

## A NEWTON METHOD FOR SHAPE-PRESERVING SPLINE INTERPOLATION\*

ASEN L. DONTCHEV<sup>†</sup>, HOU-DUO QI<sup>‡</sup>, LIQUN QI<sup>§</sup>, AND HONGXIA YIN<sup>¶</sup>

*This work is dedicated to Professor Jochem Zowe*

**Abstract.** In 1986, Irvine, Marin, and Smith proposed a Newton-type method for shape-preserving interpolation and, based on numerical experience, conjectured its quadratic convergence. In this paper, we prove local quadratic convergence of their method by viewing it as a semismooth Newton method. We also present a modification of the method which has global quadratic convergence. Numerical examples illustrate the results.

**Key words.** shape-preserving interpolation, splines, semismooth equation, Newton's method, quadratic convergence

**AMS subject classifications.** 41A29, 65D15, 49J52, 90C25

**PII.** S1052623401393128

**1. Introduction.** Given nodes  $a = t_1 < t_2 < \dots < t_{N+2} = b$  and values  $y_i = f(t_i), i = 1, \dots, N + 2, N \geq 3$ , of an unknown function  $f : [a, b] \rightarrow \mathbf{R}$ , the standard interpolation problem consists of finding a function  $s$  from a given set  $S$  of interpolants such that  $s(t_i) = y_i, i = 1, \dots, N + 2$ . When  $S$  is the set of twice continuously differentiable piecewise cubic polynomials across  $t_i$ , we deal with cubic spline interpolation. The problem of cubic spline interpolation can be viewed in various ways; the closest to this paper is the classical Holladay variational characterization, according to which the natural cubic interpolating spline can be defined as the unique solution of the following optimization problem:

$$(1) \quad \min \|f''\|_2 \quad \text{subject to} \quad f(t_i) = y_i, \quad i = 1, \dots, N + 2,$$

where  $\|\cdot\|$  denotes the norm of  $L^2[a, b]$ . With a simple transformation, this problem can be written as a nearest point problem in  $L^2[a, b]$ : find the projection of the origin on the intersection of the hyperplanes

$$\left\{ u \in L^2[a, b] \mid \int_a^b u(t)B_i(t)dt = d_i, \quad i = 1, \dots, N \right\},$$

where  $B_i$  are the piecewise linear normalized B-splines with support  $[t_i, t_{i+2}]$  and  $d_i$  are the second divided differences.

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\*Received by the editors July 29, 2001; accepted for publication (in revised form) March 26, 2002; published electronically October 1, 2002.

<http://www.siam.org/journals/siopt/13-2/39312.html>

<sup>†</sup>Mathematical Reviews, Ann Arbor, MI 48107 (ald@ams.org).

<sup>‡</sup>School of Mathematics, University of New South Wales, Sydney 2052, NSW, Australia (hdqi@maths.unsw.edu.au). The research of this author was supported by the Australian Research Council.

<sup>§</sup>Department of Applied Mathematics, Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong (maqilq@polyu.edu.hk). This author's work is supported by the Hong Kong Research Grant Council, grant PolyU 5296/02P.

<sup>¶</sup>Department of Mathematics, Graduate School, Chinese Academy of Sciences, P.O. Box 3908, Beijing 100039, People's Republic of China (hxyin@maths.unsw.edu.au). This author's work was done while the author was visiting the University of New South Wales, Australia, and was supported by the Australian Research Council.

Since the mid '80s, after the ground-breaking paper of Micchelli et al. [15], the attention of a number of researchers has been attracted to spline interpolation problems with constraints. For example, if we add to problem (1) the additional constraint  $f'' \geq 0$ , we obtain a convex interpolation problem; provided that the data are “convex,” then a convex interpolant “preserves the shape” of the data. If we add the constraint  $f' \geq 0$ , we obtain a monotone interpolation problem. Central to our analysis here is a subsequent paper by Irvine, Marin, and Smith [11], who rigorously defined the problem of shape-preserving spline interpolation and laid the groundwork for its numerical analysis. In particular, they proposed a Newton-type method and, based on numerical examples, conjectured its fast (quadratic) theoretical convergence. In the present paper we prove this conjecture.

We approach the problem of Irvine, Marin, and Smith [11] in a new way, by using recent advances in optimization. It is now well understood that, in general, the traditional methods based on standard calculus may not work for optimization problems with constraints; however, such problems can be reformulated as nonsmooth problems that need special treatment. The corresponding theory emerged already in the '70s, championed by the works of R. T. Rockafellar and his collaborators, and is now becoming a standard tool for more and more theoretical and practical problems. The present paper is an example of how nonsmooth analysis can be applied to solve a problem from numerical analysis that hasn't been solved for quite a while.

Before stating the problem of shape-preserving interpolation that we consider in this paper, we briefly review the result of nonsmooth analysis which provides the basis for this work.

For a locally Lipschitz continuous function  $G : \mathbf{R}^n \rightarrow \mathbf{R}^n$ , the generalized Jacobian  $\partial G(x)$  of  $G$  at  $x$  in the sense of Clarke [2] is the convex hull of all limits obtained along sequences on which  $G$  is differentiable:

$$\partial G(x) = \text{co} \left\{ \lim_{x^j \rightarrow x} \nabla G(x^j) \mid G \text{ is differentiable at } x^j \in \mathbf{R}^n \right\}.$$

The generalized Newton method for the (nonsmooth) equation  $G(x) = 0$  has the following form:

$$(2) \quad x^{k+1} = x^k - V_k^{-1}G(x^k), \quad V_k \in \partial G(x^k).$$

A function  $G : \mathbf{R}^n \rightarrow \mathbf{R}^m$  is strongly semismooth at  $x$  if it is locally Lipschitz and directionally differentiable at  $x$ , and for all  $h \rightarrow 0$  and  $V \in \partial G(x + h)$  one has  $G(x + h) - G(x) - Vh = O(\|h\|^2)$ .

The local convergence of the generalized Newton method for strongly semismooth equations is summarized in the following fundamental result, which is a direct generalization of the classical theorem of quadratic convergence of the Newton method.

