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*Implementing Fragments of ZFC within
an r.e. Universe*

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Implementing Fragments of ZFC within an r.e. Universe^{*}

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Abstract. Rabin showed that there is no r.e. model of the axioms of Zermelo and Fraenkel of set theory. In the present work, it is investigated to which extent natural models of a sufficiently rich fragment of set theory exist. Such models, called Friedberg models in the present work, are built as a class of subsets of the natural numbers, together with the element-relation “ x is in y ” given by $x \in A_y$ where A_0, A_1, A_2, \dots is a Friedberg numbering of all r.e. sets of natural numbers; a member A_x of this numbering is then considered to be a set in the given model iff the downward closure of the induced element-ordering from x is well-founded. For each axiom and basic property of set theory, it is shown whether or not that axiom or property holds in such a model. Comprehension and replacement need to be properly adapted, as not all functions and objects definable using first-order logic exist in the model. The validity of the power set axiom, in an adequate formulation, depends on the model chosen. The other axioms hold in every Friedberg model. Furthermore, it is shown that there is a least Friedberg model which contains exactly those sets from the von Neumann universe which exist in all Friedberg models while there is no greatest Friedberg model. The complexity of the theory of a Friedberg model depends much on the model and ranges from the ω -jump of the halting problem to the ω -jump of a Π_1^1 -complete set.

1 Introduction

Rabin [8, 11] showed that there is no r.e. model of set theory. The current study therefore does not aim at building a full model of set theory but at investigating structures which incorporate some aspects of models of set theory without satisfying all usual conditions. Indeed, the structures

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studied satisfy most of the axioms of Zermelo and Fraenkel [1, 4, 13] but have the following nonstandard features:

- Set-theoretic operations like union and intersection exist but they are not effective. For example, one cannot compute the index of the union of two sets from the indices of these sets.
- Set-difference might not exist as the property of being recursively enumerable is not closed under complementation.
- Full comprehension is not obtained, that is, one cannot define sets by arbitrary levels of quantifications. Instead one has only the restricted variant that every r.e. subset of a set also exists in the model as a set; one might call it comprehension along enumeration reducibility.
- The power set axiom is satisfied only in some but not all models considered and power sets cannot be obtained in a uniform way (by computing the corresponding indices). Furthermore, power set means in the context of the current work only the collection of all r.e. subsets of a given set as non-r.e. objects are not considered.
- In the usual universe of classes and sets, the key hallmark of proper classes is not that they are “larger than sets”; instead in the present work, the hallmark of proper classes is that they are not well-founded. The collection of all sets is hence neither a set nor a class but just does not exist at all in the model under consideration.

So the overall aim of this paper is to investigate to which extent one can overcome the problems spotted by Rabin [8] and to introduce a natural class of models, called Friedberg models, which behave as outlined above. The general idea of a Friedberg model is captured by the following:

Definition 1. A *Friedberg numbering* A_0, A_1, A_2, \dots is a one-one numbering of all r.e. subsets of \mathbb{N} [2]; such a numbering is called a *Friedberg model* if B_k defined as $\{x : |A_x| = k\}$ is recursive for every $k \in \mathbb{N}$.

Furthermore, the A_x are referred to as *classes* and they are investigated with respect to the “element-relation” defined by “ y is an element of x ” iff $y \in A_x$ with respect to this numbering. A_x is called a *set* iff A_x is well-founded, that is, iff there is no function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that $f(0) = x$ and $\forall n [f(n+1) \in A_{f(n)}]$. If A_x is not a set, it is called a *proper class*.

Note that the sets within a Friedberg model might be very difficult to determine; but it will be shown below that there are Friedberg models for which the indices of the proper classes are recursively enumerable. The indices of the sets themselves cannot be recursively enumerable as otherwise the collection of these indices would again form a set whose index is not enumerated into this collection. Note that for most proofs in this paper it is sufficient to assume that B_1 and B_2 are recursive; the postulate that B_3, B_4, \dots are also recursive is just a natural generalization and simplifies the setting. Theorem 2 below proves that there is a Friedberg model. The interested reader is referred to the books of Odifreddi [6, 7], Rogers [9] and Soare [12] for explanations on recursion theory.

Section 2 gives an overview to the extent to which the Axioms of Zermelo and Fraenkel are satisfied. The existence of Friedberg models is shown and Theorem 4 provides an example of a

model where all recursive ordinals exist. On the other hand, the set V_ω of all hereditarily finite sets does not exist in any Friedberg model.

Section 3 deals with the complexity of the predicate $Set(x)$ which is true iff A_x is a set. It is shown that the complexity of this predicate ranges from Π_1^0 to Π_1^1 .

Section 4 provides the existence of a least Friedberg model which contains exactly those sets from the von Neumann universe which exist in all Friedberg models; these can be characterized as those members of the universe which exist in some Friedberg model and in addition satisfy that all but finitely many members of the transitive closure have a cardinality bounded by some constant k . Furthermore, it is shown that there is no greatest model, that is, for every Friedberg model there is a set from the von Neumann universe which exists in some other Friedberg model but not in the given one.

Section 5 provides a methodology to investigate the theory of a Friedberg model and its complexity. It is shown that two natural approaches to define the theory can be translated one into the other and that therefore the complexity in terms of Turing degrees is in both cases the same. The complexity itself is mainly determined by the complexity of the predicate Set .

Section 6 deals with the power set axiom which is the only axiom whose validity depends on the chosen Friedberg model. While most Friedberg models will not satisfy the axiom, Theorem 29 provides a specific Friedberg model where the axiom holds. As here the power set is only the set of all r.e. subsets, this construction is in Friedberg models not strong enough to separate out various infinite cardinalities; indeed, there is always a recursive onto function from an infinite r.e. set to its power set, whenever the latter exists in the model.

2 Friedberg models and the axioms of Zermelo and Fraenkel

The main idea of the paper is to investigate to which extent a Friedberg model resembles the von Neumann universe of Zermelo Fraenkel set theory. One basic idea of the von Neumann universe is that one builds sets of sets and sets of sets of sets and so on in order to represent objects; for example the first ordinals (natural numbers) are

- 0 represented by \emptyset ;
- 1 represented by $\{0\} = \{\emptyset\}$;
- 2 represented by $\{0, 1\} = \{\emptyset, \{\emptyset\}\}$;
- 3 represented by $\{0, 1, 2\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}$.

This structured hierarchy has now to be represented by a model and the general idea is to base the representation on the following equivalence:

y is an element of x when viewed as a model (of some fragment) of set theory iff $y \in A_x$.

So an example of the above is that 0 is represented by the index x_0 of the empty set, 1 by the index x_1 of the set $\{x_0\}$, 2 by the index x_2 of the set $\{x_0, x_1\}$ and 3 by the index x_3 of the set $\{x_0, x_1, x_2\}$. The first transfinite ordinal would then be represented by the index x_ω of the set $\{x_0, x_1, x_2, \dots\}$ provided that $\{x_0, x_1, x_2, \dots\}$ is r.e., that is, exists in the model. In general, a

set X exists in the model iff every element of X exists in the model and the collection of all elements of X is r.e.; note that this condition implies that a set X exists as a class iff it exists as a set. So the proper classes in a Friedberg model are objects which cannot be sets.

The following gives an overview of the extent to which a Friedberg model satisfies the axioms of Zermelo and Fraenkel.

1. Axiom of extensionality: If $x \neq y$ then $A_x \neq A_y$. This property directly follows from imposing that A_0, A_1, A_2, \dots has to be a Friedberg numbering.
2. Axiom of regularity: For sets by definition of “set” above; for classes no kind of regularity is required and there is an x representing a class with $x \in A_x$, for example the index x of \mathbb{N} .
3. Axiom schema of comprehension: First it should be noted from the definition that a class is a set iff all of its members are sets. So the main difference between sets and classes in this model is that the classes contain members which are not well-founded and hence are by themselves not well-founded. The main property is the following one: Whenever A_x is a set and A_y is a class, then $A_x \cap A_y$ is a set again; that is, one can cut out of a set along a class. Furthermore, whenever E is e -reducible to A_x then there is a y with $A_y = E$; if $A_y \subseteq A_x$ and A_x is a set then A_y is a set as well. Equivalently, one can say that every r.e. subset of a set in the model or a union of sets in the model also exists in the model. Hence one can form intersections and unions of sets in the model and also do other operations which can be expressed by e -enumerability. What is missing is that for sets A_x, A_y the difference $A_x - A_y$ is also a set; this fails in some cases and then $A_x - A_y$ is not a set. Furthermore, universal quantification in formulas will fail to define sets while existential quantification is safe as it defines a new set in a positive way.
4. Axiom of pair: For all x, y such that A_x, A_y are sets there is another set z such that $A_z = \{x, y\}$. This axiom is satisfied in an effective way: there is a recursive function f such that $A_{f(x,y)} = \{x, y\}$. From that it follows that for every partial-recursive function g there is a class such that $A_z = \{\{u, \{u, g(u)\}\} : g(u) \text{ is defined}\}$ and hence g is represented in the model. As a consequence, one can invoke functions in the model whenever they can be constructed in an effective way.
5. Axiom of union: If A_x is a set then there is a set A_y with $A_y = \{z : \exists u \in A_x [z \in A_u]\}$. The index y exists as the Friedberg numbering covers all r.e. sets but it is impossible to compute the index y from x as otherwise V_ω would be a set in contradiction to Theorem 5.
6. Axiom of replacement: If A_x is a class defining a function f which maps indices of sets to indices of sets and A_y is a set then there is a z with A_z being a set and $A_z = \{f(u) : u \in A_y\}$. Again the existence of z follows from the property that a Friedberg numbering covers all r.e. subsets of \mathbb{N} . So the axiom of replacement applies to functions defined by classes; but the axiom does not apply to functions obtained by first-order definitions as those might fail to be partial-recursive.
7. Axiom of infinity: There is an x such that A_x is an infinite set. This set can be constructed by search inside B_1 : First let b_0 be the index with $A_{b_0} = \emptyset$ and then let b_{n+1} be inductively the unique element of B_1 found with $b_n \in A_{b_{n+1}}$. Note that this procedure only gives the set A_x but not its index x .

8. Axiom of power set: For every set A_x there is a set A_y such that $A_y = \{z : A_z \subseteq A_x\}$. This cannot be done in a uniform way as then there would be an x with $A_x = V_\omega$ by uniformly taking the power sets of \emptyset , the power set of \emptyset , the power set of the power set of \emptyset and so on; at the end V_ω is the union of these sets. This then contradicts Theorem 5 below. But it is shown in Theorem 29 below that there exists a Friedberg model in which every set has a power set and also every class has a power class.
9. Axiom of choice: There is a class ch which maps every x with $A_x \neq \emptyset$ to the first value $ch(x)$ found such that $ch(x) \in A_x$; ch is undefined on only one place.

Next it is shown that there is a Friedberg model at all.

Theorem 2. *There is a Friedberg model; that is, there is a Friedberg numbering A_0, A_1, A_2, \dots of all r.e. subsets of \mathbb{N} such that for each k the set $B_k = \{x : |A_x| = k\}$ is recursive.*

Proof. Friedberg [2] showed the existence of a one-one numbering of all r.e. sets and an easy modification of his argument gives that there is a one-one numbering E_0, E_1, E_2, \dots of all r.e. subsets of \mathbb{N} with at least 2 nonelements. Now let $A_0 = \emptyset$, $A_1 = \mathbb{N}$ and assign for $x = 2, 3, \dots$ the set A_x according that of the below cases which applies and for which the priority is highest, that is, has the lowest numerical value. In the following, $A_{y,s}$ denotes the elements enumerated into A_y within s steps and let the d -th finite set be the unique finite set X with $\sum_{a \in X} 2^a = d$. Furthermore, a set A_x follows (at stage s) a set Y iff it enumerates all elements of Y_s into A_x and is explicitly declared to be a follower of Y . Hence, one can always check in the construction which set A_x is currently following.

