International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems
Vol. 0, No. 0 (1997) 000—000

(Leave $1\frac{1}{2}$ inch blank space for Publisher)

Possibilistic Residuated Implication Logics with Applications*

Churn-Jung Liau
Institute of Information Science
Academia Sinica, Taipei, Taiwan
Fax: (+)886-2-7824814
E-mail: liaucj@iis.sinica.edu.tw

Received (received date)
Revised (revised date)

In this paper, we will develop a class of logics for reasoning about qualitative and quantitative uncertainty. The semantics of the logics is uniformly based on possibility theory. Each logic in the class is parameterized by a t-norm operation on [0,1], and we express the degree of implication between the possibilities of two formulas explicitly by using residuated implication with respect to the t-norm. The logics are then shown to be applicable to possibilistic reasoning, approximate reasoning, and nonmonotonic reasoning.

Keywords: Possibility theory, t-norms, possibilistic reasoning, similarity-based reasoning, default reasoning.

1. Introduction

Knowledge representation and reasoning is fundamental to knowledge based systems. Due to the imprecision and incompleteness of acquired knowledge, uncertain reasoning is a key issue in knowledge representation. To accommodate different types of incomplete knowledge, many uncertainty reasoning methods have been proposed and extensively studied. Most methods focus exclusively on the quantitative or the qualitative aspects of uncertainty. The former includes probabilistic methods[2], Dempster-Shafer theory[3], possibilistic and fuzzy logics[4,5,6], etc., while the latter includes rough set theory[7], and various nonmonotonic reasoning mechanisms[8,9]. Though focusing on a particular aspect of uncertain information is

*This is a revised version of Ref. 1 .

useful in the development of efficient reasoning algorithms, it is sometimes necessary to manage both types of uncertainty simultaneously. For example, we may want to summarize the numerical uncertain information in some qualitative words. On the other hand, it may be desirable to represent degrees of strength of defaults in nonmonotonic reasoning as some quantitative uncertainty.

In [10], a logic based on conditional possibility(LCP) has been proposed to integrate some of the above-mentioned logics. However, since the semantics of LCP is based on the Dempster's conditioning rule[3], it does not take into account some t-norm-based reasoning methods. For example, in [10], it is shown that LCP can model similarity-based consequence relation only when the similarity relation is sup-min transitive.

In this paper, we will develop a class of alternative logics for reasoning about qualitative and quantitative uncertainty. The semantics of the logics is uniformly based on possibility theory. Each logic in the class is parameterized by a t-norm operation on [0,1], and we express the degree of implication between the possibilities of two formulas explicitly by using residuated implication with respect to the t-norm. The logics are then shown to be applicable to possibilistic reasoning, approximate reasoning, and default reasoning.

2. Possibilistic Residuated Implication Logics

In this section, we will introduce the possibilistic residuated implication logics (PRIL). The syntax of the logics is an extension of the classical propositional logic. The alphabet of the logics consists of a set of propositional symbols $PV = \{p, q, r, \ldots\}$, the logical constants $\top$ (*verum* or *truth* constant) and $\bot$ (*falsum* or *false* constant), the classical connectives $\neg$ (negation) and $\vee$ (or), and two classes of residuated implication operators, $\stackrel{c}{\Longrightarrow}$ and $\stackrel{c^+}{\Longrightarrow}$ for all $c \in [0,1]$. the formation rules of well-formed formulas(wffs) for PRIL is as follows:

- All propositional variables and propositional constants are wffs, also called atomic formulas.
- If f and g are wffs and $c \in [0,1]$, so are $\neg f$, $f \vee g$, $f \stackrel{c}{\Longrightarrow} g$, and $f \stackrel{c^+}{\Longrightarrow} g$.
- Nothing except those determined by above are wffs.

The usual abbreviations for classical logics are used, i.e., $f \wedge g = \neg(\neg f \vee \neg g)$, $f \supset g = \neg f \vee g$, and $(f \equiv g) = (f \supset g) \wedge (g \supset f)$. We will use $\mathcal{L}$ to denote the subset of all classical wffs.

The semantics of PRIL is based on possibility theory, so we review the theory first. The t-norm (triangular norm) operations have been widely used in fuzzy set theory to define generalized intersection between fuzzy sets. Here, we use the notion of residuated implication with respect to a t-norm to define the degree of implication between the possibilities of two formulas. A binary operation $\otimes : [0,1] \times [0,1] \to [0,1]$ is a t-norm iff it is associative, commutative, and increasing in both places, and $1 \otimes a = a$ and $0 \otimes a = 0$ for all $a \in [0,1]$. The residuated implication w.r.t. $\otimes$, denoted by $\otimes\!\!\rightarrow : [0,1] \times [0,1] \rightarrow [0,1]$, is defined as

$$a \otimes\!\!\rightarrow b = \sup\{x : a \otimes x \leq b\}.$$

According to possibility theory[11], a *possibility distribution* on a universe W is a function $\pi : W \rightarrow [0,1]$. A possibility distribution π is said to be normalized if $\sup_{w \in W} \pi(w) = 1$. Obviously, π is a characteristic function of a fuzzy subset of W. Two measures on W can be derived from π. They are called possibility and necessity measures and denoted by Π and N respectively. Formally, $\Pi, N : 2^W \rightarrow [0,1]$ are defined as

$$\Pi(A) = \sup_{w \in A} \pi(w),$$

$$N(A) = 1 \perp \Pi(\overline{A}),$$

where $\overline{A}$ is the complement of A with respect to W.

A possible world model for PRIL w.r.t. $\otimes$ is then a triplet $M = \langle W, R, V \rangle$, where W is a set of possible worlds, $R \subseteq W \times W$ is a binary fuzzy relation called *accessibility relation* on W, and $V : PV \rightarrow 2^W$ assigns to each propositional symbol in PV a subset of W. Sometimes, we will also abuse the symbol V to denote a mapping from W to the set of classical interpretations such that for each $w \in W$, $V(w) : PV \rightarrow \{0,1\}$ satisfies $V(w)(p) = 1$ iff $w \in V(p)$. For each $w \in W$, a possibility distribution π_w can be defined as $\pi_w(u) = R(w,u)$ for all $u \in W$. Let Π_w denote the possibility measure corresponding to π_w for each $w \in W$.

Given a model $M = \langle W, R, V \rangle$, we can define the truth relation as follows. For each world w and wff f, let $|f| = \{w \in W \mid w \models_M f\}$ and $\Pi_w(f) = \Pi_w(|f|)$, and define

(1) $w \models_M p \Leftrightarrow w \in V(p), \forall p \in PV$,

(2) $w \models_M \top$ and $w \not\models_M \perp$,

(3) $w \models_M \neg f \Leftrightarrow w \not\models_M f$,

(4) $w \models_M f \vee g \Leftrightarrow w \models_M f$ or $w \models_M g$,

(5) $w \models_M f \stackrel{c}{\Longrightarrow} g \Leftrightarrow \Pi_w(f) \otimes\!\!\rightarrow \Pi_w(g) \geq c$,

(6) $w \models_M f \stackrel{c^+}{\Longrightarrow} g \Leftrightarrow \Pi_w(f) \otimes\!\!\rightarrow \Pi_w(g) > c$.

Note that in the definitions, every possible world may have different local possibility distribution and the residuated implication formulas are evaluated with respect to the local distributions. This is a main characteristic of our logic. The feature facilitates the interpretation of higher order (or nested) uncertain beliefs in the possibilistic reasoning framework.

A formula f is satisfiable iff there exists M and w such that $w \models_M f$, and is valid in M iff $w \models_M f$ for all $w \in W$. The subscript M will be dropped when it is clear from the context. We use $M \models f$ to denote f is valid in M and $M \models S$ to mean $M \models f$ for all $f \in S$. Let S be a set of wffs, then $S \models_{\otimes} f$ iff for all models M w.r.t. $\otimes$, if $M \models S$ implies $M \models f$.

2.1. *A comparison with LCP*

In [10], a logic for conditional possibility(LCP) is introduced which can be seen as a precedent of PRIL. In LCP, two kinds of graded conditional connectives $\overset{\langle c \rangle}{\perp\!\!\!\rightarrow}$ and $\overset{\langle c \rangle^+}{\perp\!\!\!\rightarrow}$ for $c \in [0,1]$ are provided in addition to the classical ones. The formation rules for the wffs of LCP are those for propositional logic with the following one:

- If f and g are wffs and let $c \in [0,1]$, then $f \overset{\langle c \rangle}{\perp\!\!\!\rightarrow} g$ and $f \overset{\langle c \rangle^+}{\perp\!\!\!\rightarrow} g$ are, too.