**THEOREM 1.1** (see [16, Theorem 3.2]). *Let  $G : \mathbf{R}^n \rightarrow \mathbf{R}^n$  be strongly semismooth at  $x^*$  and let  $G(x^*) = 0$ . Assume that all elements  $V$  of the generalized Jacobian  $\partial G(x^*)$  are nonsingular matrices. Then every sequence generated by the method (2) is  $q$ -quadratically convergent to  $x^*$ , provided that the starting point  $x^0$  is sufficiently close to  $x^*$ .*

In the remaining part of the introduction we review the method of Irvine, Marin, and Smith [11] for shape-preserving cubic spline interpolation and also briefly discuss the contents of this paper. Let  $\{(t_i, y_i)\}_1^{N+2}$  be given interpolation data and let  $d_i, i = 1, 2, \dots, N$ , be the associated second divided differences. Throughout the

paper we assume that  $d_i \neq 0$  for all  $i = 1, \dots, N$ ; we will discuss this assumption later. Define the following subsets  $\Omega_i, i = 1, 2, 3$ , of  $[a, b]$ :

$$\begin{aligned} \Omega_1 &:= \{[t_i, t_{i+1}] \mid d_{i-1} > 0 \text{ and } d_i > 0\}, \\ \Omega_2 &:= \{[t_i, t_{i+1}] \mid d_{i-1} < 0 \text{ and } d_i < 0\}, \\ \Omega_3 &:= \{[t_i, t_{i+1}] \mid d_{i-1}d_i < 0\}. \end{aligned}$$

Also, let

$$[t_1, t_2] \subset \begin{cases} \Omega_1 & \text{if } d_1 > 0, \\ \Omega_2 & \text{if } d_1 < 0, \end{cases} \quad [t_{N+1}, t_{N+2}] \subset \begin{cases} \Omega_1 & \text{if } d_N > 0, \\ \Omega_2 & \text{if } d_N < 0. \end{cases}$$

The problem of shape-preserving interpolation as stated by Micchelli et al. [15] is as follows:

$$(3) \quad \text{minimize } \|f''\|_2$$

$$\begin{aligned} \text{subject to } f(t_i) &= y_i, \quad i = 1, 2, \dots, N + 2, \\ f''(t) &\geq 0, \quad t \in \Omega_1, \quad f''(t) \leq 0, \quad t \in \Omega_2, \\ f &\in W^{2,2}[a, b]. \end{aligned}$$

Here  $W^{2,2}[a, b]$  denotes the Sobolev space of functions with absolutely continuous first derivatives and second derivatives in  $L^2[a, b]$ . The inequality constraint on the set  $\Omega_1$  (resp.,  $\Omega_2$ ) means that the interpolant preserves the convexity (resp., concavity) of the data; for more details, see [11, p. 137].

Micchelli et al. [15, Theorem 4.3] showed that the solution of the problem (3) exists and is unique, and its second derivative has the following form:

$$(4) \quad \begin{aligned} f''(t) &= \left( \sum_{i=1}^N \lambda_i B_i(t) \right)_+ \mathcal{X}_{\Omega_1}(t) - \left( \sum_{i=1}^N \lambda_i B_i(t) \right)_- \mathcal{X}_{\Omega_2}(t) \\ &\quad + \left( \sum_{i=1}^N \lambda_i B_i(t) \right) \mathcal{X}_{\Omega_3}(t), \end{aligned}$$

where  $\lambda = (\lambda_1, \dots, \lambda_N)^T$  is a vector in  $\mathbf{R}^N$ ,  $a_+ = \max\{0, a\}$ ,  $(a)_- = (-a)_+$ , and  $\mathcal{X}_\Omega$  is the characteristic function of the set  $\Omega$ . This result can also be deduced, as shown first in [4], from duality in optimization; specifically, here  $\lambda$  is the vector of the Lagrange multipliers associated with the equality (interpolation) constraints. For more on duality in this context, see the discussion in our previous paper [5]. In short, the optimality condition of the problem dual to (3) has the form of the nonlinear equation

$$(5) \quad F(\lambda) = d,$$

where  $d = (d_1, \dots, d_N)^T$  and the vector function  $F : \mathbf{R}^N \rightarrow \mathbf{R}^N$  has components

$$F_i(\lambda) = \int_{[t_i, t_{i+2}] \cap \Omega_1} \left( \sum_{l=1}^N \lambda_l B_l(t) \right)_+ B_i(t) dt - \int_{[t_i, t_{i+2}] \cap \Omega_2} \left( \sum_{l=1}^N \lambda_l B_l(t) \right)_- B_i(t) dt$$

$$(6) \quad + \int_{[t_i, t_{i+2}] \cap \Omega_3} \left( \sum_{l=1}^N \lambda_l B_l(t) \right) B_i(t) dt, \quad i = 1, 2, \dots, N.$$

Irvine, Marin, and Smith [11] proposed the following method for solving equation (5): Given  $\lambda^0 \in \mathbf{R}^N$ ,  $\lambda^{k+1}$  is a solution of the linear system

$$(7) \quad M(\lambda^k)(\lambda^{k+1} - \lambda^k) = -F(\lambda^k) + d,$$

where  $M(\lambda) \in \mathbf{R}^{N \times N}$  is the tridiagonal symmetric matrix with components

$$(M(\lambda))_{ij} = \int_a^b P(\lambda, t) B_i(t) B_j(t) dt.$$

Here

$$(8) \quad P(\lambda, t) := \left( \sum_{l=1}^N \lambda_l B_l(t) \right)_+^0 \mathcal{X}_{\Omega_1}(t) + \left( \sum_{l=1}^N \lambda_l B_l(t) \right)_-^0 \mathcal{X}_{\Omega_2}(t) + \mathcal{X}_{\Omega_3}(t),$$

where

$$(\tau)_+^0 := \begin{cases} 1 & \text{if } \tau > 0, \\ 0 & \text{otherwise,} \end{cases} \quad (\tau)_-^0 := (-\tau)_+^0.$$

Since the matrix  $M$  resembles the Jacobian of  $F$  (which may not exist for some  $\lambda$ , and then  $M$  is a kind of “directional Jacobian,” more precisely, as we will see later, an element of the generalized Jacobian), the method (7) has been named the Newton method. It was also observed in [11] that the Newton-type iteration (7) reduces to  $M(\lambda^k)\lambda^{k+1} = d$ ; that is, no evaluations of the function  $F$  are needed during iterations.

In our previous paper [5], we considered the problem of convex spline interpolation, that is, with  $\Omega_1 = [a, b]$ , and proved local superlinear convergence of the corresponding version of the Newton method (7). In a subsequent paper [6], by a more detailed analysis of the geometry of the dual problem, we obtained local quadratic convergence of the Newton method, again for convex interpolation. In this paper, we consider the shape-preserving interpolation problem originally stated in Irvine, Marin, and Smith [11] and prove their conjecture that the method is locally quadratically convergent. As a side result, we observe that the solution of the problem considered is Lipschitz continuous with respect to the interpolation values. In section 3 we give a modification of the method which has global quadratic convergence. Results of extensive numerical experiments are presented in section 4.

As for related results, the conjecture of Irvine, Marin, and Smith [11] was proved in [1] under an additional condition which turned out to be equivalent to smoothness of the function  $F$  in (5). Also, a positive answer to this conjecture without additional assumptions was announced in [10], but a proof was never made available to us.

**2. Local quadratic convergence.** For notational convenience, we introduce a “dummy” node  $t_0$  with corresponding  $\lambda_0 = 0$  and  $B_0(t) = 0$ ; then, for every  $i$ , the sum  $\sum_{l=1}^N \lambda_l B_l(t)$  restricted to  $[t_i, t_{i+1}]$  has the form  $\lambda_{i-1} B_{i-1}(t) + \lambda_i B_i(t)$ . Our first result concerns continuity and differentiability properties of the function  $F$  defined in (6).