- Priority $3d$: This case applies if $A_{y,x} \neq X$ for all $y < x$ where X is the d -th finite set. If this case is selected then A_x follows the set X as long as $E_{e,s} \neq X$ for all $e < |X|$. In the case that there is $e < |X|$ and a stage $s > x$ with $E_{e,s} = X$ at stage s , A_x stops following X and starts to follow $\mathbb{N} - \{u\}$ forever where u is the least number such that $u \notin X$ and no other set is currently following $\mathbb{N} - \{u\}$.
- Priority $3d + 1$: This case applies if $|E_{d,x}| > d$ and E_d does not yet have a follower. If this case is selected then $A_x = E_d$; that is, A_x follows E_d forever.
- Priority $3d + 2$: This case applies if no A_y with $1 < y < x$ is currently following $\mathbb{N} - \{d\}$. If this case is selected then $A_x = \mathbb{N} - \{d\}$; that is, A_x follows $\mathbb{N} - \{d\}$ forever.

Note that the priorities of different choices are different and therefore it is always clear which choice is taken to initialize A_x . It follows from the priorities that every set of the form $\mathbb{N} - \{u\}$ receives eventually exactly one follower A_x and no other A_y with $y \neq x$ will follow $\mathbb{N} - \{u\}$. Furthermore, each set E_d with at least $d + 1$ elements receives eventually exactly one follower. As $E_c \neq E_d$ whenever $c \neq d$, no infinite set has eventually two followers. But if E_d is finite, there might be some A_x initially following E_d ; this A_x will then stop following E_d and instead start to follow a set of the form $\mathbb{N} - \{u\}$. Hence all the E_d have eventually a unique follower A_x and all A_y with $y \neq x$ are different from E_d . Furthermore, the protocol for the finite A_x created by the first case make sure that each X exists in the case that there is no $d < |X|$ with $E_d = X$. Hence the resulting numbering is one-one, that is, a Friedberg numbering. Furthermore, one can

see that for every $k \in \{1, 2, 3, \dots\}$ there are only finitely many x such that A_x has initially the cardinality k and receives later more elements; hence each B_k is a finite variant of $\{x : A_x \text{ has at the initialization } k \text{ elements}\}$ and so each B_k is recursive. This completes the proof that the constructed model is a Friedberg model. ■

Remark 3. Harrison [3] showed that there is a recursive linear ordering with an initial segment of type ω_{CK} where ω_{CK} is the first nonrecursive ordinal and is named after Church and Kleene. This recursive linear ordering \sqsubset is given as the Kleene-Brower ordering on a tree $T \subseteq \mathbb{N}^*$ which has an infinite branch but no hyperarithmetical infinite branch.

This ordering and the underlying tree have a further interesting property: if W is an r.e. bounded set then W has a least upper bound b . Harrison [3] showed this property even for any hyperarithmetical set in place of W . Without loss of generality one can assume that W is closed downward under \sqsubset and a node is an upper bound for W iff it is outside W .

To see this, let a_n be the least upper bound of W among all the nodes up to height n in the tree; note that all a_n exist as a_0 is the root of T and an upper bound of W . For each n , one has either $a_{n+1} = a_n$ or $a_{n+1} \sqsubset a_n$. In the latter case the height of a_{n+1} is $n + 1$ and a_{n+1} is a successor of a_n as a node in T — the reason is that otherwise a_{n+1} would be a successor of a node a'_n of height n with $a'_n \sqsubset a_n$ in contradiction to the choice of a_n as the least upper bound of W within the set of nodes up to height n . Note that either almost all a_n are the same node b or they describe an infinite branch F in the tree. In the first case the node b is a least upper bound of W , in the second case one can compute F inductively from K as one would have that a_{n+1} is the least successor of a_n which is not in W . As T has no infinite K -recursive branch, the first case applies and the downward closure of W under \sqsubset is $\{x : x \sqsubset b\}$.

Theorem 4. *There is a Friedberg model in which every recursive ordinal is represented.*

Proof. Let \sqsubset be the recursive linear ordering from Remark 3 and define $A_{2x} = \{2y : y \sqsubset x\}$ for all x . Furthermore, let $A_1 = \{0, 2, 4, 6, \dots\}$ and $A_3 = \mathbb{N}$.

For the remaining members of the Friedberg model, one adapts the proof of Theorem 2 in order to complete this construction to a Friedberg model. Let E_0, E_1, E_2, \dots be a one-one numbering of all r.e. subsets of \mathbb{N} with at least 2 nonelements. The main modification in the construction is that one has to add some witnesses in order to keep the new sets different from those with even index built so far. In the following, $A_{y,s}$ denotes the elements enumerated into A_y within s steps and let the d -th finite set X be the unique set such that $\sum_{a \in X} 2^a = d$. The set A_{2x+5} is selected from the following options:

- Priority $3d$: This case applies if $A_{2y+5,x} \neq X$ for all $y < x$ and either X contains an odd element or there are v, w found in time x with $v \sqsubset w$, $v \notin X$ and $w \in X$ where X is the d -th finite set. If this case is selected, A_{2x+5} follows the set X as long as $E_{e,s} \neq X$ for all $e < |X|$. If at stage $s > x$ there is an index $e < |X|$ with $E_{e,s} = X$ then A_{2x+5} starts to follow $\mathbb{N} - \{u\}$ for the least u such that $u \notin X$ and $\mathbb{N} - \{u\}$ has no other follower so far.
- Priority $3d + 1$: This case applies if $|E_{d,x}| > d$ and E_d does not yet have a follower and either an odd element has been found in E_d at time x or there are v, w found in time x with $v \sqsubset w$,

$v \notin E_{d,x}$ and $w \in E_{d,x}$; in the latter case, v is kept fixed from now on. If this case is selected, A_{2x+5} follows the set E_d until a stage s is reached with $\{v\} \subseteq E_{d,s} \subseteq \{0, 2, 4, \dots\}$. If such a stage s is reached, then A_{2x+5} starts to follow $\mathbb{N} - \{u\}$ for the least u such that u is not yet enumerated into A_{2x+5} and no other set is following $\mathbb{N} - \{u\}$ so far.

- Priority $3d + 2$: This case applies iff no set A_{2y+5} with $y < x$ is following $\mathbb{N} - \{d\}$ so far. If this case is selected, then A_{2x+5} follows $\mathbb{N} - \{d\}$ forever.

First it should be noted that all r.e. subsets of \mathbb{N} which are closed downward along \sqsubset are either \mathbb{N} itself or of the form $\{v : v \sqsubset w\}$. Hence, in the construction, all the r.e. proper subsets W of $\{0, 2, 4, \dots\}$ which are not covered by A_0, A_2, A_4, \dots satisfy that there exist v, w with $v \sqsubset w$, $2v \notin W$ and $2w \in W$. Hence each such set W will eventually qualify with priority $3d$ or $3d + 1$ when the corresponding witnesses are found. There might be prior witnesses which are wrong and lead to the follower to be cancelled, but eventually the right witnesses will be found. If W is r.e., W is a proper subset of $\{0, 2, 4, \dots\}$ and $\{w : 2w \in W\}$ is closed downward under \sqsubset then W is equal to some set A_{2x} and all attempts to follow W by priority $3d + 1$ will eventually fail since a correct pair v, w of witnesses cannot be found.

Second, the remaining verification is similar to the verification in Theorem 2. In order to see that the numbering is one-one, one has to note that explicit collisions can only happen between finite sets X and sets E_d where the follower of the finite set X then gets destroyed. Each set E_d can destruct only one follower for each of the cardinalities $d + 1, d + 2, \dots$ and hence for each k at most k followers of a finite set get destroyed. Furthermore, each follower of E_d can be destroyed by enumerating the corresponding v ; but when doing so the number of elements in E_d increases and hence this effect does not lead to multiple destruction of the followers of the same finite set X . These arguments can be used to verify that A_0, A_1, A_2, \dots is a one-one numbering of all r.e. sets and that each B_k is recursive since B_k is a finite variant of $\{2x + 5 : A_{2x+5} \text{ was initialized as a follower of an } X \text{ with } k \text{ elements}\}$. ■

Recall that in set theory V_α is the set of all sets with rank up to α where the rank ρ is defined inductively by $\rho(\emptyset) = 0$ and $\rho(X) = \sup\{\rho(Y) + 1 : Y \in X\}$ for every nonempty set X . In particular V_ω is a natural concept as it consists of all sets whose transitive closure consists of finite sets only. The next result shows that V_ω does not exist in any Friedberg model. This in particular implies that there is no function representing the rank of sets in a Friedberg model.

Theorem 5. *There is no Friedberg model in which the set V_ω of all hereditarily finite sets exists.*

Proof. Assume by way of contradiction that V_ω exists in the model, that is, $V_\omega = \{x : A_x \text{ is hereditarily finite}\}$ is recursively enumerable. Furthermore, let E be an infinite recursive subset of $\{b_0, b_1, b_2, \dots\}$ which is defined by b_0 being the unique index of the empty set and b_{n+1} being the unique index of $\{b_n\}$. As B_1 is recursive, one can find b_{n+1} by searching in B_1 for the unique member of B_1 which contains b_n .

Now one constructs inductively the following infinite subset $\tilde{E} = \{e_0, e_1, e_2, \dots\}$ of E : e_y is the first element $s \in E$ strictly greater than all e_z with $z < y$ such that at least one of the following two conditions holds:

- y is enumerated into V_ω within s steps;
- there is an $u \in A_{y,s} - \{e_z : z < y\}$ with $u < s$.

Note that the search for s terminates as E is infinite and one of the following conditions holds for all sufficiently large $s \in E$:

- A_y is hereditarily finite and y shows up in V_ω within s steps;
- A_y is finite but not hereditarily finite and A_y contains a member outside E which shows up within s steps and is strictly below s ;
- A_y is infinite and contains some element $u \notin \{e_z : z < y\}$ which shows up within s steps and is strictly below s .

So \tilde{E} is an infinite and recursive subset of E . Furthermore, \tilde{E} differs from all infinite subsets A_y of E as there is a $u < e_y$ such that $u \notin \{e_z : z < y\}$ while \tilde{E} contains exactly the e_z with $z < y$ as elements below e_y . It follows that \tilde{E} is an infinite recursive set not occurring in the list A_0, A_1, A_2, \dots in contradiction to the assumption on the model. ■

This result establishes that one cannot enumerate all indices of sets of finite rank; hence the rank of sets is not represented in this model at all. It is natural to ask whether one can access the cardinality. In other words, one might ask whether there is a function ψ such that for finite sets A_x the function $\psi(x)$ is defined and returns $|A_x|$ and for infinite sets A_x the function $\psi(x)$ is undefined. The answer is “no”; assuming the answer would be “yes”, one can enumerate V_ω inductively as one can first enumerate the index of the empty set into V_ω and later add at every stage s all x such that $\psi(x)$ has converged within s steps to some value k and all k members of A_x are already enumerated into A_x and into V_ω . Hence ψ cannot exist.

Corollary 6. *There is no Friedberg model in which there is a partial-recursive function ψ such that $\psi(x) = k$ iff $|A_x| = k$. So the cardinality even of finite sets (and classes) cannot be computed and B_0, B_1, B_2, \dots cannot be uniformly recursively enumerable.*

Remark 7. A key difference with the traditional setting is that in a Friedberg model all infinite sets have the same cardinality. This follows from the fact that there is for any two infinite r.e. A_x, A_y a partial-recursive bijection from A_x onto A_y with domain A_x .