For the semantics, the Dempster's rule[3] is used to define the conditional possibility. According to the rule, if π is a possibility distribution and Π is its corresponding possibility measure, then the conditional possibility distribution on a subset of the universe A

$$\pi(x|A) = \begin{cases} \frac{\pi(x)}{\Pi(A)} & \text{if } x \in A, \\ 0 & \text{if } x \notin A, \end{cases} \tag{1}$$

when $\Pi(A) \neq 0$ and if $\Pi(A) = 0$, it is defined $\pi(x|A) = 1$ for all $x \in W$. The conditional possibility measure $\Pi(B|A)$ defined by

$$\Pi(B|A) = \sup_{x \in B} \pi(x|A)$$

is then easily shown to be equal to $\frac{\Pi(A \cap B)}{\Pi(A)}$ when $\Pi(A) \neq 0$. Consequently, the semantic clauses (5) and (6) for PRIL are replaced by the following:

(5') $w \models_M f \overset{\langle c \rangle}{\perp\!\!\!\rightarrow} g \Leftrightarrow \Pi_w(|g| \mid |f|) \geq c$,

(6') $w \models_M f \overset{\langle c \rangle^+}{\perp\!\!\!\rightarrow} g \Leftrightarrow \Pi_w(|g| \mid |f|) > c$.

From the semantics, it can be seen that $f \overset{\langle c \rangle}{\perp\!\!\!\rightarrow} g$ and $f \overset{\langle c \rangle^+}{\perp\!\!\!\rightarrow} g$ are respectively equivalent to $f \overset{c}{\Longrightarrow} f \wedge g$ and $f \overset{c^+}{\Longrightarrow} f \wedge g$ when the t-norm is the algebraic product. Thus, PRIL can do all reasoning that LCP can do. However, PRIL is more flexible since it is parameterized by a t-norm and not restricted to the family of algebraic product.

3. Possibilistic Reasoning

We can do possibilistic reasoning in PRIL by considering a special subclass of models. A model $M = \langle W, R, V \rangle$ is called *serial* if R satisfies the condition that for all $w \in W$, $\sup_{u \in W} R(w,u) = 1$, i.e., π_w is normalized for all w. We will write $S \models_{\otimes(D)} f$ if for all serial models M w.r.t. $\otimes$, $M \models S$ implies $M \models f$.[†]

[†] We will adopt the naming convention in modal logics[12] for systems of PRIL.

To do possibilistic reasoning, we introduce the following abbreviations.

$$\langle c\rangle f = \top \stackrel{c}{\Longrightarrow} f,$$

$$\langle c\rangle^{+} f = \top \stackrel{c^{+}}{\Longrightarrow} f,$$

$$[c]f = \neg\langle 1 \perp c\rangle^{+}\neg f,$$

$$[c]^{+} f = \neg\langle 1 \perp c\rangle\neg f.$$

Then $\langle c\rangle$, $\langle c\rangle^{+}$, $[c]$, and $[c]^{+}$ possess the properties of the modal operators $\Diamond$ and $\Box$ in modal logics[12]. Furthermore, by the semantic clauses (5) and (6), we have

(7) $w \models_M \langle c\rangle f \Leftrightarrow \Pi_w(f) \geq c$,

(8) $w \models_M \langle c\rangle^{+} f \Leftrightarrow \Pi_w(f) > c$,

(9) $w \models_M [c]f \Leftrightarrow N_w(f) \geq c$,

(10) $w \models_M [c]^{+} f \Leftrightarrow N_w(f) > c$,

where N_w is the necessity measure associated with π_w. This is exactly the semantics for quantitative modal logics (QML) introduced in [13]. Thus the intuitive meaning of $[c]f$ (resp. $[c]^{+}f$) is that an agent believes f with certainty at least c (resp. larger than c). Essentially, the wffs of QML are exactly those of PRIL composed only from the classical connectives and the modal operators $\langle c\rangle$, $\langle c\rangle^{+}$, $[c]$, and $[c]^{+}$ and the QML models are just PRIL models with the QML wffs being interpreted by clauses (1)-(4) and (7)-(10). The subclass of serial QML models is denoted by QD. Let $\mathcal{L}_{\Box}$ denote the set of QML wffs, then for $S \cup \{f\} \subseteq \mathcal{L}_{\Box}$, $S \models_{QD} f$ means the logical consequence relation with respect to the model class QD, i.e. for each M in QD, $M \models S$, then $M \models f$.

Since the class QD coincides with the serial PRIL models and the semantic clauses for wffs in $\mathcal{L}_{\Box}$ are the same in both kinds of models, we have the following result.

Theorem 1 *Let $S \cup \{f\} \subseteq \mathcal{L}_{\Box}$, then $S \models_{\otimes(D)} f$ iff $S \models_{QD} f$.*

Possibilistic logic (PL) is proposed by Dubois and Prade[4,6,5] for reasoning about uncertainty of classical formulas. The wffs of PL based on $\mathcal{L}$ are one of the forms $(f\ Nc)$ or $(f\ \Pi c)$, where $f \in \mathcal{L}$ and $c \in (0,1]$. The semantics of PL is based on possibility theory. Let Ω be the set of all classical interpretations for $\mathcal{L}$, then a PL model is just a possibility distribution on Ω. For a PL model π, we will write $\pi \models (f\ Nc)$ (resp. $\pi \models (f\ \Pi c)$) iff $N(f) \geq c$ (resp. $\Pi(f) \geq c$). Let $S \cup \{f\}$ be a set of wffs in PL, then $S \models_{PL} f$ iff for any PL models π, $\pi \models S$ implies $\pi \models f$. Let $\mathcal{L}_{\Pi}$ denote the set of PL wffs, then each PL wff can be represented in PRIL (or QML) by the following translation mapping $\tau : \mathcal{L}_{\Pi} \rightarrow PRIL$:

$$\tau((f\ Nc)) = [c]f, \quad \tau((f\ \Pi c)) = \langle c\rangle f.$$

The mapping preserves the logical consequence relation in the following precise sense.

Theorem 2 *Let $S \cup \{f\} \subseteq \mathcal{L}_\Pi$, then $S \models_{PL} f$ iff $\tau(S) \models_{\otimes(D)} \tau(f)$.*

Proof: If $S \not\models_{PL} f$, then we can find a possibility distribution π on Ω such that $\pi \models S$ but $\pi \not\models f$. Let us turn the possibility distribution into a PRIL model $M = \langle \Omega, R, V \rangle$, where for each $w \in \Omega$, $p \in V(w)$ iff $w(p) = 1$ and $R(w,u) = \pi(u)$ for all $u \in \Omega$. Thus $\pi_w = \pi$ for all $w \in \Omega$ in this PRIL model, and consequently $M \models g$ for all $g \in \tau(S)$ and $w \not\models_M \tau(f)$ for an arbitrary $w \in \Omega$, i.e. $\tau(S) \not\models_{\otimes(D)} \tau(f)$.

On the other hand, if $\tau(S) \not\models_{\otimes(D)} \tau(f)$, then we have a model $M = \langle W, R, V \rangle$ such that $M \models \tau(S)$ and a $w \in W$ such that $w \not\models_M \tau(f)$. Recall the definition of π_w from R, we can define $\pi : \Omega \to [0,1]$ as follows

$$\pi(x) = \sup\{\pi_w(u) | V(u) = x\}.$$

Note here we abuse V as an assignment function so that $V(u)$ is a classical interpretation. That is, we take π_w as the base of π, however, since π_w is a possibility distribution on W and there may be two different possible worlds in W with the same classical interpretation being assigned to them, we must take the supremum when it is changed into one on Ω. From the definition, we can show that $\pi \models S$ but $\pi \not\models f$ by the restriction that all wffs in $\tau(S)$ and $\tau(f)$ are all of the forms $[c]g$ or $\langle c \rangle g$ for some classical wff g. $\Box$

Note that the expressive power of PRIL is strictly more general than that of possibilistic logic. For example, we may want to represent the following sentence about a reflective agent:

The agent consider it completely possible that what he believes with half certainty is wrong.