LEMMA 2.1. *The function  $F$  with components defined in (6) is strongly semi-smooth.*

*Proof.* The claim is merely an extension of [6, Proposition 2.4], where it is proved that the functions

$$\int_{t_i}^{t_{i+1}} \left( \sum_{l=1}^N \lambda_l B_l(t) \right)_+ B_i(t) dt, \quad \int_{t_{i+1}}^{t_{i+2}} \left( \sum_{l=1}^N \lambda_l B_l(t) \right)_+ B_i(t) dt,$$

and

$$\int_{t_i}^{t_{i+2}} \left( \sum_{l=1}^N \lambda_l B_l(t) \right)_+ B_i(t) dt$$

are strongly semismooth. Hence the function

$$\int_{[t_i, t_{i+2}] \cap \Omega_1} \left( \sum_{l=1}^N \lambda_l B_l(t) \right)_+ B_i(t) dt$$

is strongly semismooth by noticing that

$$[t_i, t_{i+2}] \cap \Omega_1 \in \{[t_i, t_{i+1}], [t_{i+1}, t_{i+2}], [t_i, t_{i+2}], \emptyset\}.$$

We note that the function

$$\int_{[t_i, t_{i+2}] \cap \Omega_3} \left( \sum_{l=1}^N \lambda_l B_l(t) \right) B_i(t) dt$$

is linear and therefore is strongly semismooth. Since either  $[t_i, t_{i+2}] \cap \Omega_1 = \emptyset$  or  $[t_i, t_{i+2}] \cap \Omega_2 = \emptyset$ ,  $F_i$  is given either by

$$(9) \quad \begin{aligned} F_i(\lambda) &= \int_{[t_i, t_{i+2}] \cap \Omega_1} \left( \sum_{l=1}^N \lambda_l B_l(t) \right)_+ B_i(t) dt \\ &\quad + \int_{[t_i, t_{i+2}] \cap \Omega_3} \left( \sum_{l=1}^N \lambda_l B_l(t) \right) B_i(t) dt \end{aligned}$$

or by

$$(10) \quad \begin{aligned} F_i(\lambda) &= - \int_{[t_i, t_{i+2}] \cap \Omega_2} \left( \sum_{l=1}^N \lambda_l B_l(t) \right)_- B_i(t) dt \\ &\quad + \int_{[t_i, t_{i+2}] \cap \Omega_3} \left( \sum_{l=1}^N \lambda_l B_l(t) \right) B_i(t) dt. \end{aligned}$$

A composite of strongly semismooth functions is strongly semismooth [8, Theorem 19]. Hence the function  $F_i$  by (9) is strongly semismooth. If  $F_i$  is given by (10), then

$$F_i(\lambda) = - \int_{[t_i, t_{i+2}] \cap \Omega_2} \left( - \sum_{l=1}^N \lambda_l B_l(t) \right)_+ B_i(t) dt + \int_{[t_i, t_{i+2}] \cap \Omega_3} \left( \sum_{l=1}^N \lambda_l B_l(t) \right) B_i(t) dt.$$

Again from [8, Theorem 19], the first part of  $F_i$  is strongly semismooth, which in turn implies the strong semismoothness of  $F_i$ . We conclude that  $F$  is strongly semismooth since each component of  $F$  is strongly semismooth.  $\square$

If the integral over  $[a, b]$  of the piecewise linear function  $(\sum_{l=1}^N \lambda_l B_l(t))_+$  in  $\lambda$  were piecewise smooth, then one would automatically obtain that  $F$  is strongly semismooth. Furthermore, in this case quadratic convergence of the Newton method would follow directly from [13]. The following example of dimension 2 shows that such an argument does not work. Let

$$f(\lambda_1, \lambda_2) = \int_0^1 ((1-t)\lambda_1 + t\lambda_2)_+ dt.$$

Direct calculation shows that  $f$  is continuously differentiable everywhere except at the origin  $(0, 0)$ . A result due to Rockafellar [17] says that any function from  $\mathbf{R}^n$  to  $\mathbf{R}$  with  $n \geq 2$ , which is continuously differentiable everywhere but one point, could not be piecewise smooth. Hence the function above is not piecewise smooth.

In order to apply Theorem 1.1, we next prove that  $M(\lambda) \in \partial F(\lambda)$  for any  $\lambda \in \mathbf{R}^N$  and that  $V$  is nonsingular for any  $V \in \partial F(\lambda^*)$ , where  $\lambda^*$  is the unique solution of (5).

LEMMA 2.2. For any  $\lambda \in \mathbf{R}^N$ ,  $M(\lambda) \in \partial F(\lambda)$ .

Proof. Let  $\lambda \in \mathbf{R}^N$  be arbitrarily chosen (but fixed) and let

$$T(\lambda) := \left\{ t \in \Omega_1 \cup \Omega_2 \mid \sum_{l=1}^N \lambda_l B_l(t) = 0 \right\}, \quad \bar{T}(\lambda) := (\Omega_1 \cup \Omega_2) \setminus T(\lambda).$$

Suppose  $[t_i, t_{i+1}] \subset \Omega_1 \cup \Omega_2$  for some  $i$ . Due to the form of  $B_i$ , the restriction of  $(\sum_{l=1}^N \lambda_l B_l(t))$  to  $[t_i, t_{i+1}]$  becomes  $(\lambda_{i-1} B_{i-1}(t) + \lambda_i B_i(t))$ , i.e.,

$$\sum_{l=1}^N \lambda_l B_l(t) |_{[t_i, t_{i+1}]} = \lambda_{i-1} B_{i-1}(t) + \lambda_i B_i(t).$$

Then

$$(11) \quad T(\lambda) |_{[t_i, t_{i+1}]} = \begin{cases} [t_i, t_{i+1}] & \text{if } \lambda_{i-1} = \lambda_i = 0, \\ t_i^* & \text{otherwise,} \end{cases}$$

where  $t_i^*$  is a point in  $[t_i, t_{i+1}]$ . Hence  $T(\lambda)$  contains closed intervals of the form  $[t_i, t_{i+1}]$  and finitely many isolated points. For  $i = 1, \dots, N$ , define

$$\begin{aligned} F_i^-(\xi) &:= \int_{T(\lambda) \cap \Omega_1} \left( \sum_{l=1}^N \xi_l B_l(t) \right)_+ B_i(t) dt - \int_{T(\lambda) \cap \Omega_2} \left( \sum_{l=1}^N \xi_l B_l(t) \right)_- B_i(t) dt, \\ F_i^+(\xi) &:= \int_{\bar{T}(\lambda) \cap \Omega_1} \left( \sum_{l=1}^N \xi_l B_l(t) \right)_+ B_i(t) dt - \int_{\bar{T}(\lambda) \cap \Omega_2} \left( \sum_{l=1}^N \xi_l B_l(t) \right)_- B_i(t) dt \\ &\quad + \int_{\Omega_3} \left( \sum_{l=1}^N \xi_l B_l(t) \right) B_i(t) dt, \end{aligned}$$

and let  $F^-(\xi) := (F_1^-(\xi), \dots, F_N^-(\xi))^T$ ,  $F^+(\xi) := (F_1^+(\xi), \dots, F_N^+(\xi))^T$ . Then for any  $\xi \in \mathbf{R}^N$ , we have

$$F(\xi) = F^-(\xi) + F^+(\xi),$$

and it follows from (11) that  $F^+$  is continuously differentiable in a neighborhood of  $\lambda$ , say  $U(\lambda)$ . From the definition of the generalized Jacobian we obtain that for any  $\xi \in U(\lambda)$ ,

$$(12) \quad \partial F(\xi) = \partial F^-(\xi) + \nabla F^+(\xi),$$

where  $\nabla F^+(\xi)$  is the Jacobian of  $F^+$  at  $\xi \in U(\lambda)$  given by

$$\begin{aligned}
 (\nabla F^+(\xi))_{ij} &= \int_{\bar{T}(\lambda)} \left[ \left( \sum_{l=1}^N \xi_l B_l(t) \right)_+^0 \mathcal{X}_{\Omega_1}(t) + \left( \sum_{l=1}^N \xi_l B_l(t) \right)_-^0 \mathcal{X}_{\Omega_2}(t) \right] B_i(t) B_j(t) dt \\
 &+ \int_a^b B_i(t) B_j(t) \mathcal{X}_{\Omega_3}(t) dt.
 \end{aligned}
 \tag{13}$$

