A Comparison Principle for State-Constrained Differential Inequalities and Its Application to Time-Optimal Control

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Abstract

In this paper, we present a comparison principle that characterizes the maximal solutions of state-constrained differential inequalities in terms of solutions of certain differential equations with discontinuous right-hand sides. For the sake of completeness, we show through some set-valued analysis that the differential equations determining the maximal solutions have the unique solutions in the Carathéodory sense, in spite of discontinuity of their right-hand sides. We apply our comparison principle to the explicit characterization of the solution to a time-optimal control problem for a class of state-constrained second-order systems which includes the dynamic equations of robotic manipulators with geometric path constraints as well as single-degree-of-freedom mechanical systems with friction. Specifically, we show that the time-optimal trajectory is uniquely determined by two curves that can be constructed by solving two scalar ordinary differential equations with continuous right-hand sides. Hence, the time-optimal trajectory can be found in a computationally efficient way through the direct use of the well-known Euler or Runge-Kutta methods. Another interesting feature is that our method to solve the time-optimal control problem works even when there exist an infinite number of switching points. Finally, some simulation results using a two-DOF robotic manipulator are presented to demonstrate the practical use of our complete characterization of the time-optimal solution.

Key words: comparison principle, differential equations, differential inequalities, minimum time, optimal control.

1 Introduction

Inequalities have played a fundamental role in the development of all branches of pure and applied mathematics. They have also proved to be of immense use in many other fields of science and engineering including system and control theory; see [1]-[7] and the vast literature therein. Indeed, (differential) linear matrix inequalities have been used as powerful formulation and design techniques for a variety of control problems ranging from robust control to (possibly singular) linear quadratic problems [3], [4]. Moreover, differential and integral inequalities such as the Gronwall-Bellman inequality and its numerous generalizations have provided invaluable tools in qualitative and quantitative analysis of differential equations, integral equations, integro-differential equations, impulse differential equations, and so on [1], [2]. Among these differential and integral inequalities, the well-known comparison principle [10], which characterizes maximal solutions of first-order differential inequalities without state constraints, has been of particular use in the Lyapunov analysis of ordinary differential equations in the control literature [8]-[10].

On the other hand, the systematic study of optimal control dates back to the late 1950s, during which times two important advances were made: one is Pontryagin's maximum principle (PMP) [11] that states a set of necessary equations for state trajectories to be optimal, and the other is the dynamic programming (DP) [12] that reduces optimal control problems to the partial differential equations known as Hamilton-Jacobi equations. In particular, the problem of transferring a given system from one state to another in minimum time is known as the *time-optimal control problem.* During the past forty years, it has been one of the basic concerns of optimal control theory [13]-[15]. As far as second-order systems without state constraints are concerned, extensive results on the structure and structural stability of the time-optimal trajectories are now available in the literature [16]-[19]. On the contrary, in the case of state-constrained second-order systems, it is still difficult except for some special cases even to check whether the solution of the two-point boundary value problem (TPBVP) resulting from the PMP is indeed time-optimal.

In this context, the time optimal control problem of robotic manipulators with geometric path constraints, which can be reduced to that of certain state-constrained second-order systems, was solved by using a kind of phase-plane method that iteratively searches switching points between maximum acceleration and maximum deceleration rather than resorting to the PMP [20]-[27]. However, the phase-plane method works only when there exist a finite number of switching points, which does not necessarily hold for general state-constrained second-order systems. Moreover, the required computational load becomes proportionally increasing with the number of switching points, which may be arbitrarily large.

In this paper, we develop a comparison principle for state-constrained differential inequalities, which can be viewed as a natural extension of the comparison principle for differential inequalities without state constraints. The new comparison principle characterizes the maximal solutions of state-constrained differential inequalities in terms of solutions of some differential equations with discontinuous right-hand sides. For the sake of completeness, we show through some set-valued analysis that these discontinuous differential equations indeed have the unique solutions in the Carathéodory sense [31].

Using the new comparison principle, we then attempt to solve the time-optimal control problem for a class of state-constrained second-order systems which includes the dynamic equations of robotic manipulators with geometric path constraints [20]-[27] as well as single-degree-of-freedom (DOF) mechanical systems with friction [28]. Specifically speaking, we show that the time-optimal trajectory is uniquely determined by two curves: forward and backward velocity limitation curves.

The forward (respectively, backward) velocity limitation curve stands for the curve beyond which, under given control input and state constraints, the state cannot be steered forward in time from the initial state (respectively, backward in time from the final state). Moreover, these two curves can be constructed by solving two scalar ordinary differential equations with continuous right-hand sides. Hence, their numerical construction can be done in a computationally efficient way through direct use of the well-known Euler or Runge-Kutta methods [37]. Another interesting feature is that our method developed to solve the time-optimal control problem works regardless of the presence of boundary arcs and moreover works even when there exist an infinite number of switching points. Finally, some simulation results using a two-DOF robotic manipulator are presented to demonstrate the practical use of our complete characterization of the time-optimal solution.

This paper is organized as follows. In Section II, we present a comparison principle for state-constrained differential inequalities. In Section III, a time-optimal control problem for a state-constrained second-order system is formulated, and its time-optimal trajectory is explicitly characterized. In Section IV, the conclusions are summarized. The proofs of our main results are collected in the appendices.

2 A Comparison Principle for State-Constrained Differential Inequalities

We begin by introducing some definitions and notations for the two scalar ordinary differential equations

$$v'(x) = f(x,v(x)) \in \mathbb{R}, \quad x \geq x_0 \text{ with } v(x_0) = v_0, \tag{1}$$
$$v'(x) = f(x,v(x)) \in \mathbb{R}, \quad x \leq x_f \text{ with } v(x_f) = v_f, \tag{2}$$

where f is a function from $\mathbb{R}^2$ into $\mathbb{R}$. A function $\tilde{v}$ defined on an interval $[x_0, x_1) \subset \mathbb{R}$ (respectively, $(x_2, x_f] \subset \mathbb{R}$) is said to be a solution to the differential equation in (1) (respectively, (2)) in the Carathéodory sense [31], if $\tilde{v}$ is absolutely continuous on each compact subset of $[x_0, x_1)$ (respectively, $(x_2, x_f]$) and the differential equation in (1) (respectively, (2)) is satisfied with $v = \tilde{v}$ almost everywhere (a.e.) on the interval $[x_0, x_1)$ (respectively, $(x_2, x_f]$). If the ordinary differential equation in (1) (respectively, (2)) has the unique solution in the Carathéodory sense, we then denote the unique solution extended over the maximal interval of existence by $v_F(\cdot; x_0, v_0, f)$ (respectively, $v_B(\cdot; x_f, v_f, f)$) to emphasize its dependence on the function f and the initial condition $v(x_0) = v_0$ (respectively, the final condition $v(x_f) = v_f$). The solutions of the ordinary differential equations in (1) and (2) in the Carathéodory sense are often called (absolutely continuous) solutions or trajectories for short. Similar notational conventions are applied to other differential equations which take the form in (1) and (2) in terms of variables other than x and v.

Consider the two state-constrained differential inequalities:

$$\Sigma : \begin{cases} Dv(x) \leq f(x, v(x)), & x \geq x_0 \\ v(x) \leq g(x), & x \geq x_0 \\ v(x_0) = v_0 \end{cases} \tag{3}$$

$$\Omega : \begin{cases} Dv(x) \geq f(x, v(x)), & x \leq x_f \\ v(x) \leq g(x), & x \leq x_f \\ v(x_f) = v_f \end{cases} \tag{4}$$

where $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. Here and elsewhere, $Dv(x)$ stands for either the ordinary derivative $v'(x)$ or any of the four Dini derivatives [29] (the upper right derivative $D^+v(x)$, the lower right derivative $D_+v(x)$, the upper left derivative $D^-v(x)$, and the lower left derivative $D_-v(x)$). Recall that for any function $\tilde{v} : \mathbb{R} \to \mathbb{R}$,

$$D_+\tilde{v}(x) \leq D^+\tilde{v}(x), \quad D_-\tilde{v}(x) \leq D^-\tilde{v}(x), \quad \forall\, x \in \mathbb{R} \tag{5}$$

and that if $D_+\tilde{v}(x_1) = D^+\tilde{v}(x_1) = D^-\tilde{v}(x_1) = D_-\tilde{v}(x_1)$, then the function $\tilde{v}$ is differentiable at $x = x_1$ and the common value is the ordinary derivative $\tilde{v}'(x_1)$ of $\tilde{v}$ at $x = x_1$.

A continuous function $\tilde{v}$ defined on an interval $[x_0, x_1) \subset \mathbb{R}$ (respectively, $(x_2, x_f] \subset \mathbb{R}$) is said to a *C-solution* of the differential inequality Σ (respectively, Ω), if $D\tilde{v}(x)$ exists for *all* $x \in [x_0, x_1)$ (respectively, $(x_2, x_f] \subset \mathbb{R}$) and the differential inequality Σ (respectively, Ω) is satisfied with $v = \tilde{v}$ for *all* $x \in [x_0, x_1)$ (respectively, $(x_2, x_f]$). Throughout the paper, $\mathcal{C}(\Sigma)$ (respectively, $\mathcal{C}^+(\Sigma)$, $\mathcal{C}_+(\Sigma)$, $\mathcal{C}^-(\Sigma)$, and $\mathcal{C}_-(\Sigma)$) denotes the set of C-solutions of the differential inequality Σ with $Dv(x) = v'(x)$ (respectively, $Dv(x) = D^+v(x)$, $Dv(x) = D_+v(x)$, $Dv(x) = D^-v(x)$, and $Dv(x) = D_-v(x)$). The sets $\mathcal{C}(\Omega)$, $\mathcal{C}^+(\Omega)$, $\mathcal{C}_+(\Omega)$, $\mathcal{C}^-(\Omega)$, and $\mathcal{C}_-(\Omega)$ are defined analogously. On the other hand, an *absolutely continuous* function $\tilde{v}$ defined on an interval $[x_0, x_1) \subset \mathbb{R}$ (respectively, $(x_2, x_f] \subset \mathbb{R}$) is said to an *A-solution* of the differential inequality Σ (respectively, Ω), if the differential inequality Σ (respectively, Ω) is satisfied with $v = \tilde{v}$ *a.e.* on $[x_0, x_1)$ (respectively, $(x_2, x_f]$). Throughout this paper, the set of A-solutions of the differential inequality Σ (respectively, Ω) is denoted by $\mathcal{A}(\Sigma)$ (respectively, $\mathcal{A}(\Omega)$). Recall that any absolutely continuous function $\tilde{v}$ from $I \subset \mathbb{R}$ into $\mathbb{R}$, has a derivative a.e. on I. Thus, the definitions of the sets $\mathcal{A}(\Sigma)$ and $\mathcal{A}(\Omega)$ are independent of any specific choice of $Dv(x)$ among the ordinary derivative and the Dini derivatives. Finally, when the set $\mathcal{C}(\Sigma)$ (respectively, $\mathcal{A}(\Sigma)$) contains a function $v^\star$ that satisfies $\tilde{v}(x) \leq v^\star(x)$ on the common interval of existence of $v^\star$ and $\tilde{v}$ for any $\tilde{v} \in \mathcal{C}(\Sigma)$, (respectively, $\tilde{v} \in \mathcal{A}(\Sigma)$), the function $v^\star$ is called the *maximal C-solution* (respectively, *A-solution*) of the differential inequality Σ. Maximal A- and C-solutions of the differential inequality Ω are defined analogously. When no confusion arises, maximal A- and C-solutions are simply called *maximal solutions*.

The following lemma will be frequently used in our development.

Lemma 1 *Suppose that the functions $\overline{f}$ and $\underline{f}$ are continuous from $\mathbb{R}$ into $\mathbb{R}$. If an absolutely continuous function $\tilde{v} : \mathbb{R} \to \mathbb{R}$ satisfies the inequality $\underline{f}(x) \leq \tilde{v}'(x) \leq \overline{f}(x)$ a.e. on the interval $[a, b] \subset \mathbb{R}$, then*

$$\underline{f}(x) \leq D_+\tilde{v}(x) \leq D^+\tilde{v}(x) \leq \overline{f}(x), \quad \forall\, x \in [a, b), \tag{6}$$

$$\underline{f}(x) \leq D_-\tilde{v}(x) \leq D^-\tilde{v}(x) \leq \overline{f}(x), \quad \forall\, x \in (a, b]. \tag{7}$$

□

The proof of Lemma 1 is given in Appendix A.

The following lemma clarifies the relationship between C-solutions and A-solutions of the differential inequalities in (3) and (4).

Lemma 2 *Suppose that the function f in Σ and Ω is continuous. Then,*

$$\mathcal{A}(\Sigma) \subset \mathcal{C}^+(\Sigma) \subset \mathcal{C}_+(\Sigma),\ \mathcal{A}(\Sigma) \subset \mathcal{C}^-(\Sigma) \subset \mathcal{C}_-(\Sigma), \tag{8}$$

$$\mathcal{A}(\Omega) \subset \mathcal{C}_+(\Omega) \subset \mathcal{C}^+(\Omega),\ \mathcal{A}(\Omega) \subset \mathcal{C}_-(\Omega) \subset \mathcal{C}^-(\Omega). \tag{9}$$

□

The claim in this lemma follows directly from Lemma 1 and the inequalities in (5).

Suppose that the function f is continuous with respect to the first argument and is locally Lipschitz with respect to the second argument. Then, the differential equation in (1) has the unique solution $v_F(\cdot; x_0, v_0, f)$ on an interval $[x_0, x_1)$ such that $v_F'(x; x_0, v_0, f) = f(x, v_F(x; x_0, v_0, f))$ for all $x \in [x_0, x_1)$ [31]. The well-known comparison principle [1] implies that if $v_F(x; x_0, v_0, f) < g(x)$ on the common interval, say, I of existence of $v_F(\cdot; x_0, v_0, f)$ and $\tilde{v} \in \mathcal{C}(\Sigma) \cup \mathcal{C}^+(\Sigma) \cup \mathcal{C}_+(\Sigma) \cup \mathcal{C}^-(\Sigma) \cup \mathcal{C}_-(\Sigma)$, then $\tilde{v}(x) \leq v_F(x; x_0, v_0, f)$ for all $x \in I$. Thus, the differential inequality Σ always has the maximal C-solution given by $v_F(\cdot; x_0, v_0, f)$, independent of any specific choice of $Dv(x)$ among the ordinary derivative and the Dini derivatives. Similar arguments are also applied to the differential inequality Ω.

Now, this observation along with Lemma 2 leads to the following lemma.

Lemma 3 *Suppose that the function f is continuous with respect to the first argument and is locally Lipschitz with respect to the second argument. Suppose further that*

$$g(x) > v_F(x; x_0, v_0, f) \ (\textit{respectively, } v_B(x; x_f, v_f, f)) \tag{10}$$

on the common interval, say, I of existence of $\tilde{v} \in \mathcal{A}(\Sigma)$ (respectively, $\mathcal{A}(\Omega)$) and $v_F(\cdot; x_0, v_0, f)$ (respectively, $v_B(\cdot; x_f, v_f, f)$). Then, for all $x \in I$,

$$\tilde{v}(x) \leq v_F(x; x_0, v_0, f) \ (\textit{respectively, } v_B(x; x_f, v_f, f)).$$

□

Let the function $\psi : [0, 1] \to \mathbb{R}$ denote the celebrated Lebesgue's singular function [29]. The function ψ is differentiable a.e. on $[0, 1]$ such that $\psi'(x) = 0$ a.e. on $[0, 1]$ and $\psi(1) > \psi(0)$ [29]. Observe that the unique solution of the differential equation $v'(x) = 0$, $x \geq 0$ with $v(0) = 0$ is $\tilde{v} \equiv 0$. This counterexample therefore implies that if the function $\tilde{v}$ in the statement of Lemma 3 is merely continuous, then the claim in Lemma 3 does not necessarily hold.

In the sequel, we attempt to extend the comparison principle without the restrictive assumption in (10). To do this, we need to introduce some definitions. For any function $f : \mathbb{R}^2 \to \mathbb{R}$ and any differentiable function $g : \mathbb{R} \to \mathbb{R}$, we define the functions $\Sigma_{f,g} : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$, $\Omega_{f,g} : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$, and $H_{f,g} : \mathbb{R} \to \mathbb{R}$ as

$$\Sigma_{f,g}(x, v) \triangleq \begin{cases} f(x, v), & v < g(x), \\ \min\{f(x, v), g'(x)\}, & v \geq g(x), \end{cases} \tag{11}$$

$$\Omega_{f,g}(x, v) \triangleq \begin{cases} f(x, v), & v < g(x), \\ \max\{f(x, v), g'(x)\}, & v \geq g(x), \end{cases} \tag{12}$$

$$H_{f,g}(x) \triangleq f(x, g(x)). \tag{13}$$

Under quite natural assumptions on the functions f and g, we will show soon that the maximal solutions of the differential inequalities Σ, Ω are given, respectively, by the solutions of the scalar ordinary differential equations

$$\begin{cases} v'(x) = \Sigma_{f,g}(x, v(x)), \ x \geq x_0, \\ v(x_0) = v_0 \leq g(x_0), \end{cases} \tag{14}$$

$$\begin{cases} v'(x) = \Omega_{f,g}(x, v(x)), \ x \leq x_f, \\ v(x_f) = v_f \leq g(x_f). \end{cases} \tag{15}$$

Observe that the functions $\Sigma_{f,g}$ and $\Omega_{f,g}$ are not necessarily continuous at each point $(x, v) \in \mathbb{R}^2$, even when the functions f and g are real-analytic. Hence, the classical existence and uniqueness theorem [31] for the differential equations in (1) and (2)−which requires that the function f is continuous with respect to the first argument and is locally Lipschitz with respect to the second argument−does not work for the differential equations in (14) and (15). Accordingly, a natural question arises as to the existence and uniqueness of solutions of these differential equations.