The sentence can be represented as $\langle 1 \rangle([\frac{1}{2}]p \wedge \neg p)$, however it is not in the range of $\mathcal{L}_\Pi$. Moreover, we emphasize again that the local possibility distributions associated with each world in the semantics make the interpretation of such sentences possible by noticing that if $\pi_w = \pi$ for all $w \in W$, then the sentence is unsatisfiable.

From the proof above, we also note that the notion of possible worlds is different from that of classical interpretations. Indeed, a classical interpretation is an important component of a possible world by the truth assignment function V. However, a possible world contains more. Two possible worlds u and v may have the same classical truth assignment, i.e. $V(u) = V(v)$, but have different distributions associated with them, i.e. $\pi_u \neq \pi_v$, and they may be assigned different possibility degrees by another world w, i.e. $\pi_w(u) \neq \pi_w(v)$. The preceding theorem and its proof show that the difference is irrelevant when only first-degree modal formulas are considered. However, when nested modality is involved, we can not identify possible worlds with classical interpretations any more. To see this, suppose that $\mathcal{L}$ is a *finite* propositional language, then for any classical interpretation ω, the characteristic formula of ω is defined as

$$\chi_\omega = \bigwedge_{p \in PV | \omega(p)=1} p \wedge \bigwedge_{p \in PV | \omega(p)=0} \neg p.$$

If we require that for any PRIL model $M = \langle W, R, V \rangle$, $V(w_1) \neq V(w_2)$ when

$w_1 \neq w_2$, then the wff $\langle 1\rangle(\chi_\omega \wedge [1]p) \wedge \langle 1\rangle(\chi_\omega \wedge \langle 1\rangle\neg p)$ is unsatisfiable whereas it is satisfiable in PRIL if such requirement on models is not made.

While possibilistic logic and QML reason about the quantitative measures directly, a logic for reasoning about comparative possibility has been proposed[15]. The logic is called qualitative possibility logic(QPL). Let us use the following abbreviation

$$f \leq g = f \stackrel{1}{\Longrightarrow} g,$$

then we have

(11) $w \models_M f \leq g \Leftrightarrow \Pi_w(f) \leq \Pi_w(g)$.

Let $\mathcal{L}_\leq$ denote the subset of wffs composed only from the classical connectives and $\leq$, then $\mathcal{L}_\leq$ is exactly the set of QPL wffs. Furthermore, a QPL model is just a serial PRIL model with the QPL wffs being interpreted by clauses (1)-(4) and (11). Then, the following result holds obviously.

Theorem 3 *Let $S \cup \{f\} \subseteq \mathcal{L}_\leq$, then $S \models_{\otimes(D)} f$ iff $S \models_{QPL} f$, where $\models_{QPL}$ is the logical consequence relation of QPL defined as usual.*

4. Similarity-based Reasoning

A fuzzy relation R on X is called a similarity relation if it satisfies the following properties:

1. Reflexivity: $R(x,x) = 1$ for all $x \in X$.
2. Symmetry: $R(x,y) = R(y,x)$ for all $x, y \in X$.
3. $\otimes$-transitivity: $R(x,y) \otimes R(y,z) \leq R(x,z)$ for all $x, y, x \in X$.

The reasoning based on similarity relation is first proposed by Ruspini[16]. Given a similarity R on the set of classical interpretations, he defines the *implication* and *consistency measures* between two wffs $f, g \in \mathcal{L}$ as

$$I_R(g|f) = \inf_{\omega_1(f)=1} \sup_{\omega_2(g)=1} R(\omega_1, \omega_2),$$

$$C_R(g|f) = \sup_{\omega_1(f)=1} \sup_{\omega_2(g)=1} R(\omega_1, \omega_2).$$

Based on the two measures, Esteva et al.[14] recently propose a graded modal logic $MS5(G,\otimes)$ for similarity-based reasoning. The set G is a denumerable subset of $[0,1]$ such that $0, 1 \in G$. The set of wffs for $MS5(G,\otimes)$ is identical to that of QML, though they use $\Diamond^c_\alpha$ and $\Diamond^o_\alpha$ instead of $\langle\alpha\rangle$ and $\langle\alpha\rangle^+$ for the graded modal operators and restrict that $\alpha \in G$. The semantics of $MS5(G,\otimes)$ is based on similarity Kripke models. A similarity Kripke model $M = \langle W, R, V\rangle$ is essentially a PRIL (or QML) model with R being a $(G,\otimes)$-similarity relation, i.e. $R : W \times W \rightarrow G$ satisfying reflexivity, symmetry, and $\otimes$-transitivity. An axiomatic system for $MS5(G,\otimes)$ is given and shown to be sound with respect to the semantics in [14]. They also show the completeness of an expanded system $MS5^{+}(G,\otimes)$ when G is finite and that of another system $MS5^{++}(G,\min)$ when $\otimes$=min.

Since PRIL is a more general framework, all reasoning made in $MS5(G,\otimes)$ and its expansions can be simulated in PRIL. Of course, the simulated reasoning is basically at the semantic level since we do not have a complete axiomatization for PRIL yet. This is still an open problem for the further research, however, we will give a tentative set of axioms below.

Among the axioms and theorems of $MS5(G,\otimes)$, some modal wffs characterizing properties of similarity relations are especially interesting. Here, we will present some of them and discuss their intuitive meaning. Intuitively, $w \models \langle c\rangle f$ means that the world w is similar to the f-worlds at least to the extent c (or in short, w is *c-similar* to f-worlds). Dually, $w \models [c]f$ means w is discernible with $\neg f$-worlds to the extent c or w is *c-characterized* by f. Let $\models_{\otimes(S5)}$ denote the logical consequence relation w.r.t. all PRIL models $\langle W,R,V\rangle$ such that R is a similarity relation on W. Then the following three characteristic axioms are valid. First, the schema **T**

$$\models_{\otimes(S5)} f \supset \langle 1\rangle f$$

says that a world satisfying f is completely similar to f-worlds. This reflects the reflexivity. Second, the schema **4**

$$\models_{\otimes(S5)} \langle c\rangle\langle d\rangle f \supset \langle c\otimes d\rangle f$$

corresponds to transitivity, that says a world c-similar to worlds d-similar to f-worlds is itself $c\otimes d$-similar to f-worlds. Finally, the schema **B** for symmetry

$$\models_{\otimes(S5)} f \supset [c]\langle 1 \perp c\rangle^{+} f$$

means that if a world satisfies f, then we can c-discernible it from those worlds that are not $(1 \perp c)$-similar to f-worlds. Putting it in more qualitative terms, this means that if f is true in a world, then it is strongly discernible from those worlds only little similar to f-worlds.

The $MS5(G,\otimes)$ logic does similarity-based reasoning mainly by a set of unary modal operators. However, to express a kind of conditional implication measure when G is not finite, Esteva et al.[14] suggest to extend $MS5(G,\otimes)$ with a kind of binary modalities $[\cdot|\cdot]_c$. According to their semantics, a wff of the form $[g|f]_c$ is precisely equivalent to $f \stackrel{c}{\Longrightarrow} g$ in PRIL. Consequently, they show (and as we will show), some binary similarity-based consequence relations defined in a recent article by Dubois et al.[17,18] can be represented in the extended $MS5(G,\otimes)$ system as well as PRIL. The work of Dubois et al.[17,18] is based on the propositional logic $\mathcal{L}$. First, let Ω denote the set of all propositional interpretations of $\mathcal{L}$ and define a similarity relation R on Ω. Then any wff $f \in \mathcal{L}$ is blurred into a fuzzy proposition f^* such that the characteristic function $\mu_{f^*} : \Omega \rightarrow [0,1]$ is defined as

$$\mu_{f^*}(\omega) = \sup\{R(\omega,\omega') \mid \omega' \models f\}.$$

Furthermore, if $f, g \in \mathcal{L}$, then another fuzzy proposition $f^* \Rightarrow g^*$ is characterized by

$$\mu_{f^* \Rightarrow g^*}(\omega) = \mu_{f^*}(\omega) \otimes\!\!\rightarrow \mu_{g^*}(\omega).$$

If A is a fuzzy proposition, then $[A]$ denotes the fuzzy subset of Ω characterized by μ_A and $[A]_c$ is the c-cut of $[A]$ for $c \in [0,1]$, i.e., $[A]_c = \{\omega \in \Omega \mid \mu_A(\omega) \geq c\}$.