Since

$$\sum_{l=1}^N \lambda_l B_l(t) = 0 \quad \text{for all } t \in T(\lambda),$$

(13) becomes

$$(\nabla F^+(\lambda))_{ij} = \int_a^b P(\lambda, t) B_i(t) B_j(t) dt.
 \tag{14}$$

We will next prove that every element in  $\partial F^-(\lambda)$  is positive semidefinite. In particular, the zero matrix belongs to  $\partial F^-(\lambda)$ . Define  $\theta : \mathbf{R}^N \rightarrow \mathbf{R}$  as

$$\theta(\xi) := \frac{1}{2} \int_{T(\lambda) \cap \Omega_1} \left( \sum_{l=1}^N \xi_l B_l(t) \right)_+^2 dt + \frac{1}{2} \int_{T(\lambda) \cap \Omega_2} \left( \sum_{l=1}^N \xi_l B_l(t) \right)_-^2 dt.$$

The function  $\theta$  is a continuously differentiable convex function, and its gradient is equal to  $F^-(\xi)$ . Then the positive semidefiniteness of the elements of  $\partial F^-(\lambda)$  follows from the fact that any matrix in the generalized Jacobian of the gradient of a convex function must be symmetric and positive semidefinite. Because isolated points make no contribution to  $\theta(\xi)$ , we assume without loss of generality that  $T(\lambda)$  contains only intervals of the form  $[t_i, t_{i+1}]$ . Let

$$\begin{aligned}
 \mathcal{I}_1 &:= \{i \in \{1, \dots, N\} \mid [t_i, t_{i+1}] \subset T(\lambda) \cap \Omega_1\}, \\
 \mathcal{I}_2 &:= \{i \in \{1, \dots, N\} \mid [t_i, t_{i+1}] \subset T(\lambda) \cap \Omega_2\}.
 \end{aligned}$$

Then

$$\theta(\xi) = \frac{1}{2} \sum_{i \in \mathcal{I}_1} \int_{t_i}^{t_{i+1}} (\xi_{i-1} B_{i-1}(t) + \xi_i B_i(t))_+^2 dt + \frac{1}{2} \sum_{i \in \mathcal{I}_2} \int_{t_i}^{t_{i+1}} (\xi_{i-1} B_{i-1}(t) + \xi_i B_i(t))_-^2 dt.$$

Now define  $e = (e_1, \dots, e_N)^T$  by

$$e_{i-1} = e_i = 1 \quad \text{for } i \in \mathcal{I}_1, \quad e_{i-1} = e_i = -1 \quad \text{for } i \in \mathcal{I}_2,$$

and zero for the remaining components. We note that  $e$  is well defined since for any  $i \in \{1, \dots, N\}$ ,  $[t_i, t_{i+2}] \cap \Omega_1 = \emptyset$  or  $[t_i, t_{i+2}] \cap \Omega_2 = \emptyset$ . Then  $F^-(\lambda - \tau e)$  is differentiable for all  $\tau > 0$  because

$$\sum_{l=1}^N (\lambda - \tau e)_l B_l(t) \begin{cases} < 0 & \text{for } t \in T(\lambda) \cap \Omega_1 \text{ and } \tau > 0, \\ > 0 & \text{for } t \in T(\lambda) \cap \Omega_2 \text{ and } \tau > 0. \end{cases}$$

Hence

$$\lim_{\tau \rightarrow 0} \nabla F^-(\lambda - \tau e) = 0 \in \partial F^-(\lambda).$$

We are ready to complete the proof of the lemma. From (14) we have  $\nabla F^+(\lambda) = M(\lambda)$ . Since the zero matrix belongs to  $\partial F^-(\lambda)$ , we get  $M(\lambda) \in \partial F(\lambda)$  from (12).  $\square$

If  $\lambda^*$  is the solution of (5), we are able to show a stronger result about the generalized Jacobian of  $F$  at  $\lambda^*$ .

LEMMA 2.3. *If  $\lambda^*$  is the solution of (5), then every element of  $\partial F(\lambda^*)$  is positive definite.*

*Proof.* We have already shown in the preceding proof that

$$\partial F(\lambda^*) = \partial F^-(\lambda^*) + \nabla F^+(\lambda^*),$$

and every element in  $\partial F^-(\lambda^*)$  is positive semidefinite. Thus, it is sufficient to prove that  $\nabla F^+(\lambda^*)$  is positive definite; that is,  $M(\lambda^*)$  is positive definite. We use a result from [11, p. 138] which says that if  $P(\lambda)$  does not vanish identically on any  $[t_i, t_{i+2}]$ ,  $i = 1, \dots, N$ , then  $M(\lambda)$  is positive definite. On the contrary, suppose that  $P(\lambda^*)$  vanishes on, say,  $[t_i, t_{i+2}]$ . Then  $[t_i, t_{i+2}] \cap \Omega_3 = \emptyset$  and

$$\begin{aligned} 0 \neq d_i = F_i(\lambda^*) &= \int_{[t_i, t_{i+2}] \cap \Omega_1} \left( \sum_{l=1}^N \lambda_l^* B_l(t) \right)_+ B_i(t) dt \\ &\quad - \int_{[t_i, t_{i+2}] \cap \Omega_2} \left( \sum_{l=1}^N \lambda_l^* B_l(t) \right)_- B_i(t) dt = 0. \end{aligned}$$

The obtained contradiction completes the proof.  $\square$

By combining the above lemmas and applying Theorem 1.1, we obtain the main result of this paper which settles the question posed in [11].

