Interval-Related Interpolation in Interval Temporal Logics

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Abstract

This paper presents a new kind of interpolation theorems about Neighbourhood Logic (NL, [10]) and Interval Temporal Logic (ITL, [8, 4]). Unlike Craig interpolation, which holds for these logics too, the new theorems treat the existence of interpolants which specify properties of selected intervals in the models of NL and ITL.

Keywords: interval temporal logic, interpolation.

1 Introduction

Interval Temporal Logic (ITL) was introduced as a tool for reasoning about the behaviour of hardware in time [8]. It is a classical first order modal logic with one binary normal modal operator $(.;.)$, known as *chop*. It has relational (Kripke) semantics, where the possible worlds are intervals in a linearly ordered set of time points. The operator $(.;.)$ in ITL is defined as follows:

$$[\tau_1, \tau_2] \models (\varphi; \psi) \text{ iff } [\tau_1, \tau] \models \varphi \text{ and } [\tau, \tau_2] \models \psi \text{ for some } \tau \in [\tau_1, \tau_2].$$

Initially, only intervals of natural numbers were considered in ITL. Abstract semantics was introduced to ITL in [4], and a proof system was shown complete with respect to this semantics.

The operator $(.;.)$ is *introspective*. That is, the satisfaction of formulas with $(.;.)$ depends only on the assignments of symbols *in subintervals* of the given one. First order Neighbourhood Logic (NL) was introduced in [10] by replacing $(.;.)$ with two unary operators, $\Diamond_l$ and $\Diamond_r$, which allow access to arbitrary intervals of time domains. Chop is definable in NL. NL has a proof system, which is complete with respect to an abstract semantics similar to that of ITL [2, 1].

An abstract model for NL (or ITL) consists of a *frame*, which represents its time domain and the measurement of durations in it, and an *interpretation* of the non-logical symbols of the corresponding language. Non-logical symbols can be either *rigid* or *flexible*. A rigid symbol stands for the same function or predicate in all the intervals of a given model. A flexible symbol's interpretation depends on the reference interval. Distinct intervals in a model may have different properties, because of the different values of flexible symbols in them.

Classical Craig interpolation can be based on the following observation. Given a valid implication $\varphi \Rightarrow \psi$, the properties of models which satisfy φ that entail the satisfaction of ψ by these models should be related to the underlying algebraic systems of these models and the interpretations of the non-logical symbols shared by φ and

ψ in these models only. Craig's theorem states that such properties can be expressed by a first-order predicate formula involving these non-logical symbols only.

Being logics with possible worlds semantics, ITL and NL enable one more way in which restrictions can be imposed on the relationship between the antecedent and the succedent in a valid implication. Namely, the possible worlds that are supposed to have the relevant properties can be restricted. In the theorems to follow we consider the case of these possible worlds being the subintervals of some interval of time. We study two subcases depending on whether the enclosing interval is bounded or semibounded. The properties in focus are expressed by the restictions of the interpretations of the flexible symbols on the chosen intervals.

Consider pairs $\langle I_1, I_2\rangle$ of interpretations of the symbols of some language of NL or ITL into the same frame F. Let σ and σ' be intervals in F. Let $\langle F, I_1\rangle, \sigma \models \varphi$ imply that $\langle F, I_2\rangle, \sigma \models \psi$ for all $\langle I_1, I_2\rangle$ which share some property of σ'. Roughly speaking, the new interpolation theorems say that in this case the property in question can be replaced by a (possibly weaker) one which can be specified by a formula of a special kind to be satisfied at σ' by both $\langle F, I_1\rangle$ and $\langle F, I_2\rangle$. The substitute property can be chosen to be a consequence of $\langle F, I_1\rangle, \sigma \models \varphi$ and to imply $\langle F, I_2\rangle, \sigma \models \psi$ for all $\langle I_1, I_2\rangle$ which satisfy it at σ'.

The papers is organised as follows. We give brief preliminaries on NL and ITL first. We include a complete proof system for NL in the preliminaries section, because this system is relevant to the syntactical constructions in subsequent proofs. Next we describe the special kinds of formulas which are involved in the formulation of the interpolation theorems. Finally we formulate our theorems and give a detailed proof of just one of them, because the proofs of the others differ from this one in details only.

