

MORITA EQUIVALENCE OF PARTIAL GROUP ACTIONS AND GLOBALIZATION

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ABSTRACT. We consider a large class of partial actions of groups on rings, called regular, which contains all s -unital partial actions as well as all partial actions on C^* -algebras. For them the notion of Morita equivalence is introduced, and it is shown that any regular partial action is Morita equivalent to a globalizable one and that the globalization is essentially unique. It is also proved that Morita equivalent s -unital partial actions on rings with orthogonal local units are stably isomorphic. In addition, we show that Morita equivalent s -unital partial actions on commutative rings must be isomorphic, and an analogous result for C^* -algebras is also established.

1. INTRODUCTION

The relationship between partial isomorphisms and global ones is relevant in several branches of mathematics, such as operator theory, topology, logic, graph theory, differential geometry, group theory and the theory of semigroups (see [43], [44]). Since partial group actions (as introduced in [25], [45], [28], [29]) naturally appear as restrictions of global ones, it is interesting to know when a given partial action can be obtained as a restriction of a global action. Partial actions which can be obtained this way are called globalizable. The problem of globalization of partial actions was studied first in the PhD Thesis [1] (see also [2]), and independently in [52] and [43]. In [1], [2] the problem was considered in the context of continuous partial group actions on topological spaces (in particular, on abstract sets) and C^* -algebras, while in [43], besides the topological spaces (and abstract sets), globalizations of partial actions on semilattices were also treated. In [52] partial actions on 2-complexes and trees and their globalizations were investigated showing some interesting parallels with Bass-Serre Theory of group actions on trees. Other globalization results were obtained in [4]-[9], [16], [17], [20], [22], [24], [32], [33], [40], [46]. The importance of the globalization problem lies in the possibility of relating partial actions with global ones and trying to use known results on global actions to obtain more general facts. In particular, globalizable partial actions were essential for the development of Galois Theory of partial group actions in [23], for the elaboration of the concept of a partial Hopf (co)action in [12], as well as in a

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series of ring theoretic and Galois theoretic investigations in [11], [13], [14], [15], [18], [34], [35], [37], [42], [47], [48].

The notion of a partial group action appeared in the theory of operator algebras as an approach to C^* -algebras generated by partial isometries, permitting, in particular, the study of their K -theory, ideal structure and representations. Amongst the prominent classes of C^* -algebras endowed with the structure of non-trivial crossed products by partial actions one may list the Bunce-Deddens and the Bunce-Deddens-Toeplitz algebras [26], the approximately finite dimensional algebras [27], the Toeplitz algebras of quasi-ordered groups, as well as the Cuntz-Krieger algebras [31], [49]. As to the purely algebraic counterpart, one can mention applications to graded algebras in [20] and [21], to Hecke algebras in [30] and to Leavitt path algebras in [41]. More details on the origin of the concept of a partial action and related notions, their development and their usefulness may be consulted in [19].

The fact that the globalization problem heavily depends on the category under consideration already became clear in [1], [2], [43] and [52]. Thus globalizations of partial actions on topological spaces (in particular, on abstract sets) always exist, however, the topological space under the global action does not preserve the properties of that under the partial one. Example 1.4 of [2] gives a partial action on a Hausdorff space whose (minimal) globalization acts on a non-Hausdorff space, and, moreover, in [2, Proposition 1.2] a criteria was given for the preservation of the Hausdorff property under globalization. Because of the categorical equivalence between locally compact Hausdorff spaces and commutative C^* -algebras, this implies that partial actions on C^* -algebras are not globalizable in general (see Proposition 2.1 in [2] for a criteria of the existence of a globalization of a partial action on commutative C^* -algebras). Nevertheless, they are globalizable “up to Morita equivalence”, as established in Theorem 6.1 of [2]. To this end the concept of Morita equivalence of partial actions of locally compact groups on C^* -algebras was introduced and studied in [2] (see also [51] for the case of discrete groups), as well as that of a Morita enveloping action, which is roughly a global action whose restriction is a partial action Morita equivalent to the initial one. It was shown that Morita equivalent partial actions have (strongly) Morita equivalent reduced crossed products. Furthermore, the reduced crossed product of a partial action is (strongly) Morita equivalent to that of the Morita enveloping action.

The purpose of this article is to define and study the abstract ring theoretic analogues of the above-mentioned concepts from [2]. Facts similar to those from [2] mentioned above are proved in the context of idempotent rings, whose Morita theory was developed in [39]. Moreover, some further Morita theoretic results are also obtained, including the behavior of Morita equivalent partial actions under the passage to matrices of infinite size with a finite number of non-zero entries. This has no C^* -algebraic analogue so far. Here our treatment heavily depends on the technique worked out in [21] to prove a ring theoretic analogue of the stabilization result for C^* -algebraic bundles from [28].

The paper is organized as follows. In Section 2 we introduce the so-called *regular* partial group actions on idempotent rings which include all partial actions on C^* -algebras, as well as all s -unital partial actions. Note that the s -unital condition on a ring generalizes all kinds of unity conditions in ring theory, including the existence of local units. The regularity condition deals with a mild restriction on the domains of the partial isomorphisms involved in a partial action: we assume

that the intersection of domains coincides with their product (see (2)). This is a suitable constraint since, on one hand, it resolves the discrepancy between the definitions of partial actions given in [20] and [21], and on the other, in almost all investigations on the subject the considered partial group actions on algebras (or rings) are regular, so that this concept provides a sufficiently general framework for the theory.

Then we proceed by defining the concept of Morita equivalent regular partial group actions (see Definition 2.8), and prove that it is an equivalence relation (see Proposition 2.12). Also in Section 2, Proposition 2.11 gives an equivalent definition of Morita equivalence of regular partial actions which is a convenient working tool. It is used, for example, in Section 3 to prove Theorem 3.1 which states that given Morita equivalent regular partial actions α and α' of a group G on algebras \mathcal{A} and \mathcal{A}' respectively, the skew group rings $\mathcal{A} \rtimes_{\alpha} G$ and $\mathcal{A}' \rtimes_{\alpha'} G$ are Morita equivalent.

Section 4 deals with globalization up to Morita equivalence. The main result in this section is Theorem 4.1, which states that a regular partial action of a group on an algebra is Morita equivalent to a globalizable regular partial action.

Section 5 is dedicated to the uniqueness of a globalization up to Morita equivalence, the key concept being that of a Morita enveloping action (see Definition 5.1). Here the main result is Theorem 5.8 which asserts that if α is a regular partial action of a group G on an algebra \mathcal{A} , then α has a Morita enveloping action, which is unique up to Morita equivalence. Moreover, for every Morita enveloping action $\beta : G \times \mathcal{B} \rightarrow \mathcal{B}$ of α , the skew group rings $\mathcal{A} \rtimes_{\alpha} G$ and $\mathcal{B} \rtimes_{\beta} G$ are Morita equivalent algebras.

One of the algebraic versions of the Brown-Green-Rieffel Theorem [10, Theorem 1.2] says that Morita equivalent rings \mathcal{R} and \mathcal{R}' with orthogonal local units become isomorphic after stabilization (see [21, Corollary 8.4]). The latter means that there exists an infinite set of indexes X such that

$$(1) \quad \text{FMat}_X(\mathcal{R}) \cong \text{FMat}_X(\mathcal{R}'),$$

where $\text{FMat}_X(\mathcal{R})$ stands for the ring of $X \times X$ -matrices with finitely many non-zero entries. The main result of Section 6 (see Theorem 6.6) states that an analogous fact holds for partial actions. More precisely, given Morita equivalent s -unital partial actions α and α' of a group G on algebras \mathcal{A} and \mathcal{A}' with orthogonal local units, there is an infinite set X of indexes such that the partial actions θ and θ' are isomorphic, where θ is the direct extension of α to $\text{FMat}_X(\mathcal{A})$ and θ' is that of α' to $\text{FMat}_X(\mathcal{A}')$. The key step is made in Theorem 6.1 by showing that (1) is a graded isomorphism if we take $\mathcal{R} = \mathcal{A} \rtimes_{\theta} G$ and $\mathcal{R}' = \mathcal{A}' \rtimes_{\theta'} G$. Then (1) implies an isomorphism of graded algebras:

$$\text{FMat}_X(\mathcal{A}) \rtimes_{\theta} G \cong \text{FMat}_X(\mathcal{A}') \rtimes_{\theta'} G.$$

Theorem 6.6 is then obtained by applying Proposition 6.5 which says that if α and α' are s -unital partial actions of G on algebras \mathcal{A} and \mathcal{A}' , respectively, such that $\mathcal{A} \rtimes_{\alpha} G$ and $\mathcal{A}' \rtimes_{\alpha'} G$ are isomorphic as graded rings, then α and α' must be isomorphic.

Finally, it is a well-known fact that Morita equivalent commutative rings with 1 are necessarily isomorphic. More generally this holds for non-degenerate idempotent rings, as established in [39, Proposition 3.2]. Theorem 7.1 in Section 7 gives an analogous result in the context of partial actions: Morita equivalent s -unital partial actions of a group G on commutative algebras must be isomorphic. A similar result

for C^* -algebras is established in Theorem 7.3: Morita equivalent partial actions of a discrete group G on commutative C^* -algebras are necessarily isomorphic.

2. MORITA EQUIVALENCE OF PARTIAL ACTIONS

Let k be a commutative associative unital ring, which will be fixed in all that follows. By an algebra we shall mean an associative k -algebra, not necessarily with 1. We recall from [20] the following definition.

Definition 2.1. A partial action α of a group G on an algebra \mathcal{A} consists of a family of two-sided ideals \mathcal{D}_g in \mathcal{A} ($g \in G$) and algebra isomorphisms $\alpha_g : \mathcal{D}_{g^{-1}} \rightarrow \mathcal{D}_g$, such that the group operation is respected in the following sense:

- (i) α_1 is the trivial isomorphism $\mathcal{A} \rightarrow \mathcal{A}$,
- (ii) if $\alpha_g \circ \alpha_h(a)$ exists for some $g, h \in G, a \in \mathcal{A}$, then necessarily $\alpha_{gh}(a)$ exists and

$$\alpha_{gh}(a) = \alpha_g \circ \alpha_h(a).$$

Remark 2.2. It is readily verified (see [20, p. 1931]) that item (ii) of the above definition can be replaced by the following two conditions, for all $g, h \in G$:

- (ii') $\alpha_g(\mathcal{D}_{g^{-1}} \cap \mathcal{D}_h) = \mathcal{D}_g \cap \mathcal{D}_{gh}$,
- (ii'') $\alpha_g(\alpha_h(x)) = \alpha_{gh}(x)$, for any $x \in \mathcal{D}_{h^{-1}} \cap \mathcal{D}_{(gh)^{-1}}$.

Remark 2.3. If \mathcal{A} is only a k -module, then in Definition 2.1 one assumes that the \mathcal{D}_g 's are submodules and the α_g 's are k -module isomorphisms.

In [21] an alternative approach to the definition was adopted in order to introduce twisted partial actions, in which instead of intersections of \mathcal{D}_g 's their products were used. It was done because it is not possible to define twisted partial actions on rings in terms of intersection of the domains \mathcal{D}_g . Thus it is reasonable to also consider the following:

Definition 2.4. A product partial action α of a group G on an algebra \mathcal{A} consists of a family of two-sided ideals \mathcal{D}_g in \mathcal{A} ($g \in G$) and algebra isomorphisms $\alpha_g : \mathcal{D}_{g^{-1}} \rightarrow \mathcal{D}_g$, such that for all $g, h \in G$ the following properties are verified:

- (i) α_1 is the trivial isomorphism $\mathcal{A} \rightarrow \mathcal{A}$,
- (ii) $\mathcal{D}_g^2 = \mathcal{D}_g, \mathcal{D}_g \cdot \mathcal{D}_h = \mathcal{D}_h \cdot \mathcal{D}_g$,
- (iii) $\alpha_g(\mathcal{D}_{g^{-1}} \cdot \mathcal{D}_h) = \mathcal{D}_g \cdot \mathcal{D}_{gh}$,
- (iv) $\alpha_g(\alpha_h(x)) = \alpha_{gh}(x)$, for any $x \in \mathcal{D}_{h^{-1}} \cdot \mathcal{D}_{(gh)^{-1}}$.

The discrepancy between the definitions can be removed by the concept of a *regular* partial action which we define now. Suppose that $\alpha = \{\alpha_g : \mathcal{D}_{g^{-1}} \rightarrow \mathcal{D}_g, g \in G\}$ is a partial action of G on an algebra \mathcal{A} whose domains satisfy the following property:

$$(2) \quad \mathcal{D}_{g_1} \cap \mathcal{D}_{g_2} \cap \dots \cap \mathcal{D}_{g_n} = \mathcal{D}_{g_1} \mathcal{D}_{g_2} \dots \mathcal{D}_{g_n}, \quad \forall g_1, \dots, g_n \in G, \quad \forall n > 0.$$

Taking, in particular, $n = 2, g_1 = g_2 = g$, we see that

$$\mathcal{D}_g^2 = \mathcal{D}_g, \quad \forall g \in G.$$

We also have that

$$\mathcal{A} \mathcal{D}_g = \mathcal{D}_g \mathcal{A} = \mathcal{D}_g, \quad \forall g \in G.$$

Note that

$$(3) \quad \alpha_g(\mathcal{D}_{g^{-1}} \mathcal{D}_{h_1} \dots \mathcal{D}_{h_n}) = \mathcal{D}_g \mathcal{D}_{gh_1} \dots \mathcal{D}_{gh_n}, \quad \forall g, h_1, \dots, h_n \in G, \quad n > 0.$$

Indeed,

$$\begin{aligned}\alpha_g(\mathcal{D}_{g^{-1}}\mathcal{D}_{h_1}\cdots\mathcal{D}_{h_n}) &= \alpha_g(\mathcal{D}_{g^{-1}}\mathcal{D}_{h_1})\alpha_g(\mathcal{D}_{g^{-1}}\mathcal{D}_{h_2})\cdots\alpha_g(\mathcal{D}_{g^{-1}}\mathcal{D}_{h_n}) \\ &= (\mathcal{D}_g\mathcal{D}_{gh_1})\cdots(\mathcal{D}_g\mathcal{D}_{gh_n}) = \mathcal{D}_g\mathcal{D}_{gh_1}\cdots\mathcal{D}_{gh_n}.\end{aligned}$$

Definition 2.5. A partial action $\alpha = \{\alpha_g : \mathcal{D}_{g^{-1}} \rightarrow \mathcal{D}_g, g \in G\}$ of a group G on an algebra \mathcal{A} satisfying property (2) shall be called regular.

Evidently if the domains \mathcal{D}_g satisfy (2), then the disagreement between Definition 2.1 and Definition 2.4 disappears.

Remark 2.6. Observe that the existence of approximate identities in C^* -algebras implies that the intersection of any two closed ideals in a C^* -algebra \mathcal{A} coincides with their product. It follows that any partial action on \mathcal{A} (in the category of C^* -algebras: \mathcal{D}_g closed, $\forall g$) is regular. The same can be said about the algebra $C_c(X)$ (or even about any algebraic inductive limit of C^* -algebras), if we understand by a closed ideal of $C_c(X)$ any ideal such that its intersection with $C_K(X)$ is closed in $C_K(X)$, for every compact subset K of X (here X stands for a locally compact Hausdorff space, $C_c(X)$ for the algebra of the complex continuous functions defined on X that have compact support, and $C_K(X)$ its subalgebra of those functions whose support is contained in the compact subset K).

