Induction by Enumeration*

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Abstract

Induction by enumeration has a clear interpretation within the numerical paradigm of inductive discovery (i.e., the one pioneered by [Gold, 1967]). The concept is less easily interpreted within the first-order paradigm discussed by [Kelly, 1996, Martin & Osherson, 1998], in which the scientist's data amount to the basic diagram of a structure. We formulate two kinds of enumerative induction that are appropriate to the first-order paradigm, and analyze their potential for discovery. Among other results, it is shown that one form of enumerative induction achieves maximum inductive competence.

1 Introduction

Enumerative induction may be illustrated by the following game pitting you against Nature. Let N be the natural numbers, $\{0, 1, \cdots\}$. Nature chooses a set S from the collection $\mathcal{A} = \{N - \{x\} \mid x \in N\}$. She then presents you with an arbitrary enumeration of S. Upon examining each newly presented number, you issue a member of $\mathcal{A}$ with the goal of stabilizing to S. One way to proceed is to make your own enumeration of $\mathcal{A}$. Then at each stage you conjecture the first member of your enumeration that includes the finite set of numbers encountered so far. (Thus, if your enumeration puts $N - \{i\}$ in the ith position, your conjecture after seeing $5, 1, 0$ would be $N - \{2\}$.) Such is "induction by enumeration" or "enumerative induction," and it is easy to see that in this case it works: no matter which member S of $\mathcal{A}$ is chosen by Nature, and no matter the order in which its numbers are presented to you, your conjectures will be right

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cofinitely often, that is, you will stabilize to S. Moreover, you will succeed in this way no matter how you enumerate $\mathcal{A}$.

The foregoing strategy appears to be the lowest form of inductive life with a semblance of intelligence. So one is curious to determine its scope and limits. There are problems like those above — in which the hypotheses form a countable collection of subsets of N — that can be solved but not by enumerative induction. On the other hand, if the hypotheses form a countable collection of total functions from N to N then enumerative induction always works. These facts are verified straightforwardly.[1] The theory of enumerative induction is more challenging when projected into a recursion theoretic setting, in which inductive strategies must be implementable via computer. Aspects of the resulting theory are presented in [Jain *et al.*, 1999].

The inductive problems evoked so far have a numerical cast since Nature's choice ranges over subsets of N or over functions from N to N. In contrast, the goal of the present work is to explore enumerative induction within the "first-order" paradigm discussed in [Kelly, 1996, Martin & Osherson, 1998] and elsewhere. The latter paradigm conceives Nature as choosing among disjoint collections of relational structures and can be shown to include the numerical framework as a special case (see [Martin & Osherson, 1998, Secs. 3.1.5, 3.5.6]). We here leave computational issues to one side, in order to focus on the pure logic of enumerative induction.

In what follows we first review the concepts needed to define the first-order paradigm of scientific (or "inductive") inquiry. Next we specify two kinds of inductive strategies, each of enumerative character. The powers of the two strategies are then analyzed and compared. Among other things we show that one of them is a canonical form for inductive inference: any solvable problem can be solved via the method.

2 Preliminaries

All the material in the present section is drawn from [Martin & Osherson, 1998, Secs. 3.1, 3.2]. The paradigm we define is similar to other classification tasks involving learning, e.g., those discussed in [Smith *et al.*, 1997, Gasarch *et al.*, 1998]. In each case, the learner must determine the category from which an underlying reality is drawn, and need not necessarily determine the specific identity of that reality.

[1] For a solvable problem that cannot be solved via enumeration, consider $\{N - \{0 \cdots n\} \mid n \in N\})$. See [Martin & Osherson, 1998, p. 30, Ex. 43] for discussion.

2.1 Language and structures

To get started we fix a countable set **Sym** consisting of predicates and function symbols of various arities, along with constants. The (denumerable) set $\{v_i \mid i \in N\}$ of variables in **Sym** is denoted by *Var*. The resulting set of first-order formulas is denoted: $\mathcal{L}_{form}$. The set of basic formulas (atomic formulas along with their negations) is denoted: $\mathcal{L}_{basic}$. Both **Sym** and *Var* should be conceived as fixed throughout our discussion. However, the theorems below sometimes require special hypotheses on **Sym** (typically, that it include predicates of various arities). When stated without such hypotheses, our results are true for any choice of (countable) **Sym**.

To interpret our language we rely on countable structures with signature appropriate to **Sym**. The countability assumption is not essential to our paradigm, but simplifies the discussion. (For extension to structures of arbitrary countability, see [Martin & Osherson, 1998, Sec. 3.7].) Once again: *structures are henceforth assumed to have countable domains (either finite or infinite).* The domain of structure $\mathcal{S}$ is denoted by $\mid \mathcal{S} \mid$.

Let $\Gamma \subseteq \mathcal{L}_{form}$ be given. We say that structure $\mathcal{S}$ is a model of Γ just in case there is an assignment $h : Var \to \mid \mathcal{S} \mid$ with $\mathcal{S} \models \Gamma[h]$; in this case $\mathcal{S}$ *satisfies* Γ. The class of all structures that satisfy Γ is denoted by $MOD(\Gamma)$.

2.2 The first-order paradigm

Before giving formal definitions, let us provide an overview of the paradigm. It is conceived as a game between Nature and a scientist. The same partition of a given collection of structures is communicated to both players. Each cell is considered to be a "proposition," that is, a collection of possible worlds (structures). Nature chooses a structure $\mathcal{S}$ from one of the propositions of the partition. She also chooses an assignment $h : Var \to \mid \mathcal{S} \mid$ onto $\mid \mathcal{S} \mid$ (that h be onto is crucial). Then she fixes an arbitrary enumeration of $\{\beta \in \mathcal{L}_{basic} \mid \mathcal{S} \models \beta[h]\}$, the set of all basic formulas made true in $\mathcal{S}$ by h. The scientist is fed this enumeration one formula at a time. After each input, she conjectures a proposition of her choice. The scientist wins the game if cofinitely many of her conjectures are accurate. She "solves" the problem posed by the game if she is guaranteed to win regardless of Nature's choices.

Discussion of the paradigm along with variants is available in [Martin & Osherson, 1998, Ch. 3]. Now for the formalities.

Propositions and problems

(1) Definition: A nonempty class of structures is a *proposition.* A *problem* is a collection of disjoint propositions.

(2) EXAMPLE: Suppose that **Sym** consists of a binary predicate R. Let T be the theory of total orders (with respect to R) with either a least point or a greatest point (but not both). Let $\theta = \exists x \forall y Rxy$ ("there is a least point") and $\mathbf{P} = \{MOD(T \cup \{\theta\}), MOD(T \cup \{\neg\theta\})\}$. Then $\mathbf{P}$ is a problem consisting of the propositions $MOD(T \cup \{\theta\})$ and $MOD(T \cup \{\neg\theta\})$.

In this example $\mathbf{P}$ is composed of propositions that are elementary classes, that is, specified by sets of sentences. Propositions are arbitrary collections of structures, however, and problems need not have elementary members.

Environments

(3) DEFINITION: Let structure $\mathcal{S}$ be given. A *full assignment* to $\mathcal{S}$ is any mapping of *Var* onto $|\, \mathcal{S} \,|$.

A full assignment h to $\mathcal{S}$ may be conceived as providing temporary names for all the elements of $|\, \mathcal{S} \,|$. These names are exploited for the purpose of presenting the "basic diagram" of $\mathcal{S}$ to the scientist. The presentation is called an "environment," defined as follows.

(4) DEFINITION: Let structure $\mathcal{S}$ and full assignment h to $\mathcal{S}$ be given.

(a) An *environment* for $\mathcal{S}$ and h is a sequence e such that $range(e) = \{\beta \in \mathcal{L}_{basic} \,|\, \mathcal{S} \models \beta[h]\}$.

(b) An *environment* for $\mathcal{S}$ is an environment for $\mathcal{S}$ and h, for some full assignment h to $\mathcal{S}$.

(c) An *environment* is an environment for some structure.

(d) An *environment* for proposition P is an environment for some $\mathcal{S} \in P$.

(e) An *environment* for problem $\mathbf{P}$ is an environment for some $P \in \mathbf{P}$.

(5) EXAMPLE: Let binary predicate R be the only member of **Sym**, and suppose that structure $\mathcal{S}$ with $|\, \mathcal{S} \,| = N$ interprets R as $<$. If full assignment g to $\mathcal{S}$ is $\{(v_{2i}, i), (v_{2i+1}, i) \,|\, i \in N\}$ then one environment for $\mathcal{S}$ and g begins this way:

$$v_2 = v_3 \quad \neg Rv_4v_5 \quad Rv_1v_9 \quad v_9 = v_9 \quad Rv_7v_{19} \quad v_0 \neq v_3 \quad \neg Rv_{33}v_2 \quad \neg Rv_{23}v_8 \quad \ldots$$

If P is the proposition containing every strict total order, then this same environment is for P. If $\mathbf{P}$ is a problem that includes P as a component proposition, then the environment is also for $\mathbf{P}$.

(6) Definition: Given environment e and $k \in N$, $e(k)$ denotes the member of e that falls in its kth position, and $e[k]$ is the initial finite segment of e of length k. Thus $e(k)$ comes right after $e[k]$ in e.

Thus, if e is the environment of Example (5), then $e(2) = Rv_1v_9$, $e[2] = \langle v_2 = v_3, \neg Rv_4v_5 \rangle$, $e(0) = (v_2 = v_3)$, and $e[0] = \emptyset$.

The following lemma provides a sense in which environments offer complete information about the structures they are for. The proof is easy (and also an immediate consequence of [Keisler, 1977, Prop 3.2(i)]).

(7) Lemma: Let structures $\mathcal{S}$ and $\mathcal{T}$ be given.

- (a) If $\mathcal{S}$ and $\mathcal{T}$ are isomorphic then the set of environments for $\mathcal{S}$ is identical to the set of environments for $\mathcal{T}$.
- (b) If some environment is for both $\mathcal{S}$ and $\mathcal{T}$ then $\mathcal{S}$ and $\mathcal{T}$ are isomorphic.

Data

(8) Definition: Let *SEQ* denote the collection of proper initial segments of any environment. The set of elements appearing in $\sigma \in SEQ$ is denoted by $range(\sigma)$.

Thus, *SEQ* is the countable set of all consistent finite sequences of basic formulas. It exhausts the potential data that can become available to scientists. Given $\sigma \in SEQ$, we denote by $\bigwedge \sigma$ the conjunction (in order of appearance in σ) of the formulas in $range(\sigma)$. If $\sigma = \emptyset$, then $\bigwedge \sigma$ is taken to be $\forall v_0(v_0 = v_0)$.

(9) Definition: Let $\sigma \in SEQ$ be given. We say that σ is *for* proposition P just in case $\bigwedge \sigma$ is satisfiable in some member of P. We say that σ is *for* problem $\mathbf{P}$ just in case σ is for some $P \in \mathbf{P}$.

Thus, σ is for a problem $\mathbf{P}$ just in case there is $\mathcal{S} \in \bigcup \mathbf{P}$ that satisfies $\bigwedge \sigma$.

Scientists A scientist is represented by any partial or total mapping of *SEQ* into subclasses of structures. That is, if scientist Ψ is defined on $\sigma \in SEQ$, then $\Psi(\sigma)$ is a collection of structures, thus a proposition.[2]

[2]More precisely, $\Psi(\sigma)$ is proposition unless $\Psi(\sigma) = \emptyset$, since by Definition (1) propositions are non-null. We have denied the status "proposition" to the empty set in order to simplify the formulation of subsequent theorems and definitions.

Success The definition that follows requires scientists to reach stable belief in the one true proposition of $\mathbf{P}$, namely, the proposition that includes the structure presented to the scientist.

(10) DEFINITION: Let scientist Ψ be given.

(a) Let environment e for proposition P be given. We say that Ψ *solves* P *in* e just in case for cofinitely many k, $\Psi(e[k]) = P$. We say that Ψ *solves* P just in case Ψ solves P in every environment for P.

(b) Let problem $\mathbf{P}$ be given. We say that Ψ *solves* $\mathbf{P}$ just in case Ψ solves every member of $\mathbf{P}$. In this case we say that $\mathbf{P}$ is *solvable*, and otherwise *unsolvable*.

Unraveling the definitions, we see that solving $\mathbf{P}$ requires solving every $P \in \mathbf{P}$ in every environment for P. Equivalently: Ψ solves $\mathbf{P}$ just in case for every $P \in \mathbf{P}$, every $\mathcal{S} \in \mathbf{P}$, and every environment e for $\mathcal{S}$, there are cofinitely many k such that $\Psi(e[k]) = P$.