Remark 8. It has been mentioned above that the union and the power set cannot be formed effectively. The same is true for the intersection: Assume by way of contradiction that there is a recursive f computing the index of the intersection and that f is correct at least on indices of sets. Let A_x be a recursive infinite set. This has then a nonrecursive infinite r.e. subset A_y . Let a be the unique index of the empty set. Given $z \in A_x$, search the unique $b \in B_1$ with $z \in A_b$; note that then $A_b = \{z\}$. Now $z \in A_y \Leftrightarrow f(y, b) \neq a$. But this algorithm does not exist as A_y is not recursive, contradicting the choice of f . Hence the intersection is not effective.

3 The complexity of recognizing the sets

In this section a Friedberg model is constructed such that the indices of the sets in the model are co-r.e. and thus much below the theoretical possible complexity Π_1^1 . In order to formalize this, a predicate Set is defined such that $Set(x)$ is true iff A_x is a set.

Theorem 9. *There is a Friedberg model where Set has the complexity Π_1^0 , that is, $\{x : A_x \text{ is a proper class}\}$ is recursively enumerable. Furthermore, the Friedberg model contains a given recursive ordinal α .*

Proof. Let \sqsubset be a recursive well-ordering on the even numbers of order type α with least element 0 and define $A_{2x} = \{2y : 2y \sqsubset 2x\}$. Furthermore, let E_0, E_1, E_2, \dots be a Friedberg numbering of all sets which are different from A_0, A_2, A_4, \dots and which contain at least 2 nonelements. The existence of such a Friedberg numbering can be concluded by a theorem of Kummer [5]: he showed that if one can make a numbering of some class \mathcal{A} of recursively enumerable sets and a one-one numbering of a class \mathcal{B} disjoint to \mathcal{A} such that every finite set has infinitely many supersets in \mathcal{B} then $\mathcal{A} \cup \mathcal{B}$ has a one-one numbering. Now one takes as \mathcal{A} the class of all r.e. sets $W - \{u, v, w\} \cup \{t\}$ where W is r.e., u, v, w, t are distinct and either u, t are even and $u \sqsubset t$ or t is odd; furthermore one adds the empty set to \mathcal{A} . \mathcal{B} contains all subsets of \mathbb{N} with exactly two nonelements and this class clearly has a one-one numbering. Thus also their union $\mathcal{A} \cup \mathcal{B}$ has the desired Friedberg numbering E_0, E_1, E_2, \dots by Kummer's result [5].

In the following construction, let U be the r.e. set of all indices x of sets A_x which are easily seen not to be well-founded in the sense that there are a function f and numbers n, m with $n > m$ such that $f(0) = x$, $f(k+1) \in A_{f(k)}$ for all $k < n$ and $f(m) = f(n)$. Note that U is r.e. and that an enumeration of U is used when defining A_x for an odd x . Let U_s be the elements enumerated into U within s steps.

So let $A_1 = \mathbb{N}$, $A_3 = \{0, 2, 4, \dots\}$ and for each odd $x > 3$, A_x is initialized according to the entry of highest priority (where the high priorities have low numerical value and vice versa):

- With priority $3d$ make A_x a follower of the d -th finite set X if $x > \max(X)$ and all sets $A_{y,x}$ with $y < x$ are different from X and $X \neq A_z$ for all even z .
- With priority $3d+1$ let $A_x = E_d$ provided that $|E_{d,x} \cap \{0, 1, 2, \dots, x-1\}| > d$ and that E_d has no current follower.
- With priority $3d+2$ let $A_x = \mathbb{N} - \{d\}$ provided that this set is not already there.

After choosing the A_x according to the priority, one updates A_y for odd y with $3 < y < x$ provided that one of the following two cases applies:

- If A_y is initialized as the d -th finite set X and there is some e with $E_{e,x} = X$ and $|E_{e,x}| > e$ then let A_y follow $\mathbb{N} - \{u\}$ for the first u found such that $u \notin X$ and $\mathbb{N} - \{u\}$ does not yet have a follower.
- If A_y is initialized as E_d for some d , there are an odd $z < x$ and $d' \neq d$ with A_z following $E_{d'}$, $A_{y,x} \cap U_x = \emptyset$ and there are a function f and $n > 0$ with $f(0) = y$, $f(n) = z$ and $f(m+1) \in A_{f(m)}$ for all $m < n$, then let A_y, A_z follow $\mathbb{N} - \{v\}$ and $\mathbb{N} - \{w\}$, respectively, where $\mathbb{N} - \{v\}$, $\mathbb{N} - \{w\}$ have not yet a follower, v is not yet enumerated into A_y , w is not yet enumerated into A_z and $v \neq w \Leftrightarrow y \neq z$.

First, one shows that each r.e. W appears exactly once in the numbering:

- One assigns for each set $\mathbb{N} - \{u\}$ exactly one follower which never abandons this set. Hence every set $\mathbb{N} - \{u\}$ equals exactly one A_x . \mathbb{N} equals A_1 and the set of even numbers equals A_3 .

- Now let W be given such that W has at least two nonelements and there is a d with $E_d = W$ and $|E_d| > d$. Then there is a stage x such that $|E_{d,x} \cap \{0, 1, 2, \dots, x\}| > d$. Hence, for stage $y \geq x$, the set A_y is made equal to E_d with priority $3d + 1$ and hence E_d will eventually appear in the list A_0, A_1, A_2, \dots created. It can happen that the first instantiations of E_d will be transformed into sets of the form $\mathbb{N} - \{u\}$; but as then some element of E_d goes into U , this happens only finitely often and E_d will have an index eventually. Furthermore, if E_d is finite and there is an A_y initialized as a finite set X equal to E_d then A_y will eventually abandon X and follow a set $\mathbb{N} - \{u\}$ for some u and hence no such A_y ends up being equal to E_d ; if E_d is infinite then it has at least two elements in its complement and it cannot be equal to any $E_{d'}$ by the choice of the numbering E_0, E_1, E_2, \dots from which the infinite members in the numbering A_0, A_1, A_2, \dots are built.
- Now assume the last case where W is finite and not equal to any E_d instantiated in the numbering. Let $d = \sum_{a \in W} 2^a$. Then a follower will be assigned to W with priority $3d$ eventually. Furthermore, each of the sets $E_{d'}$ with $d' < |W|$ can force the follower to abandon W and to follow a set of the form $\mathbb{N} - \{u\}$ instead, but at some later stage some further element will be enumerated into $E_{d'}$ and a new set will start to follow W . It follows from the way the numbering A_0, A_1, A_2, \dots is constructed that W shows there up exactly once.

Second, it is shown that all the B_k are recursive. Note that almost all sets which initially follow a finite X of cardinality k will never abandon X . Only finitely many initializations of these sets get modified by clashing with some $E_{d'}$ with $d' < k$ temporarily and only finitely many of them coincide with some $E_{d'}$ which does not get its index at the first attempt but might be moved when the first of its elements goes into U . Furthermore, note that there is only one even x such that A_x has exactly k elements. So B_k is a finite variant of the set of all odd indices x where A_x is initialized as a set of k elements in the stage where A_x becomes defined first and hence B_k is recursive.

Third, it is shown that U contains all x where A_x is not well-founded; it is obvious that all $x \in U$ are indices of classes which are not well-founded. Note that it follows from the definition that U is recursively enumerable. Now assume that $x \notin U$ but that there is a function f such that $f(0) = x$ and $f(k+1) \in A_{f(k)}$ for all k . It follows from $x \notin U$ that there are no m, n with $f(m) = f(n)$. In the case that there is an m with $A_{f(m)} = \mathbb{N} - \{u\}$ one could modify f such that $f(k) = f(m)$ for all $k > m$ if $f(m) \neq u$ and $f(k) = 1$ for all $k > m$ if $f(m) = u$. In both cases, the modified f would witness that $x \in U$, hence this case does not occur. Similarly $f(m) \neq 1$ for all m . Note that if there are m, n, d, d' such that $A_{f(m)} = E_d$, $A_{f(n)} = E_{d'}$, $m < n$, $d < d'$ and no element of E_d is in U , then the second condition to modify the set $A_{f(m)}$ would eventually be satisfied and $A_{f(m)}$ and $A_{f(n)}$ would both eventually become sets of the form $\mathbb{N} - \{v\}$ and $\mathbb{N} - \{w\}$, respectively, which would cause x to be enumerated into U in contradiction to the assumption on x . Hence there are only finitely many k such that $A_{f(k)} = E_d$ for some d . Furthermore, there is no k such that $f(k)$ is even as every A_x with even x contains only even numbers and is well-founded. Hence there is an m such that $f(n)$ is odd and $f(n)$ is an index of a finite set with elements only below $f(n)$ for all $n \geq m$; this gives then that $f(n) \leq f(m) + m - n$ for all $n \geq m$ and $f(f(m) + 1) < 0$, a contradiction. Hence A_x is well-founded iff $x \notin U$. ■

Remark 10. Note that there is no Friedberg model in which Set is r.e. which can be seen as follows. Assume by way of contradiction there is an index x with $A_x = \{y : Set(y) \text{ is true}\}$. Then A_x is a set since every element of A_x is a set. Hence $x \in A_x$ in contradiction to the choice of sets as being those classes which are well-founded.

Note that this construction follows exactly the lines of Russel's antinomy of naive set theory [10]. The following classes of ill-defined objects which are similar to Russel's construction exist: the class of all classes which contain themselves (in every Friedberg model) and the class of all proper classes (in some Friedberg model as shown above).

Proposition 11. *There is no Friedberg model where the predicate Set is hyperarithmetical and every recursive ordinal α exists.*

Proof. Assume that such a Friedberg model A_0, A_1, A_2, \dots would be given with the corresponding B_0, B_1, B_2, \dots as defined in Definition 1 such that the predicate Set is hyperarithmetical. Then also the collection O of all ordinals is hyperarithmetical for the following reason: $x \in O$ iff $Set(x)$, A_x is transitive (that is, $z \in A_x$ for all $y \in A_x$ and all $z \in A_y$) and all different elements y, z of A_x are comparable (either $y \in A_z$ or $z \in A_y$). Let $<_0, <_1, <_2, \dots$ be an acceptable numbering of all r.e. partial orders on \mathbb{N} . Here a partial ordering $<_e$ is r.e. iff $\{\langle x, y \rangle : x <_e y\}$ is an r.e. subset of \mathbb{N} .

Now one has that $<_e$ is a well-ordering iff there are a function f and an $x \in O$ such that f is an order preserving isomorphism from $(\mathbb{N}, <_e)$ to A_x with the ordering on A_x being given by the element-relation: y is below z iff $y \in A_z$.

As O is hyperarithmetical and hyperarithmetical sets are Σ_1^1 , the above expression is a Σ_1^1 formula as one can express membership in the hyperarithmetical set with a Σ_1^1 predicate. As $\{e : <_e \text{ is a well-ordering on } \mathbb{N}\}$ is Π_1^1 -complete, the above Σ_1^1 -formula cannot exist. Hence it cannot be that in a Friedberg model one can on one hand have all recursive ordinals and on the other hand have that the indices of the sets are hyperarithmetical. ■

In the following it is shown that the predicate Set can be quite complicated. But one first needs some facts about recursive trees.

Remark 12. There is a numbering S_0, S_1, S_2, \dots of trees such that each $S_e \subseteq \mathbb{N}^*$ and $\{e : S_e \text{ is well-founded}\}$ is Π_1^1 -complete. Now construct from each S_e the tree T_e given as $T_e = \{(2a_0, 2a_1, 2a_2, \dots, 2a_n), (2a_0, 2a_1, 2a_2, \dots, 2a_n, b) : (a_0, a_1, a_2, \dots, a_n) \in S_e \wedge b \text{ is odd}\} \cup \{\text{the root of } S_e\}$. It is clear that the nodes ending with an odd number are the leaves and that every node ending with an even number has infinitely many successors. Furthermore, T_e has an infinite branch if S_e has, that is, T_e is well-founded iff S_e is well-founded.