THEOREM 2.4. *Let  $\lambda^*$  be the solution of (5), and let all second divided differences  $d_i$  be nonzero. Then the method (7) is well defined, and the sequence generated by this method converges quadratically to  $\lambda^*$  if the starting point  $\lambda^0$  is sufficiently close to  $\lambda^*$ .*

*Proof.* The method (7) is a particular case of the generalized Newton method (2) for (5) inasmuch as  $M(\lambda) \in \partial F(\lambda)$  (Lemma 2.2). Moreover,  $F$  is strongly semismooth at  $\lambda^*$  (Lemma 2.1), and every element in  $\partial F(\lambda^*)$  is nonsingular (Lemma 2.3). Hence all conditions in Theorem 1.1 are satisfied, and we obtain the claim.  $\square$

Remark 2.5. As a side result, from Lemma 2.3 and the Clarke inverse function theorem [2, Theorem 7.1.1], we obtain that the solution of the problem (3) is a Lipschitz continuous function of the interpolation values  $y_i$ . Indeed, since the generalized Jacobian  $\partial F(\lambda^*)$  is nonsingular, where  $\lambda^*$  is the optimal multiplier associated with the solution  $f^*$ , the map  $F^{-1}$  is, locally around  $d^* = F(\lambda^*)$ , single-valued and Lipschitz continuous. Thus for  $d$  close to  $d^*$  there exists a unique solution  $\lambda(d)$  to (5), and the function  $d \mapsto \lambda(d)$  is Lipschitz continuous. It remains to observe that  $d$  is linear in  $y$  and, from (4),  $f''$  is a Lipschitz continuous function of  $\lambda$  in the supremum norm of  $C[a, b]$ . Thus the mapping “interpolation values  $y \mapsto$  solution of (3)” is a Lipschitz continuous function from  $y \in \mathbf{R}^{N+2}$  to the space  $C^2[a, b]$  equipped with the supremum norm. This result could be further strengthened with respect to differentiability of the solution, but we shall not go into this here.



**3. Global convergence.** In this section we give a damped version of algorithm (7) by using the following merit function:

$$(15) \quad L(\lambda) = \frac{1}{2} \int_a^b \left( \sum_{l=1}^N \lambda_l B_l(t) \right)_+^2 \mathcal{X}_{\Omega_1}(t) dt + \frac{1}{2} \int_a^b \left( \sum_{l=1}^N \lambda_l B_l(t) \right)_-^2 \mathcal{X}_{\Omega_2}(t) dt \\ + \frac{1}{2} \int_a^b \left( \sum_{l=1}^N \lambda_l B_l(t) \right)^2 \mathcal{X}_{\Omega_3}(t) dt - \sum_{l=1}^N \lambda_l d_l.$$

From the very definition, this function is convex and continuously differentiable, with  $\nabla L(\lambda) = F(\lambda) - d$ .

Recall that a function  $\varphi : \mathbf{R}^N \rightarrow \mathbf{R}$  is *coercive* (also called inf-compact) if for every  $c \in \mathbf{R}$  its level set

$$\mathcal{L}_\varphi(c) = \{x \in \mathbf{R}^N \mid \varphi(x) \leq c\}$$

is bounded. In the proposition below we will show that the function  $L$  in (15) is coercive. To begin with, we define three index sets

$$\mathcal{I}_+ := \{i \in \{1, \dots, N\} \mid [t_i, t_{i+1}] \subset \Omega_1\}, \\ \mathcal{I}_- := \{i \in \{1, \dots, N\} \mid [t_i, t_{i+1}] \subset \Omega_2\}, \\ \mathcal{I}_0 := \{i \in \{1, \dots, N\} \mid [t_i, t_{i+1}] \subset \Omega_3\}$$

and associate with them the following function:

$$\hat{L}(\lambda) := \frac{1}{2} \sum_{i \in \mathcal{I}_+} \int_{t_i}^{t_{i+1}} \left( \sum_{l=1}^N \lambda_l B_l(t) \right)_+^2 dt + \frac{1}{2} \sum_{i \in \mathcal{I}_-} \int_{t_i}^{t_{i+1}} \left( \sum_{l=1}^N \lambda_l B_l(t) \right)_-^2 dt \\ + \frac{1}{2} \sum_{i \in \mathcal{I}_0} \int_{t_i}^{t_{i+1}} \left( \sum_{l=1}^N \lambda_l B_l(t) \right)^2 dt - \sum_{l=1}^N \lambda_l d_l.$$

Observe that, from the definition of the sets  $\Omega_i, i = 1, 2, 3$ , for any  $i \in \{1, \dots, N\}$ , we have  $[t_i, t_{i+2}] \cap \Omega_1 = \emptyset$  or  $[t_i, t_{i+2}] \cap \Omega_2 = \emptyset$ . For a fixed  $i$  this implies

$$(16) \quad i \in \mathcal{I}_+ \implies \begin{cases} i - 1 \in \mathcal{I}_+ & \text{or } i - 1 \in \mathcal{I}_0, \\ i + 1 \in \mathcal{I}_+ & \text{or } i + 1 \in \mathcal{I}_0 \end{cases}$$

and

$$(17) \quad i \in \mathcal{I}_- \implies \begin{cases} i - 1 \in \mathcal{I}_- & \text{or } i - 1 \in \mathcal{I}_0, \\ i + 1 \in \mathcal{I}_- & \text{or } i + 1 \in \mathcal{I}_0. \end{cases}$$

Also, observe that

$$\hat{L}(\lambda) = \frac{1}{2} \int_a^{t_{N+1}} \left( \sum_{l=1}^N \lambda_l B_l(t) \right)_+^2 \mathcal{X}_{\Omega_1}(t) dt + \frac{1}{2} \int_a^{t_{N+1}} \left( \sum_{l=1}^N \lambda_l B_l(t) \right)_-^2 \mathcal{X}_{\Omega_2}(t) dt \\ + \frac{1}{2} \int_a^{t_{N+1}} \left( \sum_{l=1}^N \lambda_l B_l(t) \right)^2 \mathcal{X}_{\Omega_3}(t) dt - \sum_{l=1}^N \lambda_l d_l \leq L(\lambda).$$

Thus, if we show the coercivity of  $\hat{L}$ , the coercivity of  $L$  will follow. In the proposition below we use the index set

$$\bar{\mathcal{I}}_0 := \{1, \dots, N\} \setminus \cup_{i \in \mathcal{I}_0} \{i - 1, i\}$$

and the following four sets in  $\mathbf{R}^N$ :

$$\begin{aligned} V_0 &:= \{v \in \mathbf{R}^N \mid v_{i-1} = v_i = 0 \text{ for all } i \in \mathcal{I}_0\}, \\ V_+ &:= \{v \in \mathbf{R}^N \mid v_i \leq 0 \text{ for all } i \in \mathcal{I}_+ \cap \bar{\mathcal{I}}_0\}, \\ V_- &:= \{v \in \mathbf{R}^N \mid v_i \geq 0 \text{ for all } i \in \mathcal{I}_- \cap \bar{\mathcal{I}}_0\}, \quad V := V_0 \cap V_+ \cap V_- . \end{aligned}$$