We shall say that the partial action α of G on \mathcal{A} is *s-unital* if \mathcal{D}_g is an *s-unital* ring for each $g \in G$. In particular, $\mathcal{A} = \mathcal{D}_1$ is *s-unital*. We recall that a ring \mathcal{R} is called right *s-unital* if for any $a \in \mathcal{R}$ there exists an element $x \in \mathcal{R}$ such that $ax = a$. Equivalently, for arbitrary finitely many $a_1, \dots, a_n \in \mathcal{R}$ there exists $x \in \mathcal{R}$ with $a_i x = a_i$ for all $i = 1, \dots, n$ (see [24, Lemma 2.4]). A right *s-unital* ring \mathcal{R} is said to be *s-unital* if \mathcal{R} is also left *s-unital*.

Remark 2.7. It is readily seen that any *s-unital* partial action is regular.

Recall that an associative unital ring \mathcal{A} is called von Neumann regular if for each $a \in \mathcal{A}$ there exists $a' \in \mathcal{A}$ such that $aa'a = a$. It is well known that this is equivalent to saying that any finitely generated right (respectively, left) ideal in \mathcal{A} is generated by an idempotent. This immediately implies that any ideal in \mathcal{A} is an *s-unital* ring. Consequently, any partial action α on a von Neumann regular ring is *s-unital*, and, consequently, α is regular.

In view of the above remarks, and taking into account the fact that in the majority of studies the domains of considered partial actions on rings are assumed to have a unital or a generalized unital property, it is reasonable to focus our attention on regular partial actions which form a sufficiently general framework.

Let $(\mathcal{A}, \mathcal{A}', M, M', \tau, \tau')$ be a Morita context between some algebras \mathcal{A} and \mathcal{A}' . This means that M is an $(\mathcal{A}, \mathcal{A}')$ -bimodule, M' is an $(\mathcal{A}', \mathcal{A})$ -bimodule, $\tau : M \otimes_{\mathcal{A}'} M' \rightarrow \mathcal{A}$ is an $(\mathcal{A}, \mathcal{A})$ -bimodule map, and $\tau' : M' \otimes_{\mathcal{A}} M \rightarrow \mathcal{A}'$ is an $(\mathcal{A}', \mathcal{A}')$ -bimodule map, such that

$$\tau(m_1 \otimes m') m_2 = m_1 \tau'(m' \otimes m_2), \quad \forall m_1, m_2 \in M, m' \in M',$$

and

$$\tau'(m'_1 \otimes m) m'_2 = m'_1 \tau(m \otimes m'_2), \quad \forall m'_1, m'_2 \in M', m \in M.$$

Then one can construct the *context algebra* (also called the *linking algebra* or the *Morita ring*), which is the set

$$\mathcal{C} = \begin{pmatrix} \mathcal{A} & M \\ M' & \mathcal{A}' \end{pmatrix},$$

with the obvious addition of matrices and multiplication by scalars, and the matrix multiplication determined by the bimodule structures on M and M' and the maps τ and τ' , so that $m \cdot m' = \tau(m \otimes m')$ and $m' \cdot m = \tau'(m' \otimes m)$. If we have

$$(4) \quad \mathcal{A}^2 = \mathcal{A}, (\mathcal{A}')^2 = \mathcal{A}', \mathcal{A}M + M\mathcal{A}' = M \quad \text{and} \quad \mathcal{A}'M' + M'\mathcal{A} = M',$$

it is easily verified that $\mathcal{C}^2 = \mathcal{C}$.

With the above notation let

$$\alpha = \{\alpha_g : \mathcal{D}_{g^{-1}} \rightarrow \mathcal{D}_g, \quad g \in G\} \quad \text{and} \quad \theta = \{\theta_g : \bar{\mathcal{D}}_{g^{-1}} \rightarrow \bar{\mathcal{D}}_g, \quad g \in G\}$$

be partial actions of G on \mathcal{A} and \mathcal{C} respectively. We say that α is the restriction of

$$\theta \text{ to } \begin{pmatrix} \mathcal{A} & 0 \\ 0 & 0 \end{pmatrix} \text{ if for each } g \in G$$

$$\bar{\mathcal{D}}_g \cap \begin{pmatrix} \mathcal{A} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathcal{D}_g & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$\theta_g \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \alpha_g(a) & 0 \\ 0 & 0 \end{pmatrix} \quad \forall a \in \mathcal{D}_{g^{-1}}.$$

In a similar sense one can speak of the restriction of θ to $\begin{pmatrix} 0 & 0 \\ 0 & \mathcal{A}' \end{pmatrix}$. The same definition can be made if both α and θ are product partial actions.

We recall some information on the Morita Theory for idempotent rings. The reader is referred to [39] for the details. Given an idempotent ring \mathcal{A} , a left \mathcal{A} -module is said to be *unital* if $\mathcal{A}M = M$. If ${}_A M_{\mathcal{A}'}$ is a bimodule where \mathcal{A}' is also an idempotent ring, then we say that M is a unital bimodule if $\mathcal{A}M = M = M\mathcal{A}'$ (equivalently, $\mathcal{A}M\mathcal{A}' = M$). A two-sided ideal \mathcal{I} of \mathcal{A} is said to be unital if it is unital as an $(\mathcal{A}, \mathcal{A})$ -bimodule. A left module M over \mathcal{A} is called *torsion-free* (or non-degenerate) if

$$x \in M, \quad \mathcal{A}x = 0 \implies x = 0.$$

The category of left (respectively, right) unital torsion-free modules is denoted by $\mathcal{A}\text{-mod}$ (respectively, $\text{mod-}\mathcal{A}$). Two idempotent rings \mathcal{A} and \mathcal{A}' are called Morita equivalent if the categories $\mathcal{A}\text{-mod}$ and $\mathcal{A}'\text{-mod}$ are equivalent. This happens exactly when there is a Morita context $(\mathcal{A}, \mathcal{A}', M, M', \tau, \tau')$ with unital bimodules M and M' , and surjective trace maps $\tau : M \otimes_{\mathcal{A}'} M' \rightarrow \mathcal{A}$, $\tau' : M' \otimes_{\mathcal{A}} M \rightarrow \mathcal{A}'$ (see [39, Proposition 2.6, Theorem 2.7]). Moreover, the rings \mathcal{A} and \mathcal{A}' are Morita equivalent exactly when the categories $\text{mod-}\mathcal{A}$ and $\text{mod-}\mathcal{A}'$ are equivalent [39, Corollary 2.9]. If \mathcal{A} and \mathcal{A}' are Morita equivalent, then there exists a one-to-one (inclusion preserving) correspondence between the set of unital two-sided ideals of \mathcal{A} and that of \mathcal{A}' given by $\mathcal{A} \supseteq \mathcal{I} \mapsto M'\mathcal{I}M \subseteq \mathcal{A}'$, and whose inverse map is $\mathcal{A}' \supseteq \mathcal{I}' \mapsto M\mathcal{I}'M' \subseteq \mathcal{A}$ [39, Proposition 3.5]. This correspondence results in an isomorphism between the lattices of unital ideals, where the meet and join is given by the product of ideals and their sum, respectively.

Definition 2.8. Let

$$\alpha = \{\alpha_g : \mathcal{D}_{g^{-1}} \rightarrow \mathcal{D}_g, \quad g \in G\} \quad \text{and} \quad \alpha' = \{\alpha'_g : \mathcal{D}'_{g^{-1}} \rightarrow \mathcal{D}'_g, \quad g \in G\}$$

be regular partial actions of G on algebras \mathcal{A} and \mathcal{A}' , respectively. We say that α is Morita equivalent to α' if:

- (i) there exists a Morita context $(\mathcal{A}, \mathcal{A}', M, M', \tau, \tau')$ with surjective τ and τ' and unital bimodules M and M' such that $M'\mathcal{D}_gM = \mathcal{D}'_g$ for any $g \in G$;

(ii) there exists a product partial action $\theta = \{\theta_g : \bar{\mathcal{D}}_{g^{-1}} \rightarrow \bar{\mathcal{D}}_g, g \in G\}$ of G on the context algebra

$$\mathcal{C} = \begin{pmatrix} \mathcal{A} & M \\ M' & \mathcal{A}' \end{pmatrix}$$

such that the restriction of θ to $\begin{pmatrix} \mathcal{A} & 0 \\ 0 & 0 \end{pmatrix}$ is α , whereas the restriction of θ to

$$\begin{pmatrix} 0 & 0 \\ 0 & \mathcal{A}' \end{pmatrix} \text{ is } \alpha'.$$

Remark 2.9. Note that item (i) of the above definition means that the algebras \mathcal{A} and \mathcal{A}' are Morita equivalent and each ideal \mathcal{D}_g corresponds to \mathcal{D}'_g . Moreover, it is easy to check that in (i) the condition $M'\mathcal{D}_gM = \mathcal{D}'_g$ ($g \in G$) can be replaced by any of the following equivalent conditions:

- (a) $M\mathcal{D}'_gM' = \mathcal{D}_g, \forall g \in G,$
- (b) $\mathcal{D}_gM = M\mathcal{D}'_g, \forall g \in G,$
- (c) $\mathcal{D}'_gM' = M'\mathcal{D}_g, \forall g \in G.$

In order to give an alternative definition of Morita equivalence we need

Definition 2.10. Let $\alpha = \{\alpha_g : \mathcal{D}_{g^{-1}} \rightarrow \mathcal{D}_g, g \in G\}$ be a regular partial action of G on an algebra \mathcal{A} . By a left α -module we mean a left unital \mathcal{A} -module M , together with a family of k -module isomorphisms $\gamma_g : \mathcal{D}_{g^{-1}}M \rightarrow \mathcal{D}_gM$, such that the following properties are satisfied for all $g, h \in G$:

- (i) γ_1 is the trivial isomorphism $M \rightarrow M,$
- (ii) $\gamma_g \circ \gamma_h(m) = \gamma_{gh}(m), \forall m \in \mathcal{D}_{h^{-1}}\mathcal{D}_{(gh)^{-1}}M,$
- (iii) $\gamma_g(am) = \alpha_g(a)\gamma_g(m), \forall a \in \mathcal{D}_{g^{-1}}, m \in \mathcal{D}_{g^{-1}}M.$

Similarly one defines the notion of a right α -module. Now if α' is a regular partial action of G on an algebra \mathcal{A}' , then by an (α, α') -bimodule we understand a unital $(\mathcal{A}, \mathcal{A}')$ -bimodule which is a left α -module and a right α' -module.

We shall use multipliers in the proof of the next proposition, and for this purpose we recall that the multiplier algebra $\mathcal{M}(\mathcal{A})$ of \mathcal{A} is the set

$$\mathcal{M}(\mathcal{A}) = \{(R, L) \in \text{End}({}_{\mathcal{A}}\mathcal{A}) \times \text{End}(\mathcal{A}_{\mathcal{A}}) : (aR)b = a(Lb) \text{ for all } a, b \in \mathcal{A}\}$$

with component-wise addition and multiplication (see [20] or [36, 3.12.2] for more details). For a multiplier $w = (R, L) \in \mathcal{M}(\mathcal{A})$ and $a \in \mathcal{A}$ we set $aw = aR$ and $wa = La$. Thus one always has $(ab)w = a(bw), w(ab) = (wa)b$ and $(aw)b = a(wb)$ ($a, b \in \mathcal{A}$).

Proposition 2.11. *In Definition 2.8 item (ii) can be replaced by the following: there exist an (α, α') -bimodule structure on M and an (α', α) -bimodule structure on M' such that for all $m \in \mathcal{D}_{g^{-1}}M, m' \in \mathcal{D}'_{g^{-1}}M'$:*

- (1) $\alpha_g(mm') = \gamma_g(m)\gamma'_g(m'),$
- (2) $\alpha'_g(m'm) = \gamma'_g(m')\gamma_g(m).$

Proof. Suppose that the regular partial actions α and α' are such that (i) of Definition 2.8 is satisfied. Write $M_g = \mathcal{D}_gM$ and $M'_g = \mathcal{D}'_gM', (g \in G)$. Suppose furthermore that M is an (α, α') -bimodule and M' is an (α', α) -bimodule satisfying (1) and (2) of the proposition. Then setting

$$\bar{\mathcal{D}}_g = \begin{pmatrix} \mathcal{D}_g & M_g \\ M'_g & \mathcal{D}'_g \end{pmatrix} \quad \text{and} \quad \theta_g = \begin{pmatrix} \alpha_g & \gamma_g \\ \gamma'_g & \alpha'_g \end{pmatrix},$$

we have that

$$(5) \quad \bar{\mathcal{D}}_g \bar{\mathcal{D}}_h = \begin{pmatrix} \mathcal{D}_g \mathcal{D}_h & \mathcal{D}_g \mathcal{D}_h M \\ \mathcal{D}'_g \mathcal{D}'_h M' & \mathcal{D}'_g \mathcal{D}'_h \end{pmatrix}.$$

Then we readily see that the $\bar{\mathcal{D}}_g$'s are commuting idempotent ideals in the context ring and the θ_g 's form the desired product partial action, obtaining (ii).

For the converse, assume that all conditions of Definition 2.8 are satisfied and, as above, write $M_g = \mathcal{D}_g M$ and $M'_g = \mathcal{D}'_g M'$ for all $g \in G$. One can define the multipliers $e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ of the context algebra $\mathcal{C} = \begin{pmatrix} \mathcal{A} & M \\ M' & \mathcal{A}' \end{pmatrix}$, in the natural way:

$$\begin{pmatrix} a & m \\ m' & a' \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ m' & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} a & m \\ m' & a' \end{pmatrix} = \begin{pmatrix} a & m \\ 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} a & m \\ m' & a' \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & m \\ 0 & a' \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & m \\ m' & a' \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ m' & a' \end{pmatrix},$$

where $a \in \mathcal{A}, a' \in \mathcal{A}', m \in M, m' \in M'$.

Since the ideals $\bar{\mathcal{D}}_g$ are idempotent, we have that

$$e_{11} \cdot \bar{\mathcal{D}}_g \subseteq \bar{\mathcal{D}}_g \supseteq \bar{\mathcal{D}}_g \cdot e_{11}, \quad e_{22} \cdot \bar{\mathcal{D}}_g \subseteq \bar{\mathcal{D}}_g \supseteq \bar{\mathcal{D}}_g \cdot e_{22}.$$

It follows that

$$\begin{pmatrix} a & m \\ m' & a' \end{pmatrix} \in \bar{\mathcal{D}}_g \implies \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ m' & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & a' \end{pmatrix} \in \bar{\mathcal{D}}_g.$$

Recall that since the restriction of θ to $\begin{pmatrix} \mathcal{A} & 0 \\ 0 & 0 \end{pmatrix}$ is α , we have that

$$\bar{\mathcal{D}}_g \cap \begin{pmatrix} \mathcal{A} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathcal{D}_g & 0 \\ 0 & 0 \end{pmatrix},$$

and similarly,

$$\bar{\mathcal{D}}_g \cap \begin{pmatrix} 0 & 0 \\ 0 & \mathcal{A}' \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \mathcal{D}'_g \end{pmatrix}.$$

Next take any $\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \in \bar{\mathcal{D}}_g$. Since $\bar{\mathcal{D}}_g^2 = \bar{\mathcal{D}}_g$, one can write

$$\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} = e_{11} \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} e_{22} = \sum \begin{pmatrix} a_i & m_i \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & n_i \\ 0 & a'_i \end{pmatrix} = \begin{pmatrix} 0 & \sum (a_i n_i + m_i a'_i) \\ 0 & 0 \end{pmatrix},$$

showing that $m \in \mathcal{D}_g M + M \mathcal{D}'_g = \mathcal{D}_g M = M_g$. Analogously, $\begin{pmatrix} 0 & 0 \\ m' & 0 \end{pmatrix} \in \bar{\mathcal{D}}_g$

implies $m' \in M'_g$. It follows that $\bar{\mathcal{D}}_g \subseteq \begin{pmatrix} \mathcal{D}_g & M_g \\ M'_g & \mathcal{D}'_g \end{pmatrix}$. Moreover, taking $\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}$

with $m \in M_g$ we can write $\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} = \sum \begin{pmatrix} a_i & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & m_i \\ 0 & 0 \end{pmatrix}$ with $a_i \in \mathcal{D}_g, m_i \in M$.