Until now, many authors have used the well-known Filippov's continuation method [38] to study the existence and uniqueness of solutions in differential equations with discontinuous right-hand sides such as variable structure systems [35] and adaptive systems with discontinuous switching laws [36]. However, only under some restrictive conditions on the functions f and g, the Filippov's continuation method can be applied to the differential equations in (14) and (15). To see this, we define two functions λ_- and λ_+ from $\mathbb{R}^2 \times \mathbb{R}$ into $\mathbb{R}$ by $\lambda_-(x, v) \overset{\triangle}{=} f(x, v) - g'(x)$ and $\lambda_+(x, v) \overset{\triangle}{=} \min\{f(x, v), g'(x)\} - g'(x)$, respectively. In order to apply the Filippov's continuation method to the differential equation in (14), we need to make the restrictive assumption that $\lambda_-(x, g(x)) \neq \lambda_+(x, g(x))$ for all $x \in \mathbb{R}$, or equivalently,

$$f(x, g(x)) - g'(x) > 0, \quad \forall\, x \in \mathbb{R}. \tag{16}$$

Similar arguments are also applied to the differential equation in (15).

Without such a restrictive assumption on the functions f and g in (16), the following theorem provides an affirmative answer to the existence and uniqueness of absolutely continuous solutions in the differential equations in (14) and (15).

Theorem 1 *Suppose that the function f is continuous and the function g is continuously differentiable. Then, the differential equations in (14) and (15) have absolutely continuous solutions. In particular when the function f is continuous with respect to the first argument and is locally Lipschitz with respect to the second argument and the function g is continuously differentiable, the differential equation in (14) (respectively, (15)) has the unique absolutely continuous solution $v_F(\cdot; x_0, v_0, \Sigma_{f,g})$ (respectively, $v_B(\cdot; x_f, v_f, \Omega_{f,g})$) which belongs to the set $\mathcal{A}(\Sigma)$ (respectively, $\mathcal{A}(\Omega)$).* □

The proof of Theorem 1 is given in Appendix B.

We now present the following lemma that will be frequently used in our development.

Lemma 4 *Suppose that the functions $H_{f,g}$ and g are real-analytic. Then, on any compact subinterval $[x_1, x_2]$ of the interval of existence of $v_F(\cdot; x_0, v_0, \Sigma_{f,g})$ (respectively, $v_B(\cdot; x_f, v_f, \Omega_{f,g})$), there exist a finite number of intervals $[a_k, a_{k+1}]$, $k = 0, 1, \ldots, (p-1)$ with $a_0 = x_1 < a_1 < \cdots < a_p = x_2$ such that on each interval, the trajectory of $v_F(\cdot; x_0, v_0, \Sigma_{f,g})$ (respectively, $v_B(\cdot; x_f, v_f, \Omega_{f,g})$) is either a trajectory of the differential equation $v'(x) = f(x, v(x))$ or a segment of the curve $v = g(x)$.* □

The proof is given in Appendix C.

Now, we are ready to state the comparison principle for the state-constrained differential inequalities in (3) and (4).

Theorem 2 *Suppose that the function f is continuous with respect to the first argument and is locally Lipschitz with respect to the second argument and the function g is continuously differentiable. Then, for any $\tilde{v} \in \mathcal{C}_+(\Sigma) \cup \mathcal{C}_-(\Sigma)$ (respectively, $\mathcal{C}^+(\Omega) \cup \mathcal{C}^-(\Omega)$), $\tilde{v}(x) \leq v_F(x; x_0, v_0, \Sigma_{f,g})$ (respectively, $\tilde{v}(x) \leq v_B(x; x_f, v_f, \Omega_{f,g})$) on the common interval of existence of $\tilde{v}$ and $v_F(\cdot; x_0, v_0, \Sigma_{f,g})$ (respectively, $v_B(\cdot; x_f, v_f, \Omega_{f,g})$).* □

The proof of Theorem 2 is given in Appendix D.

So far, we have shown under the hypotheses of Theorem 2 that the differential inequality Σ with any one of $Dv(x) = D^+v(x)$, $Dv(x) = D_+v(x)$, $Dv(x) = D^-v(x)$, and $Dv(x) = D_-v(x)$ has both of the maximal A- and C-solutions given by $v_F(\cdot; x_0, v_0, \Sigma_{f,g})$. We also have shown under the hypotheses of Theorem 2 that the differential inequality Σ with $Dv(x) = v'(x)$ has the maximal A-solution given by $v_F(\cdot; x_0, v_0, \Sigma_{f,g})$ but does not necessarily have the maximal C-solution, since $v_F(\cdot; x_0, v_0, \Sigma_{f,g}) \in \mathcal{A}(\Sigma)$ but, in general, $v_F(\cdot; x_0, v_0, \Sigma_{f,g}) \notin \mathcal{C}(\Sigma)$. Similar arguments are also applied to the differential inequality Ω.

Let $[x_0, x_1)$ (respectively, $(x_2, x_f]$) be the interval of existence of $v_F(\cdot; x_0, v_0, \Sigma_{f,g})$ (respectively, $v_B(\cdot; x_f, v_f, \Omega_{f,g})$). Note from the definition of the functions $\Sigma_{f,g}$ and $\Omega_{f,g}$ in (11) and (12) that if $v < g(x)$, then $\Sigma_{f,g}(x, v) = \Omega_{f,g}(x, v) = f(x, v)$. Thus, if $v_F(x; x_0, v_0, \Sigma_{f,g}) < g(x)$ for all $x \in [x_0, x_1)$, then $v_F(x; x_0, v_0, \Sigma_{f,g}) = v_F(x; x_0, v_0, f)$ for all $x \in [x_0, x_1)$. Similarly, if $v_B(x; x_f, v_f, \Omega_{f,g}) < g(x)$ for all $x \in (x_2, x_f]$, then $v_B(x; x_f, v_f, \Omega_{f,g}) = v_B(x; x_f, v_f, f)$ for all $x \in (x_2, x_f]$. In this context, Theorem 2 can be viewed as a natural extension of the well-known comparison principle [1] for the differential inequalities without state constraints to those with state constraints.

The comparison principle given in Theorem 2 can be further extended to the following state-constrained differential inequality which takes a more complex form than those in (3) and (4).

$$\Lambda : \begin{cases} \underline{f}(x, v(x)) \le Dv(x) \le \overline{f}(x, v(x)), \ x \in [x_0, x_f] \\ v(x) \le g(x), \ x \in (x_0, x_f) \\ v(x_0) = v_0 \le g(x_0) \\ v(x_f) = v_f \le g(x_f) \end{cases} \tag{17}$$

where $g : \mathbb{R} \to \mathbb{R}$. Here, the functions $\overline{f}, \underline{f} : \mathbb{R}^2 \to \mathbb{R}$ are assumed to satisfy

$$\underline{f}(x, v) \le \overline{f}(x, v), \quad \forall \, (x, v) \in \mathbb{R}^2. \tag{18}$$

In the next section, we will encounter a differential inequality of this form in solving a time-optimal control problem for a state-constrained nonlinear system consisting of (33), (34), and (44).

The (maximal) A- and C-solutions of the differential inequality Λ are defined as in the differential inequalities Σ and Ω, except that the interval of existence of these solutions is fixed to be $[x_0, x_f]$. In what follows, $\mathcal{C}(\Lambda)$ (respectively, $\mathcal{C}^+(\Lambda)$, $\mathcal{C}_+(\Lambda)$, $\mathcal{C}^-(\Lambda)$, and $\mathcal{C}_-(\Lambda)$) denotes the set of C-solutions of the differential inequality Λ with $Dv(x) = D^+v(x)$ (respectively, $Dv(x) = D_+v(x)$, $Dv(x) = D^-v(x)$, and $Dv(x) = D_-v(x)$), while $\mathcal{A}(\Lambda)$ denotes the set of A-solutions of the differential inequality Λ. Then, it is easy to see from Lemma 1 that

$$\mathcal{A}(\Lambda) \subset \mathcal{C}^+(\Lambda), \ \mathcal{C}_+(\Lambda), \ \mathcal{C}^-(\Lambda), \ \mathcal{C}_-(\Lambda). \tag{19}$$

The following theorem characterizes explicitly the maximal solution of the differential inequality Λ.

Theorem 3 *Suppose that the functions $\overline{f}$ and $\underline{f}$ are continuous with respect to the first argument and are locally Lipschitz with respect to the second argument and that the function g is continuously differentiable. Suppose further that $v_F(\cdot; x_0, v_0, \Sigma_{\overline{f},g})$ and $v_B(x; x_f, v_f, \Omega_{\underline{f},g})$ exist on the interval $[x_0, x_f]$ such that*

$$v_F(x_f; x_0, v_0, \Sigma_{\overline{f},g}) \ge v_f, \quad v_B(x_0; x_f, v_f, \Omega_{\underline{f},g}) \ge v_0. \tag{20}$$

Define the function $v^\star_\Lambda : [x_0, x_f] \to \mathbb{R}$ *by*

$$v^\star_\Lambda(x) \stackrel{\triangle}{=} \min\{v_F(x; x_0, v_0, \Sigma_{\overline{f},g}), v_B(x; x_f, v_f, \Omega_{\underline{f},g})\}. \tag{21}$$

Then, $v^\star_\Lambda$ *is an absolutely continuous solution to the differential inequality* Λ *(i.e.,* $v^\star_\Lambda \in \mathcal{A}(\Lambda)$*) such that, for any* $\tilde{v} \in \mathcal{C}(\Lambda) \cup \mathcal{C}^+(\Lambda) \cup \mathcal{C}_+(\Lambda) \cup \mathcal{C}^-(\Lambda) \cup \mathcal{C}_-(\Lambda)$*,*

$$\tilde{v}(x) \leq v^\star_\Lambda(x), \quad \forall\, x \in [x_0, x_f]. \tag{22}$$

□

The proof of Theorem 3 is given in Appendix E.

Theorem 3 along with (19) implies that if the hypotheses of Theorem 3 are satisfied, the differential inequality Λ with any one of $Dv(x) = D^+v(x)$, $Dv(x) = D_+v(x)$, $Dv(x) = D^-v(x)$, and $Dv(x) = D_-v(x)$ has both of the maximal A- and C-solutions given by $v^\star_\Lambda$ in (21). On the other hand, note from Theorem 3 and Lemma 2 that for any $\tilde{v} \in \mathcal{A}(\Lambda)$, $\tilde{v}(x) \leq v^\star_\Lambda(x)$ for all $x \in [x_0, x_f]$. Thus, we see under the hypotheses of Theorem 3 that the differential inequality Λ with $Dv(x) = v'(x)$ has the maximal A-solution given by the function $v^\star_\Lambda$ in (21). However, it does not necessarily have the maximal C-solution, since the function $v^\star_\Lambda$ in (21) does not necessarily belong to $\mathcal{C}(\Lambda)$.

As in the differential inequalities Σ and Ω, under certain conditions, the maximal solution of the differential inequality Λ consists of finitely many piecewise real-analytic curves.

Corollary 1 *Suppose that the hypotheses of Theorem 3 hold. Suppose further that the functions* $H_{\overline{f},g}$*,* $H_{\underline{f},g}$*, and* g *are real-analytic. Then, there exist a finite number of intervals* $[a_k, a_{k+1}]$*,* $k = 0, 1, \ldots, (p-1)$ *with* $a_0 = x_0 < a_1 < \cdots < a_p = x_f$ *such that on each interval, the curve* $v = v^\star_\Lambda(x)$ *is one of the three: (i) a segment of the curve* $v = g(x)$*, (ii) a trajectory of the differential equation* $v'(x) = \overline{f}(x, v(x))$*, or (iii) a trajectory of the differential equation* $v'(x) = \underline{f}(x, v(x))$*.* □

The proof of Corollary 1 is given in Appendix F.

We now show by way of contradiction that the inequality in (20) is necessary for the set of C-solutions of the differential inequality Λ to be non-empty. Suppose that the inequality in (20) does not hold, i.e.,

$$v_F(x_f; x_0, v_0, \Sigma_{\overline{f},g}) < v_f \;\text{ or }\; v_B(x_0; x_f, v_f, \Omega_{\underline{f},g}) < v_0. \tag{23}$$

Suppose further that the differential inequality Λ has a C-solution, say, $\tilde{v}$. Then,

$$\tilde{v}(x_0) = v_0, \quad \tilde{v}(x_f) = v_f. \tag{24}$$

It is also clear from Theorem 2 that

$$\begin{aligned} \tilde{v}(x_f) &\leq v_F(x_f; x_0, v_0, \Sigma_{\overline{f},g}), \\ \tilde{v}(x_0) &\leq v_B(x_0; x_f, v_f, \Omega_{\underline{f},g}). \end{aligned}$$

However, this is contradictory to (23) and (24).

We next provide two conditions under which the maximal solution $v^\star_\Lambda$ in (21) can take a simple form. First, we show that if $v^\star_\Lambda$ satisfies

$$v^\star_\Lambda(x) < g(x), \;\; \forall\, x \in [x_0, x_f], \tag{25}$$

then there exists a real number $x_1 \in [x_0, x_f]$ such that

$$v_\Lambda^\star(x) = \begin{cases} v_F(x; x_0, v_0, \overline{f}), & x \in [x_0, x_1], \\ v_B(x; x_f, v_f, \underline{f}), & x \in (x_1, x_f]. \end{cases} \tag{26}$$

To show this, we need to consider the following two cases: (i) $v_F(x; x_0, v_0, \Sigma_{\overline{f},g}) < g(x)$ for all $x \in [x_0, x_f]$ and (ii) there exists a real number $x_2 \in (x_0, x_f)$ such that $v_F(x; x_0, v_0, \Sigma_{\overline{f},g}) < g(x)$ for all $x \in [x_0, x_2)$ and $v_F(x_2; x_0, v_0, \Sigma_{\overline{f},g}) = g(x_2)$. We present the arguments for the case (i) since those for the case (ii) are very similar. In the case (i), it is obvious that

$$v_F(x; x_0, v_0, \Sigma_{\overline{f},g}) = v_F(x; x_0, v_0, \overline{f}), \quad \forall\, x \in [x_0, x_f]. \tag{27}$$

Moreover, there exists a real number $x_1 \in [x_0, x_f]$ such that

$$v_F(x_1; x_0, v_0, \overline{f}) = v_B(x_1; x_f, v_f, \underline{f}). \tag{28}$$

Hence, it follows from Theorem 6.3 in [1] that

$$v_F(x; x_0, v_0, \overline{f}) \;\geq\; v_F(x; x_f, v_f, \underline{f}),\ \forall\, x \in [x_0, x_1], \tag{29}$$

$$v_F(x; x_0, v_0, \overline{f}) \;\leq\; v_F(x; x_f, v_f, \underline{f}),\ \forall\, x \in [x_1, x_f]. \tag{30}$$

Also, it is clear that $v_B(x; x_f, v_f, \Omega_{\underline{f},g}) = v_B(x; x_f, v_f, \underline{f})$ for all $x \in [x_1, x_f]$. Finally, this along with (28)-(30) leads to (26).

We next show that if

$$\underline{f}(x, g(x)) \leq g'(x) \leq \overline{f}(x, g(x)), \quad \forall\, x \in [x_0, x_f], \tag{31}$$

then the trajectory of the maximal solution $v_\Lambda^\star$ can contain at most one segment of the curve $v = g(x)$, that is, $v_\Lambda^\star$ takes the form in (26) for some $x_1 \in [x_0, x_f]$ or there exist two real numbers $x_2, x_3 \in [x_0, x_f]$ with $x_2 < x_3$ such that

$$v_\Lambda^\star(x) = \begin{cases} v_F(x; x_0, v_0, \overline{f}), & x \in [x_0, x_2], \\ g(x), & x \in (x_2, x_3), \\ v_B(x; x_f, v_f, \underline{f}), & x \in [x_3,\ x_f]. \end{cases} \tag{32}$$

To see this, we note that if the condition in (31) is satisfied, then

$$\begin{aligned} \min\{\overline{f}(x, g(x)), g'(x)\} &= \max\{\underline{f}(x, g(x)), g'(x)\} \\ &= g'(x), \quad \forall\, x \in [x_0, x_f]. \end{aligned}$$

Hence, if there exists a real number $x_2 > x_0$ such that $v_F(x_2; x_0, v_0, \overline{f}) = g(x_2)$, then $v_F(\cdot; x_0, v_0, \Sigma_{\overline{f},g})$ has the form

$$v_F(x_f; x_0, v_0, \Sigma_{\overline{f},g}) = \begin{cases} v_F(x; x_0, v_0, \overline{f}), & x \in [x_0, x_2), \\ g(x), & x \geq x_2. \end{cases}$$

Similarly, we can see that if there exists a real number $x_3 < x_f$ such that $v_F(x_3; x_f, v_f, \underline{f}) = g(x_3)$, then $v_B(\cdot; x_f, v_f, \Omega_{\underline{f},g})$ also takes the form

$$v_B(x_f; x_0, v_0, \Omega_{\underline{f},g}) = \begin{cases} g(x), & x \leq x_3, \\ v_B(x; x_f, v_f, \underline{f}), & x \in (x_3, x_f]. \end{cases}$$

It is now clear that the trajectory of the maximal solution $v_\Lambda^\star$ can contain at most one segment of the curve $v = g(x)$.

So far, we have developed several comparison results on state-constrained differential inequalities. In the next section, we use these results to solve a time-optimal control problem for state-constrained second-order systems.