For example, the problem $\mathbf{P}$ specified in Example (2) is solvable, as will be seen in Example (17) below. In contrast, it can be shown that the problem whose cosets are all structures of given (countable) cardinality is not solvable. For the latter fact, and many other examples of solvable and unsolvable problems, see [Martin & Osherson, 1998].

2.3 Tip-offs and solvability

We now state a necessary and sufficient condition for solvability that will be central to our analysis of enumerative induction. The condition requires the following definition.

(11) DEFINITION: By a π-*set* is meant any collection of $\forall$ formulas all of whose free variables are drawn from the same finite set. Let problem $\mathbf{P}$ and $P \in \mathbf{P}$ be given. A *tip-off* for P in $\mathbf{P}$ is a countable collection $\boldsymbol{t}$ of π-sets such that:

(a) for every $\mathcal{S} \in P$ and full assignment h to $\mathcal{S}$, there is $\pi \in \boldsymbol{t}$ with $\mathcal{S} \models \pi[h]$;

(b) for all $\mathcal{U}$ and P' with $P' \in \mathbf{P}$, $\mathcal{U} \in P'$, and $P' \neq P$, all full assignments g to $\mathcal{U}$, and all $\pi \in \boldsymbol{t}$, $\mathcal{U} \not\models \pi[g]$.

If every member of $\mathbf{P}$ has a tip-off in $\mathbf{P}$, then we say that $\mathbf{P}$ *has tip-offs.*

The following theorem is proved in [Martin & Osherson, 1998, Sec. 3.2].

(12) THEOREM: A problem is solvable if and only if it is countable and has tip-offs.

3 Two kinds of induction by enumeration

To implement enumerative induction in the first-order paradigm, the first idea might be as follows. Given a countable problem $\mathbf{P}$, fix an enumeration E of the propositions in $\mathbf{P}$. Then for any $\sigma \in SEQ$ for $\mathbf{P}$, conjecture the E-first proposition that is consistent with σ. It is clear, however, that this strategy fails to solve $\mathbf{P}$ of Example (2) since both members of $\mathbf{P}$ are consistent with every σ for $\mathbf{P}$.

3.1 Discrete enumerative induction

Given a problem $\mathbf{P}$, a more promising method works as follows. First, a set of formulas is enumerated. Then for any $\sigma \in SEQ$, the first formula ψ is sought such that $MOD(\bigwedge \sigma \wedge \psi) \cap \bigcup \mathbf{P}$ is a nonempty subset of some $P \in \mathbf{P}$. This P is conjectured. Intuitively, the scientist proceeds down the ordering of formulas, looking for the first one that (in conjunction with the data σ) picks out an admissible conjecture i.e., a member of $\mathbf{P}$. Generalizing this idea slightly leads to the following definition.

(13) DEFINITION: Let problem $\mathbf{P}$ be given, and let O be a well ordering of a set of formulas. We define the scientist $\Delta[\mathbf{P}, O]$ as follows. Let $\sigma \in SEQ$ be given.

Case 1: There exists a first $\psi \in O$ such that:

(*) for some $P \in \mathbf{P}$, $\emptyset \neq \{\mathcal{S} \in \bigcup \mathbf{P} \mid \bigwedge \sigma \wedge \psi \text{ is satisfiable in } \mathcal{S}\} \subseteq P$.

Then this P is unique (since the members of $\mathbf{P}$ are disjoint), and $\Delta[\mathbf{P}, O] = P$.

Case 2: There is no $\psi \in O$ that satisfies (*). Then $\Delta[\mathbf{P}, O]$ is undefined.

Let $X \subseteq \mathcal{L}_{form}$ be given. We define the *discrete X-type* of $\mathbf{P}$ as follows. If for every well-ordering O of a subset of X, the scientist $\Delta[\mathbf{P}, O]$ does not solve $\mathbf{P}$, then the discrete X-type of $\mathbf{P}$ is undefined. Otherwise, the discrete X-type of $\mathbf{P}$ is the first ordinal α such that for some well-ordering O of a subset of X,

(a) the well order type of O is α, and

(b) $\Delta[\mathbf{P}, O]$ solves $\mathbf{P}$.

The definition holds the promise of a bidimensional classification of solvable problems. Given a problem's discrete X-type, one dimension records the quantifier complexity figuring in the formulas of X. The other records the lowest ordinal (if there is one) needed to arrange X successfully. It will turn out, however, that both dimensions collapse considerably, and that not every solvable problem can be sucessfully approached by discrete enumerative induction [see Proposition (60), below].

3.2 Segmental enumerative induction

To define a more successful kind of enumerative induction we must allow the scientist to pick out propositions via infinite sets of formulas (the discrete version offers just one formula at a time). For this purpose we rely on the following notation.

(14) Definition: Let O be a well ordering of a set of formulas. Let $\sigma \in \mathit{SEQ}$ and initial segment $\mathbf{s}$ of O be given. We denote by $\mathit{satform}(\mathbf{s}, \sigma)$ the set of satisfiable formulas of the form $\bigwedge \sigma \wedge \psi$, where $\psi \in \mathbf{s}$.

Thus, $\mathit{satform}(\mathbf{s}, \sigma)$ gathers together all the formulas in the segment $\mathbf{s}$ of O that are consistent with σ. Such sets are useful because of the following (easy) fact.

(15) Lemma: Let problem $\mathbf{P}$ be given, and let O be a well ordering of a set of formulas. Let $\sigma \in \mathit{SEQ}$ also be given. Then there is at most one $P \in \mathbf{P}$ such that for some initial segment $\mathbf{s}$ of O, $\emptyset \neq \{\mathcal{S} \in \bigcup \mathbf{P} \mid \mathit{satform}(\mathbf{s}, \sigma) \text{ is satisfiable in } \mathcal{S}\} \subseteq P$.

So, given a problem $\mathbf{P}$ and an ordering O of formulas, the scientist can search in O for the first initial segment $\mathbf{s}$ such that $\mathit{satform}(\mathbf{s}, \sigma)$ picks out a proposition in $\mathbf{P}$. This is the idea behind the following definition.

(16) Definition: Let problem $\mathbf{P}$ be given, and let O be a well ordering of a set of formulas. We define the scientist $\Psi[\mathbf{P}, O]$ as follows. Let $\sigma \in \mathit{SEQ}$ be given.

Case 1: There exists an initial segment $\mathbf{s}$ of O such that:

(**) for some $P \in \mathbf{P}$, $\emptyset \neq \{\mathcal{S} \in \bigcup \mathbf{P} \mid \mathit{satform}(\mathbf{s}, \sigma) \text{ is satisfiable in } \mathcal{S}\} \subseteq P$.

Then by Lemma (15) this P is unique and we set $\Psi[\mathbf{P}, O] = P$.

Case 2: There is no initial segment $\mathbf{s}$ of O that satisfies (**). Then $\Psi[\mathbf{P}, O]$ is undefined.

Let $X \subseteq \mathcal{L}_{form}$ be given. We define the *segmental X-type* of $\mathbf{P}$ as follows. If for every well-ordering O of a subset of X, the scientist $\Psi[\mathbf{P}, O]$ does not solve $\mathbf{P}$, then the segmental X-type of $\mathbf{P}$ is undefined. Otherwise, the segmental X-type of $\mathbf{P}$ is the first ordinal α such that for some well-ordering O of a subset of X,

(a) the well order type of O is α, and

(b) $\Psi[\mathbf{P}, O]$ solves $\mathbf{P}$.

It is not immediately evident how to compare the inductive powers of discrete versus segmental methods. Given an ordering O, initial segment $\mathbf{s}$ of O, and $\sigma \in \mathit{SEQ}$, $\mathit{satform}(\mathbf{s}, \sigma)$ may be inconsistent, and thus be useless for picking out propositions. Perhaps there are solvable problems for which this difficulty arises inevitably. But in fact such is not the case; we will see that every solvable problem can be solved via segmental enumerative induction.

For notational simplicity, we denote the class of universal formulas by $\forall$, and similarly for other quantifier prefixes. Observe that both the discrete and segmental types of unsolvable problems are undefined (and if either is defined, the problem is solvable). We will see later that the discrete type of some solvable problems is also undefined.

(17) EXAMPLE: Let $\mathbf{P}$ be as in Example (2). Given $n \in N$, set $\psi_{2n} = \forall y R v_n y$ and $\psi_{2n+1} = \forall x R x v_n$. Let $E = \{\psi_n \mid n \in N\}$. Then both $\Delta[\mathbf{P}, E]$ and $\Psi[\mathbf{P}, E]$ solve $\mathbf{P}$. Indeed, let $\mathcal{S} \in MOD(T \cup \{\theta\})$, full assignment h to $\mathcal{S}$, and environment e for $\mathcal{S}$ and h be given (the proof is parallel if $\mathcal{S} \in MOD(T \cup \{\neg\theta\})$). So there is least $n_0 \in N$ such that $\mathcal{S} \models \psi_{2n_0}[h]$ and there is $k_0 \in N$ such that for all $n < 2n_0$, $\bigwedge e[k_0] \wedge \psi_n$ is inconsistent. Hence for all $k \geq k_0$, $2n_0$ is the least $n \in N$ such that $\bigwedge e[k] \wedge \psi_n$ is consistent. From this it is easy to verify that for all $k \geq k_0$, $\Delta[\mathbf{P}, E](e[k]) = \Psi[\mathbf{P}, E](e[k]) = MOD(T \cup \{\theta\})$, which proves that both $\Delta[\mathbf{P}, E]$ and $\Psi[\mathbf{P}, E]$ solve $MOD(T \cup \{\theta\})$ in e. Hence, both the discrete and segmental $\forall$-types of $\mathbf{P}$ are bounded by ω.

We can exploit the example to make a useful point about types. The formulas that fix a type might require free variables; there may be no comparable ordering of sentences (closed formulas) that allows enumerative induction to proceed. The pitfall for sentences is that free variables may be needed to refer to specific objects denoted by variables in an environment. This is illustrated in extreme form for the discrete case by the next proposition. It shows that restriction to sentences can foreclose enumerative induction for a problem where it might have been successfully applied.

(18) PROPOSITION: Suppose that **Sym** consists of a binary predicate. Then there is a problem $\mathbf{P}$ such that:

(a) the discrete $\forall$-type of $\mathbf{P}$ is defined (hence $\mathbf{P}$ is solvable);

(b) the discrete $\mathcal{L}_{sen}$-type of $\mathbf{P}$ is undefined.

Proof: Let R be the binary predicate of **Sym**. Let T be the theory of total orders (with respect to R) with either a least point or a greatest point (but not both). Let $\theta = \exists x \forall y Rxy$. We show that $\mathbf{P} = \{MOD(T \cup \{\theta\}), MOD(T \cup \{\neg\theta\})\}$, satisfies the claim of the proposition. Clause (a) has been proved in Example (17). We prove (b). Let O be any well ordering of some set X of sentences. If no sentence in X is true in any member of $\bigcup \mathbf{P}$, then $\Delta[\mathbf{P}, O] = \emptyset$ and we are done. Otherwise, let sentence ψ be the O-least member of X which is true in some $\mathcal{S} \in \bigcup \mathbf{P}$. Suppose that $\mathcal{S} \in MOD(T \cup \{\theta\})$ (the proof is parallel if $\mathcal{S} \in MOD(T \cup \{\neg\theta\})$). Let $\sigma \in SEQ$ be for $\mathbf{P}$. Because all the models of T are infinite, $\bigwedge \sigma$ is satisfiable in $\mathcal{S}$, hence

$\bigwedge \sigma \wedge \psi$ is satisfiable in $\mathcal{S}$ (since ψ is closed). This shows that for all $\sigma \in SEQ$ which are for $\mathbf{P}$, either $\Delta[\mathbf{P}, O](\sigma) = MOD(T \cup \{\theta\})$ or $\Delta[\mathbf{P}, O](\sigma)$ is undefined. Hence $\Delta[\mathbf{P}, O]$ does not solve $MOD(T \cup \{\neg\theta\})$, so $\Delta[\mathbf{P}, O]$ does not solve $\mathbf{P}$. ■

4 Tip-off bases

To launch our investigation of discrete and segmental enumerative induction, a technical tool is needed. It can be deployed when a problem enjoys tip-offs of a particularly simple character.