It follows that $\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \in \bar{\mathcal{D}}_g$, as $\bar{\mathcal{D}}_g$ is an ideal in \mathcal{C} . Thus $\begin{pmatrix} 0 & M_g \\ 0 & 0 \end{pmatrix} \subseteq \bar{\mathcal{D}}_g$, and

similarly, $\begin{pmatrix} 0 & 0 \\ M'_g & 0 \end{pmatrix} \subseteq \bar{\mathcal{D}}_g$. This yields that

$$\bar{\mathcal{D}}_g = \begin{pmatrix} \mathcal{D}_g & M_g \\ M'_g & \mathcal{D}'_g \end{pmatrix},$$

and notice that equality (5) is verified.

For any $m \in M_{g^{-1}} = \mathcal{D}_{g^{-1}}M_{g^{-1}}$, one has

$$\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} = \sum \begin{pmatrix} a_i & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & m_i \\ 0 & 0 \end{pmatrix},$$

with $a_i \in \mathcal{D}_{g^{-1}}, m_i \in M_{g^{-1}}$. This implies that

$$\theta_g \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} = \sum \theta_g \begin{pmatrix} a_i & 0 \\ 0 & 0 \end{pmatrix} \theta_g \begin{pmatrix} 0 & m_i \\ 0 & 0 \end{pmatrix} \in \begin{pmatrix} \mathcal{D}_g & 0 \\ 0 & 0 \end{pmatrix} \bar{\mathcal{D}}_g = \begin{pmatrix} \mathcal{D}_g & M_g \\ 0 & 0 \end{pmatrix}.$$

Similarly, for $m \in M_{g^{-1}} = M_{g^{-1}}\mathcal{D}'_{g^{-1}}$, we obtain that $\theta_g \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \in \begin{pmatrix} 0 & M_g \\ 0 & \mathcal{D}'_g \end{pmatrix}$.

Consequently,

$$\theta_g \begin{pmatrix} 0 & M_{g^{-1}} \\ 0 & 0 \end{pmatrix} \subseteq \begin{pmatrix} 0 & M_g \\ 0 & 0 \end{pmatrix}.$$

In view of $\theta_g^{-1} = \theta_{g^{-1}}$, this immediately yields that

$$\theta_g \begin{pmatrix} 0 & M_{g^{-1}} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & M_g \\ 0 & 0 \end{pmatrix},$$

and we can define the map $\gamma_g : M_{g^{-1}} \rightarrow M_g$ by

$$\theta_g \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \gamma_g(m) \\ 0 & 0 \end{pmatrix}, \quad m \in M_{g^{-1}}.$$

Then in view of (5) we have that

$$(\bar{\mathcal{D}}_{h^{-1}} \cdot \bar{\mathcal{D}}_{h^{-1}g^{-1}}) \cap \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathcal{D}_{h^{-1}} \cdot \mathcal{D}_{h^{-1}g^{-1}}M \\ 0 & 0 \end{pmatrix},$$

which implies that $\gamma = \{\gamma_g : g \in G\}$ satisfies (i) and (ii) of Definition 2.10. Now if we take $a \in \mathcal{D}_{g^{-1}}$ and $m \in M_{g^{-1}}$ then

$$\begin{aligned} \begin{pmatrix} 0 & \gamma_g(am) \\ 0 & 0 \end{pmatrix} &= \theta_g \begin{pmatrix} 0 & am \\ 0 & 0 \end{pmatrix} = \theta_g \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \theta_g \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \alpha_g(a) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \gamma_g(m) \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

which gives us $\gamma_g(am) = \alpha_g(a)\gamma_g(m)$, proving (iii) of Definition 2.10. Thus γ defines a structure of a left α -module on M , and similarly one checks that γ turns M into a right α' -module, so that M becomes an (α, α') -bimodule.

Analogously we obtain a family of k -module isomorphisms $\{\gamma'_g : M'_{g^{-1}} \rightarrow M'_g, g \in G\}$ which gives a structure of an (α', α) -bimodule on M' . Finally, the verification of (1) and (2) is also straightforward. \square

We proceed with the following fact:

Proposition 2.12. *Morita equivalence of regular partial actions is an equivalence relation.*

Proof. Clearly, our Morita relation is reflexive and symmetric, so we only need to show the transitivity. Let

$$\alpha = \{\alpha_g : \mathcal{D}_{g^{-1}} \rightarrow \mathcal{D}_g, \quad g \in G\}, \quad \alpha' = \{\alpha'_g : \mathcal{D}'_{g^{-1}} \rightarrow \mathcal{D}'_g, \quad g \in G\}$$

and

$$\alpha'' = \{\alpha''_g : \mathcal{D}''_{g^{-1}} \rightarrow \mathcal{D}''_g, \quad g \in G\}$$

be regular partial actions of G on algebras $\mathcal{A}, \mathcal{A}', \mathcal{A}''$ respectively, and assume that α and α' are Morita equivalent and so too are α' and α'' . By Proposition 2.11 this means that there exist two Morita contexts $(\mathcal{A}, \mathcal{A}', M, M')$ and $(\mathcal{A}', \mathcal{A}'', L', L'')$ with surjective trace maps and unital bimodules such that $M'\mathcal{D}_gM = \mathcal{D}'_g$ and $L''\mathcal{D}'_gL' = \mathcal{D}''_g$ for any $g \in G$, and families of k -module isomorphisms

$$\gamma = \{\gamma_g : M_{g^{-1}} \rightarrow M_g, \quad g \in G\}, \quad \gamma' = \{\gamma'_g : M'_{g^{-1}} \rightarrow M'_g, \quad g \in G\},$$

$$\mu' = \{\mu'_g : L'_{g^{-1}} \rightarrow L'_g, \quad g \in G\}, \quad \mu'' = \{\mu''_g : L''_{g^{-1}} \rightarrow L''_g, \quad g \in G\},$$

with $M_g = \mathcal{D}_gM, M'_g = \mathcal{D}'_gM', L'_g = \mathcal{D}'_gL', L''_g = \mathcal{D}''_gL''$, such that γ and γ' give an (α, α') -bimodule structure on M, M' is an (α', α) -bimodule by means of μ' and μ'' , and, moreover, properties (1) and (2) of Proposition 2.11 hold for both bimodules.

We readily obtain the epimorphisms $(M \otimes_{\mathcal{A}'} L') \otimes_{\mathcal{A}''} (L'' \otimes_{\mathcal{A}'} M') \rightarrow M \otimes_{\mathcal{A}'} \mathcal{A}' \otimes_{\mathcal{A}'} M' \rightarrow \mathcal{A}$ and $(L'' \otimes_{\mathcal{A}'} M') \otimes_{\mathcal{A}} (M \otimes_{\mathcal{A}'} L') \rightarrow \mathcal{A}''$, which give surjective trace maps for the Morita context $(\mathcal{A}, \mathcal{A}'', M \otimes_{\mathcal{A}'} L', L'' \otimes_{\mathcal{A}'} M')$ whose bimodules are evidently unital. Clearly, $\mathcal{D}_g(M \otimes_{\mathcal{A}'} L') = \mathcal{D}_gM\mathcal{D}'_g \otimes_{\mathcal{A}'} L' = M_g \otimes_{\mathcal{A}'} L'_g$ and $\mathcal{D}''_g(L'' \otimes_{\mathcal{A}'} M') = L''_g \otimes_{\mathcal{A}'} M'_g$ for all $g \in G$. Set $(M \otimes_{\mathcal{A}'} L')_g = M_g \otimes_{\mathcal{A}'} L'_g$ and $(L'' \otimes_{\mathcal{A}'} M')_g = L''_g \otimes_{\mathcal{A}'} M'_g, g \in G$. Notice that

$$\gamma_g(ma') \otimes \mu'_g(l') = \gamma_g(m) \otimes \mu'_g(a'l'), \quad \forall m \in M_{g^{-1}}, a' \in \mathcal{A}', l' \in L'_{g^{-1}}.$$

Indeed, write $m = \sum m_i a'_i$ with $m_i \in M_{g^{-1}}$ and $a'_i \in \mathcal{D}'_{g^{-1}}$. Then

$$\begin{aligned} \gamma_g(ma') \otimes \mu'_g(l') &= \sum \gamma_g(m_i a'_i a') \otimes \mu'_g(l') = \sum \gamma_g(m_i) \alpha'_g(a'_i a') \otimes \mu'_g(l') \\ &= \sum \gamma_g(m_i) \otimes \alpha'_g(a'_i a') \mu'_g(l') = \sum \gamma_g(m_i) \otimes \mu'_g(a'_i a' l') \\ &= \sum \gamma_g(m_i) \otimes \alpha'_g(a'_i) \mu'_g(a' l') = \sum \gamma_g(m_i) \alpha'_g(a'_i) \otimes \mu'_g(a' l') = \gamma_g(m) \otimes \mu'_g(a' l'), \end{aligned}$$

as desired. This permits us to define the maps $\gamma_g \otimes \mu'_g : M_{g^{-1}} \otimes_{\mathcal{A}'} L'_{g^{-1}} \rightarrow M_g \otimes_{\mathcal{A}'} L'_g, (g \in G)$, and similarly the maps $\mu''_g \otimes \gamma'_g : L''_{g^{-1}} \otimes_{\mathcal{A}'} M'_{g^{-1}} \rightarrow L''_g \otimes_{\mathcal{A}'} M'_g, (g \in G)$. Thus we obtained the families of k -isomorphisms

$$\gamma \otimes \mu' = \{\gamma_g \otimes \mu'_g : M_{g^{-1}} \otimes_{\mathcal{A}'} L'_{g^{-1}} \rightarrow M_g \otimes_{\mathcal{A}'} L'_g, \quad g \in G\}$$

and

$$\mu'' \otimes \gamma' = \{\mu''_g \otimes \gamma'_g : L''_{g^{-1}} \otimes_{\mathcal{A}'} M'_{g^{-1}} \rightarrow L''_g \otimes_{\mathcal{A}'} M'_g, \quad g \in G\}.$$

Now we are going to show that they give the structure of an (α, α'') -bimodule on $M \otimes_{\mathcal{A}'} L'$ and that of an (α'', α) -bimodule on $L'' \otimes_{\mathcal{A}'} M'$, such that (1) and (2) of Proposition 2.11 are satisfied.

Notice that since the \mathcal{D}_g 's, as well as the \mathcal{D}'_g 's, are commuting idempotent ideals, and in view of Remark 2.9, we readily have

$$\mathcal{D}_{h^{-1}}\mathcal{D}_{h^{-1}g^{-1}}(M \otimes_{\mathcal{A}'} L') = (\mathcal{D}_{h^{-1}}\mathcal{D}_{h^{-1}g^{-1}}M \otimes_{\mathcal{A}'} \mathcal{D}'_{h^{-1}}\mathcal{D}'_{h^{-1}g^{-1}}L').$$

This easily implies (ii) of Definition 2.10, with (i) being trivial. Next taking $a \in \mathcal{D}_{g^{-1}}, m \in M_{g^{-1}}, l' \in L'_{g^{-1}}$, we have, using (iii) of Definition 2.10 for α and γ , that $(\gamma \otimes \mu')_g(a(m \otimes l')) = \gamma_g(am) \otimes \mu'_g(l') = \alpha_g(a) \gamma_g(m) \otimes \mu'_g(l') = \alpha_g(a) (\gamma \otimes \mu')_g(m \otimes l')$, which is (iii) of Definition 2.10 for α and $\gamma \otimes \mu'$. This shows that $M \otimes_{\mathcal{A}'} L'$ is a left α -module by means of $\gamma \otimes \mu'$, and the verification that $\gamma \otimes \mu'$ provides a structure of a right α'' -module on $M \otimes_{\mathcal{A}'} L'$ is similar. Analogously we see that $L'' \otimes_{\mathcal{A}'} M'$ is an (α'', α) -bimodule via $\mu'' \otimes \gamma'$.

To check (1) of Proposition 2.11 note first that the trace map

$$(M \otimes L') \otimes (L'' \otimes M') \rightarrow \mathcal{A}, (m \otimes l') \otimes (l'' \otimes m') \mapsto (m \otimes l') \cdot (l'' \otimes m'),$$

is defined by $(m \otimes l') \cdot (l'' \otimes m') = m \cdot [(l' \cdot l'')m']$, where $l' \cdot l''$ is the image of $l' \otimes l''$ by the trace map $L' \otimes L'' \rightarrow \mathcal{A}'$. Then using (1) of Proposition 2.11 for α, γ and γ' and (iii) of Definition 2.10 for γ' and α' we obtain

$$\begin{aligned} \alpha_g((m \otimes l') \cdot (l'' \otimes m')) &= \alpha_g(m \cdot [(l' \cdot l'')m']) = \gamma_g(m) \cdot \gamma'_g((l' \cdot l'')m') \\ &= \gamma_g(m) \cdot (\alpha'_g(l' \cdot l'') \gamma'_g(m')), \end{aligned}$$

and by applying (1) of Proposition 2.11 for $\alpha' \mu'$ and μ'' we see that this equals

$$(6) \quad \gamma_g(m) \cdot [(\mu'_g(l') \cdot \mu''_g(l'')) \gamma'_g(m')].$$

On the other hand

$$[(\gamma \otimes \mu')_g(m \otimes l')] \cdot [(\mu'' \otimes \gamma')_g(l'' \otimes m')] = [\gamma_g(m) \otimes \mu'_g(l')] \cdot [\mu''_g(l'') \otimes \gamma'_g(m')],$$

which is readily seen to be equal to the above element (6). This proves (1) of Proposition 2.11 for $\alpha, \gamma \otimes \mu'$ and $\mu'' \otimes \gamma'$. In an analogous way one checks (2) of Proposition 2.11. \square

3. SKEW GROUP RINGS BY MORITA EQUIVALENT PARTIAL ACTIONS

We recall from [20] that given a partial action α of G on \mathcal{A} , the skew group ring $\mathcal{A} \rtimes_{\alpha} G$ corresponding to α is the direct sum

$$\bigoplus_{g \in G} \mathcal{D}_g \delta_g,$$

in which the δ_g 's are symbols and the multiplication is defined by the rule

$$(a_g \delta_g) \cdot (b_h \delta_h) = \alpha_g(\alpha_g^{-1}(a_g) b_h) \delta_{gh}.$$

It follows by [20, Corollary 3.2] that the ring $\mathcal{A} \rtimes_{\alpha} G$ is associative, provided that each \mathcal{D}_g ($g \in G$) is an idempotent ideal, which clearly holds for regular partial actions. We proceed with

Theorem 3.1. *Suppose that*

$$\alpha = \{\alpha_g : \mathcal{D}_{g^{-1}} \rightarrow \mathcal{D}_g : g \in G\} \quad \text{and} \quad \alpha' = \{\alpha'_g : \mathcal{D}'_{g^{-1}} \rightarrow \mathcal{D}'_g : g \in G\}$$

are Morita equivalent regular partial actions of G on algebras \mathcal{A} and \mathcal{A}' , respectively. Then the skew group rings $\mathcal{A} \rtimes_{\alpha} G$ and $\mathcal{A}' \rtimes_{\alpha'} G$ are Morita equivalent.