Theorem 13. *There is a Friedberg model where the predicate Set is Π_1^1 -complete.*

Proof. Let $A_{3x} = \{3y : y < x\}$ for all x . For the definition of the sets A_{3x+1} , let T_0, T_1, T_2, \dots be the uniformly recursive enumeration of trees from Remark 12. Note that these trees are infinitely branching subtrees of \mathbb{N}^* , that the set of leaves is uniformly recursive, that every node is either

a leaf or has an infinite set of successors and that $\{e : T_e \text{ is well-founded}\}$ is Π_1^1 -complete.

There is a recursive one-one function g from $\{1, 4, 7, 10, \dots\}$ onto the disjoint union of T_0, T_1, T_2, \dots ; let $F_e = g^{-1}(T_e)$ and $r_e = g^{-1}$ (the root of T_e). So g induces a recursive partition of $\{1, 4, 7, 10, \dots\}$ into the sets F_0, F_1, F_2, \dots and the set $\{r_0, r_1, r_2, \dots\}$ is recursive. Furthermore, the set $\{3y + 1 : g(3y + 1) \text{ is a leaf}\}$ is recursive. Let e be the index with $3x + 1 \in F_e$. If $g(3x + 1)$ is a leaf then let $A_{3x+1} = \{6y, 6y + 3\}$ for the least y such that the set $\{6y, 6y + 3\}$ is not already used by some leaf $3z + 1$ with $z < x$ else let $g(3x + 1) = \{3y + 1 \in F_e : g(3y + 1) \text{ is a successor of } g(3x + 1) \text{ in } T_e\}$. Note that there is a function f with $f(0) = 3x + 1$ and $f(n + 1) \in A_{f(n)}$ for all n iff $g(3x + 1)$ lies on an infinite branch of T_e ; this branch is then defined by f . Hence A_{3x+1} is well-founded iff there is no infinite branch of T_e through $g(3x + 1)$. Now it has been achieved that — independent of the further constructions — $\text{Set}(r_e)$ is true iff T_e is well-founded. So Set is Π_1^1 -complete.

Hence the remaining goal will only be to define the sets A_{3x+2} such that the resulting numbering defines a Friedberg model. So let $A_2 = \mathbb{N}$ and $A_5 = \{0, 3, 6, 9, 12, \dots\}$. For the sets of the form A_{3x+8} , one adapts the proofs of Theorems 2 and 4 in order to complete this construction to a Friedberg model.

Let E_0, E_1, E_2, \dots be a one-one numbering of all r.e. subsets of \mathbb{N} with at least 2 nonelements. The main modification in the construction is that one has to add some witnesses in order to keep the new sets different from those with even index built so far. In the following, $A_{y,s}$ denotes the elements enumerated into A_y within s steps and let X denote the d -th finite set which is the unique set such that $\sum_{a \in X} 2^a = d$. The set A_{3x+8} is selected from the following options:

– Priority $3d$: This case applies if the following conditions hold:

- $A_{3y+8,x} \neq X$ for all $y < x$;
- X is not of the form $\{3y : y < z\}$ for any z ;
- X is not of the form $\{6y, 6y + 3\}$ for any y .

If this case is selected, A_{3x+8} follows the set X as long as $E_{e,s} \neq X$ for all $e < |X|$. If at stage $s > x$ there is an index $e < |X|$ with $E_{e,s} = X$ then A_{3x+8} follows $\mathbb{N} - \{u\}$ from now on forever where u is the least number such that $u \notin X$ and no set is following $\mathbb{N} - \{u\}$ so far.

– Priority $3d + 1$: This case applies if there is are v, w such that following conditions hold:

- $|E_{d,x}| > d$;
- no set A_{3y+8} with $y < x$ is currently a follower of E_d ;
- if $E_{d,x} \subseteq \{0, 3, 6, 9, 12, \dots\}$ then v is the least number with $v \notin E_{d,x} \wedge v + 3 \in E_{d,x}$;
- if there is a y with $E_{d,s} \subseteq A_{3y+1}$ then $w = \min(A_{3y+1} - E_{d,x})$.

Note that if the fourth condition applies then the y, w there are unique as $E_{d,x}$ is not empty by the first condition and so there can be at most one y with $E_{d,x} \subseteq A_{3y+1}$. If this case is selected then v and w are kept fixed as parameters of the construction for this set A_{3x+8} from now on and A_{3x+8} follows E_d until one of the following two cases occurs at some stage $s > x$:

- v is enumerated into $E_{d,s}$ and $E_{d,s} \subseteq \{0, 3, 6, 9, 12, \dots\}$;
- there is a unique y with $\{w\} \subseteq E_{d,s} \subseteq A_{3y+1}$.

If this happens then A_{3x+8} starts to follow $\mathbb{N} - \{u\}$ forever where u is the least number such that u is not yet enumerated into A_{3x+8} and no other set is currently following $\mathbb{N} - \{u\}$.

- Priority $3d + 2$: This case applies iff no set A_{3y+8} with $y < x$ is following $\mathbb{N} - \{d\}$ so far. If this case is selected, A_{3x+8} follows $\mathbb{N} - \{d\}$ forever.

First, one has to note that the algorithm for the entry of priority $3d + 1$ can be executed effectively, that is, that one can always check whether there is a y with $E_{d,x} \subseteq A_{3y+1}$. Indeed, the sets A_1, A_4, A_7, \dots have been selected such that they together with $\{3z + 1 : g(3z + 1) \text{ is the root of some tree } T_e\}$ form a partition of $\{0, 1, 3, 4, 6, 7, 9, 10, \dots\}$. Hence one can for every finite set $E_{d,x}$ and later $E_{d,s}$ determine all y with A_{3y+1} intersecting $E_{d,x}$ and $E_{d,s}$, respectively, and then see whether any of the A_{3y+1} actually does not only intersect but also contain the finite set. If this is so, then this set A_{3y+1} is unique. Furthermore, from the property of being a partition of $\{0, 1, 3, 4, 6, 7, 9, 10, \dots\}$ it follows that the sets A_{3y+1} are uniformly recursive.

Second, note that by the choice of the trees T_e , each node in T_e has either none or infinitely many successors. In the case that it has infinitely many successors the corresponding A_{3y+1} representing the set of these successors is also infinite and hence different from X in the entry for priority $3d$. Hence the conditions given there are enough to make sure that the set A_{3x+8} , if selected by this case, is different from all the sets $A_0, A_1, A_3, A_4, A_6, A_7, A_9, A_{10}, \dots$ and it is also different from the infinite sets A_2 and A_5 .

Now consider the case where a set A_{3x+8} follows some set E_d from some time on forever; note that $A_{3x+8} = E_d$. One has to show that A_{3x+8} differs from all other sets in the numbering. As E_d has at least two non-elements, A_{3x+8} differs from A_2 which is \mathbb{N} and from all sets which are of the form $\mathbb{N} - \{u\}$. If $A_{3x+8} \subseteq A_5 = \{0, 3, 6, 9, 12, \dots\}$ then the parameter v is selected such that $v \in A_5 - A_{3x+8}$ and furthermore $v \in A_{3y} - A_{3x+8}$ for all y with $A_{3x+8} \subseteq A_{3y}$. If there is a set A_{3y+1} with $A_{3x+8} \subseteq A_{3y+1}$ then this set is unique and $w \in A_{3y+1} - A_{3x+8}$. For all $y \neq x$ it holds that A_{3y+8} is different from A_{3x+8} by the following case distinction:

- If A_{3y+8} follows some finite X according to the case of priority $3d$ then the argument above shows that $A_{3x+8,s} \neq X$ for almost all stages s as otherwise A_{3y+8} would have been redirected to some set of the form $\mathbb{N} - \{u\}$.
- If there is an e such that A_{3y+8} follows E_e forever then $E_e \neq E_d$ as the given numbering E_0, E_1, E_2, \dots is one-one and hence $A_{3y+8} = E_e \neq E_d = A_{3x+8}$.
- If A_{3y+8} is eventually equal to some set $\mathbb{N} - \{u\}$ then $A_{3y+8} \neq A_{3x+8}$ as A_{3x+8} has at least two non-elements.

Furthermore, the protocols governing the process of making a set equal to $\mathbb{N} - \{u\}$ ensure that for each u there is at most one index $3x + 8$ such that $A_{3x+8} = \mathbb{N} - \{u\}$.

Third, one has to verify that every r.e. subset of \mathbb{N} occurs in the numbering. If X is finite and X differs from all sets $A_0, A_1, A_3, A_4, A_6, A_7, A_9, A_{10}, \dots$ and also from all sets E_e with $e < |X|$, then the case of priority $3d$ with $d = \sum_{c \in X} 2^c$ will eventually qualify and produce a set A_{3x+8} . Note that finitely many of these trials might be redirected to sets of the form $\mathbb{N} - \{u\}$ for some u as the set X is temporarily equal to some $E_{e,s}$ with $e < |X|$, but eventually the case with priority $3d$ will qualify and get a follower which will no longer be redirected.

Now consider any set E_d such that $|E_d| > d$ and E_d is different from all sets $A_0, A_1, A_3, A_4, A_6, A_7, A_9, A_{10}, \dots$ and from the sets A_2 and A_5 . If $E_d \subseteq \{0, 3, 6, 9, 12, \dots\}$ then v will for all

sufficiently large x take the least multiple of 3 such that $v \notin E_d \wedge v + 3 \in E_d$ else $E_{d,x} \not\subseteq \{0, 3, 6, 9, 12, \dots\}$ for all large enough x . If $E_d \subseteq A_{3y+1}$ for some y then w as defined in the description of priority $3d + 1$ will be the minimum of $A_{3y+1} - E_d$ for all sufficiently large stages x else $E_{d,x} \not\subseteq A_{3y+1}$ for any y for all sufficiently large x . Hence, if x is sufficiently large and E_d has just abandoned its follower then there will again be a new follower $A_{3x'+8}$ for some $x' \geq x$ and the parameters v, w will be chosen such that this follower will not be abandoned. Hence E_d will appear in the numbering.

For the sets of the form $\mathbb{N} - \{d\}$, the case with priority $3d + 2$ makes sure that these sets show up in the numbering eventually. Furthermore, $A_2 = \mathbb{N}$. Hence all sets with at most 1 nonelement occur in the numbering.

It remains to show that every B_k is recursive. The case $k = 2$ is a special case, here B_2 is a finite variant of $\{6\} \cup \{3y + 1 : g(3y + 1) \text{ is a leaf of some tree } T_e\} \cup \{3y + 8 : A_{3y+8} \text{ is initialized by case } 3d \text{ as an } X \text{ with 2 elements}\}$. For $k \neq 2$, B_k is a finite variant of $\{3k\} \cup \{3y + 8 : A_{3y+8} \text{ is initialized by case } 3d \text{ as an } X \text{ with } k \text{ elements}\}$. The verification is as in Theorems 2 and 4. ■

4 A least model

There is a Friedberg model which contains only those sets which are definable in every Friedberg model. In other words, this model has as few sets as possible. There is also a characterization on what sets exist in this least model.

Definition 14. Say a set X is *hereditarily k -bounded* iff it is well-founded and only finitely many sets in the transitive closure of X have more than k elements.

Theorem 15. *There exists a Friedberg model A_0, A_1, A_2, \dots such that for every set X — given up to isomorphism by the transitive structure of the elements of X and elements of elements and so on — the following conditions are equivalent:*

- X is isomorphic to some A_x in this model;
- X is hereditarily k -bounded for some k and has an isomorphic copy in some Friedberg model;
- X has an isomorphic copy in every Friedberg model.