3 Application to Time-Optimal Control

Consider the second-order system

$$\ddot{x} = u \tag{33}$$

subject to the control input constraint

$$u_m(x, \dot{x}) \leq u \;\leq\; u_M(x, \dot{x}), \tag{34}$$

where the functions u_m and u_M from $\mathbb{R}^2$ into $\mathbb{R}$ are assumed to be locally Lipschitz with respect to both arguments such that

$$u_m(x, 0) < 0 < u_M(x, 0), \quad \forall\, x \in \mathbb{R}. \tag{35}$$

For instance, the system governed by

$$\ddot{x} = F(x, \dot{x}, u), \quad |u| \leq 1 \tag{36}$$

can be equivalently represented by the system in (33) and (34) with

$$u_m(x, \dot{x}) \stackrel{\triangle}{=} \min_{|u|\leq 1} F(x, \dot{x}, u), \quad u_M(x, \dot{x}) \stackrel{\triangle}{=} \max_{|u|\leq 1} F(x, \dot{x}, u).$$

In particular when the function F is affine with respect to u, i.e., there exist two functions f and g from $\mathbb{R}^2$ into $\mathbb{R}$ such that

$$F(x, \dot{x}, u) = f(x, \dot{x}) + g(x, \dot{x})u, \tag{37}$$

then u_m and u_M are given, respectively, by

$$u_m(x, \dot{x}) = f(x, \dot{x}) - |g(x, \dot{x})| \tag{38}$$

and

$$u_M(x, \dot{x}) = f(x, \dot{x}) + |g(x, \dot{x})|. \tag{39}$$

It is obvious that the Lipschitz continuity of the functions f and g implies that of the functions u_m and u_M.

We denote by $\mathcal{X}$ the set of all absolutely continuous trajectories $(\tilde{x}, \dot{\tilde{x}}) : [0, \infty) \to \mathbb{R}$ satisfying the constraint

$$u_m(\tilde{x}(t), \dot{\tilde{x}}(t)) \leq \ddot{\tilde{x}}(t) \leq u_M(\tilde{x}(t), \dot{\tilde{x}}(t)), \text{ a.e. on } t \geq 0 \tag{40}$$

along with the initial condition

$$(\tilde{x}(0), \dot{\tilde{x}}(0)) = (x_0, 0). \tag{41}$$

For each trajectory $(\tilde{x}, \dot{\tilde{x}}) \in \mathcal{X}$, we denote by $t_f(\tilde{x}, \dot{\tilde{x}})$ its traversal time from the initial state $(x_0, 0)$ to the final state $(x_f, 0)$, that is,

$$t_f(\tilde{x}, \dot{\tilde{x}}) = \inf\{t > 0 \ : \ (\tilde{x}(t), \dot{\tilde{x}}(t)) = (x_f, 0)\}. \tag{42}$$

Here, we set $t_f(\tilde{x}, \dot{\tilde{x}}) = \infty$ if the trajectory $(\tilde{x}, \dot{\tilde{x}})$ does not arrive at the final state $(x_f, 0)$ within a finite time. Then, we define the subset $\mathcal{X}_f$ of $\mathcal{X}$ as the set of the trajectories $(\tilde{x}, \dot{\tilde{x}}) \in \mathcal{X}$ satisfying $t_f(\tilde{x}, \dot{\tilde{x}}) < \infty$.

In what follows, we consider only the case of

$$x_0 < x_f, \tag{43}$$

since the other case of $x_0 > x_f$ can be handled via the transform $s : x \to -x$. We impose the following state constraint on the system in (33):

$$0 \leq \dot{\tilde{x}}(t) \leq \alpha(\tilde{x}(t)), \quad \forall\, t \geq 0. \tag{44}$$

We assume that the function $\alpha : \mathbb{R} \to \mathbb{R}$ is piecewise continuously differentiable (that is, its derivative is piecewise continuous) such that

$$\alpha(x) > 0, \quad \forall\, x \in \mathbb{R}. \tag{45}$$

From now on, the functions u_m, u_M, and α will be called *constraint functions*. In particular, α is also called the *boundary curve*.

We can now formulate the time-optimal control problem we attempt to solve as follows:

$$(\mathrm{P}) : \quad \begin{array}{ll} \text{minimize} & t_f(\tilde{x}, \dot{\tilde{x}}) \\ \text{subject to} & (\tilde{x}, \dot{\tilde{x}}) \in \mathcal{P}. \end{array} \tag{46}$$

Here $\mathcal{P}$ stands for the set of the trajectories $(\tilde{x}, \dot{\tilde{x}})$ in $\mathcal{X}_f$ satisfying the state constraint in (44). Any trajectory $(x^\star(t), \dot{x}^\star(t)) \in \mathcal{P}$ that satisfies $t_f(x^\star(t), \dot{x}^\star(t)) = \min_{(\tilde{x}, \dot{\tilde{x}}) \in \mathcal{P}} \ t_f(\tilde{x}, \dot{\tilde{x}})$ is called a *time-optimal trajectory* or *solution*. This kind of time-optimal control problem arises in the time-optimal path planning of many practical systems such as robotic manipulators with geometric path constraints, where the non-negativity constraint on $\dot{\tilde{x}}$ in (44) implies that an admissible trajectory should move along the geometric path in the forward direction from the starting point to the final point.

From now on, we characterize completely and explicitly the solution to the time-optimal control problem (P), using the comparison results for the state-constrained differential inequalities developed in Section II. We begin by establishing an important property of the time-optimal trajectory.

Lemma 5 *Let $(\hat{x}, \dot{\hat{x}}) \in \mathcal{P}$. Suppose that the trajectory $(\hat{x}, \dot{\hat{x}})$ has an intermediate zero-velocity point before arriving at the final state $(x_f, 0)$. Then, the trajectory $(\hat{x}, \dot{\hat{x}})$ is not time-optimal.* □

The proof is given in Appendix G.

As the direct consequence of Lemma 5, the time-optimal control problem (P) is reduced to the problem:

$$(\mathrm{P}_+) : \quad \begin{array}{ll} \text{minimize} & t_f(\tilde{x}, \dot{\tilde{x}}) \\ \text{subject to} & (\tilde{x}, \dot{\tilde{x}}) \in \mathcal{P}_+. \end{array} \tag{47}$$

Here, $\mathcal{P}_+$ is the subset of $\mathcal{P}$ defined by

$$\mathcal{P}_+ \stackrel{\triangle}{=} \{(\tilde{x}, \dot{\tilde{x}}) \in \mathcal{P} \;:\; \dot{\tilde{x}}(t) \neq 0,\ \forall\, t \in (0, t_f(\tilde{x}, \dot{\tilde{x}}))\}. \tag{48}$$

That is, any trajectory in $\mathcal{P}_+$ has no intermediate zero-velocity points before arriving at the final state $(x_f, 0)$.

As will be seen soon, the solution to the time-optimal control problem (P_+) is closely related to the maximal solution of the differential inequality

$$\bar{\Lambda} : \begin{cases} a_m(x, v(x)) \leq v'(x) \leq a_M(x, v(x)),\ x \in [x_0, x_f] \\ 0 < v(x) \leq \alpha(x),\ x \in (x_0, x_f) \\ v(x_0) = v(x_f) = 0, \end{cases}$$

where the functions $a_m, a_M : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ are defined by

$$a_m(x, v) \stackrel{\triangle}{=} \frac{u_m(x, v)}{v}, \quad a_M(x, v) \stackrel{\triangle}{=} \frac{u_M(x, v)}{v}, \quad v \neq 0.$$

Here, we do not specify the values of the functions u_m and u_M at $v = 0$, since the results developed soon do not depend on their values at $v = 0$.

We now characterize the maximal solution of the differential inequality $\bar{\Lambda}$ in terms of the solutions of the differential equations

$$y'(x) = \Sigma_{y_M,\beta}(x, y(x)),\ x \geq x_0 \text{ with } y(x_0) = 0, \tag{49}$$

$$y'(x) = \Omega_{y_m,\beta}(x, y(x)),\ x \leq x_f \text{ with } y(x_f) = 0, \tag{50}$$

where

$$\begin{aligned} y_m(x, y) &\stackrel{\triangle}{=} 2u_m\left(x, \sqrt{|y|}\right), \\ y_M(x, y) &\stackrel{\triangle}{=} 2u_M\left(x, \sqrt{|y|}\right), \\ \beta(x) &\stackrel{\triangle}{=} |\alpha(x)|^2. \end{aligned}$$

Using the comparison results for differential inequalities in the preceding section, we can characterize explicitly the maximal solution of the differential inequality $\bar{\Lambda}$, which is summarized in the following corollary.

Corollary 2 *The above differential equations in (49) and (50) have the unique positive solutions $y_F(\cdot; x_0, 0, \Sigma_{y_M,\beta})$ and $y_B(\cdot; x_f, 0, \Omega_{y_m,\beta})$, respectively, over $[x_0, x_f]$. Furthermore, the differential inequality $\bar{\Lambda}$ has the maximal solution $v^\star_{\bar{\Lambda}} : [x_0, x_f] \to \mathbb{R}$ given by*

$$v^\star_{\bar{\Lambda}}(x) \stackrel{\triangle}{=} \min\{v_F(x), v_B(x)\}, \quad \forall\, x \in [x_0, x_f], \tag{51}$$

where

$$\begin{aligned} v_F(x) &\stackrel{\triangle}{=} \left(y_F(x; x_0, 0, \Sigma_{y_M,\beta})\right)^{1/2}, \\ v_B(x) &\stackrel{\triangle}{=} \left(y_B(x; x_f, 0, \Sigma_{y_m,\beta})\right)^{1.2}. \end{aligned}$$

□

The proof of Corollary 2 is given in Appendix H.

From now on, we clarify the connection between the time-optimal solution $(x^\star, \dot{x}^\star)$ and the maximal solution $v^\star_{\bar{\Lambda}}$ of the differential inequality $\bar{\Lambda}$. If $(\tilde{x}, \dot{\tilde{x}}) \in \mathcal{P}_+$, then $\tilde{x}$ is strictly increasing over the interval $[0, t_f(\tilde{x}, \dot{\tilde{x}})]$, since $\dot{\tilde{x}}(t) > 0$ for all $t \in (0, t_f(\tilde{x}, \dot{\tilde{x}}))$. Therefore, there exists a function $\tilde{s} : [x_0, x_f] \to \mathbb{R}$ such that

$$\tilde{s}(\tilde{x}(t)) = t, \quad \forall\, t \in [0, t_f(\tilde{x}, \dot{\tilde{x}})]. \tag{52}$$

Define the function $\tilde{v} : [x_0, x_f] \to \mathbb{R}$ by

$$\tilde{v}(x) \overset{\triangle}{=} \dot{\tilde{x}}(\tilde{s}(x)), \quad \forall\, x \in [x_0, x_f]. \tag{53}$$

We are now ready to state the following lemma.

Lemma 6 *For each $(\tilde{x}, \dot{\tilde{x}}) \in \mathcal{P}_+$, the function $\tilde{v}$ defined in (52) and (53) is an A-solution of the differential inequality $\bar{\Lambda}$. Moreover, the mapping K, defined by $K(\tilde{x}, \dot{\tilde{x}}) \overset{\triangle}{=} \tilde{v}$, is a bijection from $\mathcal{P}_+$ onto $\mathcal{A}(\bar{\Lambda})$ such that*

$$t_f(\tilde{x}, \dot{\tilde{x}}) = \int_{x_0}^{x_f} \frac{1}{\tilde{v}(x)} dx. \tag{54}$$

□

The proof of Lemma 6 is given in Appendix I.

As a consequence of Lemma 6, the time-optimal control problem (P_+) can be reduced to the optimization problem

$$(\bar{\mathrm{P}}): \quad \begin{array}{l} \text{minimize } \int_{x_0}^{x_f} \dfrac{1}{\tilde{v}(x)} dx \\ \text{subject to } \tilde{v} \in \mathcal{A}(\bar{\Lambda}). \end{array} \tag{55}$$

Note that $\int_{x_0}^{x_f} 1/\tilde{v}(x)dx \geq \int_{x_0}^{x_f} 1/v^\star_{\bar{\Lambda}}(x)dx$ and hence that $\int_{x_0}^{x_f} 1/v^\star_{\bar{\Lambda}}(x)dx = \min\limits_{\tilde{v} \in \mathcal{A}(\bar{\Lambda})} \int_{x_0}^{x_f} 1/\tilde{v}(x)\ dx$. The solution $(x^\star, \dot{x}^\star)$ to the optimization problem $(\bar{\mathrm{P}})$ is therefore uniquely given by

$$(x^\star, \dot{x}^\star) = K^{-1}(v^\star_{\bar{\Lambda}}). \tag{56}$$

Moreover, it is easy to see from the definition of K that whenever $\tilde{v} \in \mathcal{A}(\bar{\Lambda})$, $K^{-1}(\tilde{v})$ is the trajectory that traverses from the initial state $(x_0, 0)$ to the final state $(x_f, 0)$ along the curve $\dot{x} = \tilde{v}(x)$. This observation along with (56) leads to the complete characterization of the time-optimal solution $(x^\star, \dot{x}^\star)$, which is summarized in the following theorem.

Theorem 4 *The time-optimal control problem (P) has the unique time-optimal trajectory $(x^\star, \dot{x}^\star)$ that traverses from the initial state $(x_0, 0)$ to the final state $(x_f, 0)$ along the maximal solution $v^\star_{\bar{\Lambda}}$ of the differential inequality $\bar{\Lambda}$ with the minimum traversal time $t_f^\star$ given by*

$$t_f^\star = t_f(x^\star, \dot{x}^\star) = \int_{x_0}^{x_f} \frac{1}{v^\star_{\bar{\Lambda}}(x)} dx. \tag{57}$$

□

Note from Corollary 2 that the function v_F (respectively, v_B) in (51) is in fact the unique solution $v_F(\cdot; x_0, 0, \Sigma_{a_M,\alpha})$ (respectively, $v_B(\cdot; x_f, 0, \Sigma_{a_m,\alpha})$) of the differential equation $v'(x) = \Sigma_{a_M,\alpha}(x, v(x))$, $x \geq x_0$ with $v(x_0) = 0$ (respectively, $v'(x) = \Sigma_{a_m,\alpha}(x, v(x))$, $x \leq x_f$ with $v(x_f) = 0$). Then, it is straightforward to see that the curve $\dot{x} = v_F(x)$ (respectively, the curve $\dot{x} = v_B(x)$) stands for the curve in the $(x, \dot{x})$ phase plane beyond which the trajectory $(x, \dot{x})$ of the system in (33) cannot be steered forward in time from the initial state $(x_0, 0)$ (respectively, backward in time from the final state $(x_f, 0)$) under the control input constraint in (40) and the state constraint in (44). In this context, the curve $\dot{x} = v_F(x)$ (respectively, $\dot{x} = v_B(x)$) in (51) is called the *forward velocity limitation curve* (respectively, *backward velocity limitation curve*). According to Theorem 4, the time-optimal trajectory $(x^\star, \dot{x}^\star)$ traverses from the initial state $(x_0, 0)$ to the final state $(x_f, 0)$ along the minimum of the forward and backward velocity limitation curves.

Observe that the right-hand sides of the differential equations in (49) and (50) are continuous. Hence, we can obtain the curve $\dot{x} = v^\star_{\bar{\Lambda}}(x)$ in a numerically efficient way by applying directly the well-known Euler or Runge-Kutta methods [37] to the scalar ordinary differential equations in (49) and (50) and then by using (51). Furthermore, the computational load required by the proposed method is independent of the number of switching points. Another noticeable feature is that our method works regardless of the presence of boundary arcs and, moreover, works even if there exist an infinite number of switching points. On the other hand, the previously known methods developed for solving time-optimal control of robotic manipulators with geometric path constraints [20]-[27] can be applied only to the case where the time-optimal solution $(x^\star, \dot{x}^\star)$ consists of a finite number of trajectories of $\ddot{x} = u_M(x, \dot{x})$ and $\ddot{x} = u_m(x, \dot{x})$, which does not necessarily hold in general. Moreover, the required computational load becomes proportionally increasing with the number of switching points, which may be arbitrarily large.

From now on, the curve $\dot{x} = v^\star_{\bar{\Lambda}}(x)$ defined in (51) is called the *time-optimal curve.* In general, the time-optimal curve may consist of an infinite number of segments of the boundary curve $\dot{x} = \alpha(x)$, maximum acceleration curves (any trajectories of the differential equation $v'(x) = a_M(x, v(x))$), and maximum deceleration curves (any trajectories of the differential equation $v'(x) = a_m(x, v(x))$). The following Corollary 3 states under some restrictive conditions on the constraint functions u_m, u_M, and α that the time-optimal curve consists only of a finite number of those curves. In what follows, a real-valued function h from $D \subset \mathbb{R}$ into $\mathbb{R}$ is said to be *piecewise real-analytic*, if on each compact subinterval $[a, b]$ of D, there exist a finite number of points c_i, $i = 0, 1, \ldots, p$ with $c_0 = a < c_1 < \cdots < c_p = b$ such that h is real-analytic on each of the intervals $[c_i, c_{i+1}]$, $i = 0, 2, \ldots, (p-1)$. A basic property of piecewise real-analytic functions is that the composite function $H_{f,g}$ in (13) is piecewise real-analytic, if the function $f : \mathbb{R}^2 \to \mathbb{R}$ is real-analytic and the function $g : \mathbb{R} \to \mathbb{R}$ is piecewise real-analytic [39].

Corollary 3 *Suppose that the functions $H_{u_m,\alpha}$, $H_{u_M,\alpha}$, and α are piecewise real-analytic on the compact interval $[x_0, x_f]$. Then, there exist a finite number of time intervals $[t_k, t_{k+1}]$, $k = 0, 1, \ldots, (p-1)$ with $t_0 = 0 < t_1 < \cdots < t_p = t^\star_f$ such that on each time interval, the time-optimal trajectory $(x^\star, \dot{x}^\star)$ moves along one of the following curves: (i) a maximum acceleration curve ($\ddot{x}^\star = u_M(x^\star, \dot{x}^\star)$), (ii) a maximum deceleration curve ($\ddot{x}^\star = u_m(x^\star, \dot{x}^\star)$), or (iii) the boundary curve ($\dot{x}^\star = \alpha(x^\star)$).* □

The proof of Corollary 3 is given in Appendix J.