PROPOSITION 3.1. *The function  $L$  is coercive.*

*Proof.* In view of the above, it is sufficient to prove that the level sets

$$\mathcal{L}(c) := \{\lambda \in \mathbf{R}^N \mid \hat{L}(\lambda) \leq c\}$$

are bounded for every  $c \in \mathbf{R}$ . Note that, for every  $c \in \mathbf{R}$ , the set  $\mathcal{L}(c)$  is closed and convex. Assume on the contrary that  $\mathcal{L}(c_0)$  is unbounded for some  $c_0 \in \mathbf{R}$  and let, without loss of generality,  $c_0 > 0$ . We first show that there exists a vector  $s \in \mathbf{R}^N$ ,  $s \neq 0$ , such that  $\beta s \in \mathcal{L}(c_0)$  for every  $\beta \geq 0$ . Suppose that for every  $s \in \mathbf{R}^N$  there exists  $\beta_s \geq 0$  such that  $\beta_s s \notin \mathcal{L}(c_0)$ . From the convexity of  $\mathcal{L}(c_0)$  and  $0 \in \mathcal{L}(c_0)$ , it follows that  $\beta s \notin \mathcal{L}(c_0)$  whenever  $\beta \geq \beta_s$ . Let

$$\beta(s) := \max\{\beta \mid \beta \geq 0, \beta s \in \mathcal{L}(c_0)\}.$$

Then  $\beta(s) < \infty$  since  $\mathcal{L}(c_0)$  is closed and  $\beta(\cdot)$  is an upper semicontinuous function over  $\mathbf{R}^N$ . Then

$$\beta^* := \sup\{\beta(s) : \|s\| = 1\} < \infty.$$

Hence  $\mathcal{L}(c_0)$  is contained in a ball centered at the origin with radius  $\beta^* + 1$ . This contradiction establishes the existence of a vector  $s \in \mathbf{R}^N$ ,  $s \neq 0$ , such that  $\beta s \in \mathcal{L}(c_0)$  for all  $\beta \geq 0$ . Now for such  $s$  we define

$$\begin{aligned} \kappa(\beta) &:= \hat{L}(\beta s) = \frac{1}{2} \sum_{i \in \mathcal{I}_+} \int_{t_i}^{t_{i+1}} \beta^2 \left( \sum_{l=1}^N s_l B_l(t) \right)_+^2 dt + \frac{1}{2} \sum_{i \in \mathcal{I}_-} \int_{t_i}^{t_{i+1}} \beta^2 \left( \sum_{l=1}^N s_l B_l(t) \right)_-^2 dt \\ &\quad + \frac{1}{2} \sum_{i \in \mathcal{I}_0} \int_{t_i}^{t_{i+1}} \beta^2 \left( \sum_{i=1}^N s_l B_l(t) \right)^2 dt - \beta \sum_{l=1}^N s_l d_l. \end{aligned}$$

A more explicit form of  $\kappa(\beta)$  is

$$\begin{aligned} \kappa(\beta) &= \frac{1}{2} \sum_{i \in \mathcal{I}_+} \int_{t_i}^{t_{i+1}} \beta^2 (s_{i-1} B_{i-1} + s_i B_i)_+^2 dt + \frac{1}{2} \sum_{i \in \mathcal{I}_-} \int_{t_i}^{t_{i+1}} \beta^2 (s_{i-1} B_{i-1} + s_i B_i)_-^2 dt \\ &\quad + \frac{1}{2} \sum_{i \in \mathcal{I}_0} \int_{t_i}^{t_{i+1}} \beta^2 (s_{i-1} B_{i-1} + s_i B_i)^2 dt - \beta \sum_{l=1}^N s_l d_l. \end{aligned}$$

Now we consider the following cases.

*Case 1.*  $s \in V$ . Consider three subcases corresponding to the three quadratic terms of  $\kappa(\beta)$ , respectively.

*Subcase 1.1.*  $i \in \mathcal{I}_0$ . By the definition of  $V_0$ , we have  $s_{i-1} = 0, s_i = 0$ .

*Subcase 1.2.*  $i \in \mathcal{I}_+$ . It follows from (16) that  $(i - 1) \in \mathcal{I}_+$  or  $(i - 1) \in \mathcal{I}_0$ , and  $(i + 1) \in \mathcal{I}_+$  or  $(i + 1) \in \mathcal{I}_0$ . In particular, we have from  $s \in V$  and the definitions of  $V_0$  and  $V_+$  that

$$i - 1 \in \mathcal{I}_0 \implies \begin{cases} s_{i-1} = 0, \\ i \in \mathcal{I}_+ \cap \bar{\mathcal{I}}_0 \implies s_i \leq 0 \\ s_i = 0 \end{cases} \quad \begin{array}{l} \text{if } i + 1 \in \mathcal{I}_+, \\ \text{if } i + 1 \in \mathcal{I}_0 \end{array}$$

and

$$i - 1 \in \mathcal{I}_+ \implies \begin{cases} i - 1 \in \mathcal{I}_+ \cap \bar{\mathcal{I}}_0 \implies s_{i-1} \leq 0, \\ i \in \mathcal{I}_+ \cap \bar{\mathcal{I}}_0 \implies s_i \leq 0 \\ s_i = 0 \end{cases} \quad \begin{array}{l} \text{if } i + 1 \in \mathcal{I}_+, \\ \text{if } i + 1 \in \mathcal{I}_0. \end{array}$$

Hence for this subcase we have  $s_{i-1} \leq 0, s_i \leq 0$ .

*Subcase 1.3.*  $i \in \mathcal{I}_-$ . Then it follows from (17) that  $(i - 1) \in \mathcal{I}_-$  or  $(i - 1) \in \mathcal{I}_0$  and  $(i + 1) \in \mathcal{I}_-$  or  $(i + 1) \in \mathcal{I}_0$ . In particular, we have again from  $s \in V$  and the definitions of  $V_0$  and  $V_-$  that

$$i - 1 \in \mathcal{I}_0 \implies \begin{cases} s_{i-1} = 0, \\ i \in \mathcal{I}_- \cap \bar{\mathcal{I}}_0 \implies s_i \geq 0 \\ s_i = 0 \end{cases} \quad \begin{array}{l} \text{if } i + 1 \in \mathcal{I}_-, \\ \text{if } i + 1 \in \mathcal{I}_0 \end{array}$$

and

$$i - 1 \in \mathcal{I}_- \implies \begin{cases} i - 1 \in \mathcal{I}_- \cap \bar{\mathcal{I}}_0 \implies s_{i-1} \geq 0, \\ i \in \mathcal{I}_- \cap \bar{\mathcal{I}}_0 \implies s_i \geq 0 \\ s_i = 0 \end{cases} \quad \begin{array}{l} \text{if } i + 1 \in \mathcal{I}_-, \\ \text{if } i + 1 \in \mathcal{I}_0. \end{array}$$

Hence for this case we have  $s_{i-1} \geq 0, s_i \geq 0$ .

It follows from the three subcases that the first three terms of  $\kappa(\beta)$  (the quadratic part) vanish. Taking  $s \in V$  into account, we have

$$\kappa(\beta) = -\beta \sum_{l=1}^N s_l d_l = -\beta \sum_{l \in \mathcal{I}_+ \cap \bar{\mathcal{I}}_0} s_l d_l - \beta \sum_{l \in \mathcal{I}_- \cap \bar{\mathcal{I}}_0} s_l d_l.$$

Note that  $d_l > 0, s_l \leq 0$  for any  $l \in \mathcal{I}_+ \cap \bar{\mathcal{I}}_0$ , and  $d_l < 0, s_l \geq 0$  for any  $l \in \mathcal{I}_- \cap \bar{\mathcal{I}}_0$ . Hence the fact that there exists at least one  $s_l \neq 0$  (this  $l$  must belong to  $\mathcal{I}_+ \cap \bar{\mathcal{I}}_0$  or  $\mathcal{I}_- \cap \bar{\mathcal{I}}_0$ ) implies  $\kappa(\beta) \rightarrow +\infty$  as  $\beta \rightarrow +\infty$ , contradicting  $\hat{L}(\beta s) \leq c_0$ .