Proof. Let $(\mathcal{A}, \mathcal{A}', M, M', \tau, \tau')$ be a Morita context giving the Morita equivalence of α with α' , let $\theta = \{\theta_g : \mathcal{D}_{g^{-1}} \rightarrow \mathcal{D}_g, g \in G\}$ be the product partial action of G on the context algebra

$$\mathcal{C} = \begin{pmatrix} \mathcal{A} & M \\ M' & \mathcal{A}' \end{pmatrix},$$

given by Definition 2.8, and let $\gamma = \{\gamma_g : M_{g^{-1}} \rightarrow M_g, g \in G\}$ and $\gamma' = \{\gamma'_g : M'_{g^{-1}} \rightarrow M'_g, g \in G\}$ be the families of k -module isomorphisms provided by Proposition 2.11, where $M_g = \mathcal{D}_g M, M'_g = \mathcal{D}'_g M' (g \in G)$. We are going to produce a Morita context involving the partial skew group rings $\mathcal{A} \rtimes_{\alpha} G$ and $\mathcal{A}' \rtimes_{\alpha'} G$.

Consider the skew group ring $\mathcal{C} \rtimes_{\theta} G$ and the k -submodules

$$\overline{\mathcal{A} \rtimes G} = \bigoplus_{g \in G} \begin{pmatrix} \mathcal{D}_g & 0 \\ 0 & 0 \end{pmatrix} \delta_g \subseteq \mathcal{C} \rtimes_{\theta} G, \quad \overline{M \rtimes G} = \bigoplus_{g \in G} \begin{pmatrix} 0 & M_g \\ 0 & 0 \end{pmatrix} \delta_g \subseteq \mathcal{C} \rtimes_{\theta} G,$$

$$\overline{M' \rtimes G} = \bigoplus_{g \in G} \begin{pmatrix} 0 & 0 \\ M'_g & 0 \end{pmatrix} \delta_g \subseteq \mathcal{C} \rtimes_{\theta} G, \quad \overline{\mathcal{A}' \rtimes G} = \bigoplus_{g \in G} \begin{pmatrix} 0 & 0 \\ 0 & \mathcal{D}'_g \end{pmatrix} \delta_g \subseteq \mathcal{C} \rtimes_{\theta} G.$$

Then evidently $\overline{\mathcal{A} \rtimes G}$ and $\overline{\mathcal{A}' \rtimes G}$ are closed under the multiplication in $\mathcal{C} \rtimes_{\gamma} G$ and one has the k -algebra isomorphisms

$$(7) \quad \overline{\mathcal{A} \rtimes G} \cong \mathcal{A} \rtimes_{\alpha} G \quad \text{and} \quad \overline{\mathcal{A}' \rtimes G} \cong \mathcal{A}' \rtimes_{\alpha'} G.$$

Using (7) one can define a map

$$(\mathcal{A} \rtimes_{\alpha} G) \otimes_k (\overline{M \rtimes G}) \rightarrow \overline{M \rtimes G}$$

as follows:

$$\begin{aligned} \mathcal{D}_g \delta_g \cdot \begin{pmatrix} 0 & M_h \\ 0 & 0 \end{pmatrix} \delta_h &= \begin{pmatrix} \mathcal{D}_g & 0 \\ 0 & 0 \end{pmatrix} \delta_g \begin{pmatrix} 0 & M_h \\ 0 & 0 \end{pmatrix} \delta_h = \theta_g(\theta_g^{-1} \{ \begin{pmatrix} \mathcal{D}_g & 0 \\ 0 & 0 \end{pmatrix} \}) \begin{pmatrix} 0 & M_h \\ 0 & 0 \end{pmatrix} \delta_{gh} \\ &= \theta_g \{ \begin{pmatrix} \alpha_{g^{-1}}(\mathcal{D}_g) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & M_h \\ 0 & 0 \end{pmatrix} \} \delta_{gh} = \theta_g \{ \begin{pmatrix} 0 & \mathcal{D}_{g^{-1}} M_h \\ 0 & 0 \end{pmatrix} \} \delta_{gh} \\ &= \begin{pmatrix} 0 & \gamma_g(M_{g^{-1}} \mathcal{D}'_{g^{-1}} \mathcal{D}'_h) \\ 0 & 0 \end{pmatrix} \delta_{gh} = \begin{pmatrix} 0 & \gamma_g(M_{g^{-1}}) \alpha'_g(\mathcal{D}'_{g^{-1}} \mathcal{D}'_h) \\ 0 & 0 \end{pmatrix} \delta_{gh} \\ &\subseteq \begin{pmatrix} 0 & M_g \mathcal{D}'_{gh} \\ 0 & 0 \end{pmatrix} \delta_{gh} = \begin{pmatrix} 0 & \mathcal{D}_g M_{gh} \\ 0 & 0 \end{pmatrix} \delta_{gh} \subseteq \begin{pmatrix} 0 & M_{gh} \\ 0 & 0 \end{pmatrix} \delta_{gh}, \end{aligned}$$

in view of the equalities $M_h = M \mathcal{D}'_h, \mathcal{D}_{g^{-1}} M = M_{g^{-1}}$, Definition 2.10 and (ii') of Remark 2.2. Thanks to the associativity of $\mathcal{C} \rtimes_{\theta} G$ this gives a structure of a left $\mathcal{A} \rtimes_{\alpha} G$ -module on $\overline{M \rtimes G}$. Taking $g = 1$ above we obtain that

$$(\mathcal{A} \rtimes_{\alpha} G) \cdot (\overline{M \rtimes G}) = \overline{M \rtimes G},$$

so that the left $\mathcal{A} \rtimes_{\alpha} G$ -module $\overline{M \rtimes G}$ is unital. Similarly one defines a structure of a right $\mathcal{A}' \rtimes_{\alpha'} G$ -module on $\overline{M \rtimes G}$ with $(\overline{M \rtimes G})(\mathcal{A}' \rtimes_{\alpha'} G) = \overline{M \rtimes G}$, and again using the associativity of $\mathcal{C} \rtimes_{\gamma} G$, we have that $\overline{M \rtimes G}$ is a unital $(\mathcal{A} \rtimes_{\alpha} G, \mathcal{A}' \rtimes_{\alpha'} G)$ -bimodule.

Analogously, we obtain that $\overline{M' \rtimes G}$ is a unital $(\mathcal{A}' \rtimes_{\alpha'} G, \mathcal{A} \rtimes_{\alpha} G)$ -bimodule.

Next observe that

$$\begin{aligned} \begin{pmatrix} 0 & M_g \\ 0 & 0 \end{pmatrix} \delta_g \begin{pmatrix} 0 & 0 \\ M'_h & 0 \end{pmatrix} \delta_h &= \theta_g(\theta_g^{-1} \{ \begin{pmatrix} 0 & M_g \\ 0 & 0 \end{pmatrix} \}) \begin{pmatrix} 0 & 0 \\ M'_h & 0 \end{pmatrix} \delta_{gh} \\ &= \theta_g \{ \begin{pmatrix} 0 & \gamma_g^{-1}(M_g) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ M'_h & 0 \end{pmatrix} \} \delta_{gh} = \theta_g \{ \begin{pmatrix} M_{g^{-1}} M'_h & 0 \\ 0 & 0 \end{pmatrix} \} \delta_{gh} \\ &= \theta_g \{ \begin{pmatrix} \mathcal{D}_{g^{-1}} \mathcal{D}_h & 0 \\ 0 & 0 \end{pmatrix} \} \delta_{gh} = \begin{pmatrix} \mathcal{D}_g \mathcal{D}_{gh} & 0 \\ 0 & 0 \end{pmatrix} \delta_{gh} \subseteq \begin{pmatrix} \mathcal{D}_{gh} & 0 \\ 0 & 0 \end{pmatrix} \delta_{gh}, \end{aligned}$$

in view of the equalities $M'_h = M'\mathcal{D}_h$, $M \cdot M' = \mathcal{A}$ and $\mathcal{A}\mathcal{D}_h = \mathcal{D}_h$. Then using (7) one obtains a map

$$\bar{\tau} : \overline{M \rtimes G} \otimes \overline{M' \rtimes G} \rightarrow \mathcal{A} \rtimes_{\alpha} G.$$

Taking $g = 1$ in the computation above we see that $\bar{\tau}$ is surjective. In addition, again using the associativity of $\mathcal{C} \rtimes_{\theta} G$, we have that $\bar{\tau}$ is an $\mathcal{A}' \rtimes_{\alpha'} G$ -balanced epimorphism of $(\mathcal{A} \rtimes_{\alpha} G, \mathcal{A} \rtimes_{\alpha} G)$ -bimodules. Similarly one obtains an $\mathcal{A} \rtimes_{\alpha} G$ -balanced epimorphism of $(\mathcal{A}' \rtimes_{\alpha'} G, \mathcal{A}' \rtimes_{\alpha'} G)$ -bimodules

$$\bar{\tau}' : \overline{M' \rtimes G} \otimes \overline{M \rtimes G} \rightarrow \mathcal{A}' \rtimes_{\alpha'} G.$$

Finally, since $\mathcal{A}\mathcal{D}_g = \mathcal{D}_g$ and $\mathcal{A}'\mathcal{D}'_g = \mathcal{D}'_g$ ($g \in G$) we readily see that the multiplication maps

$$\mathcal{A} \rtimes G \otimes \mathcal{A} \rtimes G \rightarrow \mathcal{A} \rtimes G \quad \text{and} \quad \mathcal{A}' \rtimes G \otimes \mathcal{A}' \rtimes G \rightarrow \mathcal{A}' \rtimes G$$

are also surjective, i.e. the algebras $\mathcal{A} \rtimes G$ and $\mathcal{A}' \rtimes G$ are idempotent. □

With respect to the above proof observe that setting

$$M \rtimes G = \bigoplus_{g \in G} M_g \delta_g \quad \text{and} \quad M' \rtimes G = \bigoplus_{g \in G} M'_g \delta_g,$$

we have the k -module isomorphisms

$$M \rtimes G \cong \overline{M \rtimes G} \quad \text{and} \quad M' \rtimes G \cong \overline{M' \rtimes G}.$$

This permits us to define on $M \rtimes G$ a structure of a unital $(\mathcal{A} \rtimes_{\alpha} G, \mathcal{A}' \rtimes_{\alpha'} G)$ -bimodule and on $M' \rtimes G$ that of a unital $(\mathcal{A}' \rtimes_{\alpha'} G, \mathcal{A} \rtimes_{\alpha} G)$ -bimodule and surjective trace maps

$$\tau \rtimes G : M \rtimes G \otimes M' \rtimes G \rightarrow \mathcal{A} \rtimes_{\alpha} G \quad \text{and} \quad \tau' \rtimes G : M' \rtimes G \otimes M \rtimes G \rightarrow \mathcal{A}' \rtimes_{\alpha'} G.$$

More precisely, the left $\mathcal{A} \rtimes_{\alpha} G$ -module structure on $M \rtimes G$ is given by

$$a \delta_g \cdot m \delta_h = \gamma_g(\alpha_{g^{-1}}(a)m) \delta_{gh},$$

with $a \in \mathcal{D}_g, m \in M_h$, whereas the right $\mathcal{A}' \rtimes_{\alpha'} G$ -module structure is defined by

$$m \delta_g \cdot a' \delta_h = \gamma_g(\gamma_{g^{-1}}(m)a') \delta_{gh},$$

$m \in M_g, a' \in \mathcal{D}'_h, g, h \in G$. Analogously $M' \rtimes G$ is a $(\mathcal{A}' \rtimes_{\alpha'} G, \mathcal{A} \rtimes_{\alpha} G)$ -bimodule with

$$a' \delta_g \cdot m'_1 \delta_h = \gamma'_g(\alpha'_{g^{-1}}(a')m'_1) \delta_{gh},$$

$$m'_2 \delta_g \cdot a \delta_h = \gamma'_g(\gamma'_{g^{-1}}(m'_2)a) \delta_{gh},$$

$a' \in \mathcal{D}'_g, m'_1 \in M'_h, m'_2 \in M'_g, a \in \mathcal{D}_h, g, h \in G$. The trace maps are given by

$$(\tau \rtimes G)(m \delta_g \otimes m' \delta_h) = \alpha_g(\gamma_{g^{-1}}(m) \cdot m') \delta_{gh},$$

$m \in M_g, m' \in M'_h, g, h \in G$, and

$$(\tau' \rtimes G)(m' \delta_g \otimes m \delta_h) = \alpha'_g(\gamma'_{g^{-1}}(m') \cdot m) \delta_{gh},$$

$m' \in M'_g, m \in M_h, g, h \in G$. Thus $(\mathcal{A} \rtimes_{\alpha} G, \mathcal{A}' \rtimes_{\alpha'} G, M \rtimes G, M' \rtimes G, \tau \rtimes G, \tau' \rtimes G)$ is a Morita context with surjective traces.

4. GLOBALIZATION UP TO MORITA EQUIVALENCE

Let $\mathcal{B} = \bigoplus_{g \in G} \mathcal{B}_g$ be a G -graded algebra. Denote by $\text{FMat}_G(\mathcal{B})$ the algebra of all $G \times G$ -matrices over \mathcal{B} with only a finite number of non-zero entries, whose rows and columns are labeled by the elements of G . Set

$$\mathcal{E} = \text{FMat}_{g,h \in G}(\mathcal{B}_{g^{-1}h}) \subseteq \text{FMat}_G(\mathcal{B}),$$

the subalgebra of those matrices $a \in \text{FMat}_G(\mathcal{B})$, whose (g, h) -entry $a_{(g,h)}$ belongs to $\mathcal{B}_{g^{-1}h}$ ($g, h \in G$). Also set

$$\mathcal{A}' = \text{FMat}_{g,h \in G}(\mathcal{B}_{g^{-1}\mathcal{B}_h}) = \{a \in \mathcal{E} \mid a_{(g,h)} \in \mathcal{B}_{g^{-1}\mathcal{B}_h}\}.$$

Evidently, \mathcal{A}' is a two-sided ideal in \mathcal{E} . Observe that $\mathcal{A}' = \mathcal{E}$ exactly when \mathcal{B} is strictly graded, i.e. $\mathcal{B}_g \mathcal{B}_h = \mathcal{B}_{gh}$, for all $g, h \in G$.

The first row of the matrices from \mathcal{A}' forms the $(\mathcal{B}_1, \mathcal{A}')$ -bimodule

$$M = \bigoplus_{h \in G} \mathcal{B}_1 \mathcal{B}_h.$$

Similarly, the first column forms the $(\mathcal{A}', \mathcal{B}_1)$ -bimodule

$$M' = \bigoplus_{g \in G} \mathcal{B}_{g^{-1}} \mathcal{B}_1.$$

Thus one can form a Morita context $(\mathcal{B}_1, \mathcal{A}', M, M', \tau, \tau')$, where the trace maps τ and τ' are given by multiplication in \mathcal{A}' . Note that if each \mathcal{B}_g is unital as a left \mathcal{B}_1 -module, then τ and τ' are surjective and the algebras \mathcal{B}_1 and \mathcal{A}' are idempotent. Moreover,

$$M = \bigoplus_{h \in G} \mathcal{B}_h,$$

and thus M is evidently a unital left \mathcal{B}_1 -module. Furthermore, the g -part of $M\mathcal{A}'$ is $\sum_h \mathcal{B}_h \mathcal{B}_{h^{-1}} \mathcal{B}_g \supseteq \mathcal{B}_1 \mathcal{B}_g = \mathcal{B}_g$ so that $M\mathcal{A}' = M$ and M is unital as a bimodule. We also see that the fact that \mathcal{B}_1 is idempotent implies that M' is a unital $(\mathcal{A}', \mathcal{B}_1)$ -bimodule. Consequently, \mathcal{B}_1 and \mathcal{A}' are Morita equivalent. This is also true if instead of imposing $\mathcal{B}_1 \mathcal{B}_g = \mathcal{B}_g$ one assumes that the right \mathcal{B}_1 -modules \mathcal{B}_g are unital.