2 Preliminaries

Here we briefly introduce NL languages, frames, models, satisfaction and complete proof system. ITL with abstract semantics is defined similarly [4]. Neighbourhood logic is a classical first order predicate logic with equality and two unary normal modal operators.

2.1 Language

A language of NL is determined by a countable set of *individual variables* x, y, $\ldots$, and several other sets of symbols. These are *constant* symbols c, d, $\ldots$, *function* symbols f, g, $\ldots$ and *relation* symbols R, S, $\ldots$. Symbols of every kind are either *rigid* or *flexible*.

Given the sets of symbols, the *terms* t and the *formulas* φ of the corresponding NL languages are defined by the BNFs:

$t ::= c|x|f(t,\ldots,t)$

$\varphi ::= \bot|R(t,\ldots,t)|(\varphi \Rightarrow \varphi)|\exists x\varphi|\Diamond_l\varphi|\Diamond_r\varphi$

Function symbols and relation symbols are assigned *arity* to denote the number of arguments they admit. Every NL language contains the rigid constant symbol 0, the rigid binary function symbol $+$, the rigid binary relation symbols $=$ and $\leq$ and the flexible constant ℓ.

Individual variables are regarded as rigid. Formulas and terms which contain no flexible symbols, are called *rigid* too. The set of individual variables that have *free occurrences* in a formula φ is denoted by $FV(\varphi)$.

2.2 Frames, models and satisfaction

Definition 2.1 A NL *time domain* is a linearly ordered set. A NL *duration domain* is a linearly ordered commutative group.

We denote time and duration domains by $\langle T, \leq \rangle$ and $\langle D, +, 0, \leq \rangle$, respectively. Given a time domain $\langle T, \leq \rangle$, the set of the closed bounded *intervals* $\{[\tau_1, \tau_2] : \tau_1, \tau_2 \in T, \tau_1 \leq \tau_2\}$ in T is denoted by $\mathbf{I}(T)$.

Definition 2.2 Given a time domain $\langle T, \leq \rangle$ and a duration domain $\langle D, +, 0, \leq \rangle$, a *measure function* m is a surjective function of type $\mathbf{I}(T) \rightarrow \{d \in D : 0 \leq d\}$, which satisfies the axioms:

$(M1)$ $m(\sigma) = m(\sigma') \wedge \min \sigma = \min \sigma' \Rightarrow \max \sigma = \max \sigma'$

$(M2)$ $\max \sigma_1 = \min \sigma_2 \Rightarrow m(\sigma_1) + m(\sigma_2) = m(\sigma_1 \cup \sigma_2)$

$(M3)$ $0 \leq x \Rightarrow \exists \sigma'(\min \sigma' = \max \sigma \wedge m(\sigma') = x).$

$0 \leq x \Rightarrow \exists \sigma'(\max \sigma' = \min \sigma \wedge m(\sigma') = x).$

Definition 2.3 A tuple of the kind $\langle \langle T, \leq \rangle, \langle D, +, 0, \leq \rangle, m \rangle$, where $\langle T, \leq \rangle$ is a time domain, $\langle D, +, 0, \leq \rangle$ is a duration domain, and m is a measure from $\mathbf{I}(T)$ to D, is called NL *frame*.

Clearly, if a measure function from $I(T)$ to D exists, then $\langle D, \leq \rangle$ is isomorphic to $\langle T, \leq \rangle$. For this reason NL is usually regarded as having just duration domains in its frames. We keep the two components of NL frames distinct for the sake of compatibility with ITL semantics. Due to the introspectivity of $(.;.)$, ITL admits bounded time domains, while NL duration domains with $D \neq \{0\}$ are always unbounded.

Let $\mathbf{L}$ be an NL language.