The second-order system given in (33) and (34) satisfies the hypothesis of Corollary 3, if the functions u_m and u_M are real-analytic and the function α is piecewise real-analytic. Moreover, in the case of the affine second-order system in (36) with the function f in (37), the hypothesis of Corollary 3 is satisfied under much weaker conditions. To see this, first note from (38) and (39) that

$$\begin{aligned} H_{u_m,\alpha}(x) &= p(x,\alpha(x)) - |g(x,\alpha(x))| \\ &= H_{g,\alpha}(x) - s(x), \quad \forall\, x \in [x_0, x_f], \end{aligned} \tag{58}$$

$$\begin{aligned} H_{u_M,\alpha}(x) &= g(x,\alpha(x)) + |g(x,\alpha(x))| \\ &= H_{g,\alpha}(x) + s(x), \quad \forall\, x \in [x_0, x_f]. \end{aligned} \tag{59}$$

where the function $s : \mathbb{R} \to \mathbb{R}$ is defined by $s(x) \overset{\triangle}{=} |H_{g,\alpha}(x)|$. Suppose that the functions p and g are real-analytic and the function α is piecewise real-analytic. Then, we can see that the functions $H_{p,\alpha}$ and $H_{g,\alpha}$ are piecewise real-analytic [39]. Moreover, it can be readily verified that the function s is also piecewise real-analytic. Consequently, the hypothesis of Corollary 3 holds. In fact, it holds under a little more general condition that the functions p and g are piecewise real-analytic on the interval $[x_0, x_f] \times \mathbb{R}$. Finally, it is not difficult to see that the dynamic equations of the end-effector of a robotic manipulator which moves along a piecewise real-analytic path in the workspace take the form in (33) and (34) and furthermore satisfy the hypothesis of Corollary 3 [20].

We next present some simulation results using a two-DOF robotic manipulator, in order to demonstrate the practical use of our method developed so far to solve the time-optimal control problem (P). Specifically, we consider the time-optimal path planning of an X–Y gantry system along the circular path C of radius r, which is illustrated in Figure 1. The dynamic equations of the X–Y gantry system are given by

$$m_x \ddot{x} = u_x, \quad m_y \ddot{y} = u_y, \quad x(0) = y(0) = \dot{x}(0) = \dot{y}(0) = 0, \tag{60}$$

where (x, y) denotes the position E of the end-effector in the Euclidean coordinate system, while, for each $z = x, y$, m_z and u_z denote, respectively, the mass and the generated input force of the z-axis actuator. Here, the input forces u_x and u_y are subject to

$$|u_x| \le U_x, \quad |u_y| \le U_y, \tag{61}$$

where the positive constants U_x and U_y stand for the physical limitation in the generated forces of the x-axis and y-axis actuators. We denote by $\mathcal{S}$ the set of all absolutely continuous circular trajectories $(\tilde{x}, \tilde{y})$ of the end-effector that traverse in the counterclockwise direction along the circular path C and satisfy the constraints in (60) and (61) a.e. on $t \ge 0$ with $x = \tilde{x}$ and $y = \tilde{y}$. Then, the time-optimal path planning of the X–Y gantry system along the circular path C is to find the circular trajectory $(x^\star, y^\star)$ in the set $\mathcal{S}$ whose return time to the starting point A is minimal.

Let θ denote the angle between the x-axis and the position of the end-effector, that is, $x = r\cos\theta, \quad y = r\sin\theta$. The second derivatives of x and y are given by

$$\ddot{x} = -r\ddot{\theta}\sin\theta - r|\dot{\theta}|^2\cos\theta, \quad \ddot{y} = r\ddot{\theta}\cos\theta - r|\dot{\theta}|^2\sin\theta. \tag{62}$$

This along with (61) implies that

$$-U_x \le -r m_x(\ddot{\theta}\sin\theta + |\dot{\theta}|^2\cos\theta) \le U_x, \tag{63}$$

$$-U_y \le r m_y(\ddot{\theta}\cos\theta - |\dot{\theta}|^2\sin\theta) \le U_y. \tag{64}$$

Through some manipulations of trigonometric functions, we can show that for each $\theta \in [0, 2\pi]$, $\dot{\theta}$ and $\ddot{\theta}$ satisfy the above inequalities in (63) and (64) if and only if

$$u_m(\theta, \dot{\theta}) \leq \ddot{\theta} \leq u_M(\theta, \dot{\theta}) \quad \text{and} \quad |\dot{\theta}| \leq \alpha(\theta) \tag{65}$$

where the functions $u_m, u_M : [0, 2\pi] \times \mathbb{R} \to \mathbb{R}$ and $\alpha : [0, 2\pi] \to \mathbb{R}$ are given by

$$u_m(\theta, \dot{\theta}) \triangleq \begin{cases} -\frac{U_y}{rm_y}, & \theta = 0 \\ \max\{-|\dot{\theta}|^2 \cot\theta - \frac{U_x}{rm_x}\csc\theta, |\dot{\theta}|^2 \tan\theta - \frac{U_y}{rm_y}\sec\theta\}, & 0 < \theta < \frac{\pi}{2}, \\ -\frac{U_x}{rm_x} & \theta = \frac{\pi}{2}, \\ \max\{-|\dot{\theta}|^2 \cot\theta - \frac{U_x}{rm_x}\csc\theta, |\dot{\theta}|^2 \tan\theta + \frac{U_y}{rm_y}\sec\theta\}, & \frac{\pi}{2} < \theta < \pi, \\ -\frac{U_y}{rm_y}, & \theta = \pi, \\ \max\{-|\dot{\theta}|^2 \cot\theta + \frac{U_x}{rm_x}\csc\theta, |\dot{\theta}|^2 \tan\theta + \frac{U_y}{rm_y}\sec\theta\}, & \pi < \theta < \frac{3\pi}{2}, \\ -\frac{U_x}{rm_x}, & \theta = \frac{3\pi}{2}, \\ \max\{-|\dot{\theta}|^2 \cot\theta + \frac{U_x}{rm_x}\csc\theta, |\dot{\theta}|^2 \tan\theta - \frac{U_y}{rm_y}\sec\theta\}, & \frac{3\pi}{2} < \theta < 2\pi, \\ -\frac{U_y}{rm_y}, & \theta = 2\pi, \end{cases} \tag{66}$$

$$u_M(\theta, \dot{\theta}) \triangleq \begin{cases} \frac{U_y}{rm_y}, & \theta = 0, \\ \min\{-|\dot{\theta}|^2 \cot\theta + \frac{U_x}{rm_x}\csc\theta, |\dot{\theta}|^2 \tan\theta + \frac{U_y}{rm_y}\sec\theta\}, & 0 < \theta < \frac{\pi}{2}, \\ \frac{U_x}{rm_x} & \theta = \frac{\pi}{2}, \\ \min\{-|\dot{\theta}|^2 \cot\theta + \frac{U_x}{rm_x}\csc\theta, |\dot{\theta}|^2 \tan\theta - \frac{U_y}{rm_y}\sec\theta\}, & \frac{\pi}{2} < \theta < \pi, \\ \frac{U_y}{rm_y}, & \theta = \pi, \\ \min\{-|\dot{\theta}|^2 \cot\theta - \frac{U_x}{rm_x}\csc\theta, |\dot{\theta}|^2 \tan\theta - \frac{U_y}{rm_y}\sec\theta\}, & \pi < \theta < \frac{3\pi}{2}, \\ \frac{U_x}{rm_x}, & \theta = \frac{3\pi}{2}, \\ \min\{-|\dot{\theta}|^2 \cot\theta - \frac{U_x}{rm_x}\csc\theta, |\dot{\theta}|^2 \tan\theta + \frac{U_y}{rm_y}\sec\theta\}, & \frac{3\pi}{2} < \theta < 2\pi, \\ \frac{U_y}{rm_y}, & \theta = 2\pi, \end{cases} \tag{67}$$

$$\alpha(\theta) \triangleq \begin{cases} [\frac{U_x}{rm_x}\cos\theta + \frac{U_y}{rm_y}\sin\theta]^{\frac{1}{2}}, & 0 \leq \theta < \frac{\pi}{2}, \\ [\frac{U_x}{rm_x}\cos\theta - \frac{U_y}{rm_y}\sin\theta]^{\frac{1}{2}}, & \frac{\pi}{2} \leq \theta < \pi, \\ [-\frac{U_x}{rm_x}\cos\theta - \frac{U_y}{rm_y}\sin\theta]^{\frac{1}{2}}, & \pi \leq \theta < \frac{3\pi}{2}, \\ [-\frac{U_x}{rm_x}\cos\theta + \frac{U_y}{rm_y}\sin\theta]^{\frac{1}{2}}, & \frac{3\pi}{2} \leq \theta \leq 2\pi. \end{cases} \tag{68}$$

The derivation of the above functions is given in Appendix K.

Now, finding the time-optimal circular trajectory $(x^\star, y^\star)$ reduces to solving the time-optimal control problem (P) with $x = \theta$, $x_0 = 0$, $x_f = 2\pi$, and the constraint functions in (66)-(68). Here, it is clear from the circular path constraint that $\dot{\tilde{\theta}}(t) \geq 0$ for all $t \geq 0$ for any $(\tilde{x}, \tilde{y}) \in \mathcal{S}$, where $\tilde{\theta} \triangleq \tan^{-1} \dfrac{\tilde{y}}{\tilde{x}}$. That is, the non-negativity constraint in (44) holds. From now on, the time-optimal solution $\theta^\star$ will be called the *time-optimal angular trajectory*. Then, we have $x^\star = r\cos\theta^\star$ and $y^\star = r\sin\theta^\star$. Furthermore, it is clear that the corresponding non-negativity constraint in (44) follows directly from the constraint that all the trajectories $(\tilde{x}, \dot{y})$ in the set $\mathcal{S}$ traverse in the counterclockwise direction along the circular path. follows directly from the constraint that any admissible trajectory rotates counterclockwise along the circular path. all the trajectories $(\tilde{\theta}, \dot{\theta})$ in the set $\mathcal{P}$ corresponding to this time-optimal control problem satisfy the non-negativity constraint

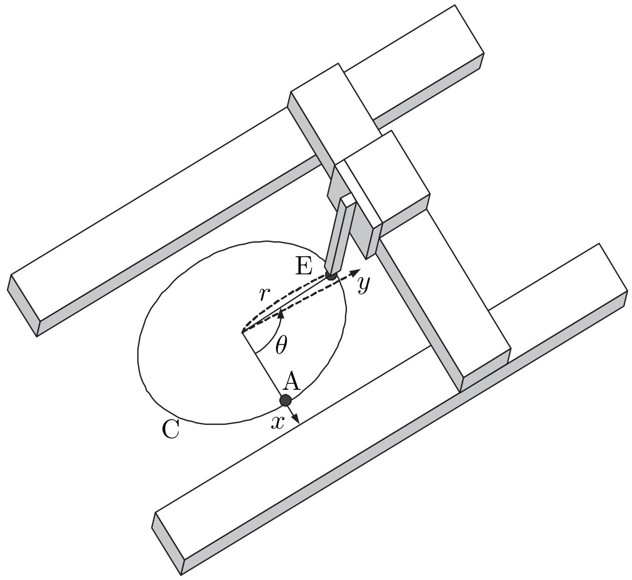

Figure 1: Circular motion of an X–Y gantry system.

in (44), that is, $\dot{\hat{\theta}} \geq 0$ for all $t \geq 0$. follows directly from the constraint that any admissible trajectory rotates counterclockwise along the circular path.

The parameter values used in our simulation work are taken as $m_x = 1$kg, $m_y = 2$kg, $U_x = 50$N, $U_y = 50$N, and $r = 0.1$m. Figure 2 shows the graphic plot of the boundary curve $\dot{\theta} = \alpha(\theta)$ in the $(\theta, \dot{\theta})$-phase plane corresponding to these parameter values. The forward velocity limitation curve $\dot{\theta} = v_F(\theta)$, the backward velocity limitation curve $\dot{\theta} = v_B(\theta)$, and the time-optimal curve $\dot{\theta} = v^\star(\theta)$ in the $(\theta, \dot{\theta})$-phase plane corresponding to these parameter values are plotted in Figures 3, 4, and 5, respectively. Evidently, the functions $H_{u_m,\alpha}$, $H_{u_M,\alpha}$, and α that correspond to the X–Y gantry system are piecewise real-analytic. Corollary 3 implies that the time-optimal angular trajectory $\theta^\star$ consists of a finite number of maximum acceleration curves, maximum deceleration curves, and segments of the boundary curve $\dot{\theta} = \alpha(\theta)$. Indeed, as can be seen from Figure 5, the time-optimal angular trajectory $\theta^\star$ undergoes five switchings between maximum acceleration and maximum deceleration. The time-optimal circular trajectory $(x^\star, y^\star)$ is simply found by solving the two differential equations with continuous right-hand sides that determine the forward and backward velocity limitation curves. The minimum time $t^\star$ for the end-effector to traverse the circular path is found to be 0.43s. The time-optimal angular trajectory $\theta^\star$ and the time-optimal circular trajectory $(x^\star, y^\star)$ of the end-effector are plotted in Figures 6 and 7, respectively.

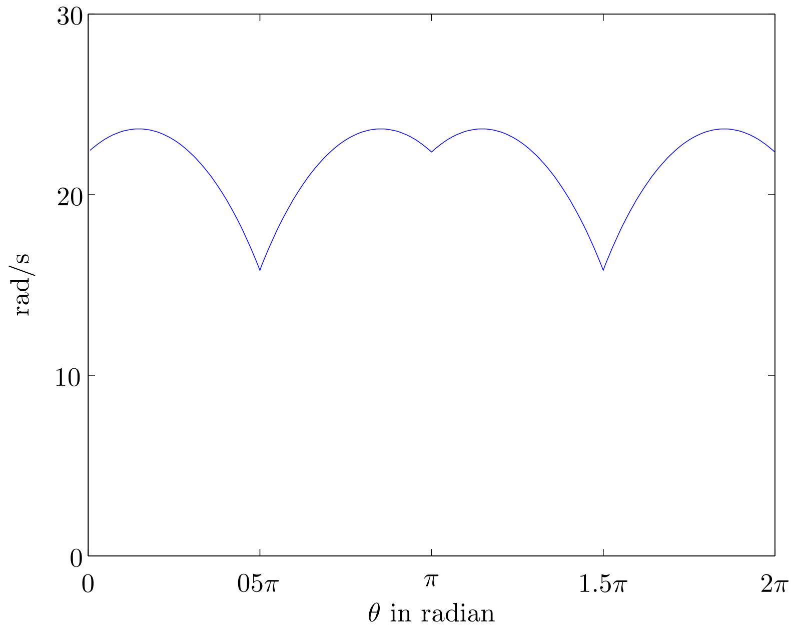

Figure 2: Boundary curve α.

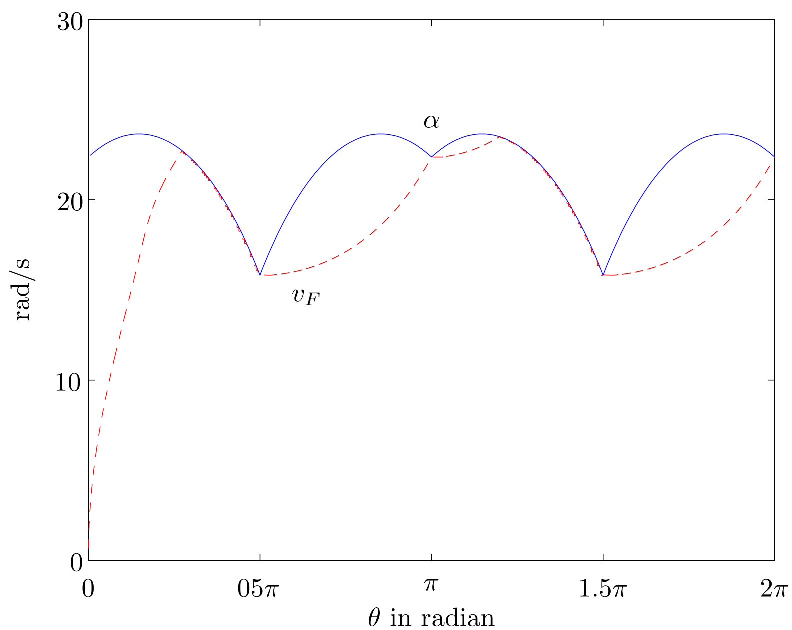

Figure 3: Forward velocity limitation curve v_F.

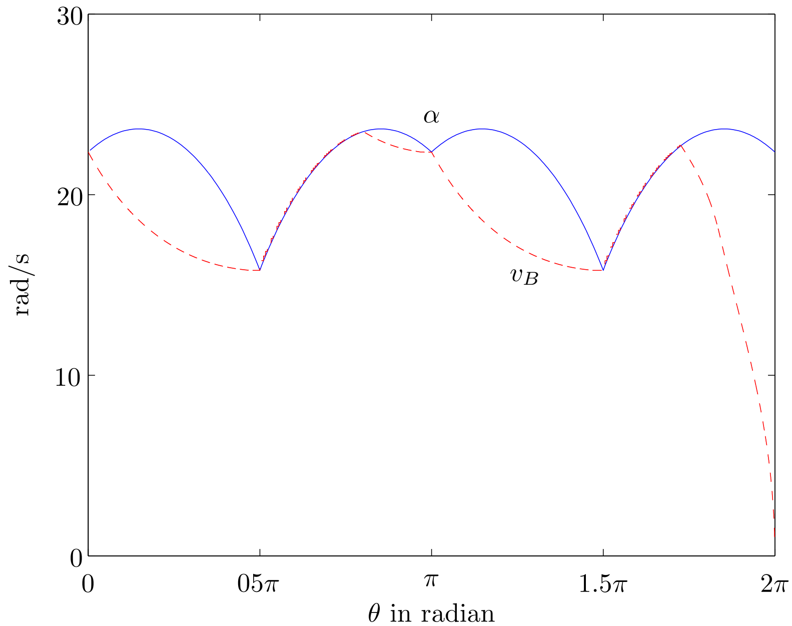

Figure 4: Backward velocity limitation curve v_B.