(19) Definition: Let set X of $\forall$ formulas and problem $\mathbf{P}$ be given. We say that X is a *tip-off base for* $\mathbf{P}$ just in case:

(a) for every $\mathcal{S} \in \bigcup \mathbf{P}$ and full assignment h to $\mathcal{S}$, there is $\varphi \in X$ such that $\mathcal{S} \models \varphi[h]$;

(b) for all $\varphi \in X$, there is at most one $P \in \mathbf{P}$ such that φ is satisfiable in some member of P.

There are two natural classes of problems that often arise in the first-order paradigm. When solvable they turn out to be distinguishable in terms of tip-off bases. The classes are defined as follows.

(20) Definition:

(a) Let problem $\mathbf{P}$, $T \subseteq \mathcal{L}_{sen}$, and $\theta_0 \ldots \theta_n \in \mathcal{L}_{sen}$ be given. We say that $\mathbf{P}$ has the form $(T, \{\theta_0 \ldots \theta_n\})$ just in case:

i. for every model $\mathcal{S}$ of T there is exactly one $i \in \{0 \ldots n\}$ such that $\mathcal{S} \models \theta_i$;

ii. $\mathbf{P} = \{MOD(T \cup \{\theta_i\}) \,|\, 0 \leq i \leq n\}$.

(b) Let $T \subseteq \mathcal{L}_{sen}$ be given. We say that problem $\mathbf{P}$ has the form $(T, \{P_0, P_1, \ldots\})$ just in case $\mathbf{P} = \{P_0, P_1, \ldots\}$ and $\bigcup \mathbf{P} = MOD(T)$.

Problems of both forms are discussed extensively in [Martin & Osherson, 1998]. For present purposes the relevant facts are given in the next two lemmas.

(21) Lemma: Every solvable problem of form $(T, \{\theta_0 \ldots \theta_n\})$ has a tip-off base.

Proof: Let a solvable problem of form $(T, \{\theta_0 \ldots \theta_n\})$ be given. By [Martin & Osherson, 1998, Theorem (55), p. 81], for all $m \leq n$, θ_m is equivalent in T to an $\exists\forall$ sentence. It follows easily that the problem has a tip-off base. ■

(22) LEMMA: No infinite problem of form $(T, \{P_0, P_1, \ldots\})$ has a tip-off base.

Proof: Let $T \subseteq \mathcal{L}_{sen}$ and disjoint propositions $P_0, P_1 \ldots$ be such that $MOD(T) = \bigcup\{P_i \mid i \in N\}$. Suppose for a contradiction that set $X = \{\varphi_i \mid i \in N\}$ of $\forall$ formulas is a tip-off base for $\mathbf{P} = \{P_0, P_1 \ldots\}$. For all $i \in N$, let ψ_i be the conjunction of the universal closure of $\neg\varphi_0 \wedge \ldots \wedge \neg\varphi_{i-1}$ with the existential closure of φ_i. Trivially:

(23) For all $i, j \in N$, if $i \neq j$ then $MOD(\psi_i) \cap MOD(\psi_j) = \emptyset$.

Let $P \in \mathbf{P}$ and $\mathcal{S} \in P$ be given. By Definition (19)a, there is least $i \in N$ such that φ_i is satisfiable in $\mathcal{S}$. Hence ψ_i is true in $\mathcal{S} \in P$. By Definition (19)b, φ_i is satisfiable in no $P' \in \mathbf{P}$ with $P' \neq P$. Hence ψ_i is false in all members of $P' \in \mathbf{P}$, $P' \neq P$. Since $\mathbf{P}$ is infinite, this proves that:

(24) There are infinitely many $i \in N$ with $MOD(T \cup \{\psi_i\}) \neq \emptyset$.

Moreover, since $\bigcup \mathbf{P} = MOD(T)$, the preceding facts imply:

(25) $\bigcup\{MOD(T \cup \{\psi_i\}) \mid i \in N\} = MOD(T)$.

From (23) and (24), we infer that $T \cup \{\neg\psi_0 \ldots \neg\psi_i\}$ is satisfiable for all $i \in N$. With compactness, this implies that $T \cup \{\neg\psi_i \mid i \in N\}$ is satisfiable, which contradicts (25). ■

Now we can begin to harness the concept of tip-off base. The next proposition shows that every problem with a tip-off base lends itself easily to enumerative induction of the discrete kind.

(26) PROPOSITION: Let set X of $\forall$ formulas be a tip-off base for problem $\mathbf{P}$. Then for every enumeration E of $\forall$ formulas with $X \subseteq range(E)$, the scientist $\Delta[\mathbf{P}, E]$ solves $\mathbf{P}$.

Proof: Let X be a tip-off base for $\mathbf{P}$, and let $E = \{\varphi_i \mid i \in N\}$ be an enumeration of $\forall$ formulas with $X \subseteq range(E)$. Let $P \in \mathbf{P}$, $\mathcal{S} \in P$, full assignment h to $\mathcal{S}$, and environment e for $\mathcal{S}$ and h be given. We will show that $\Delta[\mathbf{P}, E]$ solves P in e, thus proving that $\Delta[\mathbf{P}, E]$ solves $\mathbf{P}$.

By Definition (19), there is least $i_0 \in N$ such that:

(a) $\mathcal{S} \models \varphi_{i_0}[h]$, and

(b) φ_{i_0} is satisfiable in no $P' \in \mathbf{P}$ with $P' \neq P$.

Let $k_0 \in N$ be such that for all $i \leq i_0$, if $\mathcal{S} \models \neg\varphi_i[h]$ then $\bigwedge e[k_0] \models \neg\varphi_i$. By the choice of i_0:

(a) for all $i < i_0$ and all $k \geq k_0$, if $\bigwedge e[k] \not\models \neg\varphi_i$ then φ_i is satisfiable in at least two structures taken from two distinct propositions in $\mathbf{P}$;

(b) for all $k \geq k_0$, $range(e[k]) \cup \{\varphi_{i_0}\}$ is satisfiable in $\mathcal{S} \in P$, and φ_{i_0} is satisfiable in no member of P', for all $P' \in \mathbf{P}$, $P' \neq P$.

This implies immediately that for all $k \geq k_0$, $\Delta[\mathbf{P}, E](e[k]) = P$. So $\Delta[\mathbf{P}, E]$ solves P in e. ■

From Lemma (21) and Proposition (26) it is easy to derive the following.

(27) Corollary: Let solvable problem $\mathbf{P}$ of form $(T, \{\theta_0 \ldots \theta_n\})$ be given. Then for every enumeration E of all $\forall$ formulas, the scientist $\Delta[\mathbf{P}, E]$ solves $\mathbf{P}$.

The corollary reveals that discrete enumerative induction is strikingly easy for problems of the form $(T, \{\theta_0 \ldots \theta_n\})$. There is no need for an astute choice of formulas nor for a clever way to order them. Any ω-ordering of the universal formulas does the trick.

Proposition (26) informs us that the existence of a tip-off base is a sufficient condition for the solvability of a problem via discrete enumerative induction. But it is not necessary. As shown by the next proposition (whose proof is deferred to Section 6), there are problems without a tip-off base that can nevertheless be solved via enumerative induction of the discrete kind.

(28) Proposition: Suppose that **Sym** consists of a unary function symbol and a constant. Then there is a solvable problem $\mathbf{P}$ with the following properties.

(a) $\mathbf{P}$ has no tip-off base.

(b) The discrete $\forall$-type of $\mathbf{P}$ is defined.

The problem to be exhibited in the proof of Proposition (28) has infinite discrete $\forall$-type. The next proposition shows this to be no accident.

(29) Proposition: Every problem with finite discrete $\forall$-type has a tip-off base.

Proof: Let problem $\mathbf{P}$ with finite discrete $\forall$-type be given. Let finite enumeration E of $\forall$ formulas be such that $\Delta[\mathbf{P}, E]$ solves $\mathbf{P}$. Define X to be the set of all formulas of form $\bigwedge \sigma \wedge \varphi$, $\sigma \in SEQ$, $\varphi \in E$, satisfying the following:

(30) there is at most one $P \in \mathbf{P}$ such that $\bigwedge \sigma \wedge \varphi$ is satisfiable in some member of P.

It suffices to show that X is a tip-off base for $\mathbf{P}$. By (30) this is proved if we show that for all $\mathcal{S} \in \bigcup \mathbf{P}$ and full assignments h to $\mathcal{S}$, there is $\psi \in X$ with $\mathcal{S} \models \psi[h]$. So let $P \in \mathbf{P}$, $\mathcal{S} \in P$, and full assignment h to $\mathcal{S}$ be given. Since $\Delta[\mathbf{P}, E]$ solves $\mathbf{P}$, there is $k_0 \in N$ and $\varphi \in E$ such that:

(31) for all $k \geq k_0$, $\emptyset \neq \{\mathcal{T} \in \bigcup \mathbf{P} \mid \bigwedge e[k] \wedge \varphi$ is satisfiable in $\mathcal{T}\} \subseteq P$.

We derive immediately from (31) that $\bigwedge e[k_0] \wedge \varphi$ belongs to X. Hence it suffices to show that $\mathcal{S} \models (\bigwedge e[k_0] \wedge \varphi)[h]$. From (31) again we infer that $\bigwedge e[k] \wedge \varphi$ is satisfiable for all $k \in N$. Hence by compactness, *range*$(e) \cup \{\bigwedge e[k_0] \wedge \varphi\}$ is satisfiable. Since $\bigwedge e[k_0] \wedge \varphi$ is a $\forall$ formula and e is an environment for $\mathcal{S}$ and h, this implies that $\mathcal{S} \models (\bigwedge e[k_0] \wedge \varphi)[h]$, as required. ■

5 Finite types

Discrete enumerative induction has an elementary character if it involves ordering no more than a finite number of universal formulas. We expect problems solvable by such a method to be simple in some combinatorial sense. Proposition (29) satisfies this expectation; problems with finite discrete $\forall$-type enjoy tip-off bases in the sense of Definition (19). Does the same kind of simplicity characterize problems solvable by segmental enumerative induction over a finite ordering of universal formulas? The answer is affirmative because the same class of problems is at issue. Indeed, the next proposition shows that for every $n \in N$, the discrete $\forall$-type of a problem is n if and only if the segmental $\forall$-type of the problem is also n. (We subsequently show that all of these types are populated.)

(32) Proposition: For all problems $\mathbf{P}$, the discrete $\forall$-type of $\mathbf{P}$ is finite iff the segmental $\forall$-type of $\mathbf{P}$ is finite. Moreover, if finite, they are equal.

Proof: Let problem $\mathbf{P}$ be given. The discrete $\forall$-type of $\mathbf{P}$ is equal to 0 iff the segmental $\forall$-type of $\mathbf{P}$ is equal to 0 iff $\mathbf{P} = \emptyset$. So suppose $\mathbf{P} \neq \emptyset$. If $\mathbf{P}$ consists of a sole proposition, and if this proposition is the class of all structures, then it is easy to verify that the discrete and segmental $\forall$-types of $\mathbf{P}$ are both equal to 1. So suppose otherwise. It suffices to show that if the discrete $\forall$-type of $\mathbf{P}$ is equal to $p > 0$, then the segmental $\forall$-type of $\mathbf{P}$ is at most equal to p, and if the segmental $\forall$-type of $\mathbf{P}$ is equal to $p > 0$, then the discrete $\forall$-type of $\mathbf{P}$ is at most equal to p. Let $n \in N$ and enumeration $E = \{\varphi_i \mid i \leq n\}$ of $\forall$ formulas be such that $\Delta[\mathbf{P}, E]$ solves $\mathbf{P}$. We show that $\Psi[\mathbf{P}, E]$ solves $\mathbf{P}$, thus proving that if the discrete $\forall$-type of $\mathbf{P}$ is finite, then the segmental $\forall$-type of $\mathbf{P}$ is at most equal to the latter. Let $P \in \mathbf{P}$, $\mathcal{S} \in P$, full assignment h to $\mathcal{S}$, and environment e for $\mathcal{S}$ and h be given. It suffices to show that $\Psi[\mathbf{P}, E]$ solves P in e. Since $\Delta[\mathbf{P}, E]$ solves $\mathbf{P}$ (and E is finite), there is $k_0 \in N$ and $n_0 \leq n$ such that:

(33) for all $k \geq k_0$, $\emptyset \neq \{\mathcal{T} \in \bigcup \mathbf{P} \mid \bigwedge e[k] \wedge \varphi_{n_0}$ is satisfiable in $\mathcal{T}\} \subseteq P$.