Definition 2.4 Let $F = \langle \langle T, \leq \rangle, \langle D, +, 0, \leq \rangle, m \rangle$ be a NL frame. A function I which is defined on the set of symbols of $\mathbf{L}$ and satisfies the requirements:

$I(x), I(c) \in D$	for individual variables x and rigid constants c
$I(f) \in (D^n \rightarrow D)$	for n-place rigid function symbols f
$I(R) \in (D^n \rightarrow \{0,1\})$	for n-place rigid relation symbols R
$I(c) \in (\mathbf{I}(T) \rightarrow D)$	for flexible constants c
$I(f) \in (\mathbf{I}(T) \times D^n \rightarrow D)$	for n-place flexible function symbols f
$I(R) \in (\mathbf{I}(T) \times D^n \rightarrow \{0,1\})$	for n-place flexible relation symbols R

$I(0) = 0$, $I(+) = +$, $I(\ell) = m$, $I(\leq)$ is $\leq$ and $I(=)$ is $=$

is called *interpretation of* $\mathbf{L}$ *into* F.

Definition 2.5 A *model for* $\mathbf{L}$ is a pair of the kind $\langle F, I \rangle$, where F is a frame, and I is an interpretation of $\mathbf{L}$ into F.

Given a frame F, we denote its components by $\langle T_F, \leq_F \rangle$, $\langle D_F, +_F, 0_F, \leq_F \rangle$ and m_F, respectively. The same applies to models. We denote the frame and the interpretation of a model M by F_M and I_M, respectively.

Given a symbol s from $\mathbf{L}$, interpretations I and J of $\mathbf{L}$ into frame $\mathbf{F}$ are said to *s-agree*, if $I(s') = J(s')$ for $\mathbf{L}$ symbols s' other than s.

Definition 2.6 Let M be a model for $\mathbf{L}$. Let $\sigma \in \mathbf{I}(T_M)$. The *values* $I_\sigma(t)$ of terms t from $\mathbf{L}$ are defined as follows:

$I_\sigma(x) = I_M(x)$, $I_\sigma(c) = I_M(c)$	for variables x and rigid constants c
$I_\sigma(c) = I_M(c)(\sigma)$	for flexible constants c
$I_\sigma(f(t_1, \ldots, t_n)) = I_M(f)(I_\sigma(t_1), \ldots, I_\sigma(t_n))$	for rigid n-place f
$I_\sigma(f(t_1, \ldots, t_n)) = I_M(f)(\sigma, I_\sigma(t_1), \ldots, I_\sigma(t_n))$	for flexible n-place f

The relation $M, \sigma \models \varphi$ for formulas φ from $\mathbf{L}$ is defined as follows:

$M, \sigma \not\models \bot$	
$M, \sigma \models R(t_1, \ldots, t_n)$	iff $I_M(R)(I_\sigma(t_1), \ldots, I_\sigma(t_n)) = 1$ for rigid n-place R
$M, \sigma \models R(t_1, \ldots, t_n)$	iff $I_M(R)(\sigma, I_\sigma(t_1), \ldots, I_\sigma(t_n)) = 1$ for flexible n-place R
$M, \sigma \models (\varphi \Rightarrow \psi)$	iff either $M, \sigma \models \psi$, or $M, \sigma \not\models \varphi$
$M, \sigma \models \exists x \varphi$	iff $\langle F_M, J\rangle, \sigma \models \varphi$ for some J that x-agrees with I_M
$M, \sigma \models \Diamond_l \varphi$	iff $M, \sigma' \models \varphi$ for some $\sigma' \in \mathbf{I}(T_M)$ such that $\max \sigma' = \min \sigma$
$M, \sigma \models \Diamond_r \varphi$	iff $M, \sigma' \models \varphi$ for some $\sigma' \in \mathbf{I}(T_M)$ such that $\min \sigma' = \max \sigma$

2.3 Abbreviations

Ordinary classical first order predicate logic abbreviations $\top$, $\neg$, $\vee$, $\wedge$, $\Leftrightarrow$ and $\forall$, and infix notation for terms and formulas involving $+$, $\leq$ and $=$ are used in NL. The universal closure $\forall x_1 \ldots \forall x_n \varphi$ of formula φ, where $\{x_1, \ldots, x_n\} = FV(\varphi)$, is denoted by $\forall \varphi$. The following abbreviations are NL-specific:

$\Diamond_d^c \varphi \rightleftharpoons \Diamond_d \Diamond_{\overline{d}} \varphi$
$\Box_d \varphi \rightleftharpoons \neg \Diamond_d \neg \varphi$
$\Box_d^c \varphi \rightleftharpoons \neg \Diamond_d^c \neg \varphi$

Here d stands for either l or r, $\overline{l}$ is r and $\overline{r}$ is l. The modal operator $(.;.)$ of ITL can be defined in NL by the axiom:

$$(\varphi; \psi) \Leftrightarrow \exists x \exists y (x + y = \ell \wedge \Diamond_l^c (\varphi \wedge \ell = x) \wedge \Diamond_r^c (\psi \wedge \ell = y)),$$

where x and y are supposed to have no free occurrences in φ, ψ. That is why ITL can be regarded as a proper fragment of NL. In the sequel we assume that $(.;.)$ is available in NL languages. The following abbreviations are related to $(.;.)$:

$\Diamond \varphi \rightleftharpoons ((\top; \varphi); \top)$
$\Box \varphi \rightleftharpoons \neg \Diamond \neg \varphi$

2.4 Proof system for NL

The proof system of NL consists of axioms for classical first order predicate logic with equality, axioms about duration domais as linearly ordered commutative groups, and the following axioms and rules:

$(A1)$ $\Diamond_d\varphi \Rightarrow \varphi$ for rigid φ
$(A2)$ $0 \leq \ell$
$(A3)$ $0 \leq x \Rightarrow \Diamond_d(\ell = x)$
$(A4)$ $\Diamond_d(\varphi \vee \psi) \Rightarrow \Diamond_d\varphi \vee \Diamond_d\psi$
$(A4')$ $\Diamond_d\exists x\varphi \Rightarrow \exists x\Diamond_d\varphi$
$(A5)$ $\Diamond_d(\ell = x \wedge \varphi) \Rightarrow \Box_d(\ell = x \Rightarrow \varphi)$
$(A6)$ $\Diamond_d^c\varphi \Rightarrow \Box_d\Diamond_{\overline{d}}\varphi$
$(A7)$ $\ell = x \Rightarrow (\varphi \Leftrightarrow \Diamond_d^c(\ell = x \wedge \varphi))$
$(A8)$ $0 \leq x \wedge 0 \leq y \wedge \Diamond_d(\ell = x \wedge \Diamond_d(\ell = y \wedge \Diamond_d\varphi)) \Rightarrow$
$\Rightarrow \Diamond_d(\ell = x + y \wedge \Diamond_d\varphi)$

$(Mono)\ \dfrac{\varphi \Rightarrow \psi}{\Diamond_d\varphi \Rightarrow \Diamond_d\psi}\quad (Nec)\ \dfrac{\varphi}{\Box_d\varphi}\quad (MP)\ \dfrac{\varphi\quad \varphi \Rightarrow \psi}{\psi}\quad (G)\ \dfrac{\varphi}{\forall x\varphi}$

Substitution $[t/x]\varphi$ of variable x by term t in formula φ *is allowed in proofs* only if either t is rigid, or x does not occur in the scope of a modal operator in φ.
This system is complete with respect to the above semantics [2, 1]. The completeness argument is by a canonical construction.

3 Retrospective and introspective formulas in NL

We need some classes of formulas in NL languages in order to formulate our interpolation theorems. We call formulas which do not contain $\Diamond_r$, but may contain $(.;.)$ *retrospective*. Retrospective formulas can be used to express properties of the subintervals of some interval of time that is unbounded into the past. This semibounded interval has the end of the reference interval as its upper bound. Respectively, formulas which contain neither $\Diamond_l$, nor $\Diamond_r$, but may contain $(.;.)$, are called *introspective*. Introspective formulas can be used to express the properties of the reference interval that do not depend on intervals out of it.