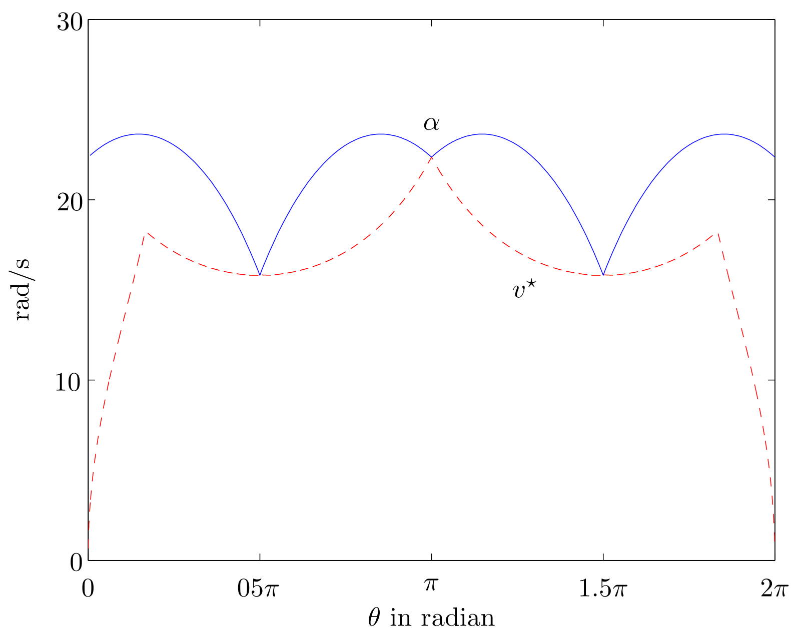

Figure 5: Time-optimal curve $v^\star$.

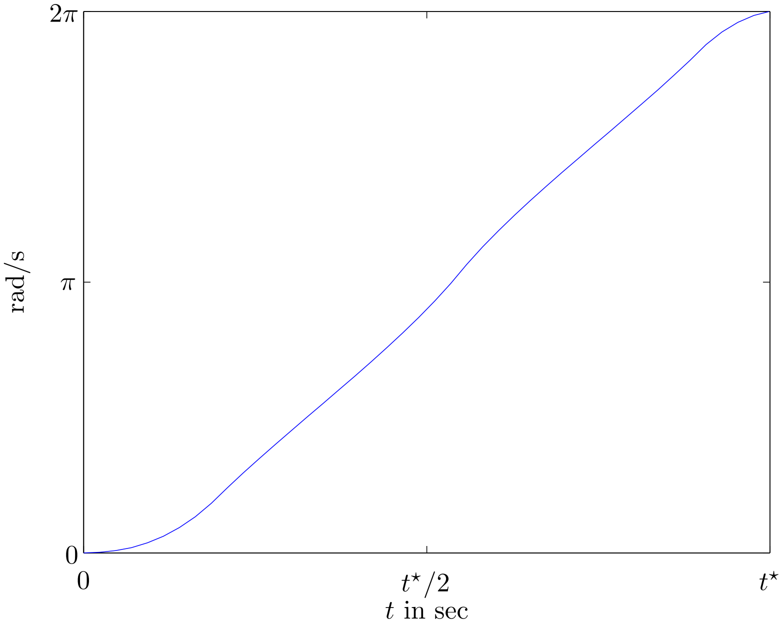

Figure 6: Time-optimal angular trajectory $\theta^\star$.

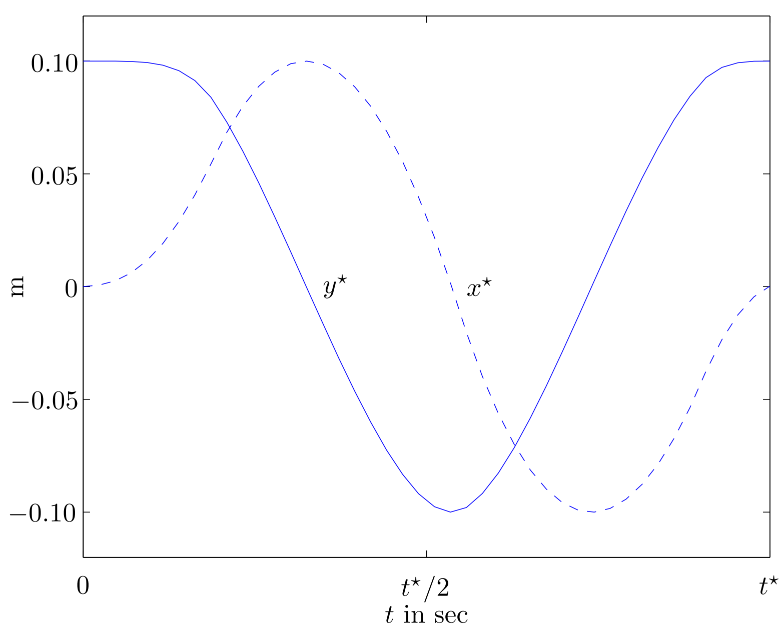

Figure 7: Time-optimal circular trajectory $(x^\star, y^\star)$.

4 Conclusions

In this paper, we have developed the comparison principle for state-constrained differential inequalities which can be viewed as a natural extension of the well-known comparison principle for differential inequalities without state constraints. We have applied it to characterize completely and explicitly the solution to the time-optimal control problem in a class of state-constrained second-order systems including the dynamic equations of robotic manipulators with geometric path constraints. The new comparison principle also may be useful for qualitative and quantitative analysis of state-constrained differential equations.

We have demonstrated the practical use of our complete characterization of the time-optimal solution, applying it to the time-optimal control of a two-DOF robotic manipulator along a circle. One of our future research topics is to develop a computationally efficient method for the time-optimal path planning of robotic manipulators with geometric path constraints, based on the complete characterization of the time-optimal solution presented in this paper. We finally point out that the method developed in this paper cannot be directly extended to the time-optimal control of robotic manipulators without geometric path constraints, i.e., the time-optimal point-to-point (PTP) path planning. Our future research will also be directed towards the extension.

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Appendix A. Proof of Lemma 1

We present only the proof for the inequality in (6), since that for the inequality in (7) is very similar. By way of contradiction, we show that the inequality in (6) holds. Suppose that there exists a real number $x_1 \in [a, b)$ such that

$$k_1 \stackrel{\triangle}{=} D^+\tilde{v}(x_1) > \overline{f}(x_1). \tag{69}$$

This along with the continuity of the function $\overline{f}$ implies that there exists a real number $x_2 \in (x_1, b)$ such that

$$\overline{f}(x) < k_1 - \epsilon, \quad x_1 \leq x \leq x_2. \tag{70}$$

Let $\{h_n \in \mathbb{R} \ : \ h_n > 0, n = 1, 2, \ldots\}$ be any sequence converging to zero. Then, there exists a positive integer p such that $x_1 < x_1 + h_n \leq x_2$ for all $n \geq p$. This along with the absolute continuity of the function $\tilde{v}$ and (70) implies that for all $n \geq p$

$$\begin{aligned} \frac{\tilde{v}(x_1 + h_n) - \tilde{v}(x_1)}{h_n} &= \frac{1}{h_n} \int_{x_1}^{x_1+h_n} \tilde{v}'(s) ds \\ &\leq \frac{1}{h_n} \int_{x}^{x_1+h_n} \overline{f}(s) ds \\ &< k_1 - \epsilon. \end{aligned}$$

By the definition of the upper right derivative [29], we also have $D^+\tilde{v}(x_1) \leq k_1 - \epsilon < k_1$. However, this is self-contradictory to the definition of the constant k_1 in (69). Thus, we have established the inequality $D^+\tilde{v}(x) \leq \overline{f}(x)$ for all $x \in [a, b)$. This along with (5) implies that $D_+\tilde{v}(x) \leq D^+\tilde{v}(x) \leq \overline{f}(x)$ for all $x \in [a, b)$. Through similar arguments, we can also show that $\underline{f}(x) \leq D_-\tilde{v}(x) \leq D^-\tilde{v}(x)$ for all $x \in [a, b)$. $\square$

Appendix B. Proof of Theorem 1

We introduce some notational convention about set-valued maps, which can be found in [32]-[34] and elsewhere. A singleton $\{a\}$ is often denoted by a by abuse of notation. The set of all subsets of a set Y is denoted by 2^Y. A set-valued map $F : X \to 2^Y$ is then a function that associates to any $x \in X$ a subset $F(x)$ of Y. The distance between a point $x \in \mathbb{R}^m$ and a set $A \subset \mathbb{R}^m$ is defined by $d(x, A) \stackrel{\triangle}{=} \inf\{\|x - a\| \ : \ a \in A\}$. Then, the *directed distance* $\vec{d}(A, B)$ between two subsets $A, B \subset \mathbb{R}^m$ is defined by $\vec{d}(A, B) \stackrel{\triangle}{=} \sup\{d(a, B) \ : \ a \in A\}$. The *Hausdorff distance* between two subsets $A, B \subset \mathbb{R}^m$ is defined by $d_H(A, B) = d_H(B, A) \stackrel{\triangle}{=} \max\{\vec{d}(A, B), \vec{d}(B, A)\}$. The ϵ-neighborhood of a set $A \subset \mathbb{R}^m$ is defined by $N_\epsilon(A) \stackrel{\triangle}{=} \{x \in \mathbb{R}^m \ : \ d(x, A) < \epsilon\}$. In particular when $A = \{x_0\}$, $N_\epsilon(\{x_0\})$ is simply denoted by $N_\epsilon(x_0)$. A set-valued map $E : \mathbb{R}^m \to 2^{\mathbb{R}^n}$ is said to be *upper semi-continuous at* x_0 if for any $\epsilon > 0$ there is a $\delta > 0$ such that $\vec{d}(E(x), E(x_0)) < \epsilon$ for all $x \in N_\delta(x_0)$, while it is said to be *continuous at* x_0 if for any $\epsilon > 0$ there is a $\delta > 0$ such that $d_H(E(x), E(x_0)) < \epsilon$ for all $x \in N_\delta(x_0)$. Finally, the set-valued map E is said to be *upper semi-continuous* if it is upper semi-continuous at each $x_0 \in \mathbb{R}^m$, while it is said to be *continuous* if E is continuous at each $x_0 \in \mathbb{R}^m$.

Consider the following two differential inclusions:

$$v'(x) \in E_{f,g}(x, v(x)), \quad \text{a.e. on } x \geq x_0 \text{ with } v(x_0) = v_0, \tag{71}$$
$$v'(x) \in F_{f,g}(x, v(x)), \quad \text{a.e. on } x \leq x_f \text{ with } v(x_f) = v_f, \tag{72}$$

where the set-valued maps $E_{f,g} : \mathbb{R}^2 \to 2^{\mathbb{R}}$ and $F_{f,g} : \mathbb{R}^2 \to 2^{\mathbb{R}}$ are defined as

$$E_{f,g}(x, v) \triangleq \begin{cases} f(x, v), & \text{if } v < g(x), \\ \{u \in \mathbb{R} \ : \ \min\{f(x, g(x)), g'(x)\} \leq u \leq f(x, g(x))\}, & \text{if } v = g(x), \\ \min\{f(x, v), g'(x)\}, & \text{if } v > g(x), \end{cases} \tag{73}$$

$$F_{f,g}(x, v) \triangleq \begin{cases} f(x, v), & \text{if } v < g(x), \\ \{u \in \mathbb{R} \ : \ f(x, g(x)) \leq u \leq \max\{f(x, g(x)), g'(x)\}\}, & \text{if } v = g(x), \\ \max\{f(x, v), g'(x)\}, & \text{if } v > g(x). \end{cases} \tag{74}$$

Note that the functions $\Sigma_{f,g}$ and $\Omega_{f,g}$ can be embedded into the set-valued maps $E_{f,g}$ and $F_{f,g}$, respectively, i.e.,

$$\Sigma_{f,g}(x, v) \in E_{f,g}(x, v), \quad \Omega_{f,g}(x, v) \in F_{f,g}(x, v). \tag{75}$$

To prove Theorem 1, we also need to establish several auxiliary lemmas.

Lemma 7 *Suppose that the function f is continuous and that the function g is continuously differentiable. Then, the set-valued maps $E_{f,g}$ and $F_{f,g}$ are upper semi-continuous.*

Proof: We present only the proof for the set-valued map $E_{f,g}$, since that for $F_{f,g}$ is nearly identical. Partition $\mathbb{R}^{n+1}$ into three regions as follows.

$$\mathbb{R}^{n+1} = M_0 \cup M_1 \cup M_2, \tag{76}$$

where

$$M_0 \triangleq \{(x, v) \mid v = g(x), \ x \in \mathbb{R}\},$$
$$M_1 \triangleq \{(x, v) \mid v < g(x), \ x \in \mathbb{R}\},$$
$$M_2 \triangleq \{(x, v) \mid v > g(x), \ x \in \mathbb{R}\}.$$

Obviously, the function $\Sigma_{f,g}$ is continuous on the open set $M_1 \cup M_2$ and

$$E_{f,g}(x, v) = \{\Sigma_{f,g}(x, v)\}, \quad \forall \ (x, v) \in M_1 \cup M_2. \tag{77}$$

Recall that a continuous set-valued map is always upper-semi-continuous [32]. The upper-semicontinuity of $E_{f,g}$ on $M_1 \cup M_2$ is therefore a direct consequence of (77) and the continuity of $\Sigma_{f,g}$ on $M_1 \cup M_2$. What remains is to show that $E_{f,g}$ is upper semi-continuous on M_0.

Define three subsets S_0, S_1, and S_2 of $\mathbb{R}$ as follows:

$$S_0 \triangleq \{x \in \mathbb{R} \mid f(x, g(x)) = g'(x)\},$$
$$S_1 \triangleq \{x \in \mathbb{R} \mid f(x, g(x)) < g'(x)\},$$
$$S_2 \triangleq \{x \in \mathbb{R} \mid f(x, g(x)) > g'(x)\}.$$

We need to show separately the upper-semicontinuity of $E_{f,g}$ at $x^\star$ in each of three cases: (i) $x^\star \in S_0$, (ii) $x^\star \in S_1$, and (iii) $x^\star \in S_2$. We present only the proof for the case (i) $x^\star \in S_0$, since those for the other two cases are nearly identical.

Suppose that $x^\star \in S_0$. Then,

$$E_{f,g}(x^\star, g(x^\star)) = f(x^\star, g(x^\star)) = g'(x^\star). \tag{78}$$

To show the upper-semicontinuity of $E_{f,g}$ at $(x^\star, g(x^\star))$, we need to consider separately the following three cases: (a) $(x, v) \in M_0$, (b) $(x, v) \in M_1$, and (c) $(x, v) \in M_2$.

(a) $(x, v) \in M_0$: In this case, we have

$$\begin{aligned}\vec{d}(E_{f,g}(x,v), E_{f,g}(x^\star, g(x^\star))) &= \sup_{u \in E_{f,g}(x,v)} |u - f(x^\star, g(x^\star))| \\ &\leq \max\{|g'(x) - f(x^\star, g(x^\star))|, |f(x, g(x)) - f(x^\star, g(x^\star))|\} \\ &= \max\{|g'(x) - g'(x^\star)|, |H_{f,g}(x) - H_{f,g}(x^\star)|\}. \end{aligned} \tag{79}$$

Here, note that the function $H_{f,g}$ is continuous. This along with the continuity of g' and (79) implies that there exists a positive constant δ_1 such that

$$\vec{d}(E_{f,g}(x,v), E_{f,g}(x^\star, g(x^\star))) \leq \epsilon, \quad \forall\, (x, v) \in N_{\delta_1}(x^\star, g(x^\star)) \cap M_0. \tag{80}$$

(b) $(x, v) \in M_1$: In this case, it holds that

$$\vec{d}(E_{f,g}(x,v), E_{f,g}(x^\star, g(x^\star))) = |f(x, v) - f(x^\star, g(x^\star))|. \tag{81}$$

By the continuity of f, there exists a positive constant δ_2 such that

$$\vec{d}(E_{f,g}(x,v), E_{f,g}(x^\star, g(x^\star))) \leq \epsilon, \quad \forall\, (x, v) \in N_{\delta_2}(x^\star, g(x^\star)) \cap M_2. \tag{82}$$

(c) $(x, v) \in M_2$: In this case, it holds that

$$E_{f,g}(x, v) = \min\{f(x, v), g'(x)\}. \tag{83}$$

From the continuity of the functions h and g', it follows that there exists a positive constant δ_3 such that $|f(x,v) - f(x^\star, g(x^\star))|,\ |g'(x) - g'(x^\star)| \leq \epsilon,\ \forall\, (x, v) \in N_{\delta_3}(x^\star, g(x^\star))$. It then follows from (78) and (83) that

$$\begin{aligned}\vec{d}(E_{f,g}(x,v), E_{f,g}(x^\star, g(x^\star))) &\leq \max\{|f(x,v) - f(x^\star, g(x^\star))|, |g'(x) - g'(x^\star)|\} \\ &\leq \epsilon, \quad \forall\, (x, v) \in N_{\delta_1}(x^\star, g(x^\star)) \cap M_2. \end{aligned} \tag{84}$$

Finally, define $\delta \overset{\triangle}{=} \min\{\delta_1,\ \delta_2,\ \delta_3\}$. Then, it is easy to see from (76), (80), (82), and (84) that $\vec{d}(E_{f,g}(x,v), E_{f,g}(x^\star, g(x^\star))) \leq \epsilon,\ \forall\ (x, v) \in N_\delta(x^\star, g(x^\star))$. This means that $E_{f,g}$ is upper semi-continuous at $(x^\star, g(x^\star))$ for each $x^\star \in S_0$. □

Lemma 8 *Suppose that all the hypotheses of Lemma 7 are satisfied. Then, there exists a function v defined on $[x_0, x_1)$ (respectively, $(x_2, x_f]$) which is absolutely continuous on each compact subset of $[x_0, x_1)$ (respectively, $(x_2, x_f]$) and satisfies (71) (respectively, (72)) a.e. on $[x_0, x_1)$ (respectively, $(x_2, x_f]$).*

Proof: Note from the definition of $E_{f,g}$ in (73) that at each $(x^\star, v^\star) \in \mathbb{R}^2$, the set $E_{f,g}(x^\star, v^\star)$ is compact and convex. This along with Lemma 7 implies that the set-valued map $E_{f,g}$ in (71) satisfies all the hypotheses of Theorem 2.4 in [32]. Hence, the differential inclusion in (71) has a solution $\tilde{v}$ in the aforementioned sense.