*Case 2.*  $s \notin V$ .

From the analysis of Case 1, for each  $i$ , at least one of the conditions  $s_{i-1}s_i = 0$  for  $i \in \mathcal{I}_0$ ,  $(s_{i-1} \leq 0, s_i \leq 0)$  for  $i \in \mathcal{I}_+$ , and  $(s_{i-1} \geq 0, s_i \geq 0)$  for  $i \in \mathcal{I}_-$  is violated. Hence

$$\begin{aligned} r &:= \frac{1}{2} \sum_{i \in \mathcal{I}_+} \int_{t_i}^{t_{i+1}} (s_{i-1}B_{i-1}(t) + s_iB_i(t))_+^2 dt + \frac{1}{2} \sum_{i \in \mathcal{I}_-} \int_{t_i}^{t_{i+1}} (s_{i-1}B_{i-1}(t) + s_iB_i(t))_-^2 dt \\ &+ \frac{1}{2} \sum_{i \in \mathcal{I}_0} \int_{t_i}^{t_{i+1}} (s_{i-1}B_{i-1}(t) + s_iB_i(t))^2 dt > 0. \end{aligned}$$

Then,  $\kappa(\beta) = r\beta^2 - \beta \sum_{l=1}^N s_l d_l \rightarrow +\infty$  as  $\beta \rightarrow +\infty$ , contradicting  $\hat{L}(\beta s) \leq c_0$ . This completes the proof.  $\square$

Since  $L(\lambda)$  is convex and coercive and  $\nabla L(\lambda) = F(\lambda) - d$ , finding a solution of (5) is equivalent to solving the following unconstrained optimization problem:

$$(18) \quad \min_{\lambda \in \mathbf{R}^N} L(\lambda).$$

Now we apply the following damped Newton method to the problem (18), which uses the Newton direction given by (7).

ALGORITHM 3.2.

(S.0) Choose  $\lambda^0 \in \mathbf{R}^N$ ,  $\rho \in (0, 1)$ ,  $\sigma \in (0, 1/2)$ , and tolerance  $tol > 0$ .  $k := 0$ .

(S.1) If  $\varepsilon_k = \|F(\lambda^k) - d\| \leq tol$ , then stop. Otherwise, go to (S.2).

(S.2) Let  $s^k$  be a solution of the linear system

$$(19) \quad (M(\lambda^k) + \varepsilon_k I)s = -\nabla L(\lambda^k).$$

(S.3) Choose  $m_k$  as the smallest nonnegative integer  $m$  satisfying

$$(20) \quad L(\lambda^k + \rho^m s^k) - L(\lambda^k) \leq \sigma \rho^m \nabla L(\lambda^k)^T s^k.$$

(S.4) Set  $\lambda^{k+1} = \lambda^k + \rho^{m_k} s^k$ ,  $k := k + 1$ ; return to step (S.1).

Assume that  $tol = 0$  and Algorithm 3.2 never stops at (S.1) (otherwise,  $\lambda^k$  would be the solution of (5)). The matrix  $M(\lambda^k)$  is always positive semidefinite because  $M(\lambda^k) \in \partial F(\lambda^k)$ ,  $F$  is monotone, and every element of the generalized Jacobian of the monotone function is positive semidefinite [12, Proposition 2.3(a)]. Hence  $M(\lambda^k) + \varepsilon_k I$  is always positive definite for  $\varepsilon_k > 0$ , and therefore the linear system (19) is uniquely solvable and  $s^k \neq 0$ . Moreover,

$$(s^k)^T \nabla L(\lambda^k) = -(s^k)^T (M(\lambda^k) + \varepsilon_k I)s^k \leq -\varepsilon_k \|s^k\|^2 < 0;$$

that is,  $s^k$  provides a descent direction for the function  $L$ . Hence the line search criterion (20) is always satisfied for some integer  $m$ . Since  $L$  is coercive, the sequence generated by the algorithm is bounded and therefore converges quadratically to the solution of (18). The proof of the latter is in line with the standard argument in these circumstances. Specifically, since locally the unit steplength is accepted, our algorithm eventually reduces to the following iteration:

$$M(\lambda^k)s^k = -(F(\lambda^k) - d) + r^k, \quad \lambda^{k+1} = \lambda^k + s^k,$$

where  $r^k = -\varepsilon_k s^k$  is the residual which measures the inaccuracy in the Newton equation

$$M(\lambda^k)\Delta\lambda^k = -(F(\lambda^k) - d).$$

Using the uniform nonsingularity of  $M(\lambda^k)$  near solution  $\lambda^*$ , it is easy to see that

$$s^k = O(\|F(\lambda^k) - d\|).$$

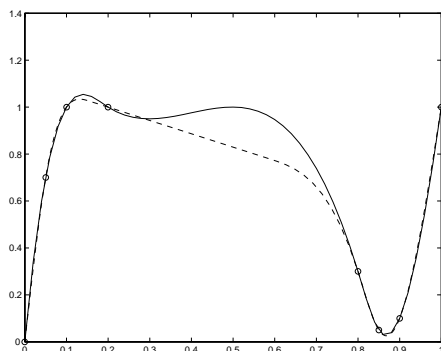
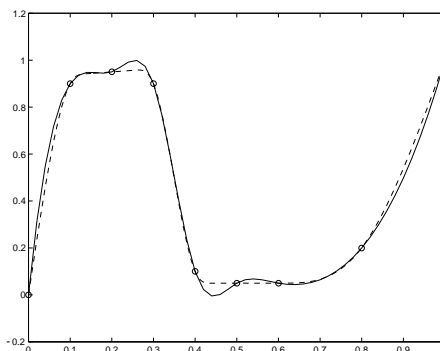
According to [3, Theorem 2.2], the accuracy  $\|r^k\| = O(\|F(\lambda^k) - d\|^2)$  is sufficient for the local quadratic convergence of the inexact Newton method. Since  $\varepsilon_k = \|F(\lambda^k) - d\|$ , we have

$$\|r^k\| = \varepsilon_k \|s^k\| = O(\|F(\lambda^k) - d\|^2).$$

For more discussion of the inexact Newton method, we refer to [3, 7, 14].

Summarizing, we have the following theorem.

**THEOREM 3.3.** *Let the sequence  $\{\lambda^k\}$  be generated by Algorithm 3.2 starting from an arbitrary  $\lambda^0 \in \mathbf{R}^N$ . Then the sequence  $\{\lambda^k\}$  converges quadratically to the solution  $\lambda^*$ .*

FIG. 1. *Example 4.1.*FIG. 2. *Example 4.2.*

**4. Numerical results.** In this section, we report on some numerical experience with Algorithm 3.2 and demonstrate its global convergence from arbitrary starting points. The typical starting point in shape-preserving algorithms is  $\text{sign}(d)$ , the sign vector of  $d$ ; see [11]. We report results with the starting point  $\mathbf{e}$ , the vector of all ones in  $\mathbf{R}^N$ , which is commonly selected as a starting point in algorithms for convex best interpolations; see [11, 6]. We also test the influence on Algorithm 3.2 of the standing assumption  $d_i \neq 0$ ,  $i = 1, \dots, N$ .