Given a NL language, introspective formulas in it constitute the corresponding ITL language. Let $\mathbf{L}$ be a language of NL and F be a NL frame. The following propositions explain the expressive power of the two kinds of formulas:

Proposition 3.1 Let $\sigma \in \mathbf{I}(T_F)$. Let the interpretations I_1 and I_2 of $\mathbf{L}$ into F coincide on rigid symbols and assign the same values to flexible symbols on subintervals of σ. Let φ be an introspective formula from $\mathbf{L}$. Let σ' be a subinterval of σ. Then $\langle F, I_1\rangle, \sigma' \models \varphi$ iff $\langle F, I_2\rangle, \sigma' \models \varphi$.

Proposition 3.2 Let $\tau \in T_F$. Let the interpretations I_1 and I_2 of $\mathbf{L}$ into F coincide on rigid symbols and assign the same values to flexible symbols on intervals σ such that $\max\sigma \leq \tau$. Let φ be a retrospective formula from $\mathbf{L}$. Let $\sigma \in \mathbf{I}(T_F)$ and $\max\sigma \leq \tau$. Then $\langle F, I_1\rangle, \sigma' \models \varphi$ iff $\langle F, I_2\rangle, \sigma' \models \varphi$.

Both propositions can be proved by straightforward induction on the length of formulas φ.

Next, consider the formulas of the kind
$\eta_{c_0,c_1,c_2,c_3} \rightleftharpoons \ell = c_0 \wedge \Diamond_r(\ell = c_1 \wedge \Diamond_r^c(\ell = c_2 \wedge \Diamond_l(\ell = c_3 \wedge R)))$
where c_0, c_1, c_2 and c_3 are rigid constants, and R is a 0-place (flexible) relation symbol. In our interpolation theorems we employ substitutions of the kind $[\varphi/R]\eta_{c_0,c_1,c_2,c_3}$. Their meaning is explained by the following proposition:

Proposition 3.3 Let $\sigma, \sigma' \in \mathbf{I}(T_F)$. Let I be an interpretation of $\mathbf{L}$ into F such that $I(c_0) = m_F(\sigma)$, $I(c_1) = m_F([\max\sigma, \tau])$, $I(c_2) = m_F([\max\sigma', \tau])$ and $I(c_3) = m_F(\sigma')$, where τ satisfies $\tau \geq \max\{\max\sigma, \max\sigma'\}$. Then $\langle F, I\rangle, \sigma \models [\varphi/R]\eta_{c_0,c_1,c_2,c_3}$ iff $\langle F, I\rangle, \sigma' \models \varphi$.

Similarly, consider formulas of the kind

$$\xi_{c_0,c_1,c_2} \rightleftharpoons ((\ell = c_1; R); \ell = c_2) \wedge \ell = c_0$$

Substitutions of the kind $[\varphi/R]\xi_{c_0,c_1,c_2}$ have the following meaning:

Proposition 3.4 Let $\sigma, \sigma' \in \mathbf{I}(T_F)$ and $\sigma' \subseteq \sigma$. Let I be an interpretation of $\mathbf{L}$ into F such that $I(c_0) = m_F(\sigma)$, $I(c_1) = m_F([\min\sigma, \min\sigma'])$ and $I(c_2) = m_F([\max\sigma', \max\sigma])$. Then $\langle F, I\rangle, \sigma \models [\varphi/R]\xi_{c_0,c_1,c_2}$ iff $\langle F, I\rangle, \sigma' \models \varphi$.

4 The interpolation theorems

Let us consider two NL languages, $\mathbf{L}$ and $\mathbf{L}'$, with the same rigid symbols and similar sets of flexible symbols. That is, if s is a flexible constant, function or relation symbol in $\mathbf{L}$, then a symbol s' of the same kind and arity is found in $\mathbf{L}'$, and vice versa. For a formula φ in $\mathbf{L}$, we denote the formula in $\mathbf{L}'$ which is obtained by replacing every flexible symbol in φ by its corresponding symbol from $\mathbf{L}'$ by φ'. Let ℓ be an exception; let it be the only flexible symbol that occurs in both languages, and not be affected by the above translation. Let $\mathbf{L}''$ be the union of $\mathbf{L}$ and $\mathbf{L}'$. Let c_0, c_1, c_2 and c_3 be rigid constants in $\mathbf{L}$.