Let $K \subseteq \mathcal{L}$ denote the background knowledge. We will identify K with $\bigwedge K$, i.e., the conjunction of all wffs in K, when K is finite. Three types of graded entailment relations are defined in [17,18]:

1. Type I:
$$f \models_1^{K,c} g \text{ iff } |K| \cap |f| \subseteq [g^*]_c$$
2. Type II:
$$f \models_2^{K,c} g \text{ iff } |K| \subseteq [f^* \Rightarrow g^*]_c$$
3. Type III:
$$f \models_3^{K,c} g \text{ iff } |K| \subseteq [f^* \Rightarrow (f \wedge g)^*]_c$$

for all $f, g \in \mathcal{L}$ and $c \in [0,1]$.

Note that all types of graded consequence relations rely implicitly on a given similarity relation R. Apparently, a similarity relation R corresponds to a possible world model $M_R = \langle \Omega, R, V \rangle$, where the set of possible worlds are identified with Ω and V assigns truth values in an obvious way. In our previous terms,

$$\mu_{f^*}(\omega) = \Pi_\omega(f),$$

and

$$\mu_{f^* \Rightarrow g^*}(\omega) = \Pi_\omega(f) \otimes\!\!\rightarrow \Pi_\omega(g),$$

so we have the following results.

Theorem 4 *1.* $f \models_1^{K,c} g$ iff $M_R \models K \wedge f \supset \langle c \rangle g$,

2. $f \models_2^{K,c} g$ iff $M_R \models K \supset (f \stackrel{c}{\Longrightarrow} g)$, *and*

3. $f \models_3^{K,c} g$ iff $M_R \models K \supset (f \stackrel{c}{\Longrightarrow} f \wedge g)$.

Although the consequence relations given above are defined with respect to a similarity relation, in general, it is difficult to have a complete specification of a similarity relation, so it is sometimes unrealistic to do similarity-based approximate reasoning for a particular similarity relation. However, by using the object level logics, we can define more realistic consequence relations in PRIL.

Definition 1 *Let* $K \subseteq \mathcal{L}$, $f, g \in \mathcal{L}$, *and* $c \in [0,1]$, *then*

1. $f \approx_1^{K,c} g$ iff $K \models_{\otimes(S5)} f \supset \langle c \rangle g$,

2. $f \approx_2^{K,c} g$ iff $K \models_{\otimes(S5)} f \stackrel{c}{\Longrightarrow} g$, *and*

3. $f \approx_3^{K,c} g$ iff $K \models_{\otimes(S5)} f \stackrel{c}{\Longrightarrow} f \wedge g$, *are respectively the type I, II, and III generic graded entailment relations.*

Note that by using $\models_{\otimes(S5)}$, we consider not only the similarity relations on Ω but also any PRIL models where W may be any subset of possible worlds. The remark after Theorem 2 have emphasized the difference between possible worlds and classical interpretations when nested modal wffs are involved. Since nested modal formulas are essential to the system $MS5(G, \otimes)$ (or $\otimes(S5)$), the difference can not be overlook. Moreover, the use of nested modal formulas in $\otimes(S5)$ also shows that it is not merely a technical generalization.

The discussion until now is mainly semantic. We have shown that PRIL is a general framework that can do possibilistic and similarity-based reasoning from the semantic viewpoint. However, to apply the logic to real problem, we would like to develop some reasoning mechanism for it. The first step will be an axiomatic system. However, unfortunately, we can not find a completeness proof for it yet. Instead of conjecturing a complete axiomatization, we present a tentative set of axioms and inference rules for PRIL. This axiomatization is inspired by the characteristic properties of the three types of consequence relations mentioned above[17,18]. (Also compare it with the axiomatic system for LCP in [10].) In the following presentation, we will write $c \wedge d$ instead of $\min(c, d)$ for $c, d \in [0, 1]$.

1. Axiom Schemata:
   (a) All instances of propositional tautologies.
   (b) Inequality constraints:
      i. Monotonicity:
      $$f \stackrel{c}{\Longrightarrow} g \supset f \stackrel{d^+}{\Longrightarrow} g \text{ if } c > d$$
      ii. Dichotomy:
      $$f \stackrel{c^+}{\Longrightarrow} g \supset f \stackrel{c}{\Longrightarrow} g$$
      iii. Boundary:
      $$f \stackrel{0}{\Longrightarrow} g \wedge \neg(f \stackrel{1^+}{\Longrightarrow} g)$$
   (c) Reflexivity:
   $$f \stackrel{1}{\Longrightarrow} f$$
   (d) Right or:
   $$(f \stackrel{c}{\Longrightarrow} g \vee h) \supset (f \stackrel{c}{\Longrightarrow} g \vee f \stackrel{c}{\Longrightarrow} h)$$
   $$(f \stackrel{c^+}{\Longrightarrow} g \vee h) \supset (f \stackrel{c^+}{\Longrightarrow} g \vee f \stackrel{c^+}{\Longrightarrow} h)$$
   (e) Right Weakening:
   $$(f \stackrel{c}{\Longrightarrow} g \wedge h) \supset (f \stackrel{c}{\Longrightarrow} g)$$
   $$(f \stackrel{c^+}{\Longrightarrow} g \wedge h) \supset (f \stackrel{c^+}{\Longrightarrow} g)$$

(f) Left Or:

$$(f \stackrel{c}{\Longrightarrow} h) \wedge (g \stackrel{d}{\Longrightarrow} h) \supset (f \vee g \stackrel{c \wedge d}{\Longrightarrow} h)$$

$$(f \stackrel{c^+}{\Longrightarrow} h) \wedge (g \stackrel{d^+}{\Longrightarrow} h) \supset (f \vee g \stackrel{(c \wedge d)^+}{\Longrightarrow} h)$$

(g) $\otimes$-transitivity

$$f \stackrel{c}{\Longrightarrow} g \wedge g \stackrel{d}{\Longrightarrow} h \supset f \stackrel{c \otimes d}{\Longrightarrow} h$$

and if $\otimes$ is strictly monotonic

$$f \stackrel{c^+}{\Longrightarrow} g \wedge g \stackrel{d}{\Longrightarrow} h \supset f \stackrel{(c \otimes d)^+}{\Longrightarrow} h$$

$$f \stackrel{c}{\Longrightarrow} g \wedge g \stackrel{d^+}{\Longrightarrow} h \supset f \stackrel{(c \otimes d)^+}{\Longrightarrow} h$$

2. Inference Rules:

(a) MP:

$$\frac{f \quad f \supset g}{g}$$

(b) Nec:

$$\frac{g}{\langle 0 \rangle^+ f \supset \neg(f \stackrel{0^+}{\Longrightarrow} \neg g)}$$

Theorem 5 *The axiomatic system is sound with respect to all PRIL models.*

Proof: We verify the validity of the $\otimes$-transitivity axiom and that the Nec rule preserves the valid formulas. The verification of other axioms and inference rule is analogous. For the $\otimes$-transitivity axiom, let $M = \langle W, R, V \rangle$ be a PRIL model and $w \in W$ such that $w \models_M f \stackrel{c}{\Longrightarrow} g$ and $w \models_M g \stackrel{d}{\Longrightarrow} h$, then by the semantic clause (6), we have

$$\Pi_w(f) \otimes\!\!\rightarrow \Pi_w(g) \geq c$$

$$\Pi_w(g) \otimes\!\!\rightarrow \Pi_w(h) \geq d.$$

Let us define the following subsets of [0,1]

$$X = \{x | \Pi_w(f) \otimes x \leq \Pi_w(g)\},$$

$$Y = \{y | \Pi_w(g) \otimes y \leq \Pi_w(h)\},$$

$$Z = \{z | \Pi_w(f) \otimes z \leq \Pi_w(h)\},$$

then it can be shown that

$$\{x \otimes y | x \in X, y \in Y\} \subseteq Z.$$

Thus,

$$\sup Z \geq \sup\{x \otimes y | x \in X, y \in Y\} = \sup X \otimes \sup Y \geq c \otimes d,$$

i.e.

$$\Pi_w(f) \otimes\!\!\rightarrow \Pi_w(h) \geq c \otimes d,$$

so $w \models_M f \overset{(c\otimes d)^+}{\Longrightarrow} h$.