In particular, (33) implies that $\bigwedge e[k] \wedge \varphi_{n_0}$ is satisfiable for all $k \in N$. Hence by compactness, *range*$(e) \cup \{\varphi_{n_0}\}$ is satisfiable. Since φ_{n_0} is a $\forall$ formula and e is an environment for $\mathcal{S}$ and h, we infer that $\mathcal{S} \models \varphi_{n_0}[h]$. Let X be the set of all $\varphi \in \{\varphi_0 \dots \varphi_{n_0}\}$ such that $\mathcal{S} \models \varphi[h]$. We have thus shown that $\varphi_{n_0} \in X$. With (33), this implies that:

(34) for all $k \geq k_0$, $\emptyset \neq \{\mathcal{T} \in \bigcup \mathbf{P} \mid \mathit{range}(e[k]) \cup X$ is satisfiable in $\mathcal{T}\} \subseteq P$.

Let $k_1 \geq k_0$ be such that for all $\varphi \in \{\varphi_0 \dots \varphi_{n_0}\} - X$, $\bigwedge e[k_1] \models \neg\varphi$. Then for all $k \geq k_1$ and for all initial segments $\mathbf{s}$ of E, $\{\bigwedge e[k] \wedge \varphi \mid \varphi \in \{\varphi_0 \dots \varphi_{n_0}\} - X\} \cap \mathit{satform}(\mathbf{s}, e[k]) = \emptyset$. With (34), this implies that for all $k \geq k_1$, $\Psi[\mathbf{P}, E](e[k]) = P$. Hence $\Psi[\mathbf{P}, E]$ solves P in e, as required.

Conversely, let $n \in N$ and enumeration $E = \{\varphi_i \mid i \leq n\}$ of $\forall$ formulas be such that $\Psi[\mathbf{P}, E]$ solves $\mathbf{P}$. For all $i \leq n$, denote by ψ_i the disjunction of all the conjunctions of subsets of $\{\varphi_0 \dots \varphi_n\}$ whose cardinality is equal to $n + 1 - i$ (conjunctions are written in ascending order of indexes of the φ_i's). For instance:

$$\psi_0 = \bigwedge_{i \leq n} \varphi_i, \quad \psi_1 = \bigvee_{i \leq n} \bigwedge_{\substack{j \leq n \\ j \neq i}} \varphi_j, \text{ and } \quad \psi_n = \bigvee_{i \leq n} \varphi_i.$$

Note that all of the ψ_i's are $\forall$ formulas. Set $F = \{\psi_i \mid i \leq n\}$. We show that $\Delta[\mathbf{P}, F]$ solves $\mathbf{P}$, thus proving that if the segmental $\forall$-type of $\mathbf{P}$ is finite, then the discrete $\forall$-type of $\mathbf{P}$ is at most equal to the latter. Let $P \in \mathbf{P}$, $\mathcal{S} \in P$, full assignment h to $\mathcal{S}$, and environment e for $\mathcal{S}$ and h be given. It suffices to show that $\Delta[\mathbf{P}, F]$ solves P in e. Since $\Psi[\mathbf{P}, E]$ solves $\mathbf{P}$ (and E is finite), there is $k_0 \in N$ and $X \subseteq \{\varphi_0 \dots \varphi_n\}$ such that:

(35) for all $k \geq k_0$, $\emptyset \neq \{\mathcal{T} \in \bigcup \mathbf{P} \mid \mathit{range}(e[k]) \cup X$ is satisfiable in $\mathcal{T}\} \subseteq P$.

Since by hypothesis $\mathbf{P}$ does not consist of a sole proposition equal to the class of all structures, it is clear from Definition (16) that:

(36) $X \neq \emptyset$.

Moreover, (35) implies that $\mathit{range}(e[k]) \cup X$ is consistent for all $k \in N$. Hence by compactness, $\mathit{range}(e) \cup X$ is satisfiable. Since X is a set of $\forall$ formulas and e is an environment for $\mathcal{S}$ and h, we infer that $\mathcal{S} \models X[h]$. Let Y^+ (respectively Y^-) be the set of all $\varphi \in \{\varphi_0 \dots \varphi_n\}$ such that $\mathcal{S} \models \varphi[h]$ (respectively $\mathcal{S} \not\models \varphi[h]$). We have thus shown that:

(37) $X \subseteq Y^+$.

Set $i_0 = \mathit{card}(Y^-)$. By (36) and (37), $i_0 \leq n$. By a finite pigeon hole argument it follows easily that:

(38) (a) for all $i < i_0$, every subset of $\{\varphi_0 \ldots \varphi_n\}$ whose cardinality is equal to $n+1-i$ has a nonempty intersection with Y^-;

(b) Y^+ is the only subset of $\{\varphi_0 \ldots \varphi_n\}$ whose cardinality is equal to $n+1-i_0$ which has an empty intersection with Y^-.

Let $k_1 \geq k_0$ be such that for all $\varphi \in Y^-$, $\bigwedge e[k_1] \models \neg\varphi$. It follows immediately from (38) and the definition of the ψ_i's, $i \leq n$, that:

(39) (a) for all $i < i_0$ and $k \geq k_1$, $\bigwedge e[k] \wedge \psi_i$ is unsatisfiable;

(b) for all $k \geq k_1$, $\bigwedge e[k] \wedge \psi_{i_0}$ is equivalent to $\bigwedge e[k] \wedge \bigwedge Y^+$.

From (35), (37) and (39), we infer that $\Delta[\mathbf{P}, F](e[k]) = P$ for all $k \geq k_1$. Hence $\Delta[\mathbf{P}, F]$ solves P in e, as required. ■

For every $n \in N$, the problems with discrete $\forall$-type equal to n are the same as those with segmental $\forall$-type equal to n. This is the content of the preceding proposition. The proposition would not be informative if there were no such problems. But the finite types are in fact rich, as the next proposition reveals.

(40) Proposition: Suppose that $\mathbf{Sym} = \emptyset$. For all $n \in N$ there is a problem whose discrete and segmental $\forall$-types are n.

Proof: The empty problem satisfies the claim of the proposition for $n = 0$. Given $n > 0$, denote by φ_n the sentence $\forall x_0 \ldots x_n (\bigvee_{0 \leq i < j \leq n} x_i = x_j)$ ("there are at most n elements"). For all $n > 0$, let P_n be the class of structures whose cardinality is n. Let $n > 0$ be given. By Proposition (32) it suffices to show that the problem $\mathbf{P} = \{P_1 \ldots P_n\}$ has discrete $\forall$-type equal to n. Set $E = \{\varphi_1 \ldots \varphi_n\}$. We first show that $\Delta[\mathbf{P}, E]$ solves $\mathbf{P}$, thus proving that the discrete $\forall$-type of $\mathbf{P}$ is at most equal to n. Let $1 \leq m \leq n$ and environment e for P_m be given. It suffices to show that $\Delta[\mathbf{P}, E]$ solves P_m in e. Let $k_0 \in N$ be such that $\bigwedge e[k_0]$ implies that there are at least m distinct elements in the domain of the underlying structure. Trivially, for all $k \geq k_0$, $\bigwedge e[k] \wedge \varphi_p$ is unsatisfiable for all $1 \leq p < m$, and $\emptyset \neq \{\mathcal{S} \in \bigcup \mathbf{P} \mid \bigwedge e[k] \wedge \varphi_m \text{ is satisfiable in } \mathcal{S}\} \subseteq P_m$. Hence for all $k \geq k_0$, $\Delta[\mathbf{P}, E](e[k]) = P_m$. So $\Delta[\mathbf{P}, E]$ solves P_m in e, as required.

We now show that the discrete $\forall$-type of $\mathbf{P}$ is at least equal to n, thus completing the proof. Let $m > 0$ and enumeration $F = \{\psi_1 \ldots \psi_m\}$ of $\forall$ formulas be such that $\Delta[\mathbf{P}, F]$ solves $\mathbf{P}$. It suffices to show that $n \leq m$. Suppose that sequence $\{\sigma_p \,|\, 1 \leq p \leq n\}$ of members of *SEQ* and function $f : \{1 \ldots n\} \to \{1 \ldots m\}$ satisfy:

(41) (a) for all $1 \leq p < n$, $\sigma_p \subseteq \sigma_{p+1}$;

(b) for all $1 \leq p \leq n$, $\emptyset \neq \{\mathcal{S} \in \bigcup \mathbf{P} \mid \bigwedge \sigma_p \wedge \psi_{f(p)} \text{ is satisfiable in } \mathcal{S}\} \subseteq P_p$.

Since the P_p's are pairwise disjoint, it is easy to see that f is one-to-one, which implies that $n \leq m$. So we only have to build a sequence $\{\sigma_p \mid 1 \leq p \leq n\}$ of members of *SEQ* and a function $f : \{1 \ldots n\} \rightarrow \{1 \ldots m\}$ that satisfy (41). We proceed by induction on p. For $p = 1$ let e_1 be an environment for P_1. Since $\Delta[\mathbf{P}, F]$ solves $\mathbf{P}$ there is $k \in N$ and $1 \leq i \leq m$ such that $\emptyset \neq \{\mathcal{S} \in \bigcup \mathbf{P} \mid \bigwedge e_1[k] \wedge \psi_i$ is satisfiable in $\mathcal{S}\} \subseteq P_1$. Set $\sigma_1 = e_1[k]$ and $f(1) = i$. Observe that σ_1 is an initial segment of an environment for P_1. For the induction step $p > 1$ suppose that $\sigma_1 \ldots \sigma_{p-1}$ and $f(1) \ldots f(p-1)$ have been defined and that σ_{p-1} is an initial segment of an environment for P_{p-1}. Let environment e for P_p be given, extending σ_{p-1}; there is such an environment since the cardinality of structures in P_p is greater than that for structures in P_{p-1}. Since $\Delta[\mathbf{P}, F]$ solves $\mathbf{P}$, there is $k > \mathit{length}(\sigma_{p-1})$ and $1 \leq i \leq m$ with $\emptyset \neq \{\mathcal{S} \in \bigcup \mathbf{P} \mid \bigwedge e[k] \wedge \psi_i$ is satisfiable in $\mathcal{S}\} \subseteq P_p$. Set $\sigma_p = e[k]$ and $f(p) = i$. We see that $\{\sigma_p \mid 1 \leq p \leq n\}$ and f satisfy (41). ■

The proofs of Propositions (32) and (40) depend on the intimate connection between universal formulas and environments for a structure; any such formula false in the structure is inconsistent with each of its environments. [This is because the assignments underlying environments are required to be onto; see Definition (4).] One might hope that enumerative induction based only on finitely many formulas reduces to the case where all the formulas involved are $\forall$. But matters are not so simple. Indeed, it will be seen below that there are problems whose discrete and segmental $\forall\exists$ and $\exists\forall$-types are 2 although their $\forall$-types are not finite. Can anything general be said about finite types in this broader context? We can report only the following fact about the coincidence of finite types across the discrete/segmental divide. Here the quantifiers of enumerated formulas are not constrained at all.

(42) Proposition: For all problems $\mathbf{P}$, if the segmental $\mathcal{L}_{\mathit{form}}$-type of $\mathbf{P}$ is finite then the discrete $\mathcal{L}_{\mathit{form}}$-type of $\mathbf{P}$ is finite.

Proof: Let problem $\mathbf{P}$ have finite segmental $\mathcal{L}_{\mathit{form}}$-type. Let enumeration $E = \{\varphi_0 \ldots \varphi_n\}$ of formulas be such that $\Psi[\mathbf{P}, E]$ solves $\mathbf{P}$. Without loss of generality we can suppose that $\varphi_0 = (x = x)$. Denote by F the enumeration of $\{\varphi_{i_0} \wedge \ldots \wedge \varphi_{i_p} \mid p \in N,\ 0 \leq i_0 < \ldots < i_p \leq n\}$ in lexicographical order. (For instance, if $n = 5$ then $\varphi_0 \wedge \varphi_2$ comes before $\varphi_0 \wedge \varphi_2 \wedge \varphi_3$, which comes before φ_2.) It suffices to show that $\Delta[\mathbf{P}, F]$ solves $\mathbf{P}$. Let $P \in \mathbf{P}$, $\mathcal{S} \in P$, full assignment h to $\mathcal{S}$, and environment e for $\mathcal{S}$ and h be given. It suffices to show that $\Delta[\mathbf{P}, F]$ solves P in e. Since $\Psi[\mathbf{P}, E]$ solves $\mathbf{P}$ [and E is finite and begins with $(x = x)$], there is $m \leq n$ and nonempty $X \subseteq \{\varphi_0 \ldots \varphi_m\}$ such that the following holds for cofinitely many k:

(43) (a) $\emptyset \neq \{\mathcal{T} \in \bigcup \mathbf{P} \mid \mathit{range}(e[k]) \cup X$ is satisfiable in $\mathcal{T}\} \subseteq P$;
(b) for all $\emptyset \neq X' \subset X$, $\{\mathcal{T} \in \bigcup \mathbf{P} \mid \mathit{range}(e[k]) \cup X'$ is satisfiable in $\mathcal{T}\} \not\subseteq P$;
(c) for all $p \leq m$, if $\varphi_p \notin X$ then $\bigwedge e[k] \wedge \varphi_p$ is not satisfiable.

