The Bochner Integral and the Weak Property (N)

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Abstract

The Russian mathematician N. N. Luzin recognized the importance of the condition which now bears his name and is also referred to as property (N). We present descriptive characterizations of Bochner integral in terms of the weak property (N).

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1. INTRODUCTION AND PRELIMINARIES

It is known that a Banach space-valued function $F : [0, 1] \rightarrow X$ is the primitive of a Bochner integrable function $f : [0, 1] \rightarrow X$ if and only if F is sAC and $F'(t) = f(t)$ almost everywhere on $[0, 1]$, Theorem 7.4.15 in [11]. For more information about Bochner integral we refer to [2], [6], [1] and [11]. In this paper, we present descriptive characterizations of Bochner integral in terms of the weak property (N), see Theorem 2.3 and Theorem 2.5.

Let X be a real Banach space with its norm $\|\cdot\|$. We denote by $B(x, \varepsilon)$ the open ball with center x and radius $\varepsilon > 0$ and by X^* the topological dual to X . We denote by $\mathcal{I}$ the family of all non-degenerate closed subintervals of $[0, 1]$, by λ the Lebesgue measure on $[0, 1]$ and by λ^* the outer Lebesgue measure. The intervals $I, J \in \mathcal{I}$ are said to be *nonoverlapping* if $\text{int}(I) \cap \text{int}(J) = \emptyset$, where $\text{int}(I)$ denotes the interior of I . If a point function $F : [0, 1] \rightarrow X$ is given, then we define the interval function $\tilde{F} : \mathcal{I} \rightarrow X$ by $\tilde{F}([u, v]) = F(v) - F(u)$, for all $[u, v] \in \mathcal{I}$.

Assume that an interval $[a, b] \subset [0, 1]$ and a function $F : [0, 1] \rightarrow X$ are given. A finite collection $\{I_i \in \mathcal{I} : i = 1, 2, \dots, m\}$ of pairwise nonoverlapping intervals is said to be a *partition of* $[a, b]$, if $\cup_{i=1}^m I_i = [a, b]$. We denote by $\Pi_{[a,b]}$ the family of all partitions of $[a, b]$. Let us define the *total variation* $\overset{b}{V}_a F$ of F on $[a, b]$ as follows

$$\overset{b}{V}_a F = \sup \left\{ \sum_{I \in \pi} \|\tilde{F}(I)\| : \pi \in \Pi_{[a,b]} \right\}.$$

If we have $\overset{1}{V}_0 F < +\infty$, then F is said to be of *strongly bounded variation* (*sBV*). If $c \in (a, b)$ and F is *sBV*, then

$$(1.1) \quad \overset{b}{V}_a F = \overset{c}{V}_a F + \overset{b}{V}_c F.$$

The last equality was proven for real valued functions in [10], Theorem VIII.3.5, but the proof works also for vector valued functions, it is enough to change the absolute value with the norm $\|\cdot\|$.

The function F is said to be *strongly absolutely continuous* (*sAC*) if for each $\varepsilon > 0$ there exists $\eta_\varepsilon > 0$ such that for each finite collection $\{I_i \in \mathcal{I} : i = 1, \dots, n\}$ of pairwise nonoverlapping intervals, we have $\sum_{i=1}^m \|\tilde{F}(I_i)\| < \varepsilon$, whenever $\sum_{i=1}^m \lambda(I_i) < \eta$. The function F is said to be *weak sAC* if for each $x^* \in X^*$ the function $x^* \circ F$ is *sAC*. The function F is said to be *weak continuous* on $[0, 1]$ if for each $x^* \in X^*$ the function $x^* \circ F$ is continuous on $[0, 1]$.

A function $f : [0, 1] \rightarrow X$ is said to be *Pettis integrable* if $x^* \circ f$ is Lebesgue integrable, for all $x^* \in X^*$, and for each $E \in \mathcal{L}$ there is a vector $\nu_f(E) \in X$ such that

$$x^*(\nu_f(E)) = (L) \int_E (x^* \circ f) d\lambda \quad \text{for all } x^* \in X^*.$$

The set function $\nu_f : \mathcal{L} \rightarrow X$ is called the *indefinite Pettis integral* of f . For information about Pettis integral we refer to [7], [8], [9] and [12].

We now extend the helpful property (N), due to N. N. Luzin, from real valued functions to Banach space-valued functions. The function $F : [0, 1] \rightarrow X$ is said to satisfy the *weak property* (N) if for each $x^* \in X^*$ the function $x^* \circ F$ has the property (N), i.e. for each $E \in \mathcal{L}$, we have $\lambda(E) = 0 \Rightarrow \lambda((x^* \circ F)(E)) = 0$.

2. THE MAIN RESULTS

The main results are Theorem 2.3 and Theorem 2.5. Let us start with a few auxiliary lemmas.

Lemma 2.1. *Let $F : [0, 1] \rightarrow X$ be a function. If F is *sBV* and weak continuous on $[0, 1]$, then F is continuous on $[0, 1]$.*

Proof. Let us define the function $\Phi : [0, 1] \rightarrow [0, +\infty)$ by $\Phi(0) = 0$ and $\Phi(t) = \int_0^t V F$ for all $t \in (0, 1]$. Since Φ increases on $[0, 1]$, we obtain by Theorem VIII.1.1 in [10] that the set Z of discontinuity points of Φ is at most countable, and since

$$\|F(t+h) - F(t)\| \leq |\Phi(t+h) - \Phi(t)|$$

it follows that each continuity point of Φ is continuity point of F . Fix an arbitrary point $t \in Z$. We are going to prove that the equality

$$(2.1) \quad \lim_{h \rightarrow 0^+} F(t+h) = F(t+0) \in X$$

holds true. First we will show that given an arbitrary sequence (h_n) of nonnegative real numbers which converges to zero, the sequence $(F(t+h_n))$ converges to an element $x \in X$. Indeed, since $(\Phi(t+h_n))$ is a Cauchy sequence and

$$\|F(t+h_n) - F(t+h_m)\| \leq |\Phi(t+h_n) - \Phi(t+h_m)|,$$

the sequence $(F(t+h_n))$ is Cauchy. Further, the sequence $(F(t+h_n))$ converges to an element $x \in X$. Secondly, let (h'_n) be another sequence of nonnegative real numbers which converges to zero. Then, the sequence $(F(t+h'_n))$ converges to an element $x' \in X$, and since F is weak continuous, we obtain $x^*(x-x') = 0$, for all $x^* \in X^*$. The last result together with Hahn-Banach Theorem yields that $x = x'$. This means that (2.1) holds true. By the same manner as above it can be proved that $\lim_{h \rightarrow 0^+} F(t-h) = F(t-0) \in X$.

We now assume by contradiction that $F(t+0) \neq F(t-0)$. Then, by applying Hahn-Banach Theorem again, there exists $x^* \in X^*$ such that $(x^* \circ F)(t+0) \neq (x^* \circ F)(t-0)$. But that contradicts with continuity of $x^* \circ F$. Consequently, $F(t+0) = F(t-0)$. By the same manner as above, we get $F(t+0) = F(t-0) = F(t)$. Since t is arbitrary, the last result yields that F is continuous on $[0, 1]$ and the proof is finished. $\square$

Lemma 2.2. *Let $F : [0, 1] \rightarrow X$ be a function which is differentiable almost everywhere on $[0, 1]$. If F is sBV and weak sAC , then F is sAC .*

Proof. Since $F : [0, 1] \rightarrow X$ is differentiable almost everywhere on $[0, 1]$, there exists a subset $Z \in [0, 1]$ with $\lambda(Z) = 0$ such that $F'(t)$ exists for all $t \in [0, 1] \setminus Z$. Thus, we can define the function $f : [0, 1] \rightarrow X$ as follows

$$f(t) = \begin{cases} F'(t) & t \in [0, 1] \setminus Z \\ 0 & t \in Z \end{cases}.$$

Since F is sBV , it can be shown in the same manner as in the proof of Lemma 2.1 in [4] that f is Pettis integrable and the countable additive vector measure $\nu : \mathcal{L} \rightarrow X$ defined by $\nu(E) = (P) \int_E f d\lambda$, for all $E \in \mathcal{L}$, is of bounded variation.

We now prove that f is strongly measurable. By virtue of Lemma 2.1, F is continuous on $[0, 1]$, and because this the set $\{F(t) : t \in [0, 1]\} \subset X$ is compact

and therefore separable. If $Y \subset X$ is the closed linear subspace spanned by the set $\{F(t) : t \in [0, 1]\}$, then Y is separable. Since

$$\frac{F(t+h) - F(t)}{h} \in Y \quad \text{for all } t \in [0, 1] \quad (h \neq 0, t+h \in [0, 1]),$$

we obtain that $f(t) \in Y$, for almost all $t \in [0, 1]$. This means that f is essentially separable valued. Hence, by Theorem II.1.2 in [1], the function f is strongly measurable. Therefore, we obtain by Theorem 4.1 and Remark 4.1 in [8] that

$$|\nu|(E) = \int_E \|f(t)\| d\lambda \quad \text{for each } E \in \mathcal{L},$$

and since ν is of bounded variation, the function $\|f(\cdot)\|$ is Lebesgue integrable on $[0, 1]$. Further, we obtain by Theorem II.2.2 in [1] that f is Bochner integrable. Since the Bochner and Pettis integrals coincide whenever they coexist, we have $\nu(E) = (B) \int_E f d\lambda$, for every $E \in \mathcal{L}$. Finally, since $\nu(I) = F(I)$, for all $I \in \mathcal{I}$, we obtain by Theorem 7.4.15 in [11] that F is sAC and the proof is finished. $\square$

Let us now present the first result.

Theorem 2.3. *Let $F : [0, 1] \rightarrow X$ be a function. Then the following are equivalent.*

(i) F is the primitive of Bochner integrable function f , i.e.,

$$\tilde{F}(I) = (B) \int_I f d\lambda \quad \text{for all } I \in \mathcal{I},$$

- (ii) F is sAC and $F'(t) = f(t)$ almost everywhere on $[0, 1]$,
- (iii) F is sBV , weak sAC and $F'(t) = f(t)$ almost everywhere on $[0, 1]$,
- (iv) F is sBV and weak continuous on $[0, 1]$, satisfies the weak property (N) and $F'(t) = f(t)$ almost everywhere on $[0, 1]$.

Proof. By Theorem 7.4.15 in [11], the equivalence (i) $\Leftrightarrow$ (ii) holds. By Banach-Zarecki Theorem 1.1 in [4], we obtain (iii) $\Leftrightarrow$ (iv). Note that if F is sAC , then F is sBV , weak sAC . This yields (ii) $\Rightarrow$ (iii). Finally, we obtain by Lemma 2.2 that (iii) $\Rightarrow$ (ii) and the proof is finished. $\square$

Corollary 2.4. *If $F : [0, 1] \rightarrow X$ is $BV_{\|\cdot\|}$ and weak sAC , then F is the primitive of a Bochner integrable function f (for the notion of $BV_{\|\cdot\|}$ we refer to [4]).*

Proof. Since F is $BV_{\|\cdot\|}$, we obtain by Theorem 2.6 in [3] that F is differentiable almost everywhere on $[0, 1]$. Hence, by (iii) in Theorem 2.3, F is the primitive of a Bochner integrable function f and the proof is finished. $\square$

We now extend Theorem 2.3 to a Frechet space X with topological dual X^* . We refer to [5] for the notions used in the following theorem. The function $F : [0, 1] \rightarrow X$ is said to be *weak sAC* if for each $x^* \in X^*$ the function $x^* \circ F$ is

sAC ; F is said to be *weak continuous* on $[0, 1]$ if for each $x^* \in X^*$ the function $x^* \circ F$ is continuous on $[0, 1]$; F is said to satisfy *the weak property (N)* if for each $x^* \in X^*$ and $E \in \mathcal{L}$, we have

$$\lambda(E) = 0 \Rightarrow \lambda((x^* \circ F)(E)) = 0.$$

We say that F is sBV if F is sBV with respect to each p_k .

Theorem 2.5. *Let X be a Frechet space and let $F : [0, 1] \rightarrow X$ be a function. Then the following are equivalent.*

- (i) F is the primitive of a Bochner integrable function f ,
- (ii) F is sBV and weak continuous on $[0, 1]$, satisfies the weak property (N) and $F'(t) = f(t)$ almost everywhere on $[0, 1]$.

Proof. (i) $\Rightarrow$ (ii) Assume that F is the primitive of a Bochner integrable function f . Then, for each $k \in \mathbb{N}$, $\phi_k \circ f$ is Bochner integrable with the primitive $\phi_k \circ F$. Hence, by Theorem 2.3, each function $\phi_k \circ F$ is sBV and weak continuous on $[0, 1]$, satisfies the weak property (N) and $(\phi_k \circ F)'(t) = (\phi_k \circ f)(t)$ almost everywhere on $[0, 1]$. Thus, for each $k \in \mathbb{N}$, there is a subset $Z_k \subset [0, 1]$ with $\lambda(Z_k) = 0$ such that

$$(\phi_k \circ F)'(t) = (\phi_k \circ f)(t) \quad \text{for all } t \in [0, 1] \setminus Z_k.$$

We have also that $X^* = \{\tilde{x}_k^* \circ \phi_k : k \in \mathbb{N}, \tilde{x}_k^* \in \tilde{X}_k^*\}$, where X^* and $\tilde{X}_k^*$ are the topological dual of the space X and $\tilde{X}_k$, respectively. Therefore, F is sBV and weak continuous on $[0, 1]$, satisfies the weak property (N) and $F'(t) = f(t)$ for all $t \in [0, 1] \setminus \bigcup_{k=1}^{+\infty} Z_k$.

(ii) $\Rightarrow$ (i) Assume that (ii) holds. Then, each function $\phi_k \circ F$ is sBV and weak continuous on $[0, 1]$, satisfies the weak property (N) and $(\phi_k \circ F)'(t) = (\phi_k \circ f)(t)$ almost everywhere on $[0, 1]$. Hence, by virtue of Theorem 2.3, each function $\phi_k \circ f$ is Bochner integrable with the primitive $\phi_k \circ F$. Thus, f is Bochner integrable with the primitive F and the proof is finished. $\square$

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