Furthermore, in this model, the collection of all indices of proper classes is recursively enumerable.

Proof. The proof is similar to that of Theorem 9. First the least model is constructed and then the equivalence of the three statements above is shown. Let E_0, E_1, E_2, \dots be a Friedberg numbering of all r.e. subsets of \mathbb{N} which contain at least 2 nonelements, let $A_0 = \emptyset$ and $A_1 = \mathbb{N}$.

In the following construction, let U be the r.e. set of all indices x of sets A_x which are easily seen not to be well-founded in the sense that there are a function f and numbers n, m with $n > m$ such that $f(0) = x$, $f(k + 1) \in A_{f(k)}$ for all $k < n$ and $f(m) = f(n)$. Note that U is r.e. and that an enumeration of U is used when defining A_x for an odd x . Let U_s be the elements enumerated into U within s steps.

So let $A_1 = \mathbb{N}$ and for each $x > 1$, A_x is initialized according to the entry of highest priority (where high priorities correspond to low numerical values):

- With priority $3d$ try to make A_x to be the d -th finite set, provided that $x > d$ and currently differs from all $A_{y,x}$ with $y < x$.
- With priority $3d + 1$ let A_x follow E_d forever, provided that $|E_{d,x} \cap \{0, 1, 2, \dots, x - 1\}| > d$ and that E_d has no current follower.
- With priority $3d + 2$ let A_x follow $\mathbb{N} - \{d\}$ forever, provided that this set does not already have a follower.

After choosing the A_x according to the priority, one updates A_y for y with $1 < y < x$ provided that one of the following three cases applies:

- If A_y is initialized as the d -th finite set and there is some e with $E_{e,x}$ being equal to this set and $|E_{e,x}| > e$ then let A_y from now on follow $\mathbb{N} - \{u\}$ for the first u found such that u is not already in A_y and no other set is currently following $\mathbb{N} - \{u\}$.
- If there are numbers k, n, z and a function f such that $A_{y,x}$ has at least k elements, $z < x$, A_z follows E_k , A_y does not follow $E_0, E_1, E_2, \dots, E_k$, $A_{z,x} \cap U_x = \emptyset$, $0 < n < x$, $f(0) = z$, $f(n) = y$ and $f(m + 1) \in A_{f(m),x}$ for all $m < n$ then let A_y follow $\mathbb{N} - \{u\}$ for the first u found such that $u \notin A_{y,x}$ and $\mathbb{N} - \{u\}$ does not have currently a follower.
- If A_y is initialized as E_d for some d , there is a $z < x$ following $E_{d'}$ with $d' \geq d$, $A_{y,x} \cap U_x = \emptyset$ and there are a function f and $n > 0$ with $f(0) = y$, $f(n) = z$ and $f(m + 1) \in A_{f(m),x}$ for all $m < n$, then let A_y, A_z follow $\mathbb{N} - \{v\}$ and $\mathbb{N} - \{w\}$, respectively, where $\mathbb{N} - \{v\}, \mathbb{N} - \{w\}$ have not yet any followers, v is not yet enumerated into A_y , w is not yet enumerated into A_z and $y \neq z \Rightarrow v \neq w$.

As in the proof of Theorem 9, one can show the following:

- every r.e. W appears exactly once in the enumeration of the A_x ;
- every $B_k = \{x : |A_x| = k\}$ is recursive;
- U contains all x where A_x is not well-founded.

Now it is shown that every set in the model is hereditarily x -bounded for some x . Consider any E_k and assume by way of contradiction that E_k is a set and is not hereditarily k -bounded. Let x be the index of E_k , it will turn out that this x can also serve as the bound. Now let y be an index of a set in the transitive closure of E_k . In the case that $|A_y| > k$ and $A_y \notin \{E_0, E_1, E_2, \dots, E_k\}$, the second condition in the modifications of sets A_y to sets of the form $\mathbb{N} - \{u\}$ is activated unless E_k has already some other element which is enumerated into U . Hence E_k becomes a proper class and is no longer a set. Consider now any other set A_x which is not following any E_k . Every index which can be reached from x transitively without going through an index of a set E_d is below x . As the index of each set E_d is at least d , the set E_k is hereditarily x -bounded.

The arguments up to now give together that all well-founded A_x are hereditarily k -bounded for some k and that every class which is not well-founded has already an index in U . Hence every set which has an (isomorphic) copy in all Friedberg models is hereditarily k -bounded. Now it is shown by induction over the finitely many exceptions that every hereditarily k -bounded set has a representative in every Friedberg model.

Assume that X is a hereditarily k -bounded set and that X has a representation in some

Friedberg model $\tilde{A}_0, \tilde{A}_1, \tilde{A}_2, \dots$ with $\tilde{B}_k = \{x : |\tilde{A}_x| = k\}$. Let $\tilde{y}_0, \tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_n$ be the indices of sets with more than k elements occurring in the construction of X (with \tilde{y}_n being the index of X). One can without loss of generality assume that the \tilde{y}_m are ordered such that in the transitive closure downwards occur below \tilde{y}_m only \tilde{y}_h with $h < m$. Now one shows by induction that for each set $\tilde{A}_{\tilde{y}_m}$ there is a counterpart A_{y_m} in any given numbering A_0, A_1, A_2, \dots of a Friedberg model. Let $f(\tilde{y}_h) = y_h$ for $h < m$. Map the unique element of \tilde{B}_0 to the unique element of B_0 , that is, the index of the empty set to the index of the empty set. For every $\tilde{x} \in \tilde{B}_1 \cup \tilde{B}_2 \cup \dots \cup \tilde{B}_k$ find the cardinality $c = |\tilde{A}_{\tilde{x}}|$ and the c elements $\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_c$ of $\tilde{A}_{\tilde{x}}$. In the case that f is defined on these elements, search in B_c the unique index x such that A_x contains $f(\tilde{a}_1), f(\tilde{a}_2), \dots, f(\tilde{a}_c)$ and let $f(\tilde{x}) = x$. With this method one can define f uniformly on all elements of $\tilde{A}_{\tilde{y}_m}$ and gets that the set $\{f(\tilde{x}) : \tilde{x} \in \tilde{A}_{\tilde{y}_m}\}$ is recursively enumerable, hence it has an index y_m and one can define $f(\tilde{y}_m) = y_m$. By this induction one can get an isomorphic copy of X into the Friedberg model A_0, A_1, A_2, \dots and has that an isomorphic copy of X exists in the given Friedberg model. As this induction did not use any property which does not hold in some Friedberg model, one has that X has an isomorphic copy in every Friedberg model. ■

Remark 16. Analyzing the structure of the last paragraph of the proof, one can see that the isomorphism f can be built iff there is an effective representation of X which allows one to compute, for each index occurring in the transitive closure of the element relation downwards, the cardinality of the corresponding set. If this is impossible, then X does not have any isomorphic copy in any Friedberg model. An example would be the following set X : X contains an inductive list x_0, x_1, x_2, \dots of elements such that x_0 represents the empty set and x_{n+1} represents $\{x_n\}$. Furthermore, X contains also a list y_n such that y_n represents $\{x_n, x_{n+2}\}$ if $n \notin K$ and y_n represents $\{x_n, x_{n+1}, x_{n+2}\}$ if $n \in K$ where K is the halting problem. Then the cardinality cannot be computed although X is hereditarily 4-bounded. Hence not every r.e. structure on hereditarily bounded sets can be realized in a Friedberg model. But if one would not postulate that the B_k are recursive in a Friedberg model, then there would be a model without any infinite set.

Remark 17. Every Friedberg model contains infinite sets and for each two infinite sets X, Y in the model, also the Cartesian product $X \times Y = \{\{x, \{x, y\}\} : x \in X, y \in Y\}$ exists. Furthermore, every partial recursive function from X to Y exists in the model. Also, if \sqsubset is a recursive ordering on $X \times X$, then the set $\{\{x, \{x, y\}\} : x \sqsubset y\}$ exists in the Friedberg model.

Note that the Friedberg model constructed in Theorem 15 does not contain any transfinite ordinal α as such an α contains for every $k \in \mathbb{N}$ an index of the set representing k and is thus not hereditarily k -bounded. But it contains for every recursive ordinal a well-ordered set of exactly this order-type, the only difference is that this well-ordered set is not isomorphic to the structure of the relation induced by $x \in A_y$ as it is usually done in the von Neumann universe.

One could ask whether a model can be embedded properly into another model via a partial recursive function. The answer is negative.

Theorem 18. *If there is a partial-recursive one-one function f which maps every set in a Friedberg model A_0, A_1, A_2, \dots to an isomorphic copy in a Friedberg model $\tilde{A}_0, \tilde{A}_1, \tilde{A}_2, \dots$ then f is surjective and both models have (up to isomorphism) the same sets.*

Proof. Assume that $A_0, A_1, A_2, \dots, \tilde{A}_0, \tilde{A}_1, \tilde{A}_2, \dots$ and f are given as in the theorem. Let X be the collection of all indices of sets in the first model and let $Y = \{f(x) : x \in X\}$; note that every member of Y is an index of a set in the second model. Assume now by way of contradiction that there is a set \tilde{A}_z with $z \notin Y$. Let Z be the transitive closure of \tilde{A}_z . As \tilde{A}_z is well-founded, there is an $y \in Z$ with $y \notin Y$ and $\tilde{A}_y \subseteq Y$. Then $\{v : f(v) \in \tilde{A}_y\}$ is r.e. and hence has an index x . For each $w \in \tilde{A}_y$ there is exactly one v with $f(v) = w$; this v is in X and hence A_v is a set in the first model and $\tilde{A}_{f(v)}$ is a set in the second model. As every $v \in A_x$ is the index of a set, A_x is a set and so is $\tilde{A}_{f(x)} = \{f(v) : v \in A_x\}$. It follows that $y = f(x)$ and $y \in Y$ in contradiction to the choice of y . Hence z cannot exist and Y is the collection of indices of sets in the second model. The restriction of f to X is then an isomorphism from the sets in the first model to the sets in the second model. So the two models are equivalent. ■

Remark 19. In general one might ask which sets can be represented in some Friedberg model. No characterization has been found so far, but the following results above give some insight: Theorem 5 shows that the set V_ω of all hereditarily finite sets cannot be represented in any Friedberg model, although the system of canonical indices is a representation of all finite sets in a uniformly recursive way. Theorem 4 above shows that there is one Friedberg model in which every recursive ordinal exists.

While there is a least Friedberg model, one might ask whether there is also a greatest Friedberg model which contains all the sets which exist in some Friedberg model. The answer to that question is negative.

Theorem 20. *There is no Friedberg model which contains all sets which exist in some Friedberg model.*

This result will be the corollary of the Propositions 21 and 22 below. For this, one first gives an abstract definition of a set $E[F]$ where F is a K -recursive function which is approximable from below. Then Proposition 21 will show that for every Friedberg model there is such a function F for which $E[F]$ does not exist in the model while Proposition 22 will show that for every such function F there is a Friedberg model which contains the set $E[F]$.

The definition of $E[F]$ is the following. Let b_0, b_1, b_2, \dots be the inductively defined sets with $b_0 = \emptyset$ and $b_{n+1} = \{b_n\}$ for all n . Let $E[F]$ consist of all triples $\{b_n, b_m, D\}$ such that $n < m \leq F(n)$, $\{b_n, b_m\} \subseteq D \subseteq \{b_n, b_{n+1}, \dots, b_m\}$ and $|D| > n+1$. Note that in this construction b_n, b_m, D are all three different.