There is a natural global action β of G on \mathcal{E} permuting rows and columns:

$$\beta_f(a)_{(g,h)} = a_{(f^{-1}g, f^{-1}h)}, \quad g, h, f \in G, a \in \mathcal{E}.$$

We know that any global action on an algebra possesses a canonical restriction on a two-sided ideal (see [20, p. 1938]), and in our case it is given by the partial isomorphisms $\alpha'_g : \mathcal{D}'_{g^{-1}} \rightarrow \mathcal{D}'_g$, where $\mathcal{D}'_g = \mathcal{A}' \cap \beta_g(\mathcal{A}')$ and α'_g is the restriction of β_g on $\mathcal{D}'_{g^{-1}}$. We shall denote this partial action by α' .

Observe also that if \mathcal{I} is an idempotent ideal in an algebra \mathcal{R} and \mathcal{J} is an ideal of \mathcal{R} , then

$$(8) \quad \mathcal{I}(\mathcal{I} \cap \mathcal{J}) = \mathcal{I} \mathcal{J}$$

for $\mathcal{I} \mathcal{J} = \mathcal{I}(\mathcal{I} \mathcal{J}) \subseteq \mathcal{I}(\mathcal{I} \cap \mathcal{J})$, while $\mathcal{I}(\mathcal{I} \cap \mathcal{J}) \subseteq \mathcal{I} \mathcal{J}$ holds for arbitrary ideals.

Our goal is to prove that if α is a regular partial action of G on an algebra \mathcal{A} and $\mathcal{B} = \mathcal{A} \rtimes_{\alpha} G$, then α and α' are Morita equivalent partial actions. Note that in this case $\mathcal{B}_g = \mathcal{D}_g \delta_g$, $g \in G$ and

$$\mathcal{B}_g \mathcal{B}_h = \mathcal{D}_g \mathcal{D}_{gh} \delta_{gh}, \quad \forall g, h \in G.$$

We immediately have that $\mathcal{B}_1\mathcal{B}_g = \mathcal{B}_g\mathcal{B}_1 = \mathcal{B}_g$, and by the above, $\mathcal{B}_1 \cong \mathcal{A}$ and \mathcal{A}' are Morita equivalent idempotent algebras. Note also that \mathcal{B} is strictly graded if and only if α is a global action.

Before formulating the main result of this section we recall that if β is a global action of a group G on an algebra \mathcal{B} and \mathcal{A} is a two-sided ideal in \mathcal{B} , then setting $\mathcal{D}_g = \mathcal{A} \cap \beta_g(\mathcal{A})$ and $\alpha_g(a) = \beta_g(a)$ for all $a \in \mathcal{D}_{g^{-1}}$ we obtain a partial action $\alpha = \{\alpha_g : \mathcal{D}_{g^{-1}} \rightarrow \mathcal{D}_g, g \in G\}$ of G on \mathcal{A} , called the *restriction* of β on \mathcal{A} . In this case we say that α is *globalizable*.¹

Theorem 4.1. *Let $\alpha = \{\alpha_g : \mathcal{D}_{g^{-1}} \rightarrow \mathcal{D}_g, g \in G\}$ be a regular partial action of a group G on a k -algebra \mathcal{A} . Then α is Morita equivalent to a globalizable regular partial action.*

Proof. We shall show that with the above notation, taking $\mathcal{B} = \mathcal{A} \rtimes G$, the partial action α' satisfies (2) and the bimodules M and M' can be used to provide a Morita equivalence of α with α' .

Since \mathcal{A} and \mathcal{A}' are Morita equivalent, there exists a lattice isomorphism between the unital ideals of \mathcal{A} and those of \mathcal{A}' , and we start by pointing out that this isomorphism takes \mathcal{D}_g to \mathcal{D}'_g for any $g \in G$ (as it is required by Definition 2.8). Observe first that

$$(9) \quad \mathcal{D}'_f = \text{FMat}_{g,h \in G}(\mathcal{B}_{g^{-1}}\mathcal{B}_f\mathcal{B}_{f^{-1}}\mathcal{B}_h).$$

Indeed, we have that

$$\beta_f(\mathcal{A}') = \text{FMat}_{g,h \in G}(\mathcal{B}_{g^{-1}}\mathcal{B}_f\mathcal{B}_{f^{-1}}\mathcal{B}_h),$$

and (9) holds if and only if

$$\mathcal{B}_{g^{-1}}\mathcal{B}_h \cap \mathcal{B}_{g^{-1}f}\mathcal{B}_{f^{-1}h} = \mathcal{B}_{g^{-1}}\mathcal{B}_f\mathcal{B}_{f^{-1}}\mathcal{B}_h, \quad \forall g, h, f \in G.$$

On one hand

$$\mathcal{B}_{g^{-1}}\mathcal{B}_h \cap \mathcal{B}_{g^{-1}f}\mathcal{B}_{f^{-1}h} = (\mathcal{D}_{g^{-1}}\mathcal{D}_{g^{-1}h} \cap \mathcal{D}_{g^{-1}f}\mathcal{D}_{g^{-1}h}) \delta_{g^{-1}h} = (\mathcal{D}_{g^{-1}}\mathcal{D}_{g^{-1}f}\mathcal{D}_{g^{-1}h}) \delta_{g^{-1}h},$$

in view of (2). On the other hand,

$$\begin{aligned} \mathcal{B}_{g^{-1}}\mathcal{B}_f\mathcal{B}_{f^{-1}}\mathcal{B}_h &= (\mathcal{D}_{g^{-1}}\mathcal{D}_{g^{-1}f} \delta_{g^{-1}f}) (\mathcal{D}_{f^{-1}}\mathcal{D}_{f^{-1}h} \delta_{f^{-1}h}) \\ &= \alpha_{g^{-1}f}(\alpha_{f^{-1}g}(\mathcal{D}_{g^{-1}}\mathcal{D}_{g^{-1}f}) \mathcal{D}_{f^{-1}}\mathcal{D}_{f^{-1}h}) \delta_{g^{-1}h} = (\mathcal{D}_{g^{-1}}\mathcal{D}_{g^{-1}f}\mathcal{D}_{g^{-1}h}) \delta_{g^{-1}h}, \end{aligned}$$

and (9) follows. Now the ideal of \mathcal{A}' which corresponds to \mathcal{D}_f by the lattice isomorphism is

$$M'(\mathcal{D}_f\delta_1)M = M'(\mathcal{B}_f\mathcal{B}_{f^{-1}})M = \text{FMat}_{g,h \in G}(\mathcal{B}_{g^{-1}}\mathcal{B}_f\mathcal{B}_{f^{-1}}\mathcal{B}_h),$$

which is \mathcal{D}'_f in view of (9). Thus

$$(10) \quad \mathcal{D}'_g = M'\mathcal{D}_gM, \quad \forall g \in G,$$

which gives (i) of Definition 2.8.

Equality (10) immediately implies

$$\mathcal{D}'_g = \mathcal{A}'\mathcal{D}'_g = \mathcal{D}'_g\mathcal{A}', \quad \forall g \in G,$$

as the bimodules M and M' are unital. Moreover, $\mathcal{A}' \cap \beta_g(\mathcal{A}') = \mathcal{D}'_g = \mathcal{A}'\mathcal{D}'_g = \mathcal{A}'(\mathcal{A}' \cap \beta_g(\mathcal{A}')) = \mathcal{A}'\beta_g(\mathcal{A}')$ in view of (8). Thus

$$(11) \quad \mathcal{A}' \cap \beta_g(\mathcal{A}') = \mathcal{A}'\beta_g(\mathcal{A}') = \beta_g(\mathcal{A}')\mathcal{A}', \quad \forall g \in G,$$

¹More generally, any partial action isomorphic (see Definition 6.4) to such α is called *globalizable*.

the second equality being obtained symmetrically. It follows that each \mathcal{D}'_g is an idempotent ideal.

Claim 1.

$$\bigcap_{g \in I} \mathcal{D}'_g = \mathcal{A}' \left(\bigcap_{g \in I} \mathcal{D}'_g \right),$$

for any finite subset I of G . We only need to show the inclusion $\bigcap_{g \in I} \mathcal{D}'_g \subseteq \mathcal{A}'(\bigcap_{g \in I} \mathcal{D}'_g)$, as the converse one is immediate. Take any $a \in \bigcap_{f \in I} \mathcal{D}'_f$. We have seen in the proof of (9) that $\mathcal{B}_{g^{-1}}\mathcal{B}_f\mathcal{B}_{f^{-1}}\mathcal{B}_h = \mathcal{D}_{g^{-1}}\mathcal{D}_{g^{-1}f}\mathcal{D}_{g^{-1}h} \delta_{g^{-1}h}$. Hence

$$(12) \quad \mathcal{D}'_f = \text{FMat}_{g,h \in G}(\mathcal{D}_{g^{-1}}\mathcal{D}_{g^{-1}f}\mathcal{D}_{g^{-1}h} \delta_{g^{-1}h}),$$

and using (2) we have that

$$a_{g,h} \in \mathcal{D}_{g^{-1}}\mathcal{D}_{g^{-1}h} \left(\prod_{f \in I} \mathcal{D}_{g^{-1}f} \right) \delta_{g^{-1}h}, \quad g, h \in G.$$

Write $a_{g,h} = \sum d_{g^{-1}b} \delta_{g^{-1}h}$ with $d_{g^{-1}} \in \mathcal{D}_{g^{-1}}$ and $b \in \mathcal{D}_{g^{-1}h}(\prod_{f \in I} \mathcal{D}_{g^{-1}f})$. Since the ideal \mathcal{D}_g is idempotent, $d_{g^{-1}} = \sum \bar{d}_{g^{-1}} \tilde{d}_{g^{-1}}$ with $\bar{d}_{g^{-1}}, \tilde{d}_{g^{-1}} \in \mathcal{D}_{g^{-1}}$. Denote by $e_{g,h}(r) \in \text{FMat}_{g,h \in G}(\mathcal{B})$ the matrix unit whose (g, h) -entry is $r \in \mathcal{B}$ and all other entries are 0. Evidently $e_{g,g}(\mathcal{D}_{g^{-1}} \delta_1) = e_{g,g}(\mathcal{B}_{g^{-1}}\mathcal{B}_g) \subseteq \mathcal{A}'$ and $e_{g,h}(\tilde{d}_{g^{-1}b} \delta_{g^{-1}h}) \in e_{g,h}(\mathcal{B}_{g^{-1}}\mathcal{B}_f\mathcal{B}_{f^{-1}}\mathcal{B}_h)$, and consequently the equality

$$e_{g,h}(d_{g^{-1}b} \delta_{g^{-1}h}) = \sum e_{g,g}(\bar{d}_{g^{-1}} \delta_1) e_{g,h}(\tilde{d}_{g^{-1}b} \delta_{g^{-1}h})$$

shows that $a \in \mathcal{A}'(\bigcap_{f \in I} \mathcal{D}'_f)$, proving the claim.

Next we show that

Claim 2. $\mathcal{D}'_{g_1} \cap \mathcal{D}'_{g_2} \cap \dots \cap \mathcal{D}'_{g_n} = \mathcal{D}'_{g_1} \mathcal{D}'_{g_2} \dots \mathcal{D}'_{g_n}$, $\forall g_1, \dots, g_n \in G, \forall n > 0$, i.e. the ideals \mathcal{D}'_g also satisfy (2). We fix n and use induction establishing first the following:

Claim 3. If Claim 2 holds for any subset of G with less than n elements, then

$$\beta_{g_1}(\mathcal{A}') \cap \beta_{g_2}(\mathcal{A}') \cap \dots \cap \beta_{g_n}(\mathcal{A}') = \beta_{g_1}(\mathcal{A}') \cdot \beta_{g_2}(\mathcal{A}') \dots \beta_{g_n}(\mathcal{A}').$$

We have

$$\begin{aligned} \beta_{g_1}(\mathcal{A}') \cap \dots \cap \beta_{g_n}(\mathcal{A}') &= \beta_{g_1}[\mathcal{A}' \cap \beta_{g_1^{-1}g_2}(\mathcal{A}') \cap \dots \cap \beta_{g_1^{-1}g_n}(\mathcal{A}')] \\ &= \beta_{g_1}[\{\mathcal{A}' \cap \beta_{g_1^{-1}g_2}(\mathcal{A}')\} \cap \dots \cap \{\mathcal{A}' \cap \beta_{g_1^{-1}g_n}(\mathcal{A}')\}] = \beta_{g_1}[\mathcal{D}'_{g_1^{-1}g_2} \cap \dots \cap \mathcal{D}'_{g_1^{-1}g_n}] \\ &= \beta_{g_1}[\mathcal{D}'_{g_1^{-1}g_2} \dots \mathcal{D}'_{g_1^{-1}g_n}] = \beta_{g_1}[\{\mathcal{A}' \beta_{g_1^{-1}g_2}(\mathcal{A}')\} \dots \{\mathcal{A}' \beta_{g_1^{-1}g_n}(\mathcal{A}')\}] \\ &= \beta_{g_1}[\mathcal{A}' \beta_{g_1^{-1}g_2}(\mathcal{A}') \dots \beta_{g_1^{-1}g_n}(\mathcal{A}')] = \beta_{g_1}(\mathcal{A}') \cdot \beta_{g_2}(\mathcal{A}') \dots \beta_{g_n}(\mathcal{A}'), \end{aligned}$$

using $(\mathcal{A}')^2 = \mathcal{A}'$ and (11), and this gives Claim 3.

Now Claim 2 we obtain by applying Claim 1, Claim 3, (8) and (11) as follows:

$$\begin{aligned} \mathcal{D}'_{g_1} \cap \mathcal{D}'_{g_2} \cap \dots \cap \mathcal{D}'_{g_n} &= \mathcal{A}' (\mathcal{D}'_{g_1} \cap \mathcal{D}'_{g_2} \cap \dots \cap \mathcal{D}'_{g_n}) \\ &= \mathcal{A}' (\mathcal{A}' \cap \beta_{g_1}(\mathcal{A}') \cap \dots \cap \beta_{g_n}(\mathcal{A}')) \\ &= \mathcal{A}' (\mathcal{A}' \cap \beta_{g_1}(\mathcal{A}') \cdot \beta_{g_2}(\mathcal{A}') \dots \beta_{g_n}(\mathcal{A}')) \\ &= \mathcal{A}' \cdot \beta_{g_1}(\mathcal{A}') \cdot \beta_{g_2}(\mathcal{A}') \dots \beta_{g_n}(\mathcal{A}') \\ &= \{\mathcal{A}' \cdot \beta_{g_1}(\mathcal{A}')\} \cdot \{\mathcal{A}' \cdot \beta_{g_2}(\mathcal{A}')\} \dots \{\mathcal{A}' \cdot \beta_{g_n}(\mathcal{A}')\} \\ &= \mathcal{D}'_{g_1} \cdot \mathcal{D}'_{g_2} \dots \mathcal{D}'_{g_n}. \end{aligned}$$

Next we define the families of k -module isomorphisms γ and γ' which will fit Proposition 2.11. Set $M_g = \mathcal{D}_g M$ and $M'_g = M' \mathcal{D}_g$, $g \in G$. By (10) and Remark 2.9 we also have that $M_g = M \mathcal{D}'_g$ and $M'_g = \mathcal{D}'_g M'$, $g \in G$. Given $m \in M = \bigoplus_{h \in G} \mathcal{B}_h$ denote by $(m)_h \in \mathcal{B}_h (= \mathcal{B}_1 \mathcal{B}_h)$ the h -entry of m . Recall that $M' = \bigoplus_{g \in G} \mathcal{B}_{g^{-1}}$ is defined as the first columns of the matrices from \mathcal{A}' , so for the g -entry of an element $m' \in M'$ we have $(m')_g \in \mathcal{B}_{g^{-1}} (= \mathcal{B}_{g^{-1}} \mathcal{B}_1)$. Since $\mathcal{B}_h = \mathcal{D}_h \delta_h$, $h \in G$, given $m \in M_{g^{-1}}$, we have

$$(13) \quad (m)_h = d_{\{g^{-1}, h\}} \delta_h, \quad d_{\{g^{-1}, h\}} \in \mathcal{D}_{g^{-1}} \mathcal{D}_h.$$

Define $\gamma_g : M_{g^{-1}} \rightarrow M_g$ by

$$(\gamma_g(m))_h = \alpha_g(d_{\{g^{-1}, g^{-1}h\}}) \delta_h.$$

Thus γ_g permutes the entries multiplying their indexes by g from the left and applies α_g to the entries. On the other hand, we define the maps $\gamma'_g : M'_{g^{-1}} \rightarrow M'_g$ by only permuting the entries whose indexes are multiplied by g from the left in the sense that the $g^{-1}h$ -entry of $m' \in M'_{g^{-1}}$ goes to the h -place. More precisely, since $M'_g = M' \mathcal{D}_g$, we have

$$M'_g = \bigoplus_{h \in G} \mathcal{B}_{h^{-1}} (\mathcal{D}_g \delta_1) = \bigoplus_{h \in G} \mathcal{D}_{h^{-1}} \delta_{h^{-1}} \mathcal{D}_g \delta_1 = \bigoplus_{h \in G} \mathcal{D}_{h^{-1}} \mathcal{D}_{h^{-1}g} \delta_{h^{-1}},$$

$g, h \in G$, so that for $m' \in M'_{g^{-1}}$ we can write

$$(14) \quad (m')_h = d'_{\{h^{-1}, h^{-1}g^{-1}\}} \delta_{h^{-1}}, \quad d'_{\{h^{-1}, h^{-1}g^{-1}\}} \in \mathcal{D}_{h^{-1}} \mathcal{D}_{h^{-1}g^{-1}}.$$

Set

$$(\gamma'_g(m'))_h = d'_{\{h^{-1}g, h^{-1}\}} \delta_{h^{-1}}.$$

It is easily seen that γ and γ' satisfy (i) and (ii) of Definition 2.10, and we are going to check that they also satisfy (iii) of Definition 2.10 (both left and right versions), as well as the properties (1) and (2) of Proposition 2.11, and this will complete the proof of the theorem.