Theorem 4.1 Let Φ be a finite set of retrospective formulas from $\mathbf{L}$. Let φ and ψ be arbitrary formulas from $\mathbf{L}$. Let

$$\models_{NL} \bigwedge_{\chi\in\Phi} [\Box_r^c \forall(\chi \Leftrightarrow \chi')/R]\eta_{c_0,c_1,c_2,c_3} \Rightarrow (\varphi \Rightarrow \psi').$$

Then there exists a retrospective formula θ from $\mathbf{L}$ such that

$$\models_{NL} \ell = c_0 \wedge 0 \leq c_1 \wedge 0 \leq c_2 \wedge 0 \leq c_3 \Rightarrow (\varphi \Rightarrow [\theta/R]\eta_{c_0,c_1,c_2,c_3})$$

and

$$\models_{NL} [\theta'/R]\eta_{c_0,c_1,c_2,c_3} \Rightarrow \psi'.$$

Theorem 4.2 Let Φ be a finite set of introspective formulas from $\mathbf{L}$. Let φ and ψ be arbitrary formulas from $\mathbf{L}$. Let

$$\models_{NL} \bigwedge_{\chi\in\Phi} [\Box \forall(\chi \Leftrightarrow \chi')/R]\eta_{c_0,c_1,c_2,c_3} \Rightarrow (\varphi \Rightarrow \psi').$$

Then there exists an introspective formula θ from $\mathbf{L}$ such that

$$\models_{NL} \ell = c_0 \wedge 0 \leq c_1 \wedge 0 \leq c_2 \wedge 0 \leq c_3 \Rightarrow (\varphi \Rightarrow [\theta/R]\eta_{c_0,c_1,c_2,c_3})$$

and

$$\models_{NL} [\theta'/R]\eta_{c_0,c_1,c_2,c_3} \Rightarrow \psi'.$$

Now let $\mathbf{L}$ and $\mathbf{L}'$ be ITL languages in the same relationship as above. All formulas in ITL languages are introspective.

Theorem 4.3 Let Φ be a finite set of formulas from $\mathbf{L}$. Let φ and ψ be arbitrary formulas from $\mathbf{L}$. Let

$$\models_{ITL} \bigwedge_{\chi\in\Phi} [\Box \forall(\chi \Leftrightarrow \chi')/R]\xi_{c_0,c_1,c_2} \Rightarrow (\varphi \Rightarrow \psi').$$

Then there exists a formula θ from $\mathbf{L}$ such that

$$\models_{ITL} \ell = c_0 \wedge c_0 \geq c_1 + c_2 \wedge c_1 \geq 0 \wedge c_2 \geq 0 \Rightarrow (\varphi \Rightarrow [\theta/R]\xi_{c_0,c_1,c_2})$$

and

$\models_{ITL} [\theta'/R]\xi_{c_0,c_1,c_2} \Rightarrow \psi'$.

The proofs of all three theorems are similar to that of Craig's interpolation theorem as presented in e.g. [3]. Craig's interpolation theorem holds for ITL and NL on the grounds of similar proofs too. We give a proof of Theorem 4.1 only as an example.

Proof. Let Γ and Γ' be sets of formulas from $\mathbf{L}$ and $\mathbf{L}'$, respectively. Let C be a countable set of fresh rigid constants. Let $\mathbf{L}(C)$ be the extension of $\mathbf{L}$ by these constants. We call Γ and Γ' *inseparable*, if for every retrospective formula θ from $\mathbf{L}(C)$ either $\Gamma \cup \{[\neg\theta/R]\eta_{c_0,c_1,c_2,c_3}\}$ is consistent or $\Gamma' \cup \{[\theta'/R]\eta_{c_0,c_1,c_2,c_3}\}$ is consistent.