Next, consider the differential inclusion:

$$v'(s) \in \tilde{E}_{f,g}(s, v(s)), \quad \text{a.e. on } s \geq -x_f \text{ with } v(-x_f) = v_f, \tag{85}$$

where the set-valued map $\tilde{E}_{f,g} : \mathbb{R}^2 \to 2^{\mathbb{R}}$ is given by $\tilde{E}_{f,g}(s, v) \stackrel{\triangle}{=} -F_{f,g}(-s, v)$. Here, through some arguments similar to those in the proof of Lemma 7, it can be shown that the set-valued map $\tilde{E}_{f,g}$ is upper-semi-continuous at each $(x^\star, v^\star) \in \mathbb{R}^2$. In addition, it is easy to see that at each $(x^\star, v^\star) \in \mathbb{R}^2$, the set $\tilde{E}_{f,g}(x^\star, v^\star)$ is compact and convex. Thus, the differential inclusion in (85) has a solution $\tilde{v}$. Then, it is easy to see that the function v, defined by $v(x) \stackrel{\triangle}{=} \tilde{v}(-x)$, is a solution of the differential inclusion in (72). □

Lemma 9 *Suppose that the hypotheses of Lemma 7 are satisfied. Let $\tilde{v}$ be a solution of the differential inclusion in (71) with $\tilde{v}(x_0) \leq g(x_0)$ and let $[x_0, x_1)$ be the interval of existence of $\tilde{v}$. Then,*

$$\tilde{v}(x) \leq g(x), \quad \forall\, x \in [x_0, x_1). \tag{86}$$

On the other hand, let $\tilde{v}$ be a solution of the differential inclusion in (72) with $\tilde{v}(x_f) \leq g(x_f)$ and let $(x_2, x_f]$ be the interval of existence of $\tilde{v}$. Then,

$$\tilde{v}(x) \leq g(x), \quad \forall\, x \in (x_2, x_f]. \tag{87}$$

Proof: We only present the proof for the inequality in (86), since that in (87) is similar. Define the set A by $A \stackrel{\triangle}{=} \{x_0 < x < x_1 \mid \tilde{v}(x) > g(x)\}$. Suppose that the set A is non-empty. Then, there exist two real numbers $a_1, b_1 \in (x_0,\ x_1)$ such that

$$\tilde{v}(a_1) = g(a_1), \tag{88}$$

$$\tilde{v}(x) > g(x), \quad x \in (a_1,\ b_1]. \tag{89}$$

Note from the definition of the set-valued map $E_{f,g}$ in (73) that $\tilde{v}'(x) = \min\{f(x, \tilde{v}(x)), g'(x)\} \leq g'(x),\ \forall\, x \in A$. This along with (88) and Lemma 4.13 in [30] implies that

$$\begin{aligned} \tilde{v}(b_1) &= \tilde{v}(a_1) + \int_{a_1}^{b_1} \tilde{v}'(s)ds \leq \tilde{v}(a_1) + \int_{a_1}^{b_1} g'(s)ds \\ &= g(a_1) + \int_{a_1}^{b_1} g'(s)ds \\ &= g(b_1). \end{aligned}$$

However, this is self-contradictory to the (89). Thus, (86) holds. □

The following Lemma 10 will be central to the proof of Theorem 1.

Lemma 10 *Suppose that a function h from a closed subset D of $\mathbb{R}$ into $\mathbb{R}$ is absolutely continuous. Then, for each $a \in \mathbb{R}$, there exist two disjoint sets P and Q such that (i) $L_h(a) \stackrel{\triangle}{=} \{x \in D\ :\ h(x) = a\} = P \cup Q$; (ii) the derivative h' always exists on P and is identically zero on P; and (iii) Q is of measure zero.*

Proof: Clearly, the set $L_h(a)$ is closed. Since h is differentiable a.e., it is clear that there exist two disjoint subsets P and Q of $L_h(a)$ satisfying (i) and (iii). Now, it remains to show (ii). Let $x \in P$. Then, since the closure of P is equal to $L_h(a)$, there exists a sequence of real numbers $\{x_n\} \subset L_h(a)$ satisfying $x_n \neq x$, $\forall\, n$ and $\lim_{n\to\infty} x_n = x$. Since $h(x_n) = h(x)$, $\forall\, n$, we obtain

$$h'(x) = \lim_{n\to\infty} \frac{h(x_n) - h(x)}{x_n - x} = 0.$$

□

We are now ready to prove Theorem 1. We present only the proof for the differential equations in (14), since that for the differential equations in (15) is similar. Let $\tilde{v} : [x_0, x_1) \to \mathbb{R}$ be the solution of the differential inclusion in (71), whose existence is guaranteed by Lemma 8. Let $\bar{x} \in (x_0, x_1)$ and define the function $h : [x_0, \bar{x}] \to \mathbb{R}$ by $h(x) \stackrel{\triangle}{=} \tilde{v}(x) - g(x)$. Clearly,

$$\text{the function } h \text{ is absolutely continuous.} \tag{90}$$

Note that the closed interval $[x_0, \bar{x}]$ can be partitioned as

$$[x_0, \bar{x}] = E_n \cup E_0,, \tag{91}$$

where E_n, E_0 are the subsets of the closed interval $[x_0, \bar{x}]$ defined, respectively, by

$$\begin{aligned} E_n &\stackrel{\triangle}{=} \{x \in [x_0, \bar{x}] \ : \ h(x) \neq 0\}, \\ E_0 &\stackrel{\triangle}{=} \{x \in [x_0, \bar{x}] \ : \ h(x) = 0\}. \end{aligned}$$

Here, it is easy to see from Lemma 9 and $v_0 \leq g(x_0)$ that if $x \in E_n$, the set $E_{f,g}(x, \tilde{v}(x))$ is the singleton $\{\Sigma_{f,g}(x, \tilde{v}(x))\}$. Thus, we have shown that

$$\tilde{v}'(x) = \Sigma_{f,g}(x, \tilde{v}(x)), \quad \text{a.e. on } E_n. \tag{92}$$

By (90) along with Lemma 10, the set E_0 is closed and

$$\tilde{v}'(x) = g'(x), \text{ a.e. on } E_0. \tag{93}$$

This, in turn, implies that $\tilde{v}'(x) = g'(x) \in E_{f,g}(x, \tilde{v}(x)) = E_{f,g}(x, g(x))$, a.e. on E_0. On the other hand, by the definition of the set-valued map $E_{f,g}$ in (73), we have

$$g'(x) \leq f(x, g(x)), \quad \text{a.e. on } E_0. \tag{94}$$

Hence, it follows from the definition of the function $\Sigma_{f,g}$ in (11) that

$$\Sigma_{f,g}(x, \tilde{v}(x)) = g'(x), \quad \text{a.e. on } E_0. \tag{95}$$

This along with (93) implies that

$$\tilde{v}'(x) = \Sigma_{f,g}(x, \tilde{v}(x)), \quad \text{a.e. on } E_0. \tag{96}$$

Since $\bar{x}$ is arbitrary, it follows from (91), (92), and (96) that any solution of the differential inclusion in (71) is indeed an absolutely continuous solution of the differential equation in (14).

We next turn to the uniqueness of the solution of the differential equation in (14). Let $\tilde{v}_1$ and $\tilde{v}_2$ be two solutions of the differential equation in (14) and let $[x_0, x_2)$ be the common interval of existence of $\tilde{v}_1$ and $\tilde{v}_2$. Define two subsets P and Q of the interval (x_0, x_2) by

$$\begin{aligned} P &\triangleq \{x \in (x_0, x_2) \ : \ \tilde{v}_1(x) = g(x)\}, \\ Q &\triangleq \{x \in (x_0, x_2) \ : \ \tilde{v}_1(x) < g(x)\}. \end{aligned}$$

Here, it is easy to see from Lemma 9 that

$$\tilde{v}_2(x) \le \tilde{v}_1(x) = g(x), \quad \forall\, x \in P. \tag{97}$$

Observe that the set Q consists of countably many disjoint open subintervals (c_k, d_k), $k = 1, 2, \ldots$ of the interval (x_0, x_2) with $c_1 \triangleq x_0$. Note from the definition of the function $\Sigma_{f,g}$ in (11) that

$$\begin{gathered} \tilde{v}_1'(x) = f(x, \tilde{v}_1(x)), \quad \text{a.e. on } (c_k, d_k), \\ \tilde{v}_1(c_1) = v_0 \text{ or } \tilde{v}_1(c_k) = g(c_k),\ k \ge 2. \end{gathered}$$

Clearly, it holds that

$$\tilde{v}_2(c_k) \le \tilde{v}_1(c_k), \quad k = 1, 2, \ldots \tag{98}$$

On the other hand, it is easy to see from the definition of the definition of the function $\Sigma_{f,g}$ given in (11) that

$$\Sigma_{f,g}(x, v) \le f(x, v), \quad \forall\, (x, v) \in \mathbb{R}^2. \tag{99}$$

Hence, the function v_2 satisfies the following differential inequality.

$$\tilde{v}_2'(x) \le f(x, \tilde{v}_2(x)), \quad \text{a.e. on } (c_k, d_k),\ k = 1, 2, \ldots \tag{100}$$

This along with (98) and Lemma 3 implies that

$$\tilde{v}_2(x) \le \tilde{v}_1(x), \quad \forall\, x \in [c_k, d_k),\ k = 1, 2, \ldots$$

So far, we have shown that $\tilde{v}_2(x) \le \tilde{v}_1(x)$, for all $x \in [x_0, x_2)$. Through some arguments similar to those used to establish the above inequality, we can also show that $\tilde{v}_2(x) \ge \tilde{v}_1(x)$, for all $x \in [x_0, x_2)$. Hence, the two solutions $\tilde{v}_1$ and $\tilde{v}_2$ are identical on the common interval of existence, i.e., the differential equation in (14) has the unique absolutely continuous solution $v_F(\cdot; x_0, v_0, \Sigma_{f,g})$.

Let $[x_0, x_1)$ be the maximal interval of existence of $v_F(\cdot; x_0, v_0, \Sigma_{f,g})$. Then, it is easy to see from Lemma 1 and (99) that

$$v_F'(x; x_0, v_0, \Sigma_{f,g}) \le f(x, v_F(x; x_0, v_0, \Sigma_{f,g})),\ \forall\, x \in [x_0, x_1). \tag{101}$$

In addition, it is clear from (86) that $v_F(x; x_0, v_0, \Sigma_{f,g}) \le g(x)$ for all $x \in [x_0, x_1)$. Finally, this along with (101) implies that $v_F(\cdot; x_0, v_0, \Sigma_{f,g})$ belongs to the set $\mathcal{A}(\Sigma)$. □

Appendix C. Proof of Lemma 4

We present only the proof for $v_F(\cdot; x_0, v_0, \Sigma_{f,g})$, since that for $v_B(\cdot; x_f, v_f, \Omega_{f,g})$ is similar. Let $[x_0, x_1)$ be the interval of existence of $v_F(\cdot; x_0, v_0, \Sigma_{f,g})$. It suffices to consider the case in which x_1 is finite. Define the function $h : \mathbb{R} \to \mathbb{R}$ by

$$h(x) \stackrel{\triangle}{=} g'(x) - H_{f,g}(x) = g'(x) - f(x, g(x)), \quad x \in \mathbb{R}. \tag{102}$$

Let $x_2 \in (x_0, x_1)$. It suffices to prove that our assertion is true on the interval $[x_0, x_2]$.

Note that the function h is real-analytic, since the derivative of a real-analytic function is real-analytic and $H_{f,g}$ is real-analytic [39]. Thus, on the compact interval $[x_0, x_2]$, it must either have a finite number of zeros or be identically zero [39]. First, we consider the case in which the function h is identically zero on the interval $[x_0, x_2]$. Then, $g'(x) = f(x, g(x))$ for all $x \in [x_0, x_2]$. Hence, $v_F(x; x_0, v_0, \Sigma_{f,g}) = v_F(x; x_0, v_0, f)$ for all $x \in [x_0, x_2]$. Thus, on the interval $[x_0, x_2]$, the trajectory of $v_F(\cdot; x_0, v_0, \Sigma_{f,g})$ consists only of the trajectory of $v_F(\cdot; x_0, v_0, f)$.

Next, we consider the case in which the function h has a finite number of zeros, say, a_i, $i = 1, \ldots, p$, $a_1 < a_2 < \cdots < a_{p-1} < a_p$ on the compact interval $[x_0, x_2]$. For later use, we define $A \stackrel{\triangle}{=} \{a_i \ : \ i = 1, \ldots, p\} \cup \{x_0, x_2\}$. In this case, on the interval $[x_0, x_2]$, there is no segment of the curve $v = g(x)$ which is a trajectory of the differential equation $v'(x) = f(x, v(x))$. (If $g'(x) = f(x, g(x))$ on a subinterval of $[x_0, x_2]$, then the function h should be identically zero on this subinterval, which is a contradiction.) Search the point $(b_1, g(b_1))$, $b_1 \in (x_0, x_2]$ at which the trajectory of $v_F(\cdot; x_0, v_0, \Sigma_{f,g})$ meets the curve $v = g(x)$ for the first time, that is, $b_1 \stackrel{\triangle}{=} \inf\{x \ : \ x \in S_1\}$, where the set S_1 is defined by $S_1 \stackrel{\triangle}{=} \{x \in (x_0, x_2] \mid v_F(x; x_0, v_0, \Sigma_{f,g}) = g(x)\}$. If S_1 is empty, the trajectory of $v_F(\cdot; x_0, v_0, \Sigma_{f,g})$ consists only of the trajectory of $v_F(\cdot; x_0, v_0, f)$.

From now on, we consider the case in which S_1 is non-empty. By the definition of b_1,

$$v_F(b_1; x_0, v_0, \Sigma_{f,g}) = g(b_1)$$

and

$$v_F(x; x_0, v_0, \Sigma_{f,g}) = v_F(x; x_0, v_0, f) < g(x), \ \forall\, x \in [x_0, b_1). \tag{103}$$

We show by way of contradiction that

$$h(b_1) \leq 0. \tag{104}$$

Suppose that this is not true, i.e., $h(b_1) = g'(b_1) - f(b_1, g(b_1)) > 0$. Then, there exists an open neighborhood D of the point $(b_1, g(b_1))$ such that $g'(x) > f(x, v)$ for all $(x, v) \in D$. Then, there exists a real number $\hat{x} > b_1$ such that the trajectory of $v_F(\cdot; b_1, g(b_1), f)$ satisfies $v'_F(x; b_1, g(b_1), f) < g'(x)$ for all $x \in (b_1, \hat{x})$. This along with $v_F(b_1; b_1, g(b_1), f) = g(b_1)$ then implies that

$$v_F(x; b_1, g(b_1), f) < g(x), \quad \forall x \in (b_1, \hat{x}). \tag{105}$$

Now, define the function $s : [x_0, \hat{x}]$ by $s(x) \stackrel{\triangle}{=} g(x) - v_F(x; x_0, v_0, f)$. Then,

$$\begin{aligned} s(b_1) &= g(b_1) - v_F(b_1; x_0, v_0, f) = 0, \\ s(x) &= g(x) - v_F(x; b_1, g(b_1), f) > 0, \quad \forall\, x \in [b_1, \hat{x}). \end{aligned}$$

This along with (103) and (105) implies that b_1 is a minimum of the function s on the interval $[x_0, \hat{x})$. Thus, $s'(b_1) = g'(b_1) - v'_F(x; x_0, v_0, f) = h(b_1) = 0$. However, this is contradictory to the hypothesis: $h(b_1) > 0$.

Due to (104), it suffices to consider the following two cases: (i) $h(x) \geq 0$ for all $x \in [b_1, x_2)$ and (ii) there exists a real number $c_1 \in [b_1, x_2)$ such that for some $\bar{x} \in (c_1, x_2)$,

$$h(c_1) = 0 \quad \text{and} \quad h(x) > 0, \;\; \forall\, x \in (c_1, \bar{x}).$$

In the former case (i), the trajectory of $v_F(\cdot; x_0, v_0, \Sigma_{f,g})$ is given by

$$v_F(\cdot; x_0, v_0, \Sigma_{f,g}) = \begin{cases} v_F(x; x_0, v_0, f), & x \in [x_0, b_1], \\ g(x), & x \in (b_1, x_2]. \end{cases}$$

Consequently, on the interval $[x_0, x_2]$, the trajectory of $v_F(\cdot; x_0, v_0, \Sigma_{f,g})$ consists only of the trajectory of $v_F(\cdot; x_0, v_0, f)$ and a segment of the curve $v = g(x)$. Then, it is not difficult to see that the claim of this lemma holds for this case.