Checking (iii) of Definition 2.10 for (α, γ) : Let $a \in \mathcal{D}_{g^{-1}}$ and assume that m is given in the form (13). Then

$$(\gamma_g(am))_h = \alpha_g(ad_{\{g^{-1}, g^{-1}h\}}) \delta_h = \alpha_g(a) \alpha(d_{\{g^{-1}, g^{-1}h\}}) \delta_h = (\alpha_g(a) \gamma_g(m))_h,$$

with arbitrary $g, h \in G$, as desired.

(iii) of Definition 2.10 for (γ, α') : Keeping the notation for $m \in M_{g^{-1}}$ in the form (13), let $a' \in \mathcal{D}'_{g^{-1}}$. By (12) we can write

$$(15) \quad (a')_{(f,h)} = \bar{d}_{\{f^{-1}, f^{-1}g^{-1}, f^{-1}h\}} \delta_{f^{-1}h}, \quad \bar{d}_{\{f^{-1}, f^{-1}g^{-1}, f^{-1}h\}} \in \mathcal{D}_{f^{-1}} \mathcal{D}_{f^{-1}g^{-1}} \mathcal{D}_{f^{-1}h},$$

$f, h \in G$. Then

$$\begin{aligned} (\gamma_g(m))_f &= \alpha_g(d_{\{g^{-1}, g^{-1}f\}}) \delta_f, \\ (a')_{(g^{-1}f, g^{-1}h)} &= \bar{d}_{\{f^{-1}g, f^{-1}, f^{-1}h\}} \delta_{f^{-1}h}, \\ (\alpha'_g(a'))_{(f,h)} &= \bar{d}_{\{f^{-1}g, f^{-1}, f^{-1}h\}} \delta_{f^{-1}h}, \end{aligned}$$

and hence

$$(16) \quad [\gamma_g(m) \alpha'_g(a')]_h = \sum_{f \in G} \alpha_g(d_{\{g^{-1}, g^{-1}f\}}) \alpha_f(\bar{d}_{\{f^{-1}g, f^{-1}, f^{-1}h\}}) \delta_h.$$

On the other hand,

$$(ma')_h = \sum_{t \in G} d_{\{g^{-1}, t\}} \alpha_t(\bar{d}_{\{t^{-1}, t^{-1}g^{-1}, t^{-1}h\}}) \delta_h,$$

and consequently,

$$(\gamma_g(ma'))_h = \sum_{t \in G} \alpha_g(d_{\{g^{-1}, t\}}) \alpha_{gt}(\bar{d}_{\{t^{-1}, t^{-1}g^{-1}, t^{-1}g^{-1}h\}}) \delta_h,$$

taking into account that $\alpha_g \circ \alpha_t = \alpha_{gt}$ on $\mathcal{D}_{t^{-1}}\mathcal{D}_{t^{-1}g^{-1}}$. Taking $f = gt$ we see that $(\gamma_g(ma'))_h$ equals (16), as desired.

(iii) of Definition 2.10 for (α', γ') : Keeping the notation of (14) and (15) we have that

$$\alpha'_g(a')_{(h,f)} = \bar{d}_{\{h^{-1}g, h^{-1}, h^{-1}f\}} \delta_{h^{-1}f}, \quad \gamma'_g(m')_f = d'_{\{f^{-1}g, f^{-1}\}} \delta_{f^{-1}},$$

and thus

$$[\alpha'_g(a')\gamma'_g(m')]_h = \sum_{f \in G} \alpha_{h^{-1}f}[\alpha_{h^{-1}f}^{-1}(\bar{d}_{\{h^{-1}g, h^{-1}, h^{-1}f\}})d'_{\{f^{-1}g, f^{-1}\}}] \delta_{h^{-1}}.$$

On the other hand

$$(a'm')_{g^{-1}h} = \sum_{t \in G} \alpha_{h^{-1}gt}[\alpha_{h^{-1}gt}^{-1}(\bar{d}_{\{h^{-1}g, h^{-1}, h^{-1}gt\}})d'_{\{t^{-1}, t^{-1}g^{-1}\}}] \delta_{h^{-1}g},$$

which equals $\gamma'_g(a'm')_h$ if we replace $\delta_{h^{-1}g}$ by $\delta_{h^{-1}}$. Taking $f = gt$ we see that this coincides with $[\alpha'_g(a')\gamma'_g(m')]_h$.

(iii) of Definition 2.10 for (γ', α) : Taking $a \in \mathcal{D}_{g^{-1}}$ and $m' \in M'_{g^{-1}}$, we have on one hand

$$[\gamma'_g(m')\alpha_g(a)]_h = d'_{\{h^{-1}g, h^{-1}\}} \delta_{h^{-1}} \alpha_g(a) \delta_1 = \alpha_{h^{-1}}[\alpha_h(d'_{\{h^{-1}g, h^{-1}\}}) \alpha_g(a)] \delta_{h^{-1}},$$

and on the other hand

$$(m'a)_{g^{-1}h} = \alpha_{h^{-1}g}[\alpha_{g^{-1}h}(d'_{\{h^{-1}g, h^{-1}\}})a] \delta_{h^{-1}g}.$$

The latter is $\gamma'_g(m'a)_h$ if we remove g from δ . Looking at the domains it is easily seen that one can split both α 's, i.e.

$$\begin{aligned} \gamma'_g(m'a)_h &= \alpha_{h^{-1}} \circ \alpha_g[\alpha_{g^{-1}} \circ \alpha_h(d'_{\{h^{-1}g, h^{-1}\}})a] \delta_{h^{-1}} \\ &= \alpha_{h^{-1}}[\alpha_h(d'_{\{h^{-1}g, h^{-1}\}}) \alpha_g(a)] \delta_{h^{-1}}, \end{aligned}$$

as desired.

(1) of Proposition 2.11: For $m \in M_{g^{-1}}$ and $m' \in M'_{g^{-1}}$ we have

$$(17) \quad \gamma_g(m)\gamma'_g(m') = \sum_{h \in G} \alpha_g(d_{\{g^{-1}, g^{-1}h\}}) \alpha_h(d'_{\{h^{-1}g, h^{-1}\}}) \delta_1.$$

On the other hand

$$\alpha_g(mm') = \sum_{f \in G} \alpha_g(d_{\{g^{-1}, f\}}) \alpha_{gf}(d'_{\{f^{-1}, f^{-1}g^{-1}\}}) \delta_1,$$

and the latter coincides with (17) by taking $f = g^{-1}h$.

Finally, we check

(2) of Proposition 2.11: For $m \in M_{g^{-1}}$ and $m' \in M'_{g^{-1}}$ we compute

$$\begin{aligned} \alpha'_g(m'm)_{(h,f)} &= \alpha_{h^{-1}g}[\alpha_{g^{-1}h}(d'_{\{h^{-1}g, h^{-1}\}}) d_{\{g^{-1}, g^{-1}f\}}] \delta_{h^{-1}f} \\ &= \alpha_{h^{-1}} \circ \alpha_g[\alpha_{g^{-1}} \circ \alpha_h(d'_{\{h^{-1}g, h^{-1}\}}) d_{\{g^{-1}, g^{-1}f\}}] \delta_{h^{-1}f} \\ &= \alpha_{h^{-1}}[\alpha_h(d'_{\{h^{-1}g, h^{-1}\}}) \alpha_g(d_{\{g^{-1}, g^{-1}f\}})] \delta_{h^{-1}f}, \end{aligned}$$

obtaining the same element on the other side:

$$\begin{aligned} [\gamma'_g(m')\gamma_g(m)]_{(h,f)} &= [\gamma'_g(m')]_h[\gamma_g(m)]_f = d'_{\{h^{-1}g, h^{-1}\}} \delta_{h^{-1}} \alpha_g(d_{\{g^{-1}, g^{-1}f\}}) \delta_f \\ &= \alpha_{h^{-1}}[\alpha_h(d'_{\{h^{-1}g, h^{-1}\}}) \alpha_g(d_{\{g^{-1}, g^{-1}f\}})] \delta_{h^{-1}f}. \end{aligned}$$

Thus we conclude by Proposition 2.11 that α and α' are Morita equivalent partial actions, completing the proof of the theorem. □

Corollary 4.2. *Any partial action of a group G on a von Neumann regular ring \mathcal{A} is Morita equivalent to a globalizable partial action.*

5. ON THE UNIQUENESS OF THE GLOBALIZATION UP TO MORITA EQUIVALENCE

In this section we show that any regular partial action possesses a so-called Morita enveloping action which is unique up to Morita equivalence. We also prove that their respective skew group rings are Morita equivalent.

We next recall the notion of an enveloping action of a partial action, and we also introduce the corresponding (weaker) notion related with globalizations up to Morita equivalence.

Definition 5.1. Let $\alpha = \{\alpha_g : \mathcal{D}_{g^{-1}} \rightarrow \mathcal{D}_g\}_{g \in G}$ be a regular partial action of G on the algebra \mathcal{A} . An enveloping action² of α is a global action $\beta : G \times \mathcal{B} \rightarrow \mathcal{B}$ such that \mathcal{A} is a two-sided ideal in \mathcal{B} , α is the restriction of β on \mathcal{A} and the linear β -orbit of \mathcal{A} is all of \mathcal{B} :

$$\mathcal{B} = \text{span}\{\beta_g(a) : g \in G, a \in \mathcal{A}\}.$$

If α' is a regular partial action Morita equivalent to α and if β' is an enveloping action of α' , we say that β' is a Morita enveloping action of α .

The following example, although perhaps the simplest possible, describes adequately the general situation when a Morita enveloping action is available.

Example 5.2. Let \mathcal{A} be any unital algebra with a non-unital but idempotent ideal J . Consider the partial action α of $G = \{1, -1\}$ on \mathcal{A} such that $\alpha_{-1} = id_J$. Since J is non-unital, α does not have an enveloping action. However, it has a Morita enveloping action, which we now show. Let

$$\mathcal{B} := \left\{ \begin{pmatrix} a & x \\ y & b \end{pmatrix} : a, b \in \mathcal{A}, x, y \in J \right\}.$$

The automorphism $\beta_{-1} : \mathcal{B} \rightarrow \mathcal{B}$ such that $\beta_{-1} \begin{pmatrix} a & x \\ y & b \end{pmatrix} = \begin{pmatrix} b & y \\ x & a \end{pmatrix}$ defines an action $\beta : G \times \mathcal{B} \rightarrow \mathcal{B}$. Consider the ideal $\mathcal{A}' = \begin{pmatrix} \mathcal{A} & J \\ J & J \end{pmatrix}$ of \mathcal{B} , and let α' be the partial action of G on \mathcal{A}' obtained by restricting β to \mathcal{A}' , so the domain of α'_{-1} is equal to

²More generally, β is called an enveloping action for any partial action isomorphic (in the sense of Definition 6.4) to α . However, we will not use this more general definition.

$J' := \mathcal{A}' \cap \beta_{-1}(\mathcal{A}') = \begin{pmatrix} J & J \\ J & J \end{pmatrix}$. It is clear that β is an enveloping action of α' . Now define $M := \begin{pmatrix} A & J \\ 0 & 0 \end{pmatrix}$, $M' := \begin{pmatrix} A & 0 \\ J & 0 \end{pmatrix} \subseteq \mathcal{A}'$. Then, identifying \mathcal{A} with $\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$, we have $M\mathcal{A}' = M = \mathcal{A}M$, $\mathcal{A}'M' = M' = M'\mathcal{A}$, $MM' = \mathcal{A}$ and $M'M = \mathcal{A}'$, so that $\mathbf{M} := (\mathcal{A}, \mathcal{A}', M, M', \tau, \tau')$, where τ and τ' are given by the product in \mathcal{A}' , is a Morita context that provides a Morita equivalence of \mathcal{A} with \mathcal{A}' . Also define $M_J = \begin{pmatrix} J & J \\ 0 & 0 \end{pmatrix}$ and $M'_J = \begin{pmatrix} J & 0 \\ J & 0 \end{pmatrix}$. Then $(J, J', M_J, M'_J, \tau, \tau')$ is a Morita context giving a Morita equivalence between J and J' . Finally, let γ and γ' be the partial actions of G on M and M' respectively, such that $\gamma_{-1} \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} y & x \\ 0 & 0 \end{pmatrix}$ and $\gamma'_{-1} = \begin{pmatrix} u & 0 \\ v & 0 \end{pmatrix} = \begin{pmatrix} v & 0 \\ u & 0 \end{pmatrix}$. It follows that \mathbf{M} together with γ and γ' (formally considered as families of k -module isomorphisms) give a Morita equivalence between α and α' . Thus β is a Morita enveloping action of α . It will follow from Theorem 5.8 that this is essentially the unique Morita enveloping action of α .

There are known results on Morita equivalence between the skew group rings of partial actions and their enveloping actions ([2, Theorem 3.3], [20, Theorem 5.4], [24, Theorem 4.1]). Our first task will be to generalize them to the case of partial actions that satisfy a condition even weaker than (2).