As for the Nec rule, assume g is a valid wff of PRIL, then for any model $M = \langle W, R, V\rangle$ and $w \in W$, we have $\Pi_w(\neg g) = 0$, so if $\Pi_w(f) > 0$, then $\Pi_w(f) \otimes\!\!\rightarrow \Pi_w(\neg g) = 0$. Thus, $w \models_M \langle 0\rangle^+ f \supset \neg(f \overset{0^+}{\Longrightarrow} \neg g)$ for any M and w. □

The PRIL system can be easily applied to interpolative reasoning. Here, let us use an example from [17] to illustrate the similarity-based reasoning in PRIL.

Example 1 If X and Y are two variables on domain U and V respectively, and we have rules of the form

if X is in A, then Y is in B,

where A and B are subsets of U and V respectively. Then the rule can be encoded into a PRIL formula $p \overset{1}{\Longrightarrow} q$, where p and q denote "X is in A" and "Y is in B" respectively. More generally, if the rule is uncertain, then it can be encoded as $p \overset{c}{\Longrightarrow} q$ with $c < 1$. Now, if it is known that $X = u_0 \notin A$ and we have a similarity relation S on the domain U. Then we can get the information $\langle d\rangle p$, where $d = \sup_{x\in A} S(u_0, x)$. Then by applying $\otimes$-transitivity axiom and propositional reasoning, we can conclude $\langle c \otimes d\rangle q$, i.e. Y is close to a point in B to the degree of $c \otimes d$. □

5. Default Reasoning

We will consider another subclass of models in this section. A model $\langle W, R, V\rangle$ is called *absolute* if there exists a normalized possibility distribution π on W such that $R(w, u) = \pi(u)$ for all $w, u \in W$. In other word, all π_w's induced from R are the same. For two absolute models $M_1 = \langle W_1, R_1, V_1\rangle$ and $M_2 = \langle W_2, R_2, V_2\rangle$, M_1 is said to be more specific than M_2, written as $M_1 \sqsubseteq M_2$, if for all $w_1 \in W_1$ there exists $w_2 \in W_2$ such that $V_1(w_1) = V_2(w_2)$ and $\pi_1(w_1) \leq \pi_2(w_2)$, where π_i is the possibility distribution corresponding to R_i for $i = 1, 2$. We will denote an absolute model as $\langle W, \pi, V\rangle$ from now on.

To represent a default, let us use the following abbreviation,

$$f \overset{c}{\perp\!\!\rightarrow} g = \neg((f \wedge g) \overset{(1-c)^+}{\Longrightarrow} (f \wedge \neg g)).$$

Let

$$\mathcal{L}_\delta = \{f \overset{c}{\perp\!\!\rightarrow} g \mid f, g \in \mathcal{L}, c > 0\}$$

be the default language and

$$\Delta = \{f_i \overset{c_i}{\perp\!\!\rightarrow} g_i \in \mathcal{L}_\delta \mid 1 \leq i \leq n\}$$

be a set of defaults. We further assume that $\otimes$ is *continuous* and *strictly monotonic* in both places. Then it can be shown that an absolute model $\langle W, \pi, V\rangle$ satisfies Δ iff

$$\Pi(f_i \wedge \neg g_i) \leq (1 \perp c_i) \otimes \Pi(f_i \wedge g_i), \tag{2}$$

for all $1 \leq i \leq n$. Note that since we restrict $c > 0$ and $\otimes$ is strictly monotonic, we have $\Pi(f_i \wedge \neg g_i) < \Pi(f_i \wedge g_i)$ from (2). This is just the traditional interpretation of qualitative default rule in the possibilistic reasoning setting[19,20]. This may also suggest an alternative way to encode $f \perp\!\!\xrightarrow{c} g$ as $\neg((f \wedge g) \stackrel{(1-c)}{\Longrightarrow} (f \wedge \neg g))$. Then (2) is replaced by

$$\Pi(f_i \wedge \neg g_i) < (1 \perp c_i) \otimes \Pi(f_i \wedge g_i) \tag{3}$$

and the restriction on c can be relaxed. However, the new set of inequalities does not have a least specific solution, so we can not define default reasoning based on the principle of minimum specificity if we adopt the alternative definition.

Let Ω denote the set of all propositional interpretations for $\mathcal{L}$ and Δ be satisfiable, then we can define a possibility distribution π^+ on Ω as follows. Intuitively, π^+ is the least specific solution to (2).

Definition 2 $\pi^+(\omega) = 1$ *if for each* i, $\omega \not\models f_i \wedge \neg g_i$ *and otherwise,*

$$\pi^+(\omega) = \min\{C_i^+ \mid \omega \models f_i \wedge \neg g_i\}$$

$$C_i^+ = \max\{\pi^+(\omega) \mid \omega \models f_i \wedge g_i\} \otimes (1 \perp c_i).$$

Though it seems that the definition is circular, it can in fact be computed in an iterative way. The procedure in Fig. 1. realizes the computation. To illustrate the computation, let us initially assign $I_0 := \emptyset$, $I_1 := \{1, \cdots, n\}$, $W_0 := \{\omega \mid \omega \not\models f_i \wedge \neg g_i, \forall i\}$, and $W_1 = \Omega \perp W_0$. Then for each $\omega \in W_0$, since $\pi^+(\omega)$ is not constrained by (2) at all, $\pi^+(\omega) = 1$ according to the principle of minimum specificity. This is also the basis case of definition 2. Next, we can assign $I_0 := \{i \mid \exists \omega \in W_0, \omega \models f_i \wedge g_i\}$ and $I_1 := I_1 \perp I_0$ (Note that I_0 is nonempty if the original Δ is satisfiable.). Then since for all $\omega \in W_0$, $\pi^+(\omega) = 1$, we have $\Pi(f_i \wedge g_i) = 1$ for all $i \in I_0$, so for all $i \in I_0$, $C_i^+ = 1 \perp c_i$, and (2) is split into two sets

$$\Pi(f_i \wedge \neg g_i) \leq (1 \perp c_i), \quad i \in I_0, \tag{4}$$

$$\Pi(f_i \wedge \neg g_i) \leq (1 \perp c_i) \otimes \Pi(f_i \wedge g_i), \quad i \in I_1. \tag{5}$$

Now, let us consider ω such that $\omega \notin W_0$, but $\omega \not\models f_i \wedge \neg g_i$ for all $i \in I_1$, then $\pi^+(\omega)$ is constrained only by the inequalities in (4), so for all such ω's, we can compute $\pi^+(\omega) := \min\{1 \perp c_i \mid \omega \models f_i \wedge \neg g_i, i \in I_0\}$, and move these ω's from W_1 to W_0. Continuing in this way, we can compute C_i^+ and $\pi^+(\omega)$ alternatively until $W_1 = \emptyset$. Note that the computation of $\pi^+(\omega)$ is impossible if the inequalities in (4) are strict. This further justifies the current definition of defaults.

Once π^+ is known, an absolute model M_Δ^+ can be defined as $\langle \Omega, \pi^+, V \rangle$, where V is such that $p \in V(\omega)$ iff $\omega \models p$ for all $p \in PV$ and $\omega \in \Omega$.

Lemma 1 *Let* $M = \langle W, \pi, V \rangle$ *be a* $\sqsubseteq$*-maximal absolute model satisfying* Δ *and assume* $f \perp\!\!\xrightarrow{c} g \in \mathcal{L}_\delta$, *then*

$$M \models f \perp\!\!\xrightarrow{c} g \;\text{ iff }\; M_\Delta^+ \models f \perp\!\!\xrightarrow{c} g.$$

Proof: By the definition of $\sqsubseteq$-maximality, we have the following two facts:

1. if M_1 and M_2 are both $\sqsubseteq$-maximal absolute model of Δ, then $M_1 \models f \perp\!\!\xrightarrow{c} g$ iff $M_2 \models f \perp\!\!\xrightarrow{c} g$,

2. if $\pi : \Omega \rightarrow [0,1]$ is the least specific solution of (2), then (Ω, π, V) is a $\sqsubseteq$-maximal absolute model of Δ.

Now, according to the computation of procedure C-rank, π^+ is indeed the least specific solution of (2)(see also the proof of theorem 7), so the lemma follows directly from these two facts.□

This lemma also shows that we can identify a set of possible worlds with the set of classical interpretations when only $\sqsubseteq$-maximal absolute models are considered. In other words, we can consider the absolute PRIL models with the set of classical interpretations as possible worlds without loss of generality since the default rule is represented by a first-degree modal formula and no nested modalities are involved. Thus, though we emphasize the difference between classical interpretations and general possible worlds in the last two sections when nested modal formulas are used in possibilistic and similarity-based reasoning, the distinction can be eliminated in the context of default reasoning represented by first-degree PRIL formulas.