Let $\psi \in F$ be the conjunction of all members of X (written in ascending order of indexes of the φ_i's). Let $\chi \in F$ come before ψ in F. By the definition of F there are two cases:

Case 1: There exists $p \leq m$ such that φ_p occurs in χ but not in ψ. Then by (43)c, $\bigwedge e[k] \wedge \chi$ is unsatisfiable for cofinitely many k.

Case 2: For all $p \leq n$, if φ_p occurs in χ then φ_p occurs in ψ. Since all members of $\mathbf{P}$ are pairwise disjoint, it then follows from (43)a,b that for cofinitely many k, $\{\mathcal{T} \in \bigcup \mathbf{P} \mid \bigwedge e[k] \wedge \chi \text{ is satisfiable in } \mathcal{T}\} \not\subseteq P'$ for all $P' \in \mathbf{P}$.

By (43)a and the definition of ψ, $\emptyset \neq \{\mathcal{T} \in \bigcup \mathbf{P} \mid \bigwedge e[k] \wedge \psi \text{ is satisfiable in } \mathcal{T}\} \subseteq P$ for cofinitely many k. We conclude from the preceding facts that $\Delta[\mathbf{P}, F](e[k]) = P$ for cofinitely many k. Hence $\Delta[\mathbf{P}, F]$ solves P in e, as required. ■

6 Infinite types

The results of the previous section show that the discrete and segmental ∀-types coincide in the finite case. Does this situation extend to the infinite, and if so, for which ordinals? We'll derive the following answer: when both are defined, the two kinds of ∀-types coincide, and they are never greater than ω. The following proposition covers much of the distance to this result, and provides other useful information.

(44) PROPOSITION: For every solvable problem $\mathbf{P}$, there exists a set X of ∀ formulas such that for every enumeration E of X, the scientist $\Psi[\mathbf{P}, E]$ solves $\mathbf{P}$.

Proof: Let solvable problem $\mathbf{P} = \{P_j \mid j \in N\}$ be given. For all $j \in N$ let $\boldsymbol{t}_j$ be a tip-off for P_j in $\mathbf{P}$. By the countability of tip-offs, let the π-sets in $\bigcup_{j \in N} \boldsymbol{t}_j$ be enumerated as $\{\pi_i \mid i \in N\}$. For all $i \in N$ fix an enumeration $\{\varphi_i^n \mid n \in N\}$ of π_i. Then set:

$$X = \{(\varphi_0^0 \wedge \ldots \wedge \varphi_0^{n_0}) \vee \ldots \vee (\varphi_i^0 \wedge \ldots \wedge \varphi_i^{n_i}) \mid i, n_0 \ldots n_i \in N\}.$$

Denote by E any enumeration of X. Since X consists of ∀ formulas, to prove the proposition it suffices to show that $\Psi[\mathbf{P}, E]$ solves $\mathbf{P}$. Let $j \in N$, $\mathcal{S} \in P_j$, full assignment h to $\mathcal{S}$, and environment e for $\mathcal{S}$ and h be given. It suffices to show that $\Psi[\mathbf{P}, E]$ solves P_j in e. Let $i_0 \in N$ be least such that $\mathcal{S} \models \pi_{i_0}[h]$. For $i < i_0$, $\mathcal{S} \not\models \pi_i[h]$, so for each $i < i_0$ we may choose $c(i) \in N$ such that:

(45) (a) $\mathcal{S} \models (\varphi_i^0 \wedge \ldots \wedge \varphi_i^{c(i)-1})[h]$;

(b) $\mathcal{S} \not\models \varphi_i^{c(i)}[h]$.

Since for all $i < i_0$, $\varphi_i^{c(i)}$ is a ∀ formula, and since h is onto $\mid \mathcal{S} \mid$, it follows from (45)b that there is $k_0 > 0$ such that:

(46) for all $k \geq k_0$ and for all $i < i_0$, $\bigwedge e[k] \wedge \varphi_i^{c(i)}$ is unsatisfiable.

Let $Y = \{(\varphi_0^0 \wedge \ldots \wedge \varphi_0^{n_0}) \vee \ldots \vee (\varphi_i^0 \wedge \ldots \wedge \varphi_i^{n_i}) \mid i < i_0,\ n_0 \geq c(0) \ldots n_i \geq c(i)\}$. We deduce from (46) that:

(47) for all $k \geq k_0$ and for all $\varphi \in Y$, $\bigwedge e[k] \wedge \varphi$ is unsatisfiable.

From (45)a and the fact that $\mathcal{S} \models \pi_{i_0}[h]$ it follows easily that $\mathcal{S} \models X - Y[h]$. So:

(48) For all $k \geq k_0$, $range(e[k]) \cup (X - Y)$ is satisfiable.

Let $Z = \{(\varphi_0^0 \wedge \ldots \wedge \varphi_0^{c(0)}) \vee \ldots \vee (\varphi_{i_0-1}^0 \wedge \ldots \wedge \varphi_{i_0-1}^{c(i_0-1)}) \vee (\varphi_{i_0}^0 \wedge \ldots \wedge \varphi_{i_0}^n) \mid n \in N\}$. From (46) we obtain $range(e[k]) \cup Z \models \{\varphi_{i_0}^n \mid n \in N\}$ $(= \pi_{i_0})$ for all $k \geq k_0$. Hence, since $Z \subseteq X - Y$:

(49) for all $k \geq k_0$, $range(e[k]) \cup (X - Y) \models \pi_{i_0}$.

For $k \in N$, let $E(e[k])$ be the set of satisfiable formulas of the form $\bigwedge e[k] \wedge \psi$, $\psi \in E$ [as in Definition (14)]. It follows immediately from (47), (48) and (49) that:

(50) for all $k \geq k_0$, $E(e[k])$ is consistent and $E(e[k]) \models \pi_{i_0}$.

We infer from (50) that for all $k \geq k_0$, there is an initial segment $\mathbf{s}$ of E such that $\emptyset \neq \{\mathcal{T} \in \bigcup \mathbf{P} \mid satform(\mathbf{s}, e[k])$ is satisfiable in $\mathcal{T}\} \subseteq P_j$. It follows immediately from Definition (16) that for all $k \geq k_0$, $\Psi[\mathbf{P}, E](e[k]) = P_j$. Hence $\Psi[\mathbf{P}, E]$ solves P_j in e, as required. ■

We see from the proposition that segmental enumerative induction using ∀ formulas is a universal strategy of inference: every solvable problem can be solved this way. Moreover, designing a successful agent of this kind requires no more than specifying the right set X of ∀ formulas. No further insight is required for their ordering since any enumeration will do the job.

A similar order-independence characterizes discrete enumerative induction using ∀ formulas; and the ordinal is again bounded by ω. But it is necessary to qualify this fact by the proviso that the discrete ∀-type be defined for the problem in question; for, we will soon see solvable problems with undefined discrete types at every level of quantifier complexity. The discrete parallel to the preceding proposition can thus be stated as follows.

(51) Proposition: Let solvable problem $\mathbf{P}$ have defined discrete ∀-type. Then there exists a set X of ∀ formulas such that for every enumeration E of X, $\Delta[\mathbf{P}, E]$ solves $\mathbf{P}$.

The proposition follows directly from the proof of the following lemma, of interest in its own right.

(52) Lemma: For all solvable problems $\mathbf{P}$, the discrete $\forall$-type of $\mathbf{P}$ is defined if and only if the following condition holds:

(*) For all $P \in \mathbf{P}$, and all $\sigma \in \mathit{SEQ}$ for P, there is a $\forall$ formula φ such that $\emptyset \neq \{\mathcal{S} \in \bigcup \mathbf{P} \mid \bigwedge \sigma \wedge \varphi \text{ is satisfiable in } \mathcal{S}\} \subseteq P$.

Proof: Let solvable problem $\mathbf{P}$ be given. For the "only if" direction, suppose that (*) does not hold. Let $P \in \mathbf{P}$ and $\sigma \in \mathit{SEQ}$ for P be such that for all $\forall$ formulas φ, either $\{\mathcal{S} \in \bigcup \mathbf{P} \mid \bigwedge \sigma \wedge \varphi \text{ is satisfiable in } \mathcal{S}\} = \emptyset$ or $\{\mathcal{S} \in \bigcup \mathbf{P} \mid \bigwedge \sigma \wedge \varphi \text{ is satisfiable in } \mathcal{S}\} \not\subseteq P$. Then for all $\tau \in \mathit{SEQ}$ extending σ:

(53) for all $\forall$ formulas φ, either $\{\mathcal{S} \in \bigcup \mathbf{P} \mid \bigwedge \tau \wedge \varphi \text{ is satisfiable in } \mathcal{S}\} = \emptyset$ or $\{\mathcal{S} \in \bigcup \mathbf{P} \mid \bigwedge \tau \wedge \varphi \text{ is satisfiable in } \mathcal{S}\} \not\subseteq P$.

Let environment e for P with $\sigma \subseteq e$ be given. Then (53) implies immediately that for every set X of $\forall$ formulas, for every well ordering O of X, and for every $k \geq \mathit{length}(\sigma)$, $\Delta[\mathbf{P}, O](e[k])$ is undefined. Hence $\Delta[\mathbf{P}, O]$ does not solve P in e, so $\Delta[\mathbf{P}, O]$ does not solve $\mathbf{P}$. It follows that the discrete $\forall$-type of $\mathbf{P}$ is not defined.

For the "if" direction, suppose that (*) holds. Suppose that $\mathbf{P} \neq \emptyset$ (otherwise the discrete $\forall$-type of $\mathbf{P}$ is trivially equal to 0). We will define a set X of $\forall$ formulas and show that for every enumeration E of X, $\Delta[\mathbf{P}, E]$ solves $\mathbf{P}$. First we define X. Let $\kappa \leq \omega$ and propositions P_j, $j < \kappa$, be such that $\mathbf{P} = \{P_j \mid j < \kappa\}$ and for all $j < j' < \kappa$, P_j and $P_{j'}$ are distinct. Since $\mathbf{P}$ is solvable, for all $j < \kappa$ let $\boldsymbol{t}_j$ be a tip-off for P_j in $\mathbf{P}$. By the countability of tip-offs, let the π-sets in $\bigcup_{j<\kappa} \boldsymbol{t}_j$ be enumerated as $\{\pi_i \mid i \in N\}$. Without loss of generality we can assume that for all $i \in N$, π_i is satisfiable in some member of $\bigcup \mathbf{P}$.

Let $i \in N$ be given. Fix an enumeration $\{\varphi_i^n \mid n \in N\}$ of π_i and an enumeration $\{\alpha_i^n \mid n \in N\}$ of all $\alpha \in \mathcal{L}_{basic}$ such that $\pi_i \models \alpha$. By the definition of tip-offs, and since P_j and $P_{j'}$ are distinct for all $j < j' < \kappa$, there is a unique $j < \kappa$ such that π_i is satisfiable in some member of P_j. Fix an enumeration $\{\sigma_i^n \mid n \in N\}$ of all $\sigma \in \mathit{SEQ}$ such that $\pi_i \cup \mathit{range}(\sigma)$ is satisfiable in some member of P_j. Let $n \in N$ be given. We define a formula ψ_i^n as follows. It is easy to verify that there is $\sigma \in \mathit{SEQ}$ such that $\mathit{range}(\sigma) = \{\alpha_i^0 \dots \alpha_i^n\} \cup \mathit{range}(\sigma_i^n)$ and σ is satisfiable in some member of P_j. Hence, σ is for P_j. By (*) there exists a $\forall$ formula φ such that $\emptyset \neq \{\mathcal{S} \in \bigcup \mathbf{P} \mid \alpha_i^0 \wedge \dots \wedge \alpha_i^n \wedge \bigwedge \sigma_i^n \wedge \varphi \text{ is satisfiable in } \mathcal{S}\} \subseteq P_j$. We choose one such $\forall$ formula φ and denote it by ψ_i^n.