Theorem 5.3. *Let $\alpha = \{\alpha_g : \mathcal{D}_{g^{-1}} \rightarrow \mathcal{D}_g\}_{g \in G}$ be a partial action of G on an algebra \mathcal{A} , such that $\mathcal{D}_g\mathcal{A} = \mathcal{D}_g = \mathcal{A}\mathcal{D}_g, \forall g \in G$. Suppose that α has an enveloping action $\alpha^e : G \times \mathcal{A}^e \rightarrow \mathcal{A}^e$. Then:*

- (1) *The enveloping algebra \mathcal{A}^e is idempotent.*
- (2) *The skew group rings $\mathcal{A} \rtimes_{\alpha} G$ and $\mathcal{A}^e \rtimes_{\alpha^e} G$ are Morita equivalent.*

Proof. Evidently, $\mathcal{A}^2 = \mathcal{A}$. To prove that \mathcal{A}^e is idempotent note that, since $\mathcal{A}^e = \text{span}\{\alpha_g^e(\mathcal{A}) : g \in G\}$,

$$(\mathcal{A}^e)^2 = \text{span}\{\alpha_g^e(\mathcal{A})\alpha_h^e(\mathcal{A}) : g, h \in G\} \supseteq \text{span}\{\alpha_g^e(\mathcal{A})^2 : g \in G\} = \mathcal{A}^e.$$

Now let $\mathcal{C} := \mathcal{A}^e \rtimes_{\alpha^e} G = \bigoplus_{g \in G} \mathcal{C}_g$, with $\mathcal{C}_g = \mathcal{A}^e \delta_g, \forall g \in G$. We have

$$\mathcal{C}_g \mathcal{C}_h = \mathcal{A}^e \delta_g \mathcal{A}^e \delta_h = \mathcal{A}^e \alpha_g^e(\mathcal{A}^e) \delta_{gh} = (\mathcal{A}^e)^2 \delta_{gh} = \mathcal{A}^e \delta_{gh} = \mathcal{C}_{gh}.$$

Define $\mathcal{M} := \bigoplus_{g \in G} \mathcal{M}_g \subseteq \mathcal{C}$ and $\mathcal{M}' := \bigoplus_{g \in G} \mathcal{M}'_g \subseteq \mathcal{C}$, with $\mathcal{M}_g = \mathcal{A} \delta_g$ and $\mathcal{M}'_g = \alpha_g^e(\mathcal{A}) \delta_g, \forall g \in G$. Since $\mathcal{A} \supseteq \mathcal{A}\mathcal{A}^e \supseteq \mathcal{A}^2 = \mathcal{A}$, so $\mathcal{A}\mathcal{A}^e = \mathcal{A}$, it follows that $\mathcal{M}_g \mathcal{C}_h = \mathcal{M}_{gh}$, and therefore $\mathcal{M}\mathcal{C} = \mathcal{M}$; thus \mathcal{M} is a unital right ideal of \mathcal{C} . On the other hand, $\mathcal{C}_g \mathcal{M}'_h = \mathcal{A}^e \alpha_{gh}^e(\mathcal{A}) \delta_{gh} \subseteq \alpha_{gh}^e(\mathcal{A}) \delta_{gh} = \mathcal{M}'_{gh}$, so \mathcal{M}' is a left ideal in \mathcal{C} . In fact, \mathcal{M}' is a unital left module, because $\mathcal{C}_1 \mathcal{M}'_g = \mathcal{M}'_g, \forall g \in G$, the latter being a consequence of the equality $\mathcal{A}^e \alpha_g^e(\mathcal{A}) = \alpha_g^e(\mathcal{A})$, which immediately follows from $\mathcal{A}^e \mathcal{A} = \mathcal{A}$. Moreover, $\mathcal{M}'\mathcal{M} = \mathcal{C}$. Indeed, $\mathcal{M}'_g \mathcal{M}_h = \alpha_g^e(\mathcal{A}^2) \delta_{gh} = \alpha_g^e(\mathcal{A}) \delta_{gh}$, whence we have that $\mathcal{M}'\mathcal{M} \supseteq \text{span}\{\alpha_l^e(\mathcal{A}) \delta_k : k, l \in G\} = \mathcal{C}$.

Observe next that $\mathcal{A}\mathcal{D}_g = \mathcal{D}_g$ and (8) imply

$$\mathcal{A} \alpha_g^e(\mathcal{A}) = \mathcal{D}_g, \quad \forall g \in G.$$

Consider now $\mathcal{B} := \mathcal{A} \rtimes_{\alpha} G = \bigoplus_{g \in G} \mathcal{B}_g$, with $\mathcal{B}_g = \mathcal{D}_g \delta_g$. Then \mathcal{B} is a graded subalgebra of \mathcal{C} , and we have

$$\begin{aligned} \mathcal{B}_g \mathcal{M}_h &= \mathcal{D}_g \alpha_g^e(\mathcal{A}) \delta_{gh} \subseteq \mathcal{M}_{gh}, \\ \mathcal{B}_1 \mathcal{M}_h &= \mathcal{M}_h, \forall h \in G, \\ \mathcal{M}'_g \mathcal{B}_h &= \alpha_g^e(\mathcal{A} \mathcal{D}_h) \delta_{gh} \subseteq \mathcal{M}'_{gh}, \\ \mathcal{M}'_g \mathcal{B}_1 &= \mathcal{M}'_g, \forall g \in G, \\ \mathcal{M}_g \mathcal{M}'_h &= \mathcal{A} \alpha_g^e(\alpha_h^e(\mathcal{A})) \delta_{gh} = \mathcal{D}_{gh} \delta_{gh} = \mathcal{B}_{gh}. \end{aligned}$$

It follows from the above equalities that \mathcal{M} is a unital left \mathcal{B} -module and \mathcal{M}' is a unital right \mathcal{B} -module, such that $\mathcal{M} \mathcal{M}' = \mathcal{B}$. We conclude that if $\tau : \mathcal{M} \otimes \mathcal{M}' \rightarrow \mathcal{B}$ and $\tau' : \mathcal{M}' \otimes \mathcal{M} \rightarrow \mathcal{C}$ are given by the product of \mathcal{C} , then $(\mathcal{B}, \mathcal{C}, \mathcal{M}, \mathcal{M}', \tau, \tau')$ is a Morita context with surjective trace maps, so \mathcal{B} and \mathcal{C} are Morita equivalent algebras. □

Given a G -graded commutative group $\mathcal{X} = \bigoplus_{g \in G} \mathcal{X}_g$, let $\text{FMat}_G(\mathcal{X})$ be the abelian group of all $(G \times G)$ -matrices over \mathcal{X} with only a finite number of non-zero entries, and define $\mathcal{E}_{\mathcal{X}} := \{\tilde{x} \in \text{FMat}_G(\mathcal{X}) : \tilde{x}_{(g,h)} \in \mathcal{X}_{g^{-1}h}, \forall g, h \in G\}$, which is a subgroup of $\text{FMat}_G(\mathcal{X})$. There is a natural action $\beta^{\mathcal{X}} : G \times \mathcal{E}_{\mathcal{X}} \rightarrow \mathcal{E}_{\mathcal{X}}$ by automorphisms of $\mathcal{E}_{\mathcal{X}}$, given by $\beta_f^{\mathcal{X}}(\tilde{x})_{(g,h)} = \tilde{x}_{(f^{-1}g, f^{-1}h)}$, $\forall \tilde{x} \in \mathcal{E}_{\mathcal{X}}$ and $f, g, h \in G$. If $\mathcal{Y} = \bigoplus_{g \in G} \mathcal{Y}_g$ is another G -graded abelian group and $\mu : \mathcal{X} \rightarrow \mathcal{Y}$ is a homomorphism of graded groups, then $\varepsilon_{\mu} : \mathcal{E}_{\mathcal{X}} \rightarrow \mathcal{E}_{\mathcal{Y}}$ such that $\varepsilon_{\mu}(\tilde{x})_{(g,h)} = \mu(\tilde{x}_{(g,h)})$ is a homomorphism of groups, which is equivariant for the natural actions $\beta^{\mathcal{X}}$ and $\beta^{\mathcal{Y}}$:

$$\varepsilon_{\mu}(\beta_f^{\mathcal{X}}(\tilde{x})) = \beta_f^{\mathcal{Y}}(\varepsilon_{\mu}(\tilde{x})), \quad \forall \tilde{x} \in \mathcal{E}_{\mathcal{X}}, f \in G.$$

Let $\mathcal{B} = \bigoplus_{g \in G} \mathcal{B}_g$ be a graded algebra over the group G . We recall that a left \mathcal{B} -module \mathcal{M} is called G -graded if $\mathcal{M} = \bigoplus_{g \in G} \mathcal{M}_g$ as abelian groups, and $\mathcal{B}_g \mathcal{M}_h \subseteq \mathcal{M}_{gh}, \forall g, h \in G$. One defines similarly graded right modules and graded bimodules over graded algebras.

If $\mathcal{B} = \bigoplus_{g \in G} \mathcal{B}_g$ is a G -graded algebra, it is clear that $\text{FMat}_G(\mathcal{B})$ and $\mathcal{E}_{\mathcal{B}}$ are algebras, and the natural action $\beta^{\mathcal{B}}$ is an action by algebra automorphisms. If $\mu : \mathcal{A} \rightarrow \mathcal{B}$ is a homomorphism of graded algebras, then $\varepsilon_{\mu} : \mathcal{E}_{\mathcal{A}} \rightarrow \mathcal{E}_{\mathcal{B}}$ is an algebra homomorphism compatible with the natural actions. Moreover, if $\mathcal{M} = \bigoplus_{g \in G} \mathcal{M}_g$ is a G -graded left \mathcal{B} -module, then $\text{FMat}_G(\mathcal{M})$ is naturally a left $\text{FMat}_G(\mathcal{B})$ -module, and it is readily seen that $\mathcal{E}_{\mathcal{M}}$ is a left $\mathcal{E}_{\mathcal{B}}$ -module. In this case, it is clear that every $\beta_g^{\mathcal{M}}$ is a group automorphism of $\mathcal{E}_{\mathcal{M}}$ such that if $\tilde{b} \in \mathcal{E}_{\mathcal{B}}, \tilde{m} \in \mathcal{E}_{\mathcal{M}}$, and $g \in G$, then $\beta^{\mathcal{M}}(\tilde{b}\tilde{m}) = \beta^{\mathcal{B}}(\tilde{b})\beta^{\mathcal{M}}(\tilde{m})$. Of course, if $\mu : \mathcal{M} \rightarrow \mathcal{N}$ is a homomorphism of G -graded modules, then $\varepsilon_{\mu} : \mathcal{E}_{\mathcal{M}} \rightarrow \mathcal{E}_{\mathcal{N}}$ is a homomorphism of modules compatible with the natural actions. Similar comments apply to graded right modules and graded bimodules over graded algebras.

Theorem 5.4. *Let $M = (\mathcal{B}, \mathcal{B}', \mathcal{M}, \mathcal{M}', \tau, \tau')$ be a Morita context between G -graded algebras \mathcal{B} and \mathcal{B}' , where \mathcal{M} is a graded $(\mathcal{B}, \mathcal{B}')$ -bimodule and \mathcal{M}' is a graded $(\mathcal{B}', \mathcal{B})$ -bimodule. Suppose that the traces τ and τ' satisfy $\tau(\mathcal{M}_g \otimes \mathcal{M}'_h) \subseteq \mathcal{B}_{gh}$ and $\tau'(\mathcal{M}'_g \otimes \mathcal{M}_h) \subseteq \mathcal{B}'_{gh}, \forall g, h \in G$. Let $\mathcal{C} = \begin{pmatrix} \mathcal{B} & \mathcal{M} \\ \mathcal{M}' & \mathcal{B}' \end{pmatrix}$ be the corresponding context*

algebra. Then:

- (1) \mathcal{C} is a G -graded algebra: $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$, where $\mathcal{C}_g = \begin{pmatrix} \mathcal{B}_g & \mathcal{M}_g \\ \mathcal{M}'_g & \mathcal{B}'_g \end{pmatrix}$.
- (2) $\mathcal{E}_{\mathcal{C}}$ can be identified with $\begin{pmatrix} \mathcal{E}_{\mathcal{B}} & \mathcal{E}_{\mathcal{M}} \\ \mathcal{E}_{\mathcal{M}'} & \mathcal{E}_{\mathcal{B}'} \end{pmatrix}$.
- (3) The restrictions of $\beta^{\mathcal{C}}$ to $\mathcal{E}_{\mathcal{B}}$ and $\mathcal{E}_{\mathcal{B}'}$ are $\beta^{\mathcal{B}}$ and $\beta^{\mathcal{B}'}$ respectively.
- (4) $\mathcal{E}_{\mathcal{M}} := (\mathcal{E}_{\mathcal{B}}, \mathcal{E}_{\mathcal{B}'}, \mathcal{E}_{\mathcal{M}}, \mathcal{E}_{\mathcal{M}'}, \tau_{\varepsilon}, \tau'_{\varepsilon})$ is a Morita context, where τ_{ε} and τ'_{ε} are the bilinear maps defined by the product of $\mathcal{E}_{\mathcal{C}}$, and if \mathcal{M} is a Morita equivalence, so is $\mathcal{E}_{\mathcal{M}}$.

Proof. It is clear that if \mathcal{C}_g is as above, then $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ as abelian groups. Moreover, by the assumptions on τ and τ' :

$$\begin{aligned} \begin{pmatrix} \mathcal{B}_g & \mathcal{M}_g \\ \mathcal{M}'_g & \mathcal{B}'_g \end{pmatrix} \begin{pmatrix} \mathcal{B}_h & \mathcal{M}_h \\ \mathcal{M}'_h & \mathcal{B}'_h \end{pmatrix} &= \begin{pmatrix} \mathcal{B}_g \mathcal{B}_h + \mathcal{M}_g \cdot \mathcal{M}'_h & \mathcal{B}_g \mathcal{M}_h + \mathcal{M}_g \mathcal{B}'_h \\ \mathcal{M}'_g \mathcal{B}_h + \mathcal{B}'_g \mathcal{M}'_h & \mathcal{M}'_g \cdot \mathcal{M}_h + \mathcal{B}'_g \mathcal{B}'_h \end{pmatrix} \\ &\subseteq \begin{pmatrix} \mathcal{B}_{gh} & \mathcal{M}_{gh} \\ \mathcal{M}'_{gh} & \mathcal{B}'_{gh} \end{pmatrix}, \end{aligned}$$

so $\mathcal{C}_g \mathcal{C}_h \subseteq \mathcal{C}_{gh}, \forall g, h \in G$, proving (1).

Observe next that it follows from the definition of the functor $\mathcal{X} \mapsto \mathcal{E}_{\mathcal{X}}$ that there is a permutation which applied to the rows and the columns produces an isomorphism

$$\mathcal{E}_{\mathcal{C}} \cong \begin{pmatrix} \mathcal{E}_{\mathcal{B}} & \mathcal{E}_{\mathcal{M}} \\ \mathcal{E}_{\mathcal{M}'} & \mathcal{E}_{\mathcal{B}'} \end{pmatrix},$$

and this isomorphism transforms $\beta^{\mathcal{C}}$ into $\begin{pmatrix} \beta^{\mathcal{B}} & \beta^{\mathcal{M}} \\ \beta^{\mathcal{M}'} & \beta^{\mathcal{B}'} \end{pmatrix}$. This proves (2) and (3).