Proposition 21. *Given any Friedberg model, there is a K -recursive function F which is approximable from below such that the set $E[F]$ is not in the model.*

Proof. Let A_0, A_1, A_2, \dots be the given Friedberg model and $B_k = \{x : |A_x| = k\}$ for all k . Using the indices in the model in place of sets, let b_0, b_1, b_2, \dots be such that $A_{b_0} = \emptyset$ and $A_{b_{n+1}} = \{b_n\}$. This sequence is recursive. Now define the following function $G(n)$: given n , search using K until either (a) or (b) or (c) below holds:

- (a) there are an index e and $m \geq n$ such that $\{b_n, b_m\} \subseteq A_e \subseteq \{b_n, b_{n+1}, \dots, b_m\}$, $|A_e| \geq n + 2$ and the index of the triple $\{b_n, b_m, e\}$ is not in A_n ;
- (b) there is a triple $\{b_n, x, y\}$ with index in A_n such that $x \notin \{b_{n+1}, b_{n+2}, \dots\}$;
- (c) there are a $k > n$ and a y such that the triple $\{b_n, b_k, y\}$ has an index in A_n and $A_y \not\subseteq \{b_n, b_{n+1}, \dots, b_k\}$.

Note that the search terminates for each n as otherwise the collection of indices of finite subsets of $\{b_n, b_{n+1}, \dots\}$ would be r.e. in contradiction to the proof of Theorem 5: one could for each $m > n$ enumerate all the triples $\{b_n, b_m, e\}$ with indices in A_n giving the indices of finite subsets of $\{b_n, b_{n+1}, \dots\}$ with at least $n + 2$ elements plus, for each $k \leq n + 1$, those indices $e \in B_k$ where all k elements of A_e are in $\{b_n, b_{n+1}, \dots\}$. As this collection is not r.e., the search must terminate. Now, if it terminates by case (a) then let $G(n) = m$ for the m found in this case; if it terminates by cases (b) or (c), let $G(n) = 2n + 2$. As $G \leq_T K$, G can be approximated in the limit by a uniformly recursive sequence of functions G_s , now let $F(n) = \max\{G_s(n) : s \in \mathbb{N}\}$; as the sequence $G_s(n)$ converges for every n , this maximum $F(n)$ exists and $F \leq_T K$.

Assume now by way of contradiction that $E[F]$ is in the model. Then there is an n with $A_n = E[F]$. Hence $G(n)$ cannot be defined by cases (b) and (c) in the definition above. So there is $m \in \{n, n + 1, \dots, G(n)\}$ and an e such that $\{b_n, b_m\} \subseteq A_e \subseteq \{b_n, b_{n+1}, \dots, b_m\}$, $|A_e| \geq n$ and the index of $\{b_n, b_m, e\}$ is not in A_n . But as $F(n) \geq G(n)$, the index of $\{b_n, b_m, e\}$ should be in A_n , a contradiction. Hence $E[F]$ cannot be A_n and $E[F]$ is not in the model. ■

Proposition 22. *Let $F \leq_T K$ be approximable from below. Then there is a Friedberg model which contains $E[F]$.*

Proof. The construction of the Friedberg model is a variation on the construction in Theorems 2 and 4. For this, one first fixes that $b_n = 3n$, $A_0 = \emptyset$ and $A_{b_{n+1}} = \{3n\}$ for all n . Furthermore, as F is approximable from below, there is a recursive enumeration of all triples $\{b_n, b_m, D\}$ in $E[F]$; given the s -th member $\{b_n, b_m, D\}$ of this enumeration, let A_{3s+1} contain all numbers 3ℓ where $b_\ell \in D$. Note that the minimum $3n$ and maximum $3m$ of A_{3s+1} can be computed from s and the A_{3s+1} are uniformly recursive; note also that for every size k there are only finitely many A_{3s+1} which have cardinality k . Furthermore, let $A_2 = \mathbb{N}$. The sets A_{3x+5} are now constructed by a priority argument as in the theorems before. Again, as in Theorem 2, let E_0, E_1, E_2, \dots be a one-one numbering of all r.e. subsets of \mathbb{N} with at least 2 nonelements. In the following, $A_{y,s}$ denotes the elements enumerated into A_y within s steps and X is the d -th finite set, that is, X is the set with $\sum_{a \in X} 2^a = d$. For each x , assign the set A_{3x+5} according that of the following cases which applies and for which the priority is highest, that is, has the lowest numerical value.

- Priority $3d$: This case applies if $A_{y,x} \neq X$ for all $y < 3x + 5$ and $X \neq \{3y\}$ for all y where X is the d -th finite set. If this case is selected then A_{3x+5} follows the set X as long as $E_{e,s} \neq X$

for all $e < |X| - 1$ and no set A_{3s+1} equal to X has been found for any s . In the case that there is $e < |X| - 1$ and a stage $s > x$ with $E_{e,s} = X$ at stage s or that X turns out to be equal to some A_{3s+1} , A_{3x+5} starts to follow $\mathbb{N} - \{u\}$ forever where u is the least number such that $u \notin X$ and no other set is currently following $\mathbb{N} - \{u\}$.

- Priority $3d + 1$: This case applies if $|E_{d,x}| > d + 1$ and E_d does currently not have a follower. If this case is selected then A_{3x+5} starts to follow E_d . If for some $t > x$ and s the equality $E_{d,t} = A_{3s+1}$ holds, then A_{3x+5} stops to follow E_d and A_{3x+5} starts to follow $\mathbb{N} - \{u\}$ forever where u is the least number such that $u \notin X$ and no other set is currently following $\mathbb{N} - \{u\}$.
- Priority $3d + 2$: This case applies if no A_y with $1 < y < x$ is currently following $\mathbb{N} - \{d\}$. If this case is selected then A_x follows $\mathbb{N} - \{d\}$ forever.

Note that the priorities of different choices are different and therefore it is always clear which choice is taken to initialize A_x . The algorithm is maintaining an explicit list of which sets of the form $\mathbb{N} - \{u\}$ it has at what stage created so that none of these sets is created twice and in order to make sure by the last case that each of these sets is created. Each set E_d with at least $d + 2$ elements which is also different from all sets A_{3s+1} receives in the limit exactly one follower; note in this context that there are only finitely many sets A_{3s+1} with $\min(E_d) = \min(A_{3s+1})$, hence it happens only at finitely many stages t that a follower is abandoned because of $E_{d,t} = A_{3s+1}$. For every finite X it is made sure that exactly one of the following cases arises (in this order of priority): (a) $X = A_{3s}$, (b) $X = A_{3s+1}$, (c) X is obtained by some A_{3x+5} following E_d with $X = E_d \wedge |E_d| > d + 1$, (d) X is obtained by an A_{3x+5} directly following X with some priority $3d$. Note that in the cases (c) and (d), whenever (b) or (c) turns out to apply, the follower is abandoned and changed to $\mathbb{N} - \{u\}$. Hence the set X exists and has a unique index. Furthermore, one can see that for every $k \in \{1, 2, 3, \dots\}$, there are only finitely many sets of cardinality k in $E[F]$ and hence, there are only finitely many x such that A_x has initially the cardinality k and receives later more elements; so each B_k is a finite variant of $\{x : A_x \text{ has at the initialization } k \text{ elements}\}$ and each B_k is recursive. Hence the constructed model is a Friedberg model.

Besides this one has to show that $E[F]$ exists in the Friedberg model constructed. It is enough to show that $E[F]$ is a set and is defined by an r.e. collection S of indices. Given s , one can compute the minimum $3n$ and maximum $3m$ of A_{3s+1} and search in B_3 an index k such that $A_k = \{3n, 3s + 1, 3m\}$. This k is then enumerated into S ; hence S is recursively enumerable. Furthermore, S is a set in the constructed Friedberg model: Every A_{3n} is a set and each A_{3s+1} contains only indices of sets of the form A_{3n} and is hence a set as well. The A_k formed contains two indices of the form $3n$ and $3m$ plus one index of the form $3s + 1$ and is a set again. So S contains only indices of sets and is the set in the Friedberg model again. ■

This result shows only that there is no greatest Friedberg model; it is open whether there is a maximal one, that is, whether there is a Friedberg model such that no other Friedberg model contains strictly more sets.

5 The theory of Friedberg models

The *pure theory of a Friedberg model* is given as follows:

- variables x, y, z, \dots ranging over indices of sets (not indices of classes);
- constants denoting some fixed sets;
- a predicate $x \in A_y$ to establish the element relation between x and y .

The theory under consideration is then the first-order theory based on these formulas and it is permitted to use parameters which are indices of certain sets. The *combined theory of a Friedberg model* considers both sets and classes. It is given as follows:

- variables x, y, z, \dots ranging over indices of classes (including sets);
- constants denoting some fixed sets;
- a predicate $x \in A_y$ to establish the element relation between x and y ;
- a predicate $Set(x)$ which tells whether x is an index of a set.

Again the theory under consideration is then the first-order theory based on these formulas and it is permitted to use indices of fixed classes as parameters.

Remark 23. There is a fixed embedding of the pure theory of Friedberg models into the combined theory of Friedberg models. The reason is that every formula quantifying over indices of sets can be transferred into a formula quantifying over indices of classes where one considers then only those indices satisfying the predicate Set . So the formula

$$\exists x \forall y \in A_x \forall z \in A_y [z \in A_x]$$

stating the existence of a transitive set would have to be translated into

$$\exists x \forall y \in A_x \forall z \in A_y [Set(x) \wedge (Set(y) \wedge Set(z) \Rightarrow z \in A_x)]$$

or, using that elements of sets are sets, one could also consider

$$\exists x \forall y \in A_x \forall z \in A_y [Set(x) \wedge z \in A_x].$$

Note that the constants or parameters might not exist in all Friedberg models, for example ω exists in some but not all models. As ω is first-order definable, this shows that the theory depends on the chosen Friedberg model.

Furthermore, there are also objects which can be defined but do not exist in any Friedberg model. The easiest example is the universe of all sets, so there is no x with $\forall y [y \in A_x]$ in the pure theory and no x with

$$\forall y [Set(y) \Rightarrow y \in A_x]$$

in the combined theory. There are also related objects which are definable but do not exist. An example is, in the pure theory of Friedberg models, the restriction \tilde{B}_k of B_k to indices of sets which is defined as follows:

$$x \in \tilde{B}_k \Leftrightarrow \exists \text{ distinct } y_1, y_2, \dots, y_k \forall z [z \in A_x \Leftrightarrow z = y_1 \vee z = y_2 \vee \dots \vee z = y_k].$$

Furthermore, one can use parameters of sets X, Y such that $X - Y$ does not exist (although it is definable). This just uses that r.e. subsets of \mathbb{N} are not closed under complementation.

Furthermore, many sets are definable although one cannot compute their index in a Friedberg model from the parameters, for example, unions, intersections, transitive closure, power set (if it exists) and so on.

Example 24. *In the pure theory of Friedberg models, there is an infinite definable set. Furthermore, the transitive closure of any set is definable.*

Proof. Let b_0 be the index of the empty set and b_{n+1} be the index of $\{b_n\}$. Then the set $X = \{b_0, b_1, b_2, \dots\}$ is r.e. as one can, using B_1 , enumerate b_0, b_1, b_2, \dots and have that the range of this enumeration is an r.e. set and hence has to exist in the given Friedberg model. It follows that X can be defined as a set satisfying

- $\exists b_0 \in X \forall y [y \notin A_{b_0}]$;
- $\forall b_n \in X \exists b_{n+1} \in X \forall y [y \in A_{b_{n+1}} \Leftrightarrow y = b_n]$;
- $\forall z [A_z \subset X \Rightarrow A_z \text{ violates one of the two conditions above}]$.

Given a set Y and an index y for Y , one can enumerate, starting with y , all elements of A_y , all elements of elements of A_y and so on. Hence one gets a set Z . Z satisfies the following conditions:

- $y \in Z$;
- $\forall z \in Z \forall x \in A_z [x \in Z]$;
- $\forall z [A_z \subset Z \Rightarrow A_z \text{ violates one of the two conditions above}]$.