Given $i, j, n_0 \dots n_i \in N$, define the *formula of form* $(i, n_0 \dots n_i)$ to be:

(54) $(\varphi_0^0 \wedge \dots \wedge \varphi_0^{n_0}) \vee \dots \vee (\varphi_{i-1}^0 \wedge \dots \wedge \varphi_{i-1}^{n_{i-1}}) \vee (\alpha_i^0 \wedge \dots \wedge \alpha_i^{n_i} \wedge \bigwedge \sigma_i^{n_i} \wedge \psi_i^{n_i})$.

Let X be the set of formulas of form $(i, n_0 \dots n_i)$, $i, n_0 \dots n_i \in N$. Note that X consists of $\forall$ formulas. Let E be any enumeration of X. We show that $\Delta[\mathbf{P}, E]$ solves $\mathbf{P}$. Let $j < \kappa$, $\mathcal{S} \in P_j$, full assignment h to $\mathcal{S}$, and environment e for $\mathcal{S}$ and h be given. It suffices to show that $\Delta[\mathbf{P}, E]$ solves P_j in e. Let $i_0 \in N$ be least such that $\mathcal{S} \models \pi_{i_0}[h]$ (there must be such an i_0 by the definition of the π_i's). For $i < i_0$, $\mathcal{S} \not\models \pi_i[h]$, π_i is a set of $\forall$ formulas and h is onto $\mid \mathcal{S} \mid$. So for each $i < i_0$ we may choose $c(i), d(i) \in N$ such that:

(55) (a) $\mathcal{S} \models (\varphi_i^0 \wedge \ldots \wedge \varphi_i^{c(i)-1})[h]$;

(b) $\mathcal{S} \not\models \varphi_i^{c(i)}[h]$;

(c) $\mathcal{S} \not\models \alpha_i^{d(i)}[h]$.

Since for all $i < i_0$, $\varphi_i^{c(i)}$, $\alpha_i^{d(i)}$ and the ψ_i^n are $\forall$ formula, and since h is onto $\mid \mathcal{S} \mid$, it follows from (55)b,c and the definition of the ψ_i^n that there is $k_0 > 0$ such that the following holds.

(56) (a) For all $k \geq k_0$ and for all $i < i_0$, $\bigwedge e[k] \wedge \varphi_i^{c(i)}$ is unsatisfiable.

(b) For all $k \geq k_0$ and for all $i < i_0$, $\bigwedge e[k] \wedge \alpha_i^{d(i)}$ is unsatisfiable.

(c) Let $i < i_0$ and (unique) $P \in \mathbf{P}$ be such that π_i is satisfiable in some member of P. If $P \neq P_j$ then for all $k \geq k_0$ and $n < d(i)$, $\bigwedge e[k] \wedge \alpha_i^0 \wedge \ldots \wedge \alpha_i^n \wedge \bigwedge \sigma_i^n \wedge \psi_i^n$ is unsatisfiable.

Let $k \geq k_0$ be given. Since $\mathcal{S} \models \pi_{i_0}[h]$, we deduce from (55)a, (56) and the definition of the $\psi_{i_0}^n$ that for all $i, n_0 \ldots n_i \in N$, the following holds.

(57) (a) If $0 < i \leq i_0$ and if $n_0 < c(0)$ or $\ldots$ or $n_{i-1} < c(i-1)$, then the conjunction of $\bigwedge e[k]$ with the formula of form $(i, n_0 \ldots n_i)$ is satisfiable in $\mathcal{S}$.

(b) If $i < i_0$ and if $n_0 \geq c(0)$ and $\ldots$ and $n_{i-1} \geq c(i-1)$ and $n_i \geq d(i)$, then the conjunction of $\bigwedge e[k]$ with the formula of form $(i, n_0 \ldots n_i)$ is not satisfiable.

(c) If $i < i_0$ and if $n_0 \geq c(0)$ and $\ldots$ and $n_{i-1} \geq c(i-1)$ and $n_i < d(i)$, then the conjunction of $\bigwedge e[k]$ with the formula of form $(i, n_0 \ldots n_i)$ is either not satisfiable or satisfiable in some member of P_j.

(d) If $i = i_0$ and if $n_0 \geq c(0)$ and $\ldots$ and $n_{i-1} \geq c(i-1)$, then the conjunction of $\bigwedge e[k]$ with the formula of form $(i, n_0 \ldots n_i)$ is either not satisfiable or satisfiable in some member of P_j.

(e) If $i > i_0$ then the conjunction of $\bigwedge e[k]$ with the formula of form $(i, n_0 \ldots n_i)$ is satisfiable in $\mathcal{S}$.

By (57), for all $\varphi \in X$, if there is $P \in \mathbf{P}$ with $\emptyset \neq \{\mathcal{T} \in \bigcup \mathbf{P} \mid \bigwedge e[k] \wedge \varphi \text{ is satisfiable in } \mathcal{T}\} \subseteq P$ then $P = P_j$. So to finish the proof it suffices to exhibit $\varphi \in X$ such that:

(58) $\emptyset \neq \{\mathcal{T} \in \bigcup \mathbf{P} \mid \bigwedge e[k] \wedge \varphi \text{ is satisfiable in } \mathcal{T}\} \subseteq P_j$.

Since $\mathcal{S} \models \pi_{i_0} \cup \mathit{range}(e[k])[h]$, there is $n \in N$ such that $e[k] = \sigma^n_{i_0}$. We infer immediately from (56)a and the definition of $\psi^n_{i_0}$ that $\varphi = (\varphi^0_0 \wedge \ldots \wedge \varphi^{c(0)}_0) \vee \ldots \vee (\varphi^0_{i_0-1} \wedge \ldots \wedge \varphi^{c(i_0-1)}_{i_0-1}) \vee (\alpha^0_{i_0} \wedge \ldots \wedge \alpha^n_{i_0} \wedge \bigwedge \sigma^n_{i_0} \wedge \psi^n_{i_0})$ satisfies (58), as required. ■

The lemma also allows us to derive Proposition (28), whose proof was deferred.

Proof of Proposition (28): Let s be the function symbol and $\overline{0}$ the constant of **Sym**. For $n \in N$, let $\overline{n}$ be the result of n applications of s to $\overline{0}$. Set:

$$P_0 = \mathit{MOD}(\{\overline{n} \neq \overline{0} \mid n > 0\}).$$

For all $n > 0$, set:

$$P_n = \mathit{MOD}(\{\overline{m} \neq \overline{0} \mid 0 < m < n\} \cup \{\overline{n} = \overline{0}\}).$$

We claim that $\mathbf{P} = \{P_0, P_1 \ldots\}$ witnesses the proposition.

Clearly, for all $i \in N$, $P_i \neq \emptyset$ and for all distinct $i, j \in N$, $P_i \cap P_j = \emptyset$, hence $\mathbf{P} = \{P_0, P_1 \ldots\}$ is an infinite problem. It is equally immediate that $\mathbf{P}$ is solvable, and that $\bigcup \mathbf{P}$ is the class of all structures. So (a) follows directly from Lemma (22).

It remains to show (b). Let $P \in \mathbf{P}$ and $\sigma \in \mathit{SEQ}$ be for P. By Lemma (52) it suffices to show that for some $\forall$ formula φ:

(59) $\emptyset \neq \{\mathcal{S} \in \bigcup \mathbf{P} \mid \bigwedge \sigma \wedge \varphi \text{ is satisfiable in } \mathcal{S}\} \subseteq P$.

If $P = P_n$ for some $n > 0$ then $\varphi = (\bigwedge_{0<m<n}(\overline{m} \neq \overline{0})) \wedge (\overline{n} = \overline{0})$ satisfies (59). Suppose that $P = P_0$. We can choose $0 < m < n$ such that $\mathit{range}(\sigma) \cup \{\overline{p} \neq \overline{q} \mid 0 \leq p < q < n\} \cup \{\overline{n} = \overline{m}\}$ is consistent. Moreover, by a simple induction on terms of the form $\overline{p}$, $\{\overline{p} \neq \overline{q} \mid 0 \leq p < q < n\} \cup \{\overline{n} = \overline{m}\} \models \{\overline{p} \neq \overline{0} \mid p > 0\}$. Hence $\varphi = (\bigwedge_{0 \leq p<q<n}(\overline{p} \neq \overline{q})) \wedge (\overline{n} = \overline{m})$ satisfies (59). ■

Let us now show that the "if defined" qualification in Proposition (51) cannot be eliminated. Indeed, the following result shows that discrete enumerative induction cannot always be made to work no matter what formulas are involved. So, unlike its segmental counterpart, discrete enumerative induction is not a universal method of inquiry within the first-order paradigm.

(60) PROPOSITION: Suppose that **Sym** consists of a denumerable set of constants. Then there exists a solvable problem whose discrete $\mathcal{L}_{form}$-type is undefined.

Proof: Let $\{\overline{n} \mid n \in N\}$ enumerate the constants of **Sym**. Set $P_0 = \mathit{MOD}(\{\overline{n} = \overline{0} \mid n \in N\})$. Let P_1 be the class of all structures that do not belong to P_0. We show that $\mathbf{P} = \{P_0, P_1\}$ satisfies the claim of the proposition. Obviously, $\mathbf{P}$ is solvable. It remains to prove that for no well ordering O of a subset of $\mathcal{L}_{form}$ does $\Delta[\mathbf{P}, O]$ solve $\mathbf{P}$. So let O well order a subset of $\mathcal{L}_{form}$. Choose $\mathcal{T} \in MOD(\{\overline{n} = \overline{0} \mid n \in N\} \cup \{\exists x(x \neq \overline{0})\}) \subset P_0$. Fix an environment e for $\mathcal{T}$. To finish the proof we show that $\Delta[\mathbf{P}, O]$ does not solve P_0 in e.

There exists $k_0 \in N$ such that $\bigwedge e[k_0] \models \exists x(x \neq \overline{0})$. Let $k \geq k_0$ be given, and let ψ be any formula such that $\bigwedge e[k] \wedge \psi$ is satisfiable. Let $n \in N$ be such that $\overline{n}$ does not appear in $\bigwedge e[k] \wedge \psi$. Since $\bigwedge e[k] \models \exists x(x \neq \overline{0})$, $\{\bigwedge e[k] \wedge \psi, \overline{n} \neq \overline{0}\}$ is consistent. Hence $\bigwedge e[k] \wedge \psi$ is satisfiable in some member of P_1. So we have shown that for every $k \geq k_0$ and for every formula ψ, if $\bigwedge e[k] \wedge \psi$ is satisfiable then $\{\mathcal{S} \in \bigcup \mathbf{P} \mid \bigwedge e[k] \wedge \psi \text{ is satisfiable in } \mathcal{S}\} \not\subseteq P_0$. By Definition (13) this implies immediately that for cofinitely many k, either $\Delta[\mathbf{P}, O](e[k])$ is undefined or $\Delta[\mathbf{P}, O](e[k]) \not\subseteq P_0$. Hence $\Delta[\mathbf{P}, O]$ does not solve P_0 in e. ■

We summarize the relation between discrete and segmental enumerative induction using universal formulas by the following consequence of Propositions (32), (44) and (51):

(61) Corollary: Let solvable problem $\mathbf{P}$ be given. The segmental $\forall$-type of $\mathbf{P}$ is ω at most. If defined, the discrete $\forall$-type of $\mathbf{P}$ is ω at most, and equal to its segmental $\forall$-type.

Proposition (60) provides an example of a solvable problem whose discrete $\mathcal{L}_{form}$-type is undefined, hence not finite. The following proposition shows that there is, in fact, a rich collection of solvable problems with nonfinite $\mathcal{L}_{form}$-types.

(62) Proposition: Let solvable problem $\mathbf{P}$ be such that every $\sigma \in SEQ$ for $\mathbf{P}$ is for at least two members of $\mathbf{P}$. Then the discrete $\mathcal{L}_{form}$-type of $\mathbf{P}$ is not finite.