As for the last assertion, the fact that $\mathcal{E}_{\mathcal{C}}$ is an algebra immediately implies that $\mathcal{E}_{\mathcal{M}}$ is a Morita context. Suppose now that \mathcal{M} is a Morita equivalence. We show next that τ_{ε} is surjective. For $b \in \mathcal{B}$, and $g, h \in G$, let $e_{(g,h)}(b)$ be the element of $\text{FMat}_G(\mathcal{B})$ whose entries are all equal to zero, except perhaps the entry corresponding to row g and column h , which is equal to b . Since $\mathcal{E}_{\mathcal{B}} = \text{span}\{e_{(g,h)}(b) : g, h \in G, b \in \mathcal{B}_{g^{-1}h}\}$, to prove that τ_{ε} is surjective it is enough to show that its image contains the elements $e_{(g,h)}(b), \forall g, h \in G, b \in \mathcal{B}_{g^{-1}h}$. Now, given $b \in \mathcal{B}_{g^{-1}h}$, there exist $m_1, \dots, m_l \in \mathcal{M}, m_j \in \mathcal{M}_{g_j}$, and $m'_1, \dots, m'_l \in \mathcal{M}', m'_j \in \mathcal{M}_{h_j}$, such that $\tau(\sum_{j=1}^l m_j \otimes m'_j) = b$. Since $b \in \mathcal{B}_{g^{-1}h}$ and $\tau(m_j \otimes m'_j) \in \mathcal{B}_{g_j} \mathcal{B}_{h_j} \subseteq \mathcal{B}_{g_j h_j}$, we may suppose that every $\tau(m_j \otimes m'_j) \in \mathcal{B}_{g^{-1}h}$, because the sum of such products that do not belong to $\mathcal{B}_{g^{-1}h}$ must be equal to zero. Thus there exist $k_1, \dots, k_l \in G$ such that $m_j \in \mathcal{M}_{g^{-1}k_j}$ and $m'_j \in \mathcal{M}'_{k_j^{-1}h}, \forall j = 1, \dots, l$. Consider the elements $\tilde{m}_j := e_{(g,k_j)}(m_j)$ and $\tilde{m}'_j := e_{(k_j,h)}(m'_j)$. Then $\tilde{m}_j \in \mathcal{E}_{\mathcal{M}}$ and $\tilde{m}'_j \in \mathcal{E}_{\mathcal{M}'}$, and

$$\tau_{\varepsilon} \left(\sum_{j=1}^l \tilde{m}_j \otimes \tilde{m}'_j \right) = \sum_{j=1}^l e_{(g,k_j)}(m_j) e_{(k_j,h)}(m'_j) = \sum_{j=1}^l e_{(g,h)}(\tau(m_j \otimes m'_j)) = e_{(g,h)}(b).$$

The proof that τ'_{ε} is surjective if so too is τ' is analogous. □

It is clear that if α is Morita equivalent to a globalizable regular partial action α' , then α has a Morita enveloping action. For if β' is a globalization of α' , then the restriction of β' to the algebra $\text{span}\{\beta'_g(a) : g \in G, a \in \mathcal{A}'\}$ is an enveloping

action of α' . This is the case of any regular partial action α , a fact that immediately follows from Theorem 4.1. More precisely:

Proposition 5.5. *Let $\alpha = \{\alpha_g : \mathcal{D}_{g^{-1}} \rightarrow \mathcal{D}_g\}_{g \in G}$ be a regular partial action of G on the algebra \mathcal{A} . Let $\mathcal{B} = \mathcal{A} \rtimes_{\alpha} G = \bigoplus_{g \in G} \mathcal{B}_g$, where $\mathcal{B}_g = \mathcal{D}_g \delta_g, \forall g \in G$. Let $\mathcal{I}_{\alpha} := \{\tilde{b} \in \mathcal{E}_{\mathcal{B}} : \tilde{b}_{(g,h)} \in \mathcal{B}_{g^{-1}} \mathcal{B}_h, \forall g, h \in G\}$, and denote by $\tilde{\alpha} := \beta^{\mathcal{B}}$ the natural action of G on $\mathcal{E}_{\mathcal{B}}$. Then $\text{span}\{\tilde{\alpha}_f(\tilde{b}) : f \in G, \tilde{b} \in \mathcal{I}_{\alpha}\} = \mathcal{E}_{\mathcal{B}}$. Thus $\tilde{\alpha}$ is a Morita enveloping action of α .*

Proof. It is enough to show that $e_{(g,h)}(b) \in \text{span}\{\tilde{\alpha}_f(\tilde{b}) : f \in G, \tilde{b} \in \mathcal{I}_{\alpha}\} = \mathcal{E}_{\mathcal{B}}, \forall g, h \in G$ and $b \in \mathcal{B}_{g^{-1}h}$. So let $b = d\delta_{g^{-1}h}, d \in \mathcal{D}_{g^{-1}h}$. Since $\mathcal{D}_{g^{-1}h}$ is idempotent, there exist $d_j, d'_j \in \mathcal{D}_{g^{-1}h} \subseteq \mathcal{D}_1, j = 1, \dots, n$, such that $d = \sum_{j=1}^n d_j d'_j$. If we let $b_j := d_j \delta_1 \in \mathcal{B}_1$ and $b'_j := d'_j \delta_{g^{-1}h} \in \mathcal{B}_{g^{-1}h}$, we have $\sum_{j=1}^n b_j b'_j = \sum_{j=1}^n d_j d'_j \delta_{g^{-1}h} = b$. For each j we have $e_{(1,g^{-1}h)}(b_j b'_j) \in \mathcal{I}_{\alpha}$, and $\sum_{j=1}^n \tilde{\alpha}_g(e_{(1,g^{-1}h)}(b_j b'_j)) = \sum_{j=1}^n e_{(g,gg^{-1}h)}(b_j b'_j) = e_{(g,h)}(\sum_{j=1}^n b_j b'_j) = e_{(g,h)}(b)$. \square

The action $\tilde{\alpha}$ will be called the *canonical Morita enveloping action* of α . In what follows we will use the notation of Proposition 5.5.

Theorem 5.6. *Let $\alpha = \{\alpha_g : \mathcal{D}_{g^{-1}} \rightarrow \mathcal{D}_g\}_{g \in G}$ and $\alpha' = \{\alpha'_g : \mathcal{D}'_{g^{-1}} \rightarrow \mathcal{D}'_g\}_{g \in G}$ be Morita equivalent regular partial actions of G on algebras \mathcal{A} and \mathcal{A}' , respectively. If $\tilde{\alpha}$ and $\tilde{\alpha}'$ are, respectively, the canonical Morita enveloping actions of α and α' (see Proposition 5.5), then $\tilde{\alpha}$ and $\tilde{\alpha}'$ are Morita equivalent (global) actions.*

Proof. Suppose that $(\mathcal{A}, \mathcal{A}', M, M', \tau, \tau')$ is a Morita context giving a Morita equivalence of α with α' . As shown at the end of Section 3, and using the same notation introduced there, we have that $(\mathcal{A} \rtimes_{\alpha} G, \mathcal{A}' \rtimes_{\alpha'} G, M \rtimes G, M' \rtimes G, \tau \rtimes G, \tau' \rtimes G)$ is a Morita context providing a Morita equivalence of $\mathcal{A} \rtimes_{\alpha} G$ with $\mathcal{A}' \rtimes_{\alpha'} G$. So, by (3) of Theorem 5.4 it will be enough to check that this Morita context satisfies the hypothesis of Theorem 5.4. Let us show first that $M \rtimes G$ is G -graded as a left $\mathcal{A} \rtimes_{\alpha} G$ -module:

$$\begin{aligned} (\mathcal{D}_g \delta_g)(M_h \delta_h) &= \gamma_g(\mathcal{D}_{g^{-1}} M_h) \delta_{gh} \\ &= \gamma_g(\mathcal{D}_{g^{-1}}^2 \mathcal{D}_h M_h) \delta_{gh} = \alpha_g((\mathcal{D}_{g^{-1}} \mathcal{D}_h) \gamma_g(\mathcal{D}_{g^{-1}} M_h) \delta_{gh}) \\ &\subseteq D_g \mathcal{D}_{gh} \gamma_g(M_{g^{-1}}) \delta_{gh} \subseteq D_{gh} M_g \delta_{gh} \subseteq D_{gh} M \delta_{gh} = M_{gh} \delta_{gh}. \end{aligned}$$

The proofs that $M \rtimes G$ is G -graded as a right $\mathcal{A}' \rtimes_{\alpha'} G$ -module and the corresponding assertions for $M' \rtimes G$ are similar. We show next that $(\tau \rtimes G)(M_g \delta_g \otimes M'_h \delta_h) \subseteq \mathcal{D}_{gh} \delta_{gh}, \forall g, h \in G$. By the definition of $\tau \rtimes G$ we have

$$\begin{aligned} (\tau \rtimes G)(M_g \delta_g \otimes M'_h \delta_h) &= \alpha_g(\gamma_{g^{-1}}(M_g) \cdot M'_h) \delta_{gh} = \alpha_g(M_{g^{-1}} \cdot M'_h) \delta_{gh} \\ &= \alpha_g(\mathcal{D}_{g^{-1}}(M \cdot M') \mathcal{D}_h) \delta_{gh} = \alpha_g(\mathcal{D}_{g^{-1}} \mathcal{D}_h) \delta_{gh} = \mathcal{D}_g \mathcal{D}_{gh} \delta_{gh} \subseteq \mathcal{D}_{gh} \delta_{gh}. \end{aligned}$$

Analogous computations show that $\tau' \rtimes G(M'_g \delta_g \otimes M_h \delta_h) \subseteq \mathcal{D}'_{gh} \delta_{gh}, \forall g, h \in G$, so we are done. \square

Proposition 5.7. *Let $\alpha = \{\alpha_g : \mathcal{D}_{g^{-1}} \rightarrow \mathcal{D}_g\}_{g \in G}$ be a regular partial action of G on \mathcal{A} . If $\alpha^e : G \times \mathcal{A}^e \rightarrow \mathcal{A}^e$ is an enveloping action of α , then α^e and the canonical Morita enveloping action $\tilde{\alpha}$ of α are Morita equivalent.*

Proof. Let $\widetilde{\alpha}^e$ be the canonical Morita enveloping action of the action α^e . By the proof of Theorem 4.1, we have that α^e is Morita equivalent to $\widetilde{\alpha}^e|_{\mathcal{I}_{\alpha^e}}$. But note that $\mathcal{I}_{\alpha^e} = \mathcal{E}_{\mathcal{C}}$, where $\mathcal{C} = \mathcal{A}^e \rtimes_{\alpha^e} G = \bigoplus_{g \in G} \mathcal{C}_g$, because $\mathcal{C}_g \mathcal{C}_h = \mathcal{C}_{gh}$, as shown in the proof of Theorem 5.3. Thus α^e is Morita equivalent to $\widetilde{\alpha}^e|_{\mathcal{I}_{\alpha^e}} = \widetilde{\alpha}^e$. Therefore, since the Morita equivalence of actions is an equivalence relation, it suffices to show that $\tilde{\alpha}$ and $\widetilde{\alpha}^e$ are Morita equivalent. To this end, let $\mathcal{B} := \mathcal{A} \rtimes_{\alpha} G = \bigoplus_{g \in G} \mathcal{B}_g$, with $\mathcal{B}_g = \mathcal{D}_g \delta_g, \forall g \in G$, and consider the Morita context $(\mathcal{B}, \mathcal{C}, \mathcal{M}, \mathcal{M}', \tau, \tau')$ defined in the proof of Theorem 5.3, which gives a Morita equivalence between \mathcal{B} and \mathcal{C} . By (3) and (4) of Theorem 5.4, we have that the natural actions $\beta^{\mathcal{B}}$ and $\beta^{\mathcal{C}}$ are Morita equivalent, that is, $\tilde{\alpha}$ and $\widetilde{\alpha}^e$ are Morita equivalent, as we wanted to prove. \square

Theorem 5.8. *Let $\alpha = \{\alpha_g : \mathcal{D}_{g^{-1}} \rightarrow \mathcal{D}_g\}_{g \in G}$ be a regular partial action of G on \mathcal{A} . Then α has a Morita enveloping action, which is unique up to Morita equivalence. Moreover, for every Morita enveloping action $\beta : G \times \mathcal{B} \rightarrow \mathcal{B}$ of α , the skew group rings $\mathcal{A} \rtimes_{\alpha} G$ and $\mathcal{B} \rtimes_{\beta} G$ are Morita equivalent algebras.*

Proof. We have already seen in Proposition 5.5 that $\tilde{\alpha}$ is a Morita enveloping action of α . Suppose now that α is Morita equivalent to the partial action α' on an algebra \mathcal{A}' and that $\beta : G \times \mathcal{B} \rightarrow \mathcal{B}$ is an enveloping action of α' . Then $\tilde{\alpha}$ and $\tilde{\alpha}'$ are Morita equivalent by Theorem 5.6. On the other hand, by Proposition 5.7 we have that β and $\tilde{\alpha}'$ are Morita equivalent actions. Since the Morita equivalence of actions is an equivalence relation, we conclude that $\tilde{\alpha}$ and β are Morita equivalent global actions. Now, by Theorem 3.1 we have that $\mathcal{A} \rtimes_{\alpha} G$ and $\mathcal{A}' \rtimes_{\alpha'} G$ are Morita equivalent. On the other hand, since α' satisfies condition (2), we have by Theorem 5.3 that $\mathcal{A}' \rtimes_{\alpha'} G$ and $\mathcal{B} \rtimes_{\beta} G$ are Morita equivalent. But Morita equivalence of algebras is an equivalence relation, and therefore $\mathcal{A} \rtimes_{\alpha} G$ and $\mathcal{B} \rtimes_{\beta} G$ are Morita equivalent. \square

6. STABILIZING MORITA EQUIVALENT PARTIAL ACTIONS AND THEIR SKEW GROUP RINGS

Let

$$\alpha = \{\alpha_g : \mathcal{D}_{g^{-1}} \rightarrow \mathcal{D}_g, g \in G\} \quad \text{and} \quad \alpha' = \{\alpha'_g : \mathcal{D}'_{g^{-1}} \rightarrow \mathcal{D}'_g, g \in G\}$$

be regular partial actions of G on algebras \mathcal{A} and \mathcal{A}' respectively, and let furthermore $(\mathcal{A}, \mathcal{A}', M, M', \tau, \tau')$ be a Morita context which establishes a Morita equivalence between α and α' . Then by Theorem 3.1, $\mathcal{R} = \mathcal{A} \rtimes_{\alpha} G$ and $\mathcal{R}' = \mathcal{A}' \rtimes_{\alpha'} G$ are Morita equivalent via the Morita context $(\mathcal{R}, \mathcal{R}', \mathcal{M}, \mathcal{M}')$, where $\mathcal{M} = M \rtimes G$ and $\mathcal{M}' = M' \rtimes G$ (see the comments after the proof of Theorem 3.1). Assume in addition that \mathcal{A} and \mathcal{A}' are k -algebras with orthogonal local units³ in the sense of [21], which means that there exist sets of (non-necessarily central) pairwise orthogonal idempotents $E \subseteq \mathcal{A}$ and $F \subseteq \mathcal{A}'$ such that

$$(18) \quad \mathcal{A} = \bigoplus_{e \in E} \mathcal{A}e = \bigoplus_{e \in E} e\mathcal{A}, \quad \mathcal{A}' = \bigoplus_{f \in F} \mathcal{A}'f = \bigoplus_{f \in F} f\mathcal{A}'.$$

Then \mathcal{R} and \mathcal{R}' are also algebras with orthogonal local units $E\delta_1 \subseteq \mathcal{R}, F\delta_1 \subseteq \mathcal{R}'$. Indeed, taking an arbitrary $g \in G$ and $a \in \mathcal{D}_g$ there exist $e_1, \dots, e_k \in E$ such that $\alpha_{g^{-1}}(a) \sum_{i=1}^k e_i = \alpha_{g^{-1}}(a)$. Then $a\delta_g = \sum_{i=1}^k a\delta_g e_i \delta_1$, and consequently

³Note that rings with (18) are also called rings with enough idempotents (see [38]); however the latter term is used in [21] in a different sense.

$\mathcal{R} = \sum_{e \in E} \mathcal{R} e \delta_1$. Since in view of the orthogonality of the e 's, $\mathcal{R} e \cap \sum_{e' \in E, e' \neq e} \mathcal{R} e' = 0$, we readily obtain that

$$(19) \quad \mathcal{R} = \bigoplus_{e \in E} \mathcal{R} e \delta_1 = \bigoplus_{