We implemented Algorithm 3.2 in MATLAB and tested it on a DEC George Server 8200 with the termination criterion  $\|F(\lambda^k) - d\| \leq \text{tol}$  and the following values of the parameters:  $\rho = 0.5$ ,  $\sigma = 0.1$ ,  $\text{tol} = 10^{-12}$ . In our implementation,  $\varepsilon_k = \min\{\delta, \|F(\lambda^k) - d\|\}$  with  $\delta = 0.01$ . The integrals involved are evaluated exactly using Simpson's rule. The testing problems are collected from the literature and are described in details as follows.

*Example 4.1.* This problem is from [11] and has the following data:

$$\begin{array}{l} t_i = 0.0 \quad 0.05 \quad 0.1 \quad 0.2 \quad 0.8 \quad 0.85 \quad 0.9 \quad 1.0. \\ y_i = 0.0 \quad 0.7 \quad 1.0 \quad 1.0 \quad 0.3 \quad 0.05 \quad 0.1 \quad 1.0. \end{array}$$

*Example 4.2.* This problem is again from [11] and has the following data:

$$\begin{array}{l} t_i = 0.0 \quad 0.1 \quad 0.2 \quad 0.3 \quad 0.4 \quad 0.5 \quad 0.6 \quad 0.8 \quad 1.0. \\ y_i = 0.0 \quad 0.9 \quad 0.95 \quad 0.9 \quad 0.1 \quad 0.05 \quad 0.05 \quad 0.2 \quad 1.0. \end{array}$$

*Example 4.3.* This problem is from [9] and has the following data:

$$\begin{array}{l} t_i = 0 \quad 4 \quad 6 \quad 10 \quad 12 \quad 14 \quad 18 \quad 20. \\ y_i = 3 \quad 4 \quad 9 \quad 10 \quad 9 \quad 5 \quad 4 \quad 3. \end{array}$$

*Example 4.4.* This problem is from [4]:  $t_1 = 0$ ,  $t_2 = 0.1$ ,  $t_3 = 0.4$ ,  $t_5 = 0.8$ ,  $t_6 = 1$ ,  $t_7 = 1.166$ ,  $t_8 = 1.333$ ,  $t_9 = 1.5$ ,  $t_{10} = 1.666$ .  $y_i = 1/((0.05 + t_i)(1.05 - t_i))$ ,  $i = 1, \dots, 4$ ,  $y_5 = 10$ ,  $y_6 = 5$ ,  $y_7 = y_8 = y_9 = 4$ ,  $y_{10} = 10$ .

In Figures 1–5, the dashed line is for the resulting shape-preserving cubic spline (using the data obtained with the starting point  $\lambda^0 = \text{sign}(d)$ ); the solid line is for the natural spline (using the MATLAB SPLINE function), and “o” stands for the original given data. In Table 1 for results of the numerical experiments we use the following notation:

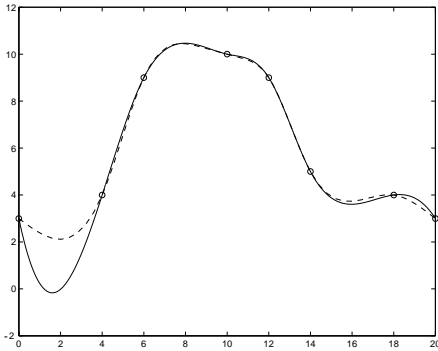


FIG. 3. Example 4.3.

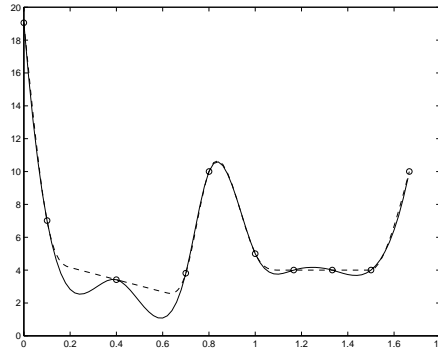


FIG. 4. Example 4.4.

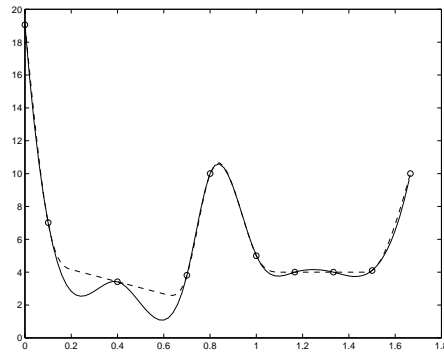


FIG. 5. Example 4.4 ( $y_9 = 4.1$ ).

TABLE 1  
Numerical results with Algorithm 3.2.

Problem	$\lambda^0$	It	Nf	$\ F(\lambda^f) - d\ $
4.1	<b>e</b>	11	17	8.57e-15
	sign( $d$ )	9	10	1.06e-14
4.2	<b>e</b>	11	15	3.02e-14
	sign( $d$ )	10	11	1.03e-14
4.3	<b>e</b>	8	11	4.59e-16
	sign( $d$ )	7	8	2.91e-16
4.4 ( $y_9 = 4$ )	<b>e</b>	30	31	1.43e-01
	sign( $d$ )	30	31	1.43e-01
4.4 ( $y_9 = 4.1$ )	<b>e</b>	24	44	2.39e-13
	sign( $d$ )	23	39	1.95e-13
4.4 ( $y_9 = 5$ )	<b>e</b>	12	13	1.01e-13
	sign( $d$ )	12	13	1.43e-13

Problem: name of the test problem.  
 $\lambda^0$ : starting point.  
 It : number of iterations.  
 Nf : number of evaluations of the function  $f(\lambda)$ .  
 $\|F(\lambda^f) - d\|$ : value of  $\|F(\lambda) - d\|$  at the last iteration.  
 From Table 1, we observe that Algorithm 3.2 converges rapidly to the solution

from both starting points for all problems except Example 4.4 ( $y_9 = 4$ ), to which the algorithm within 30 iterations failed to produce an approximate solution meeting the required accuracy. A close look at the example shows that  $d_7 = 0$ , which violates our theoretical assumption  $d_i \neq 0, i = 1, \dots, N$ . To avoid such a degeneracy in Example 4.4, we increase the value  $y_9$  from 4 to 4.1; Algorithm 3.2 now finds an approximate solution within accuracy  $10^{-13}$ , but using a relatively large number of Newton steps ( $\geq 20$ ). When we further increase the value  $y_9$  to 5, the number of Newton steps needed for the assumed tolerance is reduced considerably. These observations indicate that how far away from zero each divided difference is may make a big difference in the numerical performance of the algorithm. This is perhaps related to a property that can be regarded as conditioning. The problem is, however, nonsmooth, and here we are entering a new territory.

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