Proof: Suppose for a contradiction that there is $n \in N$ and enumeration $E = \{\varphi_0 \dots \varphi_n\}$ of formulas such that $\Delta[\mathbf{P}, E]$ solves $\mathbf{P}$. We will build by induction on i a sequence $\{\sigma_i \mid i \in N\}$ of members of SEQ that are for $\mathbf{P}$, a sequence $\{P_i \mid i \in N\}$ of members of $\mathbf{P}$, and a total function $f : N \rightarrow \{0 \dots n\}$ such that for all $i \in N$:

(63) (a) $f(i)$ is the least $m \leq n$ such that $\emptyset \neq \{\mathcal{S} \in \bigcup \mathbf{P} \mid \bigwedge \sigma_i \wedge \varphi_m \text{ is satisfiable in } \mathcal{S}\} \subseteq P$ for some $P \in \mathbf{P}$;

(b) $\sigma_i \subseteq \sigma_{i+1}$;

(c) $\emptyset \neq \{\mathcal{S} \in \bigcup \mathbf{P} \mid \bigwedge \sigma_i \wedge \varphi_{f(i)} \text{ is satisfiable in } \mathcal{S}\} \subseteq P_i$;

(d) $P_i \neq P_{i+1}$.

Before we build those items, we show how to derive a contradiction from (63), thus completing the proof. Let $i_0 \in N$ be such that $range(f) \subseteq \{f(0) \dots f(i_0)\}$. Then:

(64) for all $i_0 \leq i_1 \leq i_2$, $f(i_1) \leq f(i_2)$.

Proof of (64): Let $i_0 \leq i_1 \leq i_2$ be given, and suppose for a contradiction that $f(i_2) < f(i_1)$. By (63)a applied twice, since $f(i_2) \in \{f(0) \ldots f(i_0)\}$ there exists $j \leq i_0$ such that $\emptyset \neq \{\mathcal{S} \in \bigcup \mathbf{P} \mid \bigwedge \sigma_j \wedge \varphi_{f(j)}$ is satisfiable in $\mathcal{S}\} \subseteq P$ for some $P \in \mathbf{P}$, $\emptyset \neq \{\mathcal{S} \in \bigcup \mathbf{P} \mid \bigwedge \sigma_{i_2} \wedge \varphi_{f(i_2)}$ is satisfiable in $\mathcal{S}\} \subseteq P$ for some $P \in \mathbf{P}$, and $f(j) = f(i_2)$. Hence $\bigwedge \sigma_{i_2} \wedge \varphi_{f(i_2)}$ is satisfiable and $\emptyset \neq \{\mathcal{S} \in \bigcup \mathbf{P} \mid \bigwedge \sigma_j \wedge \varphi_{f(i_2)}$ is satisfiable in $\mathcal{S}\} \subseteq P$ for some $P \in \mathbf{P}$. Since $j \leq i_1 \leq i_2$, it follows from (63)b that $\sigma_j \subseteq \sigma_{i_1} \subseteq \sigma_{i_2}$. Thus:

(65) $\emptyset \neq \{\mathcal{S} \in \bigcup \mathbf{P} \mid \bigwedge \sigma_{i_1} \wedge \varphi_{f(i_2)}$ is satisfiable in $\mathcal{S}\} \subseteq P$ for some $P \in \mathbf{P}$.

But (63)a applied to $i = i_1$ implies that $f(i_1)$ is the least $m \leq n$ such that $\emptyset \neq \{\mathcal{S} \in \bigcup \mathbf{P} \mid \bigwedge \sigma_{i_1} \wedge \varphi_m$ is satisfiable in $\mathcal{S}\} \subseteq P$ for some $P \in \mathbf{P}$. This contradicts (65) together with the hypothesis that $f(i_2) < f(i_1)$. ∎

Now we use (64) to obtain the desired contradiction. Since all members of $\mathbf{P}$ are pairwise disjoint, we infer from (63)b–d that for all $i \in N$, $\varphi_{f(i)} \neq \varphi_{f(i+1)}$. Hence [since $range(f)$ is finite] there is $i_1 \geq i_0$ with $f(i_1 + 1) < f(i_1)$, which contradicts (64).

So to finish the proof we build $\{\sigma_i \mid i \in N\}$, $\{P_i \mid i \in N\}$ and $f : N \to \{0 \ldots n\}$ satisfying (63). Let P be any member of $\mathbf{P}$. Fix an environment e for P. Since $\Delta[\mathbf{P}, E]$ solves $\mathbf{P}$, there exists $k > 0$ such that $m \leq n$ is least with $\emptyset \neq \{\mathcal{S} \in \bigcup \mathbf{P} \mid \bigwedge e[k] \wedge \varphi_m$ is satisfiable in $\mathcal{S}\} \subseteq P'$ for some $P' \in \mathbf{P}$, and $\emptyset \neq \{\mathcal{S} \in \bigcup \mathbf{P} \mid \bigwedge e[k] \wedge \varphi_m$ is satisfiable in $\mathcal{S}\} \subseteq P$. Set $\sigma_0 = e[k]$, $P_0 = P$, and $f(0) = m$. Note that σ_0 is for $\mathbf{P}$. Let $i \in N$ be given, and suppose that $\sigma_0 \ldots \sigma_i$, $P_0 \ldots P_i$ and $f(0) \ldots f(i)$ have been defined, and that σ_i is for $\mathbf{P}$. By hypothesis there exists $P \in \mathbf{P}$ such that $P \neq P_{f(i)}$ and σ_i is for P. Let environment e for P extend σ_i. Since $\Delta[\mathbf{P}, E]$ solves $\mathbf{P}$, there exists $k \geq length(\sigma_i)$ such that $m \leq n$ is least with $\emptyset \neq \{\mathcal{S} \in \bigcup \mathbf{P} \mid \bigwedge e[k] \wedge \varphi_m$ is satisfiable in $\mathcal{S}\} \subseteq P'$ for some $P' \in \mathbf{P}$, and $\emptyset \neq \{\mathcal{S} \in \bigcup \mathbf{P} \mid \bigwedge e[k] \wedge \varphi_m$ is satisfiable in $\mathcal{S}\} \subseteq P$. Set $\sigma_{i+1} = e[k]$, $P_{i+1} = P$, and $f(i+1) = m$. Plainly this construction satisfies (63). ■

To appreciate the import of Proposition (62), consider again the problem $\mathbf{P}$ defined in our running Example (2). It is easy to verify that $\mathbf{P}$ satisfies the conditions of the proposition. Hence, $\mathbf{P}$ does not have finite discrete $\mathcal{L}_{form}$-type. From Proposition (42) it follows that $\mathbf{P}$ does not have finite segmental $\mathcal{L}_{form}$-type either. On the other hand, Example (17) shows $\mathbf{P}$ to have discrete and segmental $\forall$-types bounded by ω. The upshot is that $\mathbf{P}$ can be solved via enumerative induction of both the discrete and segmental kind, but infinitely many formulas are required for this purpose. The following proposition summarizes the situation.

(66) Proposition: Let $\mathbf{P}$ be as in Example (2). Then the discrete and segmental $\forall$-types of $\mathbf{P}$ are both ω. The same is true of their $\mathcal{L}_{form}$-types.

7 Existential, $\forall\exists$-, and $\exists\forall$-types

Once again, let $\mathbf{P}$ be as in Example (2). Although $\mathbf{P}$ can be solved by enumerative induction, infinitely many formulas are required for this purpose. In particular, Proposition (66) shows that no reduction in number is achieved by increasing the quantifier complexity of the enumerated formulas. This raises the general question: is the discrete, universal type of a problem the lowest possible, and similarly for segmental types? In the present section we provide a negative answer by exhibiting problems whose universal types are either undefined or greater than their $\forall\exists$- and $\exists\forall$-types.

As a preliminary, let us ask whether simply switching from universal to existential formulas can lower the type of a problem. In the discrete case the answer is No, as revealed by the following proposition.

(67) PROPOSITION: For all solvable problems $\mathbf{P}$, if the discrete $\exists$-type of $\mathbf{P}$ is defined then the discrete $\forall$-type of $\mathbf{P}$ is 1 at most.

Proof: Let nonempty, solvable problem $\mathbf{P}$ be given (if $\mathbf{P} = \emptyset$ then the $\forall$-type of $\mathbf{P}$ is 0). Suppose that the discrete $\exists$-type of $\mathbf{P}$ is defined. Let well ordering O of set X of $\exists$ formulas be such that $\Delta[\mathbf{P}, O]$ solves $\mathbf{P}$. Let $P \in \mathbf{P}$ and environment e for P be given. Let $k_0 \in N$ be such that $\Delta[\mathbf{P}, O](e[k_0]) = P$. Then there is $\varphi \in X$ such that $\emptyset \neq \{\mathcal{S} \in \bigcup \mathbf{P} \mid \bigwedge e[k_0] \wedge \varphi \text{ is satisfiable in } \mathcal{S}\} \subseteq P$. Since φ is an $\exists$ formula, there is $k_1 \geq k_0$ such that $\bigwedge e[k_1] \models \varphi$. This implies immediately that for all $k \geq k_1$, $\emptyset \neq \{\mathcal{S} \in \bigcup \mathbf{P} \mid \bigwedge e[k_1] \wedge (x = x) \text{ is satisfiable in } \mathcal{S}\} \subseteq P$. Hence $\Delta[\mathbf{P}, \{x = x\}]$ solves $\mathbf{P}$, and the discrete $\forall$-type of $\mathbf{P}$ is equal to 1. ■

In contrast to the debility of existential formulas, we now give a sense in which the ascent to $\forall\exists$ formulas allows maximal improvement in type. By Corollary (61), the worst universal types are ω in the segmental case, and undefined in the discrete case. Except in trivial cases, the best discrete and segmental $\forall\exists$-types are 2. The following proposition shows there to be problems that simultaneously have the worst $\forall$-types and the best $\forall\exists$-types.

(68) PROPOSITION: Suppose that **Sym** consists of a denumerable set of constants and a binary predicate. Then there is a solvable problem such that:

(a) the discrete $\forall$-type of $\mathbf{P}$ is undefined;

(b) the segmental $\forall$-type of $\mathbf{P}$ is infinite;

(c) the discrete and segmental $\forall\exists$-types of $\mathbf{P}$ are 2.

Proof: Let $\{\overline{n} \mid n \in N\}$ enumerate the constants of **Sym**, and let R be its binary predicate. Let P_0 consist of all members of $MOD(\{\overline{n} = \overline{0} \mid n \in N\})$ in which the interpretation of R is a strict total order without greatest point. Let P_1 consist of all finite members of $\bigcup_{n>0} \mathit{MOD}(\overline{n} \neq \overline{0})$ in which the interpretation of R is a strict total order. Then $\mathbf{P} = \{P_0, P_1\}$ is a solvable problem. We show that $\mathbf{P}$ satisfies the claim of the proposition, starting with (a). Let satisfiable $\forall$ formula φ be such that $\varphi \models \exists x(x \neq \overline{0})$. Let X be the set of all $n > 0$ such that $\overline{n}$ does not appear in φ. Denote by n_0 the least member of X. By the choice of φ, $\{\varphi, \overline{n_0} \neq \overline{0}\}$ is consistent. Hence $W = \{\varphi, \overline{n_0} \neq \overline{0}\} \cup \{\overline{n} = \overline{n_0} \mid n \in X\}$ is consistent. Choose structure $\mathcal{S}$ and full assigment h to $\mathcal{S}$ with $\mathcal{S} \models W[h]$. Let $A \subseteq |\mathcal{S}|$ be the union of $\{h(x) \mid x \in \mathit{Var}(\varphi)\}$ with the set of interpretations of the constants $\overline{n}$, $n \in N$. Let $\mathcal{T}$ be the restriction of $\mathcal{S}$ to A. Since $\forall$ formulas are preserved in substructures, W is satisfiable in $\mathcal{T}$. Note that $|\mathcal{T}|$ is finite, and $\mathcal{T} \models (\overline{n_0} \neq \overline{0})$. Hence $\mathcal{T} \in P_1$. So we have shown the following:

(69) Every satisfiable $\forall$ formula that implies $\exists x(x \neq \overline{0})$ is satisfiable in some member of P_1.

Now let e be an environment for a structure $\mathcal{S}$ in P_0 that satisfies $\mathcal{S} \models \exists x(x \neq \overline{0})$. Let $k_0 \in N$ be such that $\bigwedge e[k_0] \models \exists x(x \neq \overline{0})$. It follows from (69) that for all $\forall$ formulas φ and for all $k \geq k_0$, either $\bigwedge e[k] \wedge \varphi$ is unsatisfiable or $\bigwedge e[k] \wedge \varphi$ is satisfiable in some member of P_1. This implies immediately that for all well orderings O of some set of $\forall$ formulas and for all $k \geq k_0$, $\Delta[\mathbf{P}, O](e[k]) \neq P_0$. Hence for all well orderings O of some set of $\forall$ formulas, $\Delta[\mathbf{P}, O]$ does not solve P_0 in e. Hence the discrete $\forall$-type of $\mathbf{P}$ is undefined.