Let $\Delta = \{[\Box_r^c\forall(\chi \Leftrightarrow \chi')/R]\eta_{c_0,c_1,c_2,c_3} : \chi$ is a retrospective formula from $\mathbf{L}\}$. We first prove that, if Γ and Γ' are inseparable, then $\Gamma\cup\Gamma'\cup\Delta$ is a satisfiable set of formulas from $\mathbf{L}''$. Let C be a countable set of fresh rigid constants. Let $\langle\varphi_i : i < \omega\rangle$ and $\langle\psi_i : i < \omega\rangle$ be fixed enumerations of the formulas from $\mathbf{L}(C)$ and $\mathbf{L}'(C)$, respectively. We construct two ascending sequences Γ_i and Γ'_i, $i < \omega$, of consistent sets of formulas from $\mathbf{L}(C)$ and $\mathbf{L}'(C)$, respectively so that Γ_i and Γ'_i are inseparable for all $i < \omega$. Let $\Gamma_0 = \Gamma$ and $\Gamma'_0 = \Gamma'$. Given Γ_k and Γ'_k for some k, we define Γ_{k+1} by considering the cases:

1. $\Gamma_k \cup \{\varphi_k\}$ and Γ'_k are inseparable.

1a. φ_k is $\exists x\alpha$ for some α from $\mathbf{L}(C)$. We choose a $c \in C$ which does not occur in $\Gamma_k \cup \Gamma'_k$. Then $\Gamma_k \cup \{\varphi_k, [c/x]\alpha\}$ and Γ'_k are inseparable too. We put $\Gamma_{k+1} = \Gamma_k \cup \{\varphi_k, [c/x]\alpha\}$.

1b. φ_k is not existential. Then $\Gamma_{k+1} = \Gamma_k \cup \{\varphi_k\}$.

2. $\Gamma_k \cup \{\varphi_k\}$ and Γ'_k are not inseparable. Then $\Gamma_{k+1} = \Gamma_k$.

Γ'_{k+1} is defined symmetrically, using Γ_{k+1} and Γ'_k. Let $\Gamma_\omega = \bigcup\limits_{k<\omega} \Gamma_k$ and $\Gamma'_\omega = \bigcup\limits_{k<\omega} \Gamma'_k$. Γ_ω and Γ'_ω are inseparable maximal consistent sets in $\mathbf{L}(C)$ and $\mathbf{L}'(C)$, respectively. Besides, both Γ_ω and Γ'_ω have witnesses in C. That is, for every individual variable x and formula φ from $\mathbf{L}(C)$ ($\mathbf{L}'(C)$) such that $\exists x\varphi \in \Gamma_\omega$ ($\exists x\varphi \in \Gamma'_\omega$) there exists a $c \in C$ such that $[c/x]\varphi \in \Gamma_\omega$ ($[c/x]\varphi \in \Gamma'_\omega$).

Since $\alpha \doteq \alpha'$ for rigid α, Γ_ω and Γ'_ω contain the same rigid formulas.

Let $\langle F, I\rangle$ and $\langle F', I'\rangle$ be the canonical models for Γ_ω and Γ'_ω. As we know by the construction of these models [2, 1], their frames are determined by the rigid formulas from Γ_ω and Γ'_ω that are built using rigid constants from C, 0, + and $\leq$ only. Hence $F = F'$. Similarly I and I' coincide on rigid symbols from $\mathbf{L}$ and $\mathbf{L}'$. Hence, we can define an interpretation I'' of the symbols of $\mathbf{L}''$ into F which extends both I and I' so that $\langle F, I''\rangle, \sigma \models \Gamma_\omega \cup \Gamma'_\omega$ for some $\sigma \in \mathbf{I}(T_F)$.