Let $S \models_{\otimes\sqsubseteq} f$ denote that f is valid in all $\sqsubseteq$-maximal absolute models satisfying S. Then, we have

Theorem 6 *For any $\Delta \subseteq \mathcal{L}_\delta$ and $f \perp\!\!\xrightarrow{c} g \in \mathcal{L}_\delta$, $\Delta \models_{\otimes\sqsubseteq} f \perp\!\!\xrightarrow{c} g$ iff $M^+_\Delta \models f \perp\!\!\xrightarrow{c} g$.*

When $\otimes$ is the numerical product, $\models_{\otimes\sqsubseteq}$ is equivalent to 1-entailment of Goldszmidt and Pearl's system Z^+[21], so PRIL indeed provides a general framework for default reasoning with degrees of strength. The procedures in [21] can then be modified to provide an effective way to decide $\models_{\otimes\sqsubseteq}$ relation in PRIL. We now present the procedures in Fig. 1 and 2.

In the procedures, we use the following notations. Given a wff $r = f \perp\!\!\xrightarrow{c} g \in \mathcal{L}_\delta$, define $V(r) = \{f \wedge g\}$, $F(r) = \{f \wedge \neg g\}$, and $S(r) = \{f \supset g\}$. Let $\Delta \subseteq \mathcal{L}_\delta$, then $V(\Delta) = \bigcup_{r\in\Delta} V(r)$, $F(\Delta) = \bigcup_{r\in\Delta} F(r)$, and $S(\Delta) = \bigcup_{r\in\Delta} S(r)$. Let $S \subseteq \mathcal{L}$, then SAT-TEST(S) returns TRUE iff S is classical satisfiable.

It can be shown that the two procedures are correct.

Theorem 7 *1. The outputs of C-rank procedure are $C^+(r_i)(1 \leq i \leq n)$ defined in Def. 2.*

2. The output of C^+-consequence procedure is "TRUE" iff $\Delta \models_{\otimes\sqsubseteq} f \perp\!\!\xrightarrow{c} g$.

Proof:

1. We prove the following is a loop invariant with respect to the while-loop in the C-rank procedure:

$$(r_i \in \Delta_1) \Rightarrow (C(r_i) = C_i^+). \tag{6}$$

First, if $r_i \in \Delta_0$, then there exists ω_0 such that $\omega_0 \models f_i \wedge g_i$ and $\omega_0 \not\models f \wedge \neg g$ for all $f \perp\!\!\xrightarrow{c} g \in \Delta$, so by the definition of π^+, we have $\pi^+(\omega_0) = 1$ and

Procedure C**-rank**
Input: a set of satisfiable wffs $\Delta = \{r_i = f_i \stackrel{c_i}{\perp\!\!\rightarrow} g_i \mid 1 \leq i \leq n\} \subseteq \mathcal{L}_\delta$.
Output: $C(r_i)(1 \leq i \leq n)$.
Steps:

1 $\Delta_0 := \{r_i \in \Delta \mid V(r_i) \cup S(\Delta) \text{ is satisfiable}\}$

2 $\Delta_1 := \emptyset$

3 For each $r_i = f_i \stackrel{c_i}{\perp\!\!\rightarrow} g_i \in \Delta_0$ do

  3.1 $C(r_i) := 1 \perp c_i$

  3.2 $\Delta_1 := \Delta_1 \cup \{r_i\}$

4 While $\Delta_1 \neq \Delta$ do

  4.1 $\Omega_0 := \{\omega \mid \exists r \in \Delta \perp \Delta_1, \omega \models V(r) \text{ and } \forall r(\omega \models F(r) \Rightarrow r \in \Delta_1)\}$

  4.2 For each $\omega \in \Omega_0$ do

    4.2.1 $\pi(\omega) := \min\{C(r_i) \mid \omega \models F(r_i)\}$

  4.3 $\omega^* :=$ an interpretation in Ω_0 with maximum π

  4.4 For each $r_i = f_i \stackrel{c_i}{\perp\!\!\rightarrow} g_i \in \Delta \perp \Delta_1$ s.t. $\omega^* \models V(r_i)$ do

    4.4.1 $C(r_i) := \pi(\omega^*) \otimes (1 \perp c_i)$

    4.4.2 $\Delta_1 := \Delta_1 \cup \{r_i\}$

End

Figure 1: The procedure to compute C^+

Procedure C^+-consequence
Input: a satisfiable set $\Delta \subseteq \mathcal{L}_\delta$, the function C^+ on Δ, two wffs f and $g \in \mathcal{L}$, and $c > 0$.
Output: TRUE or FALSE to the question $\Delta \models_{\otimes\sqsubseteq} f \perp\!\!\xrightarrow{c} g$?
Steps:

1 $x := 1$, $c_1 := c_2 := 0$, $\Delta_1 := \Delta$

2 if $c_1 = 0$ and SAT-TEST($\{f \wedge g\} \cup S(\Delta_1)$), then $c_1 := x$

3 if $c_2 = 0$ and SAT-TEST($\{f \wedge \neg g\} \cup S(\Delta_1)$), then $c_2 := x$

4 if $(c_1 = 0 \vee c_2 = 0) \wedge \Delta_1 \neq \emptyset$, then

  4.1 $x := \max_{r_i \in \Delta_1} C_i^+$

  4.2 $\Delta_1 := \Delta_1 \perp \{r_i \mid C_i^+ = x\}$

  4.3 goto 2

5 Return $(c_2 \leq (1 \perp c) \otimes c_1)$

End

Figure 2: The procedure to test $\models_{\otimes\sqsubseteq}$ relation

$C_i^+ = \max\{\pi^+(\omega) \mid \omega \models f_i \wedge g_i\} \otimes (1 \perp c_i) = 1 \perp c_i = C(r_i)$. Thus, (6) holds at the beginning of the while-loop. This is the basis case of our proof.

Second, we show that Ω_0 is nonempty if $\Delta_1 \neq \Delta$. Because assume it is not the case, then for each $r \in \Delta \perp \Delta_1$ and $\omega \in \Omega$, $\omega \models V(r)$ implies existence of $r' \in \Delta \perp \Delta_1$ such that $\omega \models F(r')$, this means that $\bigcup_{r \in \Delta - \Delta_1} Mod(V(r)) \subseteq \bigcup_{r \in \Delta - \Delta_1} Mod(F(r))$, where $Mod(V(r))$(resp. $Mod(F(r))$) is the set of classical models for $V(r)$(resp. $F(r)$). Thus,

$$\Pi(\bigcup_{r \in \Delta - \Delta_1} Mod(V(r))) \leq \Pi(\bigcup_{r \in \Delta - \Delta_1} Mod(F(r)))$$

for any possibility measure Π. However, according to (2) and the strict monotonicity of $\otimes$, if π is a solution of (2), then we have $\Pi(F(r)) < \Pi(V(r))$ for all $r \in \Delta$, so

$$\begin{aligned}\Pi(\bigcup_{r \in \Delta - \Delta_1} Mod(F(r))) &= \max_{r \in \Delta - \Delta_1} \Pi(F(r)) \\ &< \max_{r \in \Delta - \Delta_1} \Pi(V(r)) \\ &= \Pi(\bigcup_{r \in \Delta - \Delta_1} Mod(F(r))).\end{aligned}$$

This contradicts with the satisfiability of Δ.

Third, since for each $\omega \in \Omega_0$, the condition $\forall r(\omega \models F(r) \Rightarrow r \in \Delta_1)$ is satisfied, then by the induction hypothesis, the assignment 4.2.1 results in $\pi(\omega) = \pi^+(\omega)$.

Fourth, we prove that if $r \in \Delta \perp \Delta_1$ and $\omega^* \models V(r)$, then $\Pi^+(V(r)) = \pi^+(\omega^*)$. If it is not the case, then there exists $\omega_0 \models V(r)$ and $\omega_0 \notin \Omega_0$ such that $\Pi^+(V(r)) = \pi^+(\omega_0) > \pi^+(\omega^*)$. By the definition of Ω_0 and $\omega_0 \models V(r)$, this means that $\omega_0 \models F(r')$ for some $r' \in \Delta \perp \Delta_1$. It in turn implies that $\Pi^+(\bigcup_{r \in \Delta - \Delta_1} Mod(V(r))) \leq \Pi^+(\bigcup_{r \in \Delta - \Delta_1} Mod(F(r)))$, so π^+ is not a solution of (2). This contradicts with the fact that π^+ is the least specific solution of (2).

