(S_1,F_1,F_1^c,P_1)\}$ is a pushout of **CFOL** signature morphisms, and $E'=\varphi_C(E_C)\cup\chi_1(E_1)$. Note that

- χ_1 is the extension of (S_1,F_1,F_1^c,P_1) with constants from C', and
- φ_C works like φ on (S,F,F^c,P).

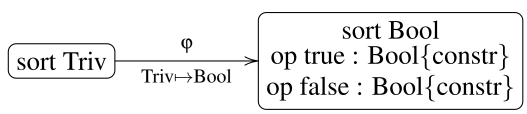
Let $M_1\in\mathbb{M}od((S_1,F_1,F_1^c,P_1),E_1)$ and $M_2\in\mathbb{M}od((S,F\cup C,F^c,P),E_C)$ such that $M_1\upharpoonright_{\varphi}=M_2\upharpoonright_{\chi}$. By Proposition 26, there exists $M'\in\mathbb{M}od(S_1,F_1\cup C',F_1^c,P_1)$ such that $M'\upharpoonright_{\chi_1}=M_1$ and $M'\upharpoonright_{\varphi_C}=M_2$. By the satisfaction condition, $M'\models E'$. It follows that $\mathbf{CHCL}^{sc}$ is $(\mathcal{L},\mathcal{R})$-semi-exact.

By Theorem 39, φ has a left adjoint. ■

Similar results can be formulated for **CHPOA** too.

The example below shows that $\mathbf{CHCL}^{sc}$ is not liberal, in general.

Example 41 Consider the following example of **CHCL** signature morphism:



where $TRIV=(\{Triv\},\emptyset,\emptyset,\emptyset)$ and $BOOL=(\{Bool\},\{true:\to Bool,false:\to Bool\},\{true:\to Bool,false:\to Bool\},\emptyset)$. Let A be a $TRIV$-model that consists of one element $(a:Triv)$. Assume that A^{φ} is the free model along φ generated by A, and $\eta_A:A\to A^{\varphi}\upharpoonright_{\varphi}$ is an universal arrow. Let T_{BOOL} be the term $BOOL$-model, and $f_1:A\to T_{BOOL}\upharpoonright_{\varphi}$ defined by $f_1(a)=true$. By the universality of η_A, there exists a model morphism $h_1:A^{\varphi}\to T_{BOOL}$ such that $\eta_A;h_1\upharpoonright_{\varphi}=f_1$. If $A^{\varphi}_{true}=A^{\varphi}_{false}$ then $h_1(A^{\varphi}_{true})=h_1(A^{\varphi}_{false})$, which implies $(T_{BOOL})_{true}=(T_{BOOL})_{false}$, a contradiction. Since A^{φ} is reachable and $A^{\varphi}_{true}\neq A^{\varphi}_{false}$, h_1 is an isomorphism. We have $\eta_A(a)=h_1^{-1}(f_1(a))=h_1^{-1}((T_{BOOL})_{true})=A^{\varphi}_{true}$. Let $f_2:A\to T_{BOOL}\upharpoonright_{\varphi}$, defined by $f_2(a)=false$. By the universality of η_A, there exists a model morphism $h_2:A^{\varphi}\to T_{BOOL}$ such that $\eta_A;h_2\upharpoonright_{\varphi}=f_2$. Since A^{φ} is reachable and $A^{\varphi}_{true}\neq A^{\varphi}_{false}$, h_2 is an isomorphism. We have $\eta_A(a)=h_2^{-1}(f_2(a))=h_2^{-1}((T_{BOOL})_{false})=A^{\varphi}_{false}$. We obtained $A^{\varphi}_{true}=\eta_A(a)=A^{\varphi}_{false}$, a contradiction. Therefore, η_A is not an universal arrow.