We next consider the latter case (ii). Since $h(c_1) = 0$ and $h(x) > 0$ for all $x \in (c_1, \hat{x})$, it is obvious that there exists a real number $\hat{x} \in (c_1, x_2]$ such that

$$v_F(x; x_0, v_0, \Sigma_{f,g}) = v_F(x; c_1, g(c_1), f), \quad \forall\, x \in [c_1, \hat{x}). \tag{106}$$

Through some arguments similar to those used to show the last claim of the preceding paragraph, we can show that there exist a real number $c_2 \in A$ with $c_2 > c_1$ such that, on the interval $[c_1, c_2]$, the trajectory of $v_F(\cdot; x_0, v_0, \Sigma_{f,g})$ consists of the trajectory of $v_F(\cdot; c_1, g(c_1), f)$ and a segment of the curve $v = g(x)$. Iteratively, we can find $c_i \in A$, $i = 1, 2, \ldots$ with $x_0 < c_1 < c_2 < \ldots$ such that on each of the intervals $[x_0, c_1]$ and $[c_i, c_{i+1}]$, $i = 1, 2, \ldots$, the trajectory of $v_F(\cdot; x_0, v_0, \Sigma_{f,g})$ consists of a trajectory of the differential equation $v'(x) = f(x, v(x))$ and a segment of the curve $v = g(x)$. However, note that the set A is finite. Therefore, the set $\{c_i \;:\; i = 1, 2, \ldots\}$ should be finite. □

Appendix D. Proof of Theorem 2

We present only the proof for the differential inequality Σ, since that for the differential inequality Ω is similar. Let $[x_0, x_1)$ be the maximal interval of existence of $v_F(x; x_0, v_0, \Sigma_{f,g})$, which is well defined by Theorem 1. For notational brevity, we temporarily write $v_F(x) \stackrel{\triangle}{=} v_F(x; x_0, v_0, \Sigma_{f,g})$. Let $\tilde{v} \in \mathcal{C}_+(\Sigma) \cup \mathcal{C}_-(\Sigma)$. Let $[x_0, x_2)$ be the common interval of existence of v_F and $\tilde{v}$. It suffices to consider the case in which x_2 is finite. Define two subsets B and C of the interval (x_0, x_2) as

$$B \stackrel{\triangle}{=} \{x \in (x_0, x_2) \;:\; v_F(x) = g(x)\}, \tag{107}$$

$$C \stackrel{\triangle}{=} \{x \in (x_0, x_2) \;:\; v_F(x) < g(x)\}. \tag{108}$$

Here, it is easy to see that

$$\tilde{v}(x) \leq v_F(x), \quad \forall\, x \in B. \tag{109}$$

Observe that the set C consists of countably many disjoint open subintervals (c_k, d_k), $k = 1, 2, \ldots$ of the interval (x_0, x_2) with $c_1 \stackrel{\triangle}{=} x_0$. Through some arguments similar to those used to show (100), we can derive the following two differential inequalities:

$$\begin{aligned} &v_F'(x) = f(x, v_F(x)), \quad \forall\, x \in (c_k, d_k), \\ &v_F(c_1) = v_0, \quad v_F(c_k) = g(c_k), \quad k \geq 2, \end{aligned}$$

and

$$D\tilde{v}(x) \le f(x, \tilde{v}(x)), \quad \forall\, x \in (c_k, d_k),$$
$$\tilde{v}(c_k) \le v_F(c_k), \quad k \ge 1.$$

This along with Theorem 6.3 in [1] implies that $\tilde{v}(x) \le v_F(x)$ for all $x \in [c_k, d_k)$, $k = 1, 2, \ldots$ and hence that

$$\tilde{v}(x) \le v_F(x), \quad \forall\, x \in C. \tag{110}$$

Finally, we can see from (109) and (110) that for any $x \in [x_0, x_2)$: $\tilde{v}(x) \le v_F(x; x_0, v_0, \Sigma_{f,g})$. □

Appendix E. Proof of Theorem 3

In what follows, we write $v_F(x) \triangleq v_F(x; x_0, v_0, \Sigma_{\overline{f},g})$ and $v_B(x) \triangleq v_B(x; x_f, v_f, \Omega_{\underline{f},g})$ for notational convenience. Let $\tilde{v} \in \mathcal{C}(\Lambda) \cup \mathcal{C}^+(\Lambda) \cup \mathcal{C}_+(\Lambda) \cup \mathcal{C}^-(\Lambda) \cup \mathcal{C}_-(\Lambda)$. Then, it can be seen from (20) and Theorem 2 that

$$\tilde{v}(x) \;\le\; v_F(x) \le g(x), \quad \forall\, x \in [x_0, x_f], \tag{111}$$
$$\tilde{v}(x) \;\le\; v_B(x) \le g(x), \quad \forall\, x \in [x_0, x_f]. \tag{112}$$

Hence, it follows from (21) that the inequality in (22) holds.

Next, we show that the function $v^\star_\Lambda$ belongs to the set $\mathcal{A}(\Lambda)$:

$$\underline{f}(x, v^\star_\Lambda(x)) \le \frac{d}{dx} v^\star_\Lambda(x) \le \overline{f}(x, v^\star_\Lambda(x)), \text{ a.e. on } [x_0, x_f], \tag{113}$$
$$v^\star_\Lambda(x) \le g(x), \ \forall\, x \in (x_0, x_f), \tag{114}$$
$$v^\star_\Lambda(x_0) = v_0, \ v^\star_\Lambda(x_f) = v_f. \tag{115}$$

Here, (114) and (115) are the direct consequences of (20), (21), (111), and (112). Moreover, it can be readily verified that the maximum $v^\star_\Lambda$ of the two absolutely continuous functions $v_F(;x_0, v_f, \Sigma_{\overline{f},g})$ and $v_B(\cdot; x_f, v_f, \Omega_{\underline{f},g})$ is also absolutely continuous. Recall that an absolutely continuous function defined on a subinterval of $\mathbb{R}$ has a derivative a.e. on the interval [29]. As the consequence,

$$\text{the derivative of the function } v^\star_\Lambda \text{ exists a.e. on } [x_0, x_f]. \tag{116}$$

Now, we note that the interval $[x_0, x_f]$ can be decomposed as

$$[x_0, x_f] = A \cup B \cup C \tag{117}$$

where

$$A \;\triangleq\; \{x \in [x_0, x_f] \;:\; v^\star_\Lambda(x) = g(x)\}, \tag{118}$$
$$B \;\triangleq\; \{x \in [x_0, x_f] \;:\; v^\star_\Lambda(x) = v_F(x) < g(x)\}, \tag{119}$$
$$C \;\triangleq\; \{x \in [x_0, x_f] \;:\; v^\star_\Lambda(x) = v_B(x) < g(x)\}. \tag{120}$$

Here, it is worthwhile to note that the set A is not necessarily an interval; it is just a closed set.

First, note that

$$x \in A \;\Rightarrow\; v_F(x) = v_B(x) = g(x). \tag{121}$$

Recall that the functions v_F, v_B, and g are absolutely continuous. By Lemma 10, we can then see that

$$\frac{d}{dx}v_\Lambda^\star(x) = v_F'(x) = v_B'(x) = g'(x), \quad \text{a.e. on } A. \tag{122}$$

Here, it is clear from (121) that

$$v_F'(x) = \Sigma_{\overline{f},g}(x, v_F(x)) = \min\{\overline{f}(x, g(x)), g'(x)\}, \text{ a.e. on } A.$$

Similarly, we can see that

$$v_B'(x) = \max\{\underline{f}(x, g(x)), g'(x)\}, \quad \text{a.e. on } A. \tag{123}$$

It then follows from (122)-(123) that

$$\underline{f}(x, v_\Lambda^\star(x)) \le g'(x) = \frac{d}{dx}v_\Lambda^\star(x) \le \overline{f}(x, v_\Lambda^\star(x)), \quad \text{a.e. on } A. \tag{124}$$

Next, it is easy to see from Lemma 10 that

$$\frac{d}{dx}v_\Lambda^\star(x) = v_F'(x) = \Sigma_{\overline{f},g}(x, v_F(x)) \quad \text{a.e. on } B, \tag{125}$$

$$\frac{d}{dx}v_\Lambda^\star(x) = v_B'(x) = \Omega_{\underline{f},g}(x, v_B(x)) \quad \text{a.e. on } C. \tag{126}$$

Here, note from (119) and (120) that

$$\Sigma_{\overline{f},g}(x, v_F(x)) = \overline{f}(x, v_F(x)) = \overline{f}(x, v_\Lambda^\star(x)), \quad \forall\, x \in B,$$
$$\Omega_{\underline{f},g}(x, v_B(x)) = \underline{f}(x, v_B(x)) = \underline{f}(x, v_\Lambda^\star(x)), \quad \forall\, x \in C.$$

This along with (125) and (126) implies that

$$\frac{d}{dx}v_\Lambda^\star(x) = \overline{f}(x, v_\Lambda^\star(x)), \text{ a.e. on } B, \tag{127}$$

$$\frac{d}{dx}v_\Lambda^\star(x) = \underline{f}(x, v_\Lambda^\star(x)), \text{ a.e. on } C. \tag{128}$$

Finally, the claim in (113) follows from (117), (124), (127), and (128). □

Appendix F. Proof of Corollary 1

Note from Lemma 4 that on the interval $[x_0, x_f]$, $v_F(\cdot; x_0, v_0, \Sigma_{\overline{f},g})$ (respectively, $v_B(\cdot; x_f, v_f, \Omega_{\underline{f},g})$) consists of a finite number of the segments of the curve $v = g(x)$ and the trajectories of the differential equations $v'(x) = \overline{f}(x, v(x))$ (respectively, $v'(x) = \underline{f}(x, v(x))$). Note that these trajectories are also real-analytic [31]. As the consequence, the function $v_\Lambda^\star$ in (21) consists of a finite number of the segments of the curve $v = g(x)$ and the trajectories of the differential equations $v'(x) = \overline{f}(x, v(x))$ and $v'(x) = \underline{f}(x, v(x))$. Furthermore, it is clear from Lemma 4 that these segments of the curve $v = g(x)$ are not the trajectories of the differential equations $v'(x) = \overline{f}(x, v(x))$ and $v'(x) = \underline{f}(x, v(x))$. □

Appendix G. Proof of Lemma 5

Suppose that a trajectory $(\hat{x}, \dot{\hat{x}})$ has a zero-velocity point. Define the function $\hat{v} : [x_0, x_f] \to \mathbb{R}$ by $\hat{v}(x) = \dot{\hat{x}}(\hat{s}(x))$ where $\hat{s}(x) \stackrel{\triangle}{=} \min\{t \geq 0 \ : \ \hat{x}(t) = x\}$. Note from (35), (44), and (45) that there exists an $\epsilon^\star$ such that $0 < \epsilon^\star < \min_{x_0 \leq x \leq x_f} \alpha(x)$ and

$$\max_{x_0 \leq x \leq x_f} u_m(x, v) < 0 < \min_{x_0 \leq x \leq x_f} u_M(x, v), \quad \forall\, v \in [0, \epsilon^\star].$$

It is not difficult to show that the function $\hat{v}$ is continuous and that for some $\epsilon \in (0, \epsilon^\star)$, there exist two real numbers x_1 and x_2 such that $\hat{v}(x_1) = \hat{v}(x_2) = \epsilon$ and

$$0 < \hat{v}(x) < \epsilon, \quad \forall\, x \in (x_0, x_1) \cup (x_2, x_f).$$

Let t_1 be an intermediate zero-velocity point of the trajectory $(\hat{x}, \dot{\hat{x}})$, which may be supposed to be unique without loss of generality. Then,

$$x_1 < \hat{x}(t_1) < x_2, \qquad \hat{v}(\hat{x}(t_1)) = 0,$$

and there exist two real numbers c_1, d_1 with $x_1 \leq c_1 < \hat{x}(t_1) < d_1 \leq x_2$ such that

$$\hat{v}(c_1) = \hat{v}(d_1) = \epsilon \ \text{ and } \ \hat{v}(x) < \epsilon,\ \forall\, x \in (c_1, d_1). \tag{129}$$

Now, define the curve $\bar{v} : [x_0, x_f] \to \mathbb{R}$ by

$$\bar{v}(x) \stackrel{\triangle}{=} \begin{cases} \hat{v}(x), & x_0 \leq x \leq c_1, \\ \epsilon, & c_1 < x < d_1, \\ \hat{v}(x), & d_1 \leq x \leq x_f. \end{cases} \tag{130}$$

Let $(\bar{x}, \dot{\bar{x}})$ be the trajectory which traverses from the initial state $(x_0, 0)$ to the final state $(x_f, 0)$ along the curve $v = \bar{v}(x)$. We will show that the trajectory $(\bar{x}, \dot{\bar{x}})$ has a shorter traversal time than $(\hat{x}, \dot{\hat{x}})$ which traverses from $(x_0, 0)$ to $(x_f, 0)$ along the curve $v = \hat{v}(x)$.

First it is not difficult to see from the definition of the function $\bar{v}$ given in (130) that the traversal time T_0 (respectively, T_f) of the trajectory $(\hat{x}, \dot{\hat{x}})$ from $(x_0, 0)$ to (c_1, ϵ) (respectively, from (d_1, ϵ) to $(0, x_f)$) is equal to that of the trajectory $(\bar{x}, \dot{\bar{x}})$ from $(x_0, 0)$ to (c_1, ϵ) (respectively, from (d_1, ϵ) to $(0, x_f)$). Let $\bar{T}_1$ (respectively, $\hat{T}_1$) denote the traversal time of the trajectory $(\bar{x}, \dot{\bar{x}})$ (respectively, $(\hat{x}, \dot{\hat{x}})$) from (c_1, ϵ) to (d_1, ϵ). Then, we can easily see that the traversal time of the trajectory $(\bar{x}, \dot{\bar{x}})$ (respectively, $(\hat{x}, \dot{\hat{x}})$) is given by $T_0 + \bar{T}_1 + T_f$ (respectively, $T_0 + \hat{T}_1 + T_f$). Furthermore, we will show soon that

$$\hat{T}_1 > \bar{T}_1. \tag{131}$$

This then implies that the trajectory $(\bar{x}, \dot{\bar{x}})$ has a shorter traversal time than the trajectory $(\hat{x}, \dot{\hat{x}})$, i.e., $(\hat{x}, \dot{\hat{x}})$ is not time-optimal.

We show (131) by way of contradiction. Suppose that (131) is not true. Let Δ denote the common traversal time of $(\bar{x}, \dot{\bar{x}})$ and $(\hat{x}, \dot{\hat{x}})$ from $(x_0, 0)$ to (c_1, ϵ). We define two functions $\bar{x}_1 : [0, \bar{T}_1] \to \mathbb{R}$ and $\hat{x}_1 : [0, \hat{T}_1] \to \mathbb{R}$ as follows:

$$\begin{aligned} (\bar{x}_1(t), \dot{\bar{x}}_1(t)) &= (\bar{x}(t + \Delta), \dot{\bar{x}}(t + \Delta)), \\ (\hat{x}_1(t), \dot{\hat{x}}_1(t)) &= (\hat{x}(t + \Delta), \dot{\hat{x}}(t + \Delta)). \end{aligned}$$

Then, we can see that the trajectories $(\hat{x}, \dot{\hat{x}})$ and $(\bar{x}, \dot{\bar{x}})$ satisfy

$$(\hat{x}_1(0), \dot{\hat{x}}_1(0)) = (c_1, \epsilon),\ (\hat{x}_1(\hat{T}_1), \dot{\hat{x}}_1(\hat{T}_1)) = (d_1, \epsilon), \tag{132}$$
$$c_1 < \hat{x}_1(t) < d_1, \quad \forall\, t \in (0, \hat{T}_1), \tag{133}$$
$$(\bar{x}_1(0), \dot{\bar{x}}_1(0)) = (c_1, \epsilon),\ (\bar{x}_1(\bar{T}_1), \dot{\bar{x}}_1(\bar{T}_1)) = (d_1, \epsilon), \tag{134}$$
$$c_1 < \bar{x}_1(t) < d_1, \quad \forall\, t \in (0, \bar{T}_1). \tag{135}$$

Here, since $\bar{v}(x) = \epsilon$, $x \in [c_1, d_1]$, we have

$$(\bar{x}_1(t), \dot{\bar{x}}_1(t)) = (\epsilon t + c_1, \epsilon), \quad \forall\, t \in [0, \bar{T}_1]. \tag{136}$$

This along with the negation of (131) and (132)-(135) implies that there exists $\hat{t} \in (0, \hat{T}_1)$ such that $\dot{\bar{x}}_1(\hat{t}) \le \dot{\hat{x}}_1(\hat{t})$ and

$$\dot{\bar{x}}_1(\hat{t}) = \epsilon, \quad c_1 < \bar{x}_1(\hat{t}) < d_1. \tag{137}$$

Then, $\hat{v}(\hat{x}_1(\hat{t})) = \dot{\hat{x}}_1(\hat{t}) \ge \dot{\bar{x}}_1(\hat{t}) = \epsilon = \bar{v}(\bar{x}_1(\hat{t}))$. However, this along with (133) is contradictory to (129). □

Appendix H. Proof of Corollary 2

We need the following lemma.

Lemma 11 *The differential equations*

$$y'(x) = y_M(x, y(x)),\ x \ge x_0 \text{ with } y(x_0) = 0, \tag{138}$$
$$y'(x) = y_m(x, y(x)),\ x \le x_f \text{ with } y(x_f) = 0 \tag{139}$$

have the unique solutions $y_F(\cdot; x_0, 0, y_M)$, $y_B(\cdot; x_f, 0, y_m)$, *respectively. Let* $[x_0, x_1)$ *(respectively,* $(x_2, x_f]$*) be the interval of existence of* $y_F(\cdot; x_0, 0, y_M)$ *(respectively,* $y_B(\cdot; x_f, 0, y_m)$*). Then,*

$$y_F(x; x_0, 0, y_M) > 0, \quad x \in (x_0, x_1), \tag{140}$$
$$y_B(x; x_f, 0, y_m) > 0, \quad x \in (x_2, x_f). \tag{141}$$

Proof: Consider the differential equation

$$\begin{bmatrix} \dot{x}(t) \\ \dot{v}(t) \end{bmatrix} = \begin{bmatrix} v(t) \\ u_M(x(t), \dot{x}(t)) \end{bmatrix},\ \begin{bmatrix} x(0) \\ v(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ 0 \end{bmatrix}. \tag{142}$$

Let a trajectory $(\tilde{x}, \dot{\tilde{x}})$ satisfy the above differential equation. Let $[0,\ t_1)$ be the maximal interval of existence of $(\tilde{x}, \dot{\tilde{x}})$. Note from (35) that $\tilde{x}$ is strictly increasing over the interval $[0,\ t_1)$. Hence, there eixists a function $\tilde{s} : [x_0,\ \infty) \to \mathbb{R}$ such that $\tilde{s}(\tilde{x}(t)) = t,\ \forall\, t \in [0,\ t_1)$.