From the third and the fourth results, it immediately follows that (6) still holds after the execution of steps 4.4.1 and 4.4.2.

Moreover, by the definition of Ω_0 and ω^* and the condition in step 4.4, the cardinality of Δ_1 at lease increases 1 for each iteration step, so the procedure will terminate since Δ is finite.

2. If SAT-TEST($\{f \wedge g\} \cup S(\Delta_1)$) is true, then there exists $\omega \models f \wedge g$ such that $\omega \not\models F(r)$ for all $r \in \Delta_1$, so $\Pi^+(f \wedge g) \geq \pi^+(\omega) > \max_{r_i \in \Delta_1} C_i^+$. The last inequality holds because $\pi^+(\omega)$ is not constrained by rules in Δ_1. This implies $\Pi^+(f \wedge g) \geq x$. On the other hand, if $c_1 = 0$, it means that SAT-TEST($\{f \wedge g\} \cup S(\Delta_1 \cup \{r_i \mid C_i^+ = x\})$) is false, so for all ω, if $\omega \models f \wedge g$, then $\omega \models F(r)$ for some $r \in \Delta_1 \cup \{r_i \mid C_i^+ = x\}$. Thus, for all ω, if $\omega \models f \wedge g$, then $\pi^+(\omega) \leq x$, so $\Pi^+(f \wedge g) = x$. Therefore, after the assignment in step 2., $c_1 = \Pi^+(f \wedge g)$. Analogously, after step 3., we have $c_2 = \Pi^+(f \wedge \neg g)$, so by theorem 6, the result follows. □

To illustrate the procedures, let us consider a familiar default reasoning example from [21].

Example 2 Let Δ be a set of defaults $\{r_1 : b \overset{c_1}{\perp\!\!\rightarrow} f, r_2 : p \overset{c_2}{\perp\!\!\rightarrow} b, r_3 : p \overset{c_3}{\perp\!\!\rightarrow} \neg f\}$ meaning "birds fly", "penguins are birds", and "penguins don't fly" respectively. and $\Omega = \{\omega_i | 0 \leq i \leq 7\}$ be the set of classical interpretations defined by the following table:

	b	p	f
ω_0	0	0	0
ω_1	0	0	1
ω_2	0	1	0
ω_3	0	1	1
ω_4	1	0	0
ω_5	1	0	1
ω_6	1	1	0
ω_7	1	1	1

then $V(r_i), F(r_i)$, and $S(r_i)$ can be represented by the following table

	V	F	S
r_1	ω_5, ω_7	ω_4, ω_6	$\omega_0, \omega_1, \omega_2, \omega_3, \omega_5, \omega_7,$
r_2	ω_6, ω_7	ω_2, ω_3	$\omega_0, \omega_1, \omega_4, \omega_5, \omega_6, \omega_7,$
r_3	ω_2, ω_6	ω_3, ω_7	$\omega_0, \omega_1, \omega_3, \omega_4, \omega_5, \omega_6,$
Δ	$\perp$	$\perp$	$\omega_0, \omega_1, \omega_5$

Here, we use the model sets to denote the wffs. For example, $F(r_1) = \{b \wedge f\}$ which corresponds to the interpretations ω_5 and ω_7.

Running the C-rank procedure step by step will produce the following results

1. $\Delta_0 = \{r_1\}$ since only $V(r_1)$ is satisfiable with $S(\Delta)$
2. $\Delta_1 = \emptyset$
3. $C(r_1) = 1 \perp c_1$ and $\Delta_1 = \{r_1\}$
4. 4.1 $\Omega_0 = \{\omega_6\}$ since it satisfies $V(r_2), V(r_3)$ and only $F(r_1)$

   4.2 $\pi(\omega_6) = C(r_1) = 1 \perp c_1$

   4.3 $\omega^* = \omega_6$

   4.4 $C(r_2) = (1 \perp c_1) \otimes (1 \perp c_2)$ and $C(r_3) = (1 \perp c_1) \otimes (1 \perp c_3)$ since ω^* satisfies both of $V(r_2), V(r_3)$

Then by running the C^+ consequence procedure to test whether $p \wedge b \perp\!\stackrel{c}{\rightarrow} \neg f$ holds, we get $x_1 = 1 \perp c_1$ and $x_2 = (1 \perp c_1) \otimes (1 \perp c_3)$ (here we use the variables x_i to replace c_i in the procedure for avoiding confusion) in two or three iterations, depending on the relative magnitude of c_2 and c_3. Thus, if $c \leq c_3$, then $p \wedge b \perp\!\stackrel{c}{\rightarrow} \neg f$ holds. This means that we can draw the conclusion that if Tweety is a penguin and a bird, then it does not fly, but with certainty not exceeding that of original rule r_3. On the other hand, $p \wedge b \perp\!\stackrel{c}{\rightarrow} f$ does not hold since $(1 \perp c_1) > (1 \perp c) \otimes (1 \perp c_1) \otimes (1 \perp c_3)$ for all $c_3 > 0$ because $\otimes$ is strictly monotonic.

Alternatively, we can compute π^+ and C^+ by definition directly according to the remark after definition 2. First, $\pi^+(\omega_0) = \pi^+(\omega_1) = \pi^+(\omega_5) = 1$ since the three interpretations don't satisfy $F(r_i)$ for $1 \leq i \leq 3$ (the (S, Δ)-cell of the above table). Then $C_1^+ = 1 \perp c_1$ since only $V(r_1)$ is satisfied by one of the three, i.e. ω_5 (the intersection of (S, Δ) and (V, r_1)-cells is not empty). Then $\pi^+(\omega_4) = \pi^+(\omega_6) = 1 \perp c_1$ (the (F, r_1)-cell) and $C_2^+ = (1 \perp c_1) \otimes (1 \perp c_2)$, and $C_3^+ = (1 \perp c_1) \otimes (1 \perp c_3)$ (the intersections of (F, r_1) and $(V, r_2), (V, r_3)$-cells are both nonempty). Finally, we have $\pi^+(\omega_2) = (1 \perp c_1) \otimes (1 \perp c_2)$, $\pi^+(\omega_7) = (1 \perp c_1) \otimes (1 \perp c_3)$ and $\pi^+(\omega_3) = \max((1 \perp c_1) \otimes (1 \perp c_2), (1 \perp c_1) \otimes (1 \perp c_3))$. To decide whether $p \wedge b \perp\!\stackrel{c}{\rightarrow} \neg f$ holds, we just test the following inequality

$$\Pi^+(p \wedge b \wedge f) \leq (1 \perp c) \otimes \Pi^+(p \wedge b \wedge \neg f),$$

i.e. $(1 \perp c_1) \otimes (1 \perp c_3) \leq (1 \perp c) \otimes (1 \perp c_1)$. □

6. Conclusion and Discussion

In this paper, we propose a class of uncertainty logics parameterized by t-norm operations called PRIL. The semantics of the logics is based on possibility theory and a residuated implication between possibilities of two wffs is defined according to the given t-norm.

When we restrict the semantics to serial models, we can simulate possibilistic reasoning in PRIL, and it is shown that the syntax of PRIL is more general than that of possibilistic logic.

When considering similarity models, we can model three types of meta-level graded consequence relations. It is also shown that the syntax of PRIL facilitates the representation of some characteristic axioms for similarity that are not expressible as mete-level construct.

Finally, we show that when the t-norm $\otimes$ is continuous and strictly monotonic, the logics can do default reasoning with degrees of strength by using absolute models.

6.1. *Related works*

To extend the perspective of the current paper, we would like to discuss some related works[‡]briefly in the final section.