Let $\sigma' \in \mathbf{I}(T_F)$ be such that $I''(c_1) = m_F([\max\sigma, \tau])$, $I''(c_2) = m_F([\max\sigma', \tau])$ and $I''(c_3) = m_F(\sigma')$, where $\tau = \max\{\max\sigma, \max\sigma'\}$. Then $\langle F, I''\rangle, \sigma \models \Delta$ iff $\langle F, I''\rangle, \sigma' \models \Box_r^c\forall(\chi \Leftrightarrow \chi')$ for all retrospective χ in $\mathbf{L}$ will follow from Proposition 3.2, if we prove that the interpretations I and I' of every flexible symbol s from $\mathbf{L}$ and its counterpart s' from $\mathbf{L}'$, respectively, coincide on intervals $\sigma'' \in \mathbf{I}(T_F)$ such that $\max\sigma'' \leq \max\sigma'$. Let s be an n-ary relation symbol and $a_1, \ldots, a_n \in D_F$. Let $d_1, \ldots, d_n, e_1, e_2 \in C$ be such that $I(d_i) = a_i$, $i = 1, \ldots, n$, $I(e_1) = m_F(\sigma'')$ and $I(e_2) = m_F([\max\sigma'', \max\sigma'])$. These rigid constants exist, because $\langle F, I\rangle$ is a canonical model. Consider the retrospective formula $\alpha \rightleftharpoons (\top; \ell = 0 \wedge \Diamond_l(\ell = e_2 \wedge \Diamond_l(\ell = e_1 \wedge s(d_1, \ldots, d_n))))$. Since Γ_ω and Γ'_ω are inseparable, either $[\alpha/R]\eta_{c_0,c_1,c_2,c_3}, [\alpha'/R]\eta_{c_0,c_1,c_2,c_3} \in \Gamma_\omega, \Gamma'_\omega$, or $[\neg\alpha/R]\eta_{c_0,c_1,c_2,c_3}, [\neg\alpha'/R]\eta_{c_0,c_1,c_2,c_3} \in$

$\Gamma_\omega, \Gamma'_\omega$. In either case $I(s)(\sigma'', d_1, \ldots, d_n) = I'(s')(\sigma'', d_1, \ldots, d_n)$. Function symbols are tackled similarly.

So, we have proven that, if Γ and Γ' are inseparable, then

$$\Gamma \cup \Gamma' \cup \{[\Box_r^c \forall(\chi \Leftrightarrow \chi')/R]\eta_{c_0,c_1,c_2,c_3} : \chi \text{ is a retrospective formula from } \mathbf{L}\}$$

is satisfiable.

If the interpolant θ from the theorem does not exist, $\{\varphi\}$ and $\{\neg\psi'\}$ are inseparable: Assume that an α in $\mathbf{L}(C)$ such that $\vdash_{NL} \varphi \Rightarrow [\alpha/R]\eta_{c_0,c_1,c_2,c_3}$ and $\vdash_{NL} [\alpha'/R]\eta_{c_0,c_1,c_2,c_3} \Rightarrow \psi'$ exists. Let β be the result of the replacement in α of all the constants from C by fresh individual variables. Then $\forall\beta$ satisfies the requirements for θ. Hence, if θ does not exist, $\{\varphi, \neg\psi'\} \cup \{[\Box_r^c \forall(\chi \Leftrightarrow \chi')/R]\eta_{c_0,c_1,c_2,c_3} : \chi \in \Phi\}$ is satisfiable. This is a contradiction. ∎

Concluding remarks

Craig interpolation and the above interval-related interpolation theorems establish some nice and natural properties of NL and ITL. The study of these theorems was motivated by the needs of a specific application. Namely, Theorem 4.1 was used in the completeness proof for a probabilistic extension of NL, where its use is related to an extensionality axiom about the probability operator [5, 6]. In that logic interpretations which satisfy the conditions of Proposition 3.1 are used to represent branching time, which is the underlying structure for modelling probabilistic behaviour.

An interesting observation to be made about the new interpolation theorems is that there is a straightforward correspondence between the form of interpolation theorem pursued and the notion of inseparability between theories to be employed in its proof. In fact each of the three theorems presented is proved using inseparability by formulas of the appropriate kind. This certainly suggests that the above theorems can be viewed as instances of a rather general and abstract separability theorem. Yet we find it more appropriate to point to the above concrete results here, because the temporal logics these results apply to have drawn substantial interest on their own right.

The proofs of the above theorems also suggest that they can be reformulated after Robinson's consistency theorem (cf. [3] again).

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