Recall that $A_v \subset A_w$ just means that $\forall u \in A_v [u \in A_w]$ and $\exists u \in A_w [u \notin A_v]$ hold. The verification that these conditions properly define X and Z is straightforward. ■

Theorem 25. *In every Friedberg model, the combined theory of the model can be reduced to the pure theory of the model. In particular, the Friedberg model contains an inner model with the predicate Set of the combined theory.*

Proof. Let $X = \{b_0, b_1, b_2, \dots\}$ be defined as in Example 24 and let $A = \{(b_n, b_m) : n \in A_m\}$ with (b_n, b_m) written as an abbreviation for $\{b_n, \{b_n, b_m\}\}$. Then A is an r.e. set and has therefore an index, so it exists in the given Friedberg model. Hence X and A define an inner model as every class A_x is represented by the set $\{b_y : y \in A_x\}$ which is equal to $\{b_y : (b_y, b_x) \in A\}$.

Now it is shown that this inner model also can be equipped with a predicate *Set* defining the sets. In other words, there is a formula Φ which is true iff b_x represents an A_x which is a set. Now $\Phi(b_x) \Leftrightarrow b_x \in X \wedge \exists u, v, w$ such that

- A_v is the transitive closure of A_u ;
- A_w is the graph of a function f with domain A_v ;
- $f(u) = b_x$ and f maps A_v to a subset of X ;
- for all $y, z \in A_v$, $y \in A_z$ iff $(f(y), f(z)) \in A$;
- for all $b_y, b_z \in X$ with b_z in the range of f and $(b_y, b_z) \in A$ it holds that b_y is also in the range of f .

What $\Phi(b_x)$ mainly does is that it verifies that $b_x \in X$ (as already indicated by using the notion b_x instead of just a letter) and that A_x is isomorphic to A_u . As one quantifies only over indices of sets, u would not come up if A_u was a proper class. Hence $\Phi(b_x)$ brings the information that A_x is a set over to the inner model. One has to use the transitive closure A_v instead of A_u in order to get hold of the elements, elements of elements and so on. Then f is an isomorphism

from A_v with the ordering given by $y \in A_z$ to the fragment of X which is the downward-closure along the relation coded by A from b_x ; here $f(u) = b_x$ in order to make the connection between these two sets. Note that f exists whenever A_v is r.e. as f takes the domain A_v and is defined by mapping every $y \in A_v$ to b_y ; thus the index w exists whenever A_u is isomorphic to the set coded by b_x , in other words, whenever $A_u = A_x$ and $u = x$. But as the index x of b_x is not given, u, v, w have to be existentially quantified in order to get hold of these three indices. ■

Due to this equivalence, the Turing degrees of the pure theory and the combined theory of a Friedberg model are the same. So one can just talk about the complexity of the theory of a Friedberg model without specifying whether the pure theory or combined theory is meant. The next result determines the minimum complexity of the theory of a Friedberg model.

Theorem 26. *The theory of a Friedberg model is at least as complicated as K^ω which is $K \oplus K' \oplus K'' \oplus \dots$; for some Friedberg models, the theory has the same many-one degree.*

Proof. As above, let $X = \{b_0, b_1, b_2, \dots\}$ with $A_{b_0} = \emptyset$ and $A_{b_{n+1}} = \{b_n\}$ for all n . Furthermore, let $Y = \{((b_i, b_j), b_k) : k = \langle i, j \rangle\}$ where $\langle i, j \rangle = \frac{(i+j)(i+j+1)}{2} + i$ is Cantor's pairing function. Furthermore, let $Z = \{b_{\langle x, e \rangle} : x \in W_e\}$. The sets X, Y, Z exist in every Friedberg model and can be used as parameters. Now let Φ be any arithmetic formula in the natural numbers, say

$$\Phi(v, w) = \exists x \forall y \exists z [\langle v, w, x, y, z \rangle \in W_e]$$

where e is a constant identifying the set W_e and tuples of more than two components are defined inductively starting with $\langle i, j, k \rangle = \langle \langle i, j \rangle, k \rangle$. Now one can translate Φ into a formula in the theory of the Friedberg model with two inputs b_v, b_w in place of v, w and with b_e used to code the parameter e :

$$\begin{aligned} \Phi(b_v, b_w) \Leftrightarrow & b_v, b_w \in X \text{ and } \exists b_x \in X \forall b_y \in X \exists b_z \in X \forall b_m, b_n, b_o, b_p, b_q \in X \\ & (((b_v, b_w), b_m) \in Y \wedge ((b_m, b_x), b_n) \in Y \wedge ((b_n, b_y), b_o) \in Y \wedge \\ & ((b_o, b_z), b_p) \in Y \wedge ((b_p, b_e), b_q) \in Y \Rightarrow b_q \in Z]. \end{aligned}$$

For each b_v, b_w, b_x, b_y, b_z there is exactly one sequence b_m, b_n, b_o, b_p, b_q for which the formula $((b_v, b_w), b_m) \in Y \wedge ((b_m, b_x), b_n) \in Y \wedge ((b_n, b_y), b_o) \in Y \wedge ((b_o, b_z), b_p) \in Y \wedge ((b_p, b_e), b_q) \in Y$ holds. The so defined q satisfies $b_q = b_{\langle v, w, x, y, z, e \rangle}$. Now $\langle v, w, x, y, z \rangle \in W_e$ iff $b_q \in Z$. Hence this example shows how to reduce every arithmetic formula into a formula in the theory of the given Friedberg model; thus K^ω is many-one reducible to this theory.

Furthermore, there is some Friedberg model where the theory is many-one reducible to K^ω . To see this, consider the Friedberg model constructed in Theorem 9. In this model, the predicate $Set(x)$ which says that A_x is a set is Π_1^0 , hence one can write $x \notin A_u$ for a fixed index u instead of $Set(x)$. This permits to transform any given formula into a quantified formula which has Boolean connectives over atomic parts of the form $x \in A_y$ and $x \notin A_y$. Such formulas are exactly those which occur in the arithmetic hierarchy and their truth-values can be established by querying K^ω at the correct place. This reduction from a formula to K^ω can be realized by a many-one reduction. ■

Remark 27. Note that it follows from Theorem 4 in combination with Proposition 11 that there is a Friedberg model where the first-order theory is not hyperarithmetical. Furthermore, Theorem 13 gives a model where the theory is Π_1^1 -hard. So see this, first recall the definitions of b_0, b_1, b_2, \dots from the proof of Example 24 and r_0, r_1, r_2, \dots from the proof of Theorem 13 and then note that the class $\{(b_0, r_0), (b_1, r_1), (b_2, r_2), \dots\}$ exists and can be used as a parameter in order to get the mapping $b_n \mapsto \text{Set}(r_n)$ which realizes a Π_1^1 -complete predicate in the combined theory. This predicate can then be used to show that the theory of this model is actually many-one equivalent to the ω -jump of the Π_1^1 -complete problem $\{n : \text{Set}(r_n)\}$. Note that this is the highest possible complexity which the theory of a Friedberg model can have; the predicate Set itself is always Π_1^1 or less as $\text{Set}(x)$ is equivalent to $\forall f \exists n [f(0) \neq x \vee f(n+1) \notin A_{f(n)}]$.

6 The power set axiom

The power set axiom is weaker than in usual set theory as one is only interested in the r.e. subsets of a set. So the question is whether for given A_x the set $A_z = \{y : A_y \subseteq A_x\}$ exists. The next remark shows that this is true only for few models, but Theorem 29 shows that such models nevertheless exist.

Remark 28. The power set axiom is more restrictive than other axioms. It in particular has the following consequence: if every set has a power set then there is no set consisting of infinitely many indices of pairwise disjoint infinite sets. Otherwise, if $\{x_0, x_1, x_2, \dots\}$ would be an r.e. set of indices of pairwise disjoint r.e. sets, then one could look at the set

$$A_y = \{z : \exists e, u [z \in A_{x_e} \wedge z < |W_{e,u}|]\}$$

and would have that x_e is in the power set of A_y iff W_e is infinite, a contradiction to the fact that the power set is assumed to be recursively enumerable. As many Friedberg models have infinite sets of indices of pairwise disjoint infinite sets, these models do not satisfy the power set axiom.

Theorem 29. *There is a Friedberg model A_0, A_1, A_2, \dots such that for every x there is a z with $A_z = \{y : A_y \subseteq A_x\}$; that is, every set has a power set in this model.*

Proof. Let E_0, E_1, E_2, \dots be a Friedberg numbering containing all r.e. subsets of \mathbb{N} . Furthermore, let f be a recursive one-one mapping from all triples (d, σ, t) to the odd numbers where σ itself is a mapping from $\{0, 1, 2, \dots, d-1\}$ to $\{0, 1\}$ and $f(d, \sigma, t+1) > f(d, \sigma, t)$ for all d, σ, t . The idea is to create indices for two types of sets:

- Following a set E_d with a parameter σ which is a finite function from $\{0, 1, 2, \dots, d-1\}$ to $\{0, 1\}$:
 1. let $s = 0$;
 2. wait until $f(d, \sigma, s)$ elements have been enumerated into E_d and let Q_s be the set of these elements;
 3. wait until the elements of Q_s have been enumerated into all sets $E_{d'}$ with $d' < d \wedge \sigma(d') = 1$;

4. choose a not yet used index x , initialize A_x as Q_s and let $t = s + 1$;
 5. wait until $f(d, \sigma, t)$ elements have been enumerated into E_d and let Q_t be the set of these elements;
 6. wait until the elements of Q_t have been enumerated into all sets $E_{d'}$ with $d' < d \wedge \sigma(d') = 1$;
 7. update A_x to Q_t ;
 8. if there is no $E_{d'}$ with $d' < d \wedge \sigma(d') = 0$ such that all elements of Q_s are already enumerated into $E_{d'}$ then let $t = t + 1$ and go to step 5;
 9. freeze the set A_x and do no longer consider x to be an index of a follower of E_d ; let $s = t + 1$ and go to step 2.
- Candidates for a finite set X ; here s is a stage counter which always goes to infinity; at every stage s only finitely many sets are initialized. There is also a variable x which is undefined before stage 0. Now do the following for every stage s :
1. if x is undefined and there is no $y < s$ with $A_{y,s} = X$ then define x as the least index of a set which is not yet initialized and let $A_{x,s} = X$;
 2. determine for each set $Z \subseteq \{0, 1, 2, \dots, s + 2|X|\}$ the state

$$st_{X,Z,s} = \sum_{d \leq s: X \cup Z \subseteq E_{d,s}} 2^{-d};$$

3. among all those $Z \subseteq \{0, 1, 2, \dots, s + 2|X|\}$ with $|Z| = X$ and $Z \cap X = \emptyset$, let Y_s be that set Z for which $st_{X,Z,s}$ is as large as possible;
4. if x is defined then test whether there is a $y \neq x$ such that A_y is already initialized and currently, $A_{y,s} = X$;
5. if so then update $A_{x,s+1} = X \cup Y_s$ and freeze A_x and make x undefined.

First, it is shown that all infinite sets occur exactly once in the numbering. So consider any d and any function σ from $\{0, 1, 2, \dots, d - 1\}$ to $\{0, 1\}$. Now consider the following cases:

- The algorithm for (d, σ) remains waiting in step 2 or step 5 forever; then the algorithm might have created finitely many sets A_x but all but at most one of them have eventually been frozen and there is at most one further set A_x which remains as Q_t for some t forever. This happens only if E_d is finite.
- The algorithm for (d, σ) remains waiting in step 3 or step 6 forever; then the algorithm is waiting for some elements of E_d to show up in some set $E_{d'}$ with $d' < d \wedge \sigma(d') = 1$ but they do not show up there, hence $E_d \not\subseteq E_{d'}$ although σ says so.
- The algorithm for (d, σ) goes through steps 2 to 7 and step 9 infinitely often; then there is a $d' < d$ with $\sigma(d') = 0$ and $E_d \subseteq E_{d'}$ although σ said that there is a non-inclusion, hence σ is again false.
- The algorithm goes through steps 5 to 8 infinitely often but through all other steps only finitely often. Then E_d is infinite and $\sigma(d') = 1$ iff $E_d \subseteq E_{d'}$ for all $d' < d$; furthermore, the last x which is assigned to be a follower is the index of a set A_x with $A_x = E_d$.