Now, we define the function $\tilde{y}$ from $[x_0,\ \infty)$ into $\mathbb{R}$ by $\tilde{y}(x) \stackrel{\triangle}{=} |\dot{\tilde{x}}(\tilde{s}(x))|^2$. Here, it is not difficult to verify that the function $\tilde{y}$ satisfies the differential equation in (138). Then,we can see that there exists a one-to-one and onto correspondence between solutions of the differential equation in (142) and the differential equation in (138). Moreover, note that the right-hand side of the differential equation in (142) is locally Lipschitz. As the consequence, the differential equation in (142) has the unique solution. This, in turn, implies that the differential equation in (138) also has the unique solution $y_F(\cdot; x_0, 0, y_M)$. Similarly, we can show that the differential equation in (139) has the unique solution $y_B(\cdot; x_f, 0, y_m)$. Furthermore, it is clear from (35) that

$$y_m(x, 0) < 0 < y_M(x, 0), \quad \forall\, x \in \mathbb{R}. \tag{143}$$

Finally, it is easy to see that the inequalities in (140) and (141) hold. □

Proof of Corollary 2: Define the new variable y by $y \overset{\triangle}{=} |v|^2$. In terms of the new variable y, the differential inequality $\bar{\Lambda}$ can be written as

$$\hat{\Lambda}: \begin{cases} y_m(x, y(x)) \le y'(x) \le y_M(x, y(x)), \ x \in [x_0, x_f] \\ 0 < y(x) \le \beta(x), \ x \in (x_0, x_f) \\ y(x_0) = y(x_f) = 0. \end{cases}$$

As will be seen soon, the differential inequality $\hat{\Lambda}$ has the maximal solution $y^\star_{\hat{\Lambda}} : [x_0, x_f] \to \mathbb{R}$ given by

$$y^\star_{\hat{\Lambda}}(x) \overset{\triangle}{=} \min\{y_F(x), y_B(x)\}, \quad \forall\, x \in [x_0, x_f], \tag{144}$$

where

$$y_F(x) \overset{\triangle}{=} y_F(x; x_0, 0, \Sigma_{y_M,\beta}), \ \ y_B(x) \overset{\triangle}{=} y_B(x; x_f 0, 0, \Omega_{y_m,\beta}).$$

Then, it is clear that the differential inequality $\bar{\Lambda}$ has the maximal solution $v^\star_{\bar{\Lambda}}$ given by (51).

From now on, we show that $y^\star_{\hat{\Lambda}}$ is the maximal solution to the differential inequality $\hat{\Lambda}$. By Lemma 11, there exists a real number $x_2 \in (x_0, x_f)$ such that the solution of the differential equation in (49) is uniquely given by $y_F(\cdot; x_0, 0, y_M)$ over the interval $[x_0, x_2]$. Consider the differential equation

$$y'(x) = \Sigma_{y_M,\beta}(x, y(x)), \ x \ge x_2, \tag{145}$$

$$y(x_2) = y_F(x_2; x_0, 0, y_M) > 0. \tag{146}$$

Note that the function $\Sigma_{y_M,\beta}$ is continuous with respect to the first argument and is locally Lipschitz with respect to the second argument on the upper-half plane $U \overset{\triangle}{=} \{(x, v) \ : \ v > 0 \text{ and } x \in \mathbb{R}\}$. Hence, the differential equation given in (145) and (146) has the unique solution $\hat{y}$ such that $\hat{y}(x) > 0$ on the maximal interval, say, $[x_2, x_3)$ of existence of $\hat{y}$. From this along with the state constraint in the differential inequality $\hat{\Lambda}$, it is not difficult to see that x_3 can be extended to the infinity.

Now, we define the function $\bar{y} : [x_0, \infty)$ by

$$\bar{y}(x) \overset{\triangle}{=} \begin{cases} y_F(x; x_0, 0, y_M), & x \in [x_0, x_2), \\ \hat{y}(x), & x \in [x_2, \infty). \end{cases} \tag{147}$$

Note that Theorem 1 still holds, even though the function $g = \beta$ is not continuously differentiable but is piecewise continuously differentiable. Thus, the above function $\bar{y}$ is indeed the unique solution y_F of the differential equation in (49) such that

$$y_F(x) > 0, \quad x \in (x_0, \infty). \tag{148}$$

Similarly, we can show that the differential equation in (50) has the unique solution $y_B : (-\infty, x_f] \to \mathbb{R}$ such that

$$y_B(x) > 0, \quad x \in (-\infty, x_f). \tag{149}$$

This along with (144) and (148) implies that

$$y^{\star}_{\hat{\Lambda}}(x) > 0, \quad \forall\, x \in (x_0, x_f). \tag{150}$$

Hence, it is not difficult to see from (148) and (149) that there exists two real numbers $\bar{x}_1, \bar{x}_2 \in (x_0, x_f)$, $\bar{x}_1 < \bar{x}_2$ such that the function $y^{\star}_{\hat{\Lambda}}$ in (144) can be written as

$$y^{\star}_{\hat{\Lambda}}(x) = \begin{cases} y_F(x; x_0, 0, y_M), & x \in [x_0, \bar{x}_1), \\ h(x), & x \in [\bar{x}_1, \bar{x}_2], \\ y_B(x; x_f, 0, y_m), & x \in (\bar{x}_2, x_f]. \end{cases} \tag{151}$$

where

$$h(x) \stackrel{\triangle}{=} \min\{y_F(x; \bar{x}_1, \bar{y}_1, \Sigma_{y_M,\beta}), y_B(x; \bar{x}_2, \bar{y}_2, \Omega_{y_m,\beta})\}$$

and $y_F(\cdot; \bar{x}_1, \bar{y}_1, \Sigma_{y_M,\beta})$ and $y_B(\cdot; \bar{x}_2, \bar{y}_2, \Omega_{y_m,\beta})$ denote, respectively, the unique solutions of the differential equations

$$y'(x) = \Sigma_{y_M,\beta}(x, y(x)),\ x \geq \bar{x}_1,$$
$$y(\bar{x}_1) = y_F(\bar{x}_1; x_0, 0, y_M) \stackrel{\triangle}{=} \bar{y}_1 > 0,$$

and

$$y'(x) = \Omega_{y_m,\beta}(x, y(x)),\ x \leq \bar{x}_2,$$
$$y(\bar{x}_2) = y_B(\bar{x}_2; x_f, 0, y_m) \stackrel{\triangle}{=} \bar{y}_2 > 0.$$

On the other hand, even when the function $g = \beta$ is piecewise continuously differentiable but not continuously differentiable, it is still true that the function $v^{\star}_{\Lambda}$ in (21) is the maximal A-solution to the differential inequality Λ in (17). Thus, it is evident from (150) and (151) that

$$y^{\star}_{\hat{\Lambda}} \in \mathcal{A}(\hat{\Lambda}). \tag{152}$$

Moreover, it is easy to see that for any $\tilde{y} \in \mathcal{A}(\hat{\Lambda})$,

$$\tilde{y}(x) \leq \min\{y_F(x; \bar{x}_1, \bar{y}_1, \Sigma_{y_M,\beta}),$$
$$y_B(x; \bar{x}_2, \bar{y}_2, \Omega_{y_m,\beta})\},\ \forall\, x \in [\bar{x}_1, \bar{x}_2].$$

Also, note from Theorem 6.3 in [1] and Lemma 1 that

$$\tilde{y}(x) \leq y_F(x; x_0, 0, y_M) = y^{\star}_{\hat{\Lambda}}(x),\ \forall\, x \in [x_0, \bar{x}_1),$$
$$\tilde{y}(x) \leq y_B(x; x_f, 0, y_m) = y^{\star}_{\hat{\Lambda}}(x),\ \forall\, x \in (\bar{x}_2, x_f].$$

Thus, we have established the inequality $\tilde{y}(x) \leq y^{\star}_{\hat{\Lambda}}(x),\ \forall\, x \in [x_0, x_f]$. Finally, this along with (152) implies that $y^{\star}_{\hat{\Lambda}}$ is the maximal solution of the differential inequality $\hat{\Lambda}$. □

Appendix I. Proof of Lemma 6

Let $m(A)$ denote the Lebesque measure of $A \subset \mathbb{R}$ and let $\tilde{x}(A)$ denote the image $\{\tilde{x}(t) \mid t \in A\}$ of A under the mapping $\tilde{x}$. Let I denote the set of points where $\tilde{x}$ is twice differentiable. Let $x_1 \in \tilde{x}(I) \cap (x_0, x_f)$. Then, $\tilde{x}$ is twice differentiable at $\tilde{s}(x_1)$. Note that $\tilde{s}$ in (52) is always differentiable on (x_0, x_f) such that

$$\tilde{s}'(x) = \frac{1}{\dot{\tilde{x}}(\tilde{s}(x))}, \quad x \in (x_0, x_f).$$

Then, $\tilde{v}$ is differentiable at x_1 such that

$$\tilde{v}'(x_1) = \ddot{\tilde{x}}(\tilde{s}(x_1))\tilde{s}'(x_1) = \frac{\ddot{\tilde{x}}(\tilde{s}(x_1))}{\dot{\tilde{x}}(\tilde{s}(x_1))} = \frac{\ddot{\tilde{x}}(\tilde{s}(x_1))}{\tilde{v}(x_1)}.$$

Note that $(\tilde{x}, \dot{\tilde{x}})$ is an absolutely continuous trajectory, since $(\tilde{x}, \dot{\tilde{x}}) \in \mathcal{P} \subset \mathcal{X}$. Thus, the set $[0, t_f(\tilde{x}, \dot{\tilde{x}})] - I$ is of measure zero. Then, it follows from Theorem 18.25 in [29] that $m(\tilde{x}([0, t_f(\tilde{x}, \dot{\tilde{x}})] - I)) = 0$. We therefore have

$$\tilde{v}'(x) = \frac{\ddot{\tilde{x}}(\tilde{s}(x))}{\dot{\tilde{x}}(\tilde{s}(x))} = \frac{\ddot{\tilde{x}}(\tilde{s}(x))}{\tilde{v}(x)}, \quad \text{a.e. on } [x_0, x_f]. \tag{153}$$

This together with (40) implies that

$$a_m(x, \tilde{v}(x)) = \frac{u_m(x, \dot{\tilde{x}}(\tilde{s}(x)))}{\tilde{v}(x)} \leq \tilde{v}'(x), \quad \text{a.e. on } [x_0, x_f],$$
$$\tilde{v}'(x) \leq \frac{u_M(x, \dot{\tilde{x}}(\tilde{s}(x)))}{\tilde{v}(x)} = a_M(x, \tilde{v}(x)), \quad \text{a.e. on } [x_0, x_f].$$

Also, note from (52) that

$$\tilde{v}(x_0) = \dot{\tilde{x}}(\tilde{s}(x_0)) = \dot{\tilde{x}}(0) = 0,$$
$$\tilde{v}(x_f) = \dot{\tilde{x}}(\tilde{s}(x_f)) = \dot{\tilde{x}}(t_f(\tilde{x}, \dot{\tilde{x}})) = 0.$$

In addition, it is easy to see from (44) that

$$0 < \tilde{v}(x) \leq \alpha(x), \quad \forall\, x \in (x_0, x_f).$$

Thus far, we have shown that $\tilde{v} \in \mathcal{A}(\bar{\Lambda})$, *i.e*, K is a mapping from $\mathcal{P}_+$ into $\mathcal{A}(\bar{\Lambda})$.

Through some arguments similar to those used to prove Lemma 5, we can show that for each $\tilde{v} \in \mathcal{A}(\bar{\Lambda})$, there exists a trajectory $(\tilde{x}, \dot{\tilde{x}}) \in \mathcal{P}_+$ such that $\tilde{v} = K(\tilde{x}, \dot{\tilde{x}})$, that is to say, K is surjective. Moreover, it is obvious from (53) that K is injective. Finally, by change of variable, we have $t_f(\tilde{x}, \dot{\tilde{x}}) = \int_{t_0}^{t_f} dt = \int_{x_0}^{x_f} \frac{dt}{dx} dx = \int_{x_0}^{x_f} \frac{1}{\dot{\tilde{x}}(\tilde{s}(x))} dx = \int_{x_0}^{x_f} \frac{1}{\tilde{v}(x)} dx.$ □

Appendix J. Proof of Corollary 3

It is easy to see from (151) that there exists two real numbers $\bar{x}_1, \bar{x}_2 \in (x_0, x_f)$, $\bar{x}_1 < \bar{x}_2$ such that the function $v^\star_{\hat{\Lambda}}$ in (51) can be written as

$$v^\star_{\hat{\Lambda}}(x) = \begin{cases} v_F(x; x_0, 0, a_M), & x \in [x_0, \bar{x}_1), \\ h(x), & x \in [\bar{x}_1, \bar{x}_2], \\ v_B(x; x_f, 0, a_m), & x \in (\bar{x}_2, x_f], \end{cases} \tag{154}$$

where $\bar{v}_1 \stackrel{\triangle}{=} v_F(\bar{x}_1; x_0, 0, a_M)$, $\bar{v}_2 \stackrel{\triangle}{=} v_B(\bar{x}_2; x_f, 0, a_m)$, and

$$h(x) \stackrel{\triangle}{=} \min\{v_F(x; \bar{x}_1, \bar{v}_1, \Sigma_{a_M,\alpha}), v_B(x; \bar{x}_2, \bar{v}_2, \Omega_{a_m,\alpha})\}.$$

On the other hand, we easily see from the definition of piecewise real-analytic functions that Lemma 4 still holds even though the functions $H_{f,g}$ and g are piecewise real-analytic. It is thus clear that the last claim in Theorem 3 holds even when the functions $H_{\overline{f},g}$, $H_{\underline{f},g}$, and g are piecewise real-analytic. Thus, on the interval $[\bar{x}_1, \bar{x}_2]$, the function $v^\star_{\hat{\Lambda}}$ consists of a finite number of the trajectories of the two differential equations $v'(x) = a_M(x, v(x))$ and $v'(x) = a_m(x, v(x))$ and a finite number of the segments of the curve $v = \alpha(x)$. Finally, this along with (154) and Theorem 4 leads to the assertion in this corollary. □

Appendix K. Derivation of (66)-(68)

We present the proof only for the case of $0 \le \theta < \frac{\pi}{2}$, since those for the other three cases of $\frac{\pi}{2} \le \theta < \pi$, $\pi \le \theta < \frac{3\pi}{2}$ and $\frac{3\pi}{2} \le \theta \le 2\pi$ are very similar. When $\theta = 0$, it is easy to that $\dot{\theta}$ and $\ddot{\theta}$ satisfy the inequalities in (63) and (64) if and only if $-\frac{U_y}{rm_y} \le \ddot{\theta} \le \frac{U_y}{rm_y}$, and $|\dot{\theta}| \le \sqrt{\frac{U_y}{rm_y}}$. Next, suppose that $0 < \theta < \frac{\pi}{2}$. Then, $\dot{\theta}$ and $\ddot{\theta}$ satisfy the two inequalities in (63) and (64) if and only if

$$-|\dot{\theta}|^2 \cot\theta - \frac{U_x}{rm_x}\csc\theta \le \ddot{\theta} \le -|\dot{\theta}|^2 \cot\theta + \frac{U_x}{rm_x}\csc\theta,$$
$$|\dot{\theta}|^2 \tan\theta - \frac{U_y}{rm_y}\sec\theta \le \ddot{\theta} \le |\dot{\theta}|^2 \tan\theta + \frac{U_y}{rm_y}\sec\theta.$$

We define

$$\begin{aligned} & g_1(\theta, \dot{\theta}) \\ \stackrel{\triangle}{=} \; & \max\{-|\dot{\theta}|^2 \cot\theta - \frac{U_x}{rm_x}\csc\theta, |\dot{\theta}|^2 \tan\theta - \frac{U_y}{rm_y}\sec\theta\}, \\ & g_2(\theta, \dot{\theta}) \\ \stackrel{\triangle}{=} \; & \min\{-|\dot{\theta}|^2 \cot\theta + \frac{U_x}{rm_x}\csc\theta, |\dot{\theta}|^2 \tan\theta + \frac{U_y}{rm_y}\sec\theta\}. \end{aligned}$$

Then, $\dot{\theta}$ and $\ddot{\theta}$ satisfy the two inequalities in (63) and (64) if and only if

$$g_1(\theta, \dot{\theta}) \le g_2(\theta, \dot{\theta}), \quad g_1(\theta, \dot{\theta}) \le \ddot{\theta} \le g_2(\theta, \dot{\theta}). \tag{155}$$

Here, note that the inequality in (155) holds if and only if

$$|\dot{\theta}|^2 \tan\theta - \frac{U_y}{rm_y}\sec\theta \le -|\dot{\theta}|^2 \cot\theta + \frac{U_x}{rm_x}\csc\theta. \tag{156}$$

After some manipulations of trigonometric polynomials, we can show that the inequality in (156) holds if and only if $|\dot{\theta}|^2 \le \frac{U_x}{rm_x}\cos\theta + \frac{U_y}{rm_y}\sin\theta$. Consequently, for each $\theta \in (0, \frac{\pi}{2})$, $\dot{\theta}$ and $\ddot{\theta}$ satisfy the two inequalities in (63) and (64) if and only if

$$g_1(\theta, \dot{\theta}) \le \ddot{\theta} \le g_2(\theta, \dot{\theta}) \;\text{ and }\; |\dot{\theta}|^2 \le \frac{U_x}{rm_x}\cos\theta + \frac{U_y}{rm_y}\sin\theta.$$

□