6 Conclusions

We investigate the existence of pushouts in the concrete category of **CFOL** signature morphisms. This research is important as the pushout construction constitutes the basis of building large specifications from smaller ones. In connection to the semantics of the complex system development we conduct an institution dependent study of the semi-exactness property in logics with constructors.

We proved the existence of initial models of any set of sentences in arbitrary 'reachable' institutions. We do not use factorization systems as [2, 33, 1] or inclusion systems as [10]. We simply require that all sets of atoms are basic and the models defining the sets of atoms as basic sets of sentences are ground reachable. Note that the restriction to reachable models can be also obtained by allowing infinitary disjunctions. For example, if $\Sigma = (\{Nat\}, \{0 :\to Nat, s\_ : Nat \to Nat\}, \emptyset)$ is a first-order signature then the restriction to the models that are reachable by the constructors $0 :\to Nat$ and $s\_ : Nat \to Nat$ can be obtained by using the infinitary sentence $(\forall x) \bigvee_{n \in \mathbb{N}at} x = s^n 0$, where $s^n 0$ is the term obtained by applying n times the function successor to 0. It follows that the class of models of an implicational theory in logics with constructors do not form a quasi-variety, and do not fall into the framework of [2, 33, 10]. Therefore new initiality results are required for 'reachable' institutions. Initiality is then extended to constructor-based institutions via sufficient completeness.

Free models along presentation morphisms provide semantics for the modules with initial denotation in algebraic specifications. In order to provide an institution independent proof of liberality, we define the diagrams for logics with constructors. Given a Σ-model A, the extension of Σ via A is obtained by adding all elements of loose sorts of A to the signature Σ. Taking into consideration that constructor-based institutions do not have all pushouts, freeness follows as in [10], and then it is instantiated to the institution of sufficient complete presentations.

Our abstract results can be applied to other frameworks such as order sorted algebra [22, 21], higher order logic [6, 24] with intensional Henkin semantics, and partial algebra [28, 5]. In the future we are planning to study interpolation in logics with constructors. Also an axiomatizability result is desired in this case.

References

[1] J. Adámek, H. Herrlich, and G. Strecker. *Abstract and concrete categories*. Wiley-Interscience, New York, NY, USA, 1990.

[2] H. Andréka and I. Németi. Generalization of the concept of variety and quasivariety to partial algebra through category theory. *Dissertationes Mathematicae*, CCIV, 1983.

[3] M. Bidoit and R. Hennicker. Constructor-based observational logic. *J. Log. Algebr. Program.*, 67(1-2):3–51, 2006.

[4] M. Bidoit, R. Hennicker, and A. Kurz. Observational logic, constructor-based logic, and their duality. *Theor. Comput. Sci.*, 3(298):471–510, 2003.

[5] P. Burmeister. *A Model Theoretic Oriented Approach to Partial Algebras*. Akademie-Verlag, 1986.

[6] A. Church. A formulation of the simple theory of types. *J. Symb. Log.*, 5(2):56–68, 1940.

[7] M. Codescu and D. Găină. Birkhoff completeness in institutions. *Logica Universalis*, 2(2):277–309, September 2008.

[8] R. Diaconescu. Extra theory morphisms for institutions: logical semantics for multi-paradigm languages. *Applied Categorical Structures*, 6(4):427–453, 1998. A preliminary version appeared as JAIST Technical Report IS-RR-97-0032F in 1997.

[9] R. Diaconescu. Institution-independent ultraproducts. *Fundamenta Informaticae*, 55(3-4):321–348, 2003.

[10] R. Diaconescu. Elementary diagrams in institutions. *Journal of Logic and Computation*, 14(5):651–674, 2004.

[11] R. Diaconescu. Herbrand theorems in arbitrary institutions. *Inf. Process. Lett.*, 90(1):29–37, 2004.

[12] R. Diaconescu. *Institution-independent Model Theory*. Birkhäuser Basel, 1st edition, 2008.