First, in [22], a Kripke structure is defined as (Ω, R), where Ω is the set of all classical interpretations and R is a fuzzy binary relation on Ω. Then four classes of Kripke structures with the relation R being respectively reflexive, $\otimes$-similarity, seperating $\otimes$-similarity[§] and serial and their corresponding logical consequence relations are compared. As we can see here, serial models correspond to possibilistic reasoning, while the other three types of models all belong to the similarity-based reasoning domain, so the results in [22] present a clear relationship between these two kinds of reassoning. Moreover, they distinguish the local and global logical consequence defined with respect to a class of models. Essentially, global logical consequence is the one we use in this paper, while the local one is defined by $S \models_l f$ iff for all models $M = (\Omega, R)$ and $\omega \in \Omega$, $\omega \models_M S$ implies $\omega \models_M f$. When S is a finite set of wffs, the local logical consequence $S \models_l f$ corresponds to the global validity of the wff $\bigwedge S \supset f$. However, when S is infinite, the local logical consequence is in general weaker than the global one. This has been observed in the classical modal logic systems[23]. The results in [22] extend the observation to the graded modal logics for possibilistic and similarity-based reasoning.

Second, while PRIL and all related logics introduced so far are based on the generalization of Kripke semantics for modal logics, and so they are non-truth-functional, a truth-functional semantics for possibilistic logic has been proposed recently[24,25]. Their logic is called *local possibilistic logic* (LPL). The language of LPL is composed from atomic proposition and a set of constant $\{\mathbf{c} \mid c \in [0,1]\}$

[‡]We thank two anonymous refrees for directing our attention to these interesting works and kindly sending us the reference [22].

[§]A binary relation R is called seperating if $R(x,y) \neq 1$ for all $x \neq y$.

by ordinary Boolean connectives and two additional binary connectives $\otimes$ and $\rightarrow$ corresponding to t-norm and its residuated implication. The crucial part of LPL is that it adopts dynamic semantics[26] instead of the traditional static view of Tarskian semantics. According to the dynamic view, each formula is interpreted as an information state instead of a truth value in $\{0, 1\}$ or $[0, 1]$, where an information state is a (possibly unnormalized) possibility distribution on the set of all classical interpretations, and the logical connectives correspond to different types of information aggregations (like conjunction, disjunction, and fusion). Each information state can be seen as an agent's epistemic state and the meaning of a wff depends on how it change the agent's epistemic state. As noted in the preceding sections, the LPL can not represent nested modalities since it use possibility distributions on the classical interpretations instead of a binary relation on possible worlds. However, this is just suitable for possibilistic reasoning, though it can not cover the similarity-based reasoning. The class of all possibility distributions on classical interpretations with the operations corresponding to LPL connectives forms a complete lattice and commutative monoid with unit. To find the corresponding algebraic structure for our semantics, we must fix some model in advance, then the truth set of each wff under the model can be seen as an element of a modal algebra[27]. A complete Gentzen-style system for LPL has been provided and its relationship with substructural logics[28] is emphasized. From a practical viewpoint, the LPL is applicable to the data fusion problem. The exploration along this direction has been made in [29].

Acknowledgements

The author would like to thank two anonymous referees for valuable comments whose input motivated improvement of the paper.

References

1. C.J. Liau. "Possibilistic residuated implication logics and applications". In *Proceedings of IPMU*, pages 935–940, 1996.
2. J. Pearl. *Probabilistic Reasoning in Intelligent Systems : Networks of Plausible Inference.* Morgan Kaufmann Publisher, 1988.
3. G. Shafer. *A Mathematical Theory of Evidence.* Princeton University Press, 1976.
4. D. Dubois and H. Prade. "An introduction to possibilistic and fuzzy logics". In P. Smets, A. Mamdani, D. Dubois, and H. Prade, editors, *Non-Standard Logics for Automated Reasoning*, pages 253–286. Academic Press, 1988.
5. D. Dubois, J. Lang, and H. Prade. "Fuzzy sets in approximate reasoning, Part 2: logical approaches". *Fuzzy Sets and Systems*, 40:203–244, 1991.
6. D. Dubois, J. Lang, and H. Prade. "Possibilistic logic". In D.M. Gabbay, C.J. Hogger, and J.A. Robinson, editors, *Handbook of Logic in Artificial Intelligence and Logic Programming, Vol 3 : Nonmonotonic Reasoning and Uncertain Reasoning*, pages 439–513. Clarendon Press - Oxford, 1994.
7. Z. Pawlak. *Rough Sets–Theoretical Aspects of Reasoning about Data.* Kluwer Academic Publishers, 1991.
8. W. Łukaszewicz. *Nonmonotonic Reasoning: Formalization of Commonsense Reasoning.* Ellis Horwood, England, 1990.

9. G. Brewka. *Nonmonotonic Reasoning: Logical Foundations of Commonsense.* Cambridge University Press, Boston, 1991.
10. C.J. Liau and I.P. Lin. "Possibilistic Reasoning—A Mini-survey and Uniform Semantics". *Artificial Intelligence*, 88:163–193, 1996.
11. L.A. Zadeh. "Fuzzy sets as a basis for a theory of possibility". *Fuzzy Sets and Systems*, 1(1):3–28, 1978.
12. B.F. Chellas. *Modal Logic : An Introduction.* Cambridge University Press, 1980.
13. C.J. Liau and I.P. Lin. "Quantitative modal logic and possibilistic reasoning". In B. Neumann, editor, *Proceedings of the 10th ECAI*, pages 43–47. John Wiley & Sons. Ltd, 1992.
14. F. Esteva, P. Garcia, L. Godo, and R. Rodriguez. "A modal account of similarity-based reasoning". *International Journal of Approximate Reasoning*, 16:235–260, 1997.
15. L. Farinas del Cerro and A. Herzig. "A modal analysis of possibility theory". In R. Kruse and P. Siegel, editors, *Proceedings of the 1st ECSQAU*, LNAI 548, pages 58–62. Springer-Verlag, 1991.
16. E. Ruspini. "On the semantics of fuzzy logic". *Int. J. of Approximate Reasoning*, 5:45–88, 1991.
17. D. Dubois, F. Esteva, P. Garcia, L. Godo, and H. Prade. "Similarity-based consequence relations". In C. Froidevaux and J. Kohlas, editors, *Symbolic and Quantitative Approaches to Uncertainty*, LNAI 946. pages 171–179, Springer-Verlag, 1995.
18. D. Dubois, F. Esteva, P. Garcia, L. Godo, and H. Prade. "A logical approach to interpolation based on similarity relations". *Int. J. of Approximate Reasoning*, 17:1–36, 1997.
19. S. Benferhat, D. Dubois, and H. Prade. "Representing default rules in possibilistic logic". In B. Nebel, C. Rich, and W. Swartout, editors, *Proceedings of the 3rd International Conference on Principle of Knowledge Representation and Reasoning*, pages 673–684. Morgan Kaufmann, 1992.
20. L. Farinas del Cerro, A. Herzig, and J. Lang. "From ordering-based nonmonotonic reasoning to conditional logics". *Artificial Intelligence*, 66:375–393, 1994.
21. M. Goldszmidt and J. Pearl. "System-Z^+: a formalism for reasoning with variable-strength defaults". In *Proceedings of the 9th National Conference on Artificial Intelligence*, pages 399–404. MIT Press, 1991.
22. F. Esteva, P. Garcia, L. Godo, and R. Rodriguez. "Approximate-based multi-modal systems and possibilistic logic: A comparative study". *Mathware and Soft Computing*, To appear, 1998.
23. M.C. Fitting. *Proof Methods for Modal and Intuitionistic Logics* , Vol.169 of Syntheses Library, D. Reidel Publishing Company, 1983.
24. L. Boldrin and C. Sossai. "An algebraic semantics for possibilistic logic". In *Proceedings of the 11th International Conference on Uncertainty in Artificial Intelligence*, pages 27–35. Morgan Kaufmann, 1995
25. L. Boldrin and C. Sossai. "Local possibilistic logic". *Journal of Applied Non-classical Logic*, 3:309–333, 1997.
26. F. Veltman. "Defaults in update semantics". *Journal of Philosophical Logic*, 25:221–261, 1996.
27. W.J.. Blok. "The lattice of modal algebras: An algebraic investigation". *Journal of Symbolic Logic*, 45:221–236, 1980.
28. P. Schroeder-Heister and K. Dosen. *Substructural Logics.* Oxford University Press, 1993.
29. P. Bison, G. Chemello, C. Sossai, and G. Trainito. "A syntactical approach to data fusion". In D. Gabbay et al. editors, *First International Joint Conference on Qualitative and Quantitative Practical Reasoning*, LNAI 1244, pages 58–70. Springer-Verlag, 1997.