Clause (b) follows immediately from clause (a) and Propositions (44) and (32). For clause (c), set $E = \{\forall x \exists y Rxy, x = x\}$. If e is an environment for P_0, then trivially $\Delta[\mathbf{P}, E](e[k]) = \Psi[\mathbf{P}, E](e[k]) = P_0$ for all $k \in N$. Let environment e for P_1 be given. Let $n > 0$ and $k_0 \in N$ be such that $\bigwedge e[k_0] \models (\overline{n} \neq \overline{0})$. Then trivially, $\Delta[\mathbf{P}, E](e[k]) = \Psi[\mathbf{P}, E](e[k]) = P_1$ for all $k \geq k_0$. So both $\Delta[\mathbf{P}, E]$ and $\Psi[\mathbf{P}, E]$ solve $\mathbf{P}$, and the discrete and segmental $\forall\exists$-types of $\mathbf{P}$ are 2 at most. They cannot be equal to 1. Indeed, let $\psi \in \mathcal{L}_{form}$ be given. Suppose that $\Delta[\mathbf{P}, \{\psi\}]$ (respectively $\Psi[\mathbf{P}, \{\psi\}]$) solves $\mathbf{P}$. Let environment e_0 for P_0 be given. Then there exists $k_0 \in N$ such that $\emptyset \neq \{\mathcal{S} \in \bigcup \mathbf{P} \mid \bigwedge e_0[k_0] \wedge \psi \text{ is satisfiable in } \mathcal{S}\} \subseteq P_0$. Let environment e for P_1 extend $e_0[k_0]$. Then for all $k \geq k_0$, $\Delta[\mathbf{P}, \{\psi\}](e[k])$ (respectively $\Psi[\mathbf{P}, \{\psi\}](e[k])$) is not equal to P_1. ■

Compared to $\forall\exists$, the ascent to $\exists\forall$ formulas does not yield quite the same improvement over universal formulas. The reason is that a problem's discrete $\forall$-type is defined whenever its $\exists\forall$-type is defined. There can thus be no strict analogy to Proposition (68) with $\exists\forall$ replacing $\forall\exists$.

(70) Proposition: For all solvable problems $\mathbf{P}$, the discrete $\exists\forall$-type of $\mathbf{P}$ is defined iff the discrete $\forall$-type of $\mathbf{P}$ is defined.

Proof: Let solvable problem $\mathbf{P}$ be given. Trivially, if the discrete $\forall$-type of $\mathbf{P}$ is defined then the discrete $\exists\forall$-type of $\mathbf{P}$ is defined. Suppose that the discrete $\exists\forall$-type of $\mathbf{P}$ is defined. Let set X of $\exists\forall$ formulas and well ordering O of X be such that $\Delta[\mathbf{P}, O]$ solves $\mathbf{P}$. Let $P \in \mathbf{P}$ and $\sigma \in \mathit{SEQ}$ for P be given. By Proposition (52) it suffices to exhibit a $\forall$ formula φ such that:

(71) $\emptyset \neq \{\mathcal{S} \in \bigcup \mathbf{P} \mid \bigwedge \sigma \wedge \varphi$ is satisfiable in $\mathcal{S}\} \subseteq P$.

Fix an environment e for P extending σ. Since $\Delta[\mathbf{P}, O]$ solves $\mathbf{P}$, there is $k \geq \mathit{length}(\sigma)$ and $\psi \in X$ such that $\emptyset \neq \{\mathcal{S} \in \bigcup \mathbf{P} \mid \bigwedge e[k] \wedge \psi$ is satisfiable in $\mathcal{S}\} \subseteq P$. Since ψ is an $\exists\forall$ formula, we can choose $p \in N$, variables $x_1 \ldots x_p$, and $\forall$ formula χ such that $\{x_1 \ldots x_p\} \cap \mathit{Var}(\bigwedge e[k] \wedge \psi) = \emptyset$ and $\models \exists x_1 \ldots x_p \chi \leftrightarrow \psi$. Then $\emptyset \neq \{\mathcal{S} \in \bigcup \mathbf{P} \mid \bigwedge e[k] \wedge \chi$ is satisfiable in $\mathcal{S}\} \subseteq P$, and $\varphi = \bigwedge e[k] \wedge \chi$ satisfies (71), as required. ■

In view of the last result and Corollary (61), what is the most drastic improvement to be hoped for by using $\exists\forall$ formulas instead of universal ones? There might turn out to be a problem whose discrete and segmental $\forall$-types are infinite but whose respective $\exists\forall$-types are 2. The following proposition reveals the existence of just such a problem.

(72) Proposition: Suppose that **Sym** consists of a binary predicate, a unary function symbol, and a constant. Then there exists a solvable problem $\mathbf{P}$ such that:

(a) the discrete and segmental $\forall$-types of $\mathbf{P}$ are infinite;

(b) the discrete and segmental $\forall\exists$-types of $\mathbf{P}$ are 2;

(c) the discrete and segmental $\exists\forall$-types of $\mathbf{P}$ are 2.

Proof: Let R be the binary predicate, s the function symbol, and $\overline{0}$ the constant of **Sym**. For $n \in N$, let $\overline{n}$ be the result of n applications of s to $\overline{0}$. Set:

$$T = \{(\overline{n} = \overline{0}) \rightarrow (\exists x \forall y Rxy \wedge \forall x \exists y Rxy) \mid n > 0\}.$$

Set:

$$P_0 = \mathit{MOD}(\{\overline{n} \neq \overline{0} \mid n > 0\}).$$

For all $n > 0$ set:

$$P_n = \mathit{MOD}(T \cup \{\overline{m} \neq \overline{0} \mid 0 < m < n\} \cup \{\overline{n} = \overline{0}\}).$$

Clearly, for all $i \in N$, $P_i \neq \emptyset$ and for all distinct $i, j \in N$, $P_i \cap P_j = \emptyset$, hence $\mathbf{P} = \{P_0, P_1 \ldots\}$ is an infinite problem. It is equally immediate that $\mathbf{P}$ is solvable. We prove that $\mathbf{P}$ satisfies the claim of the proposition, starting with (a). Since $\{\overline{n} \neq \overline{0} \mid n > 0\} \models T$, it is easy to see that $\bigcup \mathbf{P} = \mathit{MOD}(T)$. Hence, by Lemma (22) and Propositions (29) and (61), it suffices to show that the discrete $\forall$-type of $\mathbf{P}$ is defined. Let $P \in \mathbf{P}$ and $\sigma \in \mathit{SEQ}$ be for P. By Lemma (52) it suffices to show that for some $\forall$ formula φ:

(73) $\emptyset \neq \{\mathcal{S} \in \bigcup \mathbf{P} \mid \bigwedge \sigma \wedge \varphi \text{ is satisfiable in } \mathcal{S}\} \subseteq P$.

If $P = P_n$ for some $n > 0$, $\varphi = (\bigwedge_{0<m<n}(\overline{m} \neq \overline{0})) \wedge (\overline{n} = \overline{0})$ satisfies trivially (73). Suppose that $P = P_0$. We can choose $0 < m < n$ such that $range(\sigma) \cup \{\overline{p} \neq \overline{q} \mid 0 \leq p < q < n\} \cup \{\overline{n} = \overline{m}\}$ is consistent. It is easy to verify that $\varphi = (\bigwedge_{0\leq p<q<n}(\overline{p} \neq \overline{q})) \wedge (\overline{n} = \overline{m})$ satisfies (73).

We prove (b). Set $E = \{\forall x \exists y \neg Rxy, x = x\}$. Observe that every initial segment of an environment for $\mathbf{P}$ is consistent with $\forall x \exists y \neg Rxy$. Using this fact and the definition of T, it is easy to verify that if e is an environment for P_0, then $\Delta[\mathbf{P}, E](e[k]) = \Psi[\mathbf{P}, E](e[k]) = P_0$ for all $k \in N$. Let $n > 0$ and environment e for P_n be given. Let $k_0 \in N$ be such that $\bigwedge e[k_0] \models (\bigwedge_{0<m<n}(\overline{m} \neq \overline{0})) \wedge (\overline{n} = \overline{0})$. Then trivially, $\Delta[\mathbf{P}, E](e[k]) = \Psi[\mathbf{P}, E](e[k]) = P_n$ for all $k \geq k_0$. So both $\Delta[\mathbf{P}, E]$ and $\Psi[\mathbf{P}, E]$ solve $\mathbf{P}$, and the discrete and segmental $\forall\exists$-types of $\mathbf{P}$ are 2 at most. They cannot be equal to 1. Indeed, let $\psi \in \mathcal{L}_{form}$ be given. Suppose that $\Delta[\mathbf{P}, \{\psi\}]$ (respectively $\Psi[\mathbf{P}, \{\psi\}]$) solves $\mathbf{P}$. Let environment e_0 for P_0 be such that $range(e_0) \models \{\overline{m} \neq \overline{n} \mid m, n \in N,\, m \neq n\}$. Then there exists $k_0 \in N$ such that $\emptyset \neq \{\mathcal{S} \in \bigcup \mathbf{P} \mid \bigwedge e_0[k_0] \wedge \psi \text{ is satisfiable in } \mathcal{S}\} \subseteq P_0$. Let $n > 0$ be such that for all $m \geq n$, $\overline{m}$ does not occur in $\bigwedge e_0[k_0]$. Let environment e for P_n extend $e_0[k_0]$. Then for all $k \geq k_0$, $\Delta[\mathbf{P}, \{\psi\}](e[k])$ (respectively $\Psi[\mathbf{P}, \{\psi\}](e[k])$) is not equal to P_n. This ends the proof of (b). The proof of (c) is similar, using $\{\exists x \forall y \neg Rxy, x = x\}$ instead of E. ■

8 Concluding remarks

An important challenge in the theory of scientific discovery is to provide a motivated characterization of problem difficulty. Within the numerical paradigm, one approach to such characterization counts the number of times an hypothesis needs to be changed in the course of solving a problem; see, for example, [Kinber & Stephan, 1995, Lange & Zeugmann, 1993, Lange & Zeugmann, 1995]. Other approaches include [Daley & Smith, 1986, Freivalds *et al.*, 1995, Jain & Sharma, 1997]. These perspectives are ingenious and well-developed but they are best suited to paradigms built around the recursivity of scientists and problems. Such is not the case for the first-order paradigm studied in the present paper.

For the first-order paradigm, it seems natural to measure problem difficulty in terms of consumption of an information resource. Segmental $\forall$-types illustrate the idea if we take the complexity of problem $\mathbf{P}$ to be the smallest ordinal κ such that $\mathbf{P}$ can be solved via segmental enumerative induction using a κ-ordering of universal formulas. [Such is indeed the definition of $\mathbf{P}$'s segmental $\forall$-type; see Definition (16).] The difficulty for this approach is that Corollary (61) largely trivializes it. There turns out to be just one level of infinite difficulty, concealing great differences in the intelligence needed to solve its different members. In contrast, problems at finite levels all seem monotonously easy to solve. The situation is even worse for the discrete case, since $\mathcal{L}_{form}$-types in this sense may not even be defined [see Proposition (60)].

There nonetheless remain potentially interesting questions about enumerative induction. For one thing, we would like to have a revealing characterization of the class of problems with discrete $\mathcal{L}_{form}$-types. By Corollary (27), this class includes all solvable problems of form $(T, \{\theta_0 \ldots \theta_n\})$. By Lemma (21) and Proposition (28), there are yet other members, but we have no independent description of them. A second question concerns the complexity of problems that do have discrete $\mathcal{L}_{form}$-types. By Proposition (51), the complexity of problems with discrete $\forall$-types is bounded by ω. Is it similarly the case that every problem with discrete $\mathcal{L}_{form}$-type has $\mathcal{L}_{form}$-type no greater than ω?

On the more speculative side, our results are consistent with the existence of a subset X of $\mathcal{L}_{form}$ yielding a rich and interesting class of segmental X-types that embraces all solvable problems. If the set X were natural or had other interesting properties, it would provide a potentially useful measure of problem complexity. In view of Corollary (61), X could not include all $\forall$ formulas. But aside from this constraint it is presently unclear what its composition might be.

Perhaps something akin to induction by enumeration may yet prove useful for classifying problem complexity. At present, however, a key idea appears to be missing.

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