[13] R. Diaconescu and K. Futatsugi. *CafeOBJ Report: The Language, Proof Techniques, and Methodologies for Object-Oriented Algebraic Specification*, volume 6 of *AMAST Series in Computing*. World Scientific, 1998.

[14] R. Diaconescu and K. Futatsugi. Logical foundations of CafeOBJ. *Theor. Comput. Sci.*, 285(2):289–318, 2002.

[15] R. Diaconescu, J. Goguen, and P. Stefaneas. Logical support for modularisation. In G. Huet and G. Plotkin, editors, *Logical Environments*, pages 83–130. Cambridge, 1993. Proceedings of a Workshop held in Edinburgh, Scotland, May 1991.

[16] K. Futatsugi. Verifying specifications with proof scores in CafeOBJ. In *ASE*, pages 3–10. IEEE Computer Society, 2006.

[17] K. Futatsugi, J. A. Goguen, and K. Ogata. Verifying design with proof scores. In B. Meyer and J. Woodcock, editors, *VSTTE*, volume 4171 of *Lecture Notes in Computer Science*, pages 277–290. Springer, 2005.

[18] D. Găină, K. Futatsugi, and K. Ogata. Constructor-Based Logics. submitted.

[19] D. Găină, K. Futatsugi, and K. Ogata. Constructor-based institutions. In A. Kurz, M. Lenisa, and A. Tarlecki, editors, *CALCO*, volume 5728 of *Lecture Notes in Computer Science*, pages 398–412. Springer, 2009.

[20] J. A. Goguen and R. Burstall. Institutions: Abstract model theory for specification and programming. *Journal of the Association for Computing Machinery*, 39(1):95–146, January 1992.

[21] J. A. Goguen and R. Diaconescu. An Oxford survey of order sorted algebra. *Mathematical Structures in Computer Science*, 4(3):363–392, 1994.

[22] J. A. Goguen and J. Meseguer. Order-sorted algebra I: Equational deduction for multiple inheritance, overloading, exceptions and partial operations. *Theoretical Computer Science*, 105(2):217–273, 1992.

[23] J. A. Goguen and G. Rosu. Institution morphisms. *Formal Asp. Comput.*, 13(3-5):274–307, 2002.

[24] L. Henkin. Completeness in the theory of types. *J. Symb. Log.*, 15(2):81–91, 1950.

[25] S. Mac Lane. *Categories for the Working Mathematician*. Springer, second edition, 1998.

[26] K. Ogata and K. Futatsugi. Flaw and modification of the iKP electronic payment protocols. *Inf. Process. Lett.*, 86(2):57–62, 2003.

[27] K. Ogata and K. Futatsugi. Simulation-based verification for invariant properties in the OTS/CafeOBJ method. *Electr. Notes Theor. Comput. Sci.*, 201:127–154, 2008.

[28] H. Reichel. *Structural induction on partial algebras*. Akademie-Verlag (Berlin), 1984.

[29] D. Sannella and A. Tarlecki. Specifications in an arbitrary institution. *Inf. Comput.*, 76(2/3):165–210, 1988.

[30] L. Schröder, T. Mossakowski, A. Tarlecki, B. Klin, and P. Hoffman. Amalgamation in the semantics of CASL. *Theor. Comput. Sci.*, 331(1):215–247, 2005.

[31] A. Tarlecki. On the existence of free models in abstract algebraic institutions. *Theor. Comput. Sci.*, 37:269–304, 1985.

[32] A. Tarlecki. Bits and pieces of the theory of institutions. In D. Pitt, S. Abramsky, A. Poigné, and D. Rydeheard, editors, *Proceedings, Summer Workshop on Category Theory and Computer Programming, Lecture Notes in Computer Science*, volume 240, pages 334–360. Springer, 1986.

[33] A. Tarlecki. Quasi-varieties in abstract algebraic institutions. *J. Comput. Syst. Sci.*, 33(3):333–360, 1986.