Note that all sets enumerated by candidates for a finite set X are finite; hence the last set A_x produced by the last entry above is the only infinite set generated in this case-distinction and

it is equal to E_d ; furthermore, the unique σ which predicts the inclusions $E_d \subseteq E_{d'}$ correctly for all $d' < d$ produces also eventually the index x with $A_x = E_d$ but no other σ does. Hence every infinite set E_d has exactly one index and that one is given by the last x which is allocated for the correct σ when searching the candidate with (σ, d) .

Second, it is shown that every finite X is produced exactly once. Note that for each odd u there is exactly one triple (d, σ, t) such that $f(d, \sigma, t) = u$ and only when the algorithm for (d, σ) reaches the state t a candidate of the first type with u elements can be generated which might later be withdrawn in the case that neither the enumeration algorithm for this candidate becomes stuck nor the candidate is frozen with this number of indices. Only if some candidate for E_d enumerates a set of u elements, there might be a collision for sets having $u \cdot 2^r$ elements as step 4 the set A_x traced by the algorithm can be updated from X to $X \cup Y_s$ and — in a chain-reaction — the same might happen for sets $A_{x'}$ of $2|X|$ and $A_{x''}$ of $4|X|$ elements and so on. Each such chain-reaction is triggered originally by a candidate of the first type getting u elements and this happens at most once; hence there is for each odd u only one stage where sets of cardinality $u \cdot 2^r$ are updated and hence it is enough to provide one candidate set Y_s for extending and there is no need to consider other ones. The test in step 1 of each stage s makes sure that the algorithm always provides an index x with $A_x = X$ whenever currently no other set has the range X . Furthermore, step 4 explicitly resolves collisions and hence there is for each finite set X exactly one index x such that A_x is eventually equal to X .

Third, for given A_x , it is shown that $\{y : A_y \subseteq A_x\}$ is recursively enumerable and hence equal to some A_z . If A_x is finite, then the existence of A_z is clear. So assume that A_x is infinite. Now it is necessary to analyze the conditions by which the Y_s are selected a bit more closely. Let d be the unique index with $A_x = E_d$. Let $g(n)$ be the first stage $s > d$ such that for every $C \subseteq \{0, 1, 2, \dots, d\}$ the set $\bigcap_{c \in C} E_{c,s}$ has at least $\min\{2n, \bigcap_{c \in C} E_c\}$ elements below s . Note that this function g is recursive as there are only finitely many sets C and for each of them one knows whether one can find n elements or only $|\bigcap_{c \in C} E_c|$ many. Let R be the union of all finite sets of the form $\bigcap_{c \in C} E_c$ with $C \subseteq \{0, 1, 2, \dots, d\}$. Now one has that $\{y : A_y \subseteq A_x\}$ consists of the following indices:

1. All indices y where A_y was initialized by a process belonging to some (e, σ) with $e > d$ and $\sigma(d) = 1$.
2. All indices y where A_y was initialized by a process belonging to some (e, σ) and later A_y became frozen as a finite set Q_t and $Q_t \subseteq E_d$.
3. All indices y where A_y was initialized as X with $X \not\subseteq R$, A_y is updated to $X \cup Y_s$ for some $s \leq g(2|X|)$ and $X \cup Y_s \subseteq E_d$.
4. All indices y where A_y was initialized as a set X with $X \not\subseteq R$, $A_{y,s} = X$ at the stage $s = g(2|X|)$ and $X \subseteq E_d$.
5. All indices y where $A_y \subseteq E_d$ and A_y was either initialized by a process belonging to (e, σ) with $e \leq d$ without ever reaching the state of being frozen or A_y was initialized as some set $X \subseteq R$.

Note that the set in this construction is recursively enumerable as the fifth case deals only with finitely many y and the other four cases directly tell how the enumeration procedure works. But

this construction does not provide an effective way to get the enumeration procedure, it only says that this procedure exists.

Now it is shown that the y enumerated are exactly those with $A_y \subseteq A_x$. If y is an index of a candidate of the first type generated by some process belonging to (e, σ) with $\sigma(d) \downarrow = 1$ then the construction guarantees that $A_y \subseteq E_d$; furthermore, y is enumerated. If y is an index of a candidate of the first type generated by some process belonging to (e, σ) with $\sigma(d) \downarrow = 0$ then the construction guarantees that $A_y \subseteq E_d$ only if A_y is eventually frozen; indeed y is enumerated according to the second case iff A_y is eventually frozen and all of its elements belong to A_x , so again the enumeration procedure is correct.

If y is an index of a candidate of the first type generated by some process belonging to (e, σ) with $e \leq d$ and A_y is eventually frozen then again y is enumerated iff $A_y \subseteq A_x$. If y is an index of a candidate of the first type generated by some process belonging to (e, σ) with $e \leq d$ and A_y is never frozen then the fifth case applies and again y is enumerated iff $A_y \subseteq A_x$.

If y is an index belonging to a candidate of the second type and initialized as X with $X \subseteq R$ then again the fifth case applies and y is enumerated iff $A_y \subseteq A_x$. If y is an index belonging to a candidate of the second type and initialized as X with $X \not\subseteq R$ and $A_{y,g(2|X|)} = X \cup Y_s$ for some $s \leq g(2|X|)$ then the third case applies and y is enumerated iff $A_y \subseteq A_x$. If y is an index belonging to a candidate of the second type and initialized as X with $X \not\subseteq R$ and $A_y = X$ (that is, A_y is not later modified), then the fourth case of the enumeration procedure applies and y is enumerated iff A_y is a subset of A_x .

The remaining case is that y belongs to a candidate of the second type and is initialized as X such that $X \not\subseteq R$ and $A_{y,g(2|X|)} = X$ but there is some $s \geq g(2|X|)$ with $A_y = X \cup Y_s$. Let $C = \{c \leq d : X \subseteq E_{c,s}\}$. As $X \not\subseteq R$ it follows that $\bigcap_{c \in C} E_c$ is infinite. Hence there are, below s , at least $2|X|$ numbers enumerated into $\bigcap_{c \in C} E_{c,s}$ by stage s . Hence one can choose Y_s such that also $X \cup Y_s$ is a subset of $\bigcap_{c \in C} E_{c,s}$ and such an Y_s is also taken as otherwise the state $st_{X,Y_s,s}$ would not be as large as it could be. Thus, if $X \subseteq E_{d,s}$ then also $Y_s \subseteq E_{d,s}$ and hence the index y is taken into the constructed r.e. set iff $A_y \subseteq E_d$. This completes the verification that exactly the y with $\{y : A_y \subseteq E_d\}$ are enumerated by the above enumeration procedure and hence there is a z such that $A_z = \{y : A_y \subseteq A_x\}$.

Fourth, it is noted that all $B_k = \{x : |A_x| = k\}$ are recursive. This stems from the fact that for all k , there is at most one set starting as a candidate of the first type with exactly k elements and that among those sets starting as candidates of the second type, there is at most one set with k elements which was initialized with $k/2$ elements and had later $k/2$ elements enumerated into it and that among all the sets initialized with k elements, at most one receives later k further elements. Hence B_k is a finite variant of $\{x : A_x \text{ has exactly } k \text{ elements when initialized}\}$. Therefore, the constructed model is a Friedberg model. ■

So one can show that the power set axiom can be satisfied, when formulated in a way which is adequate for Friedberg models. An interesting question would be to determine the minimum complexity of finding for given x the index z with $A_z = \{y : A_y \subseteq A_x\}$. Clearly z cannot be found in a recursive way as otherwise V_ω would be representable in some Friedberg model.

Furthermore, the mapping is certainly K' -recursive. But it would be interesting to know whether it can be made K -recursive.

7 Conclusion

This paper is dedicated to the investigation of Friedberg models of fragments of set theory where a Friedberg model is given by a Friedberg numbering A_0, A_1, A_2, \dots of all r.e. subsets of \mathbb{N} with the additional constraint that every $B_k = \{x : |A_x| = k\}$ is recursive.

It is investigated to which extent the axioms of Zermelo and Fraenkel hold when defining the element-relation in this universe as x is in y iff $x \in A_y$. For this, one has to cut out the true sets as those which are well-founded with respect to the element relation. Full comprehension cannot be satisfied as the r.e. subsets of \mathbb{N} are not closed under complement, set difference and universal quantification. Furthermore, the power set axiom holds only in some Friedberg models and there also in the following adjusted form: for all x there is a z with $A_z = \{y : A_y \subseteq A_x\}$. The other axioms of Zermelo and Fraenkel can be carried over and choice can be implemented easily.

One major question is how comprehensive the models are, that is, how many sets from the von Neumann universe are represented in a Friedberg model. The answer depends on the choice of the model. It has been shown that there is a least Friedberg model which contains exactly those sets from the von Neumann universe which are in all Friedberg models. But there is no greatest Friedberg model. It has been shown that for every Friedberg model there is some set $E[F]$, as defined after Theorem 20 with a suitable K -recursive parameter function F , which does not exist in this model although it exists in some other Friedberg model. It remains open whether there is a maximal Friedberg model such that no other Friedberg model contains strictly more sets from the von Neumann universe than this model. Furthermore, there are some concrete examples of sets which exist or do not exist. So the set $X = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}, \dots\}$ exists in all Friedberg models. The set V_ω of all hereditarily finite sets does not exist in any Friedberg model. Using the ordering of Harrison [3], one can show that there is a Friedberg model containing all recursive ordinals while the least Friedberg model contains only the finite ordinals. Another open problem is to find a characterization of those sets in the von Neumann universe which exist in some Friedberg model.

The complexity of the predicate $Set(x)$ stating that “ A_x is a set in the model” (and not a proper class) can range from Π_1^0 up to Π_1^1 ; the complexity of the theory of a Friedberg model depends heavily on the complexity of the predicate and ranges from K^ω to the ω -jump of a Π_1^1 -complete set.

One might ask how natural the decisions on the choice of Friedberg models is. While the decision to take a Friedberg numbering arises quite naturally from the axiom of extensionality, the question on how effective the B_k should be is more tricky. Certainly one cannot postulate that the B_k are uniformly recursive as then V_ω would exist, what has been shown to be impossible. Furthermore, B_∞ , that is, $\{x : |A_x| = \infty\}$, cannot be recursively enumerable as otherwise there would be a recursive enumeration of all infinite subsets of \mathbb{N} which does not exist. On the other

hand, if one drops all effectiveness constraints on the B_k , one could take a Friedberg numbering in which the set of the indices x with A_x being of the form $\mathbb{N} - \{u\}$ form a simple subset of \mathbb{N} ; as these A_x are all proper classes, it would follow that no set in the model is infinite. Hence the model would just contain the sets in V_ω and nothing else. That is of course undesirable. So, the recursiveness of B_1 and B_2 is quite useful to guarantee the existence of infinite sets as well as functions with infinite domain; hence it is natural to postulate that the B_k are all recursive.

Since Rabin [8] has shown that there is no r.e. model of ZFC, the present work had to consider models of fragments of ZFC, which clearly have shortcomings. Nevertheless, the structures investigated in this paper are quite rich and the study is interesting, so the authors hope that this work still provides some motivation to study such r.e. Friedberg models based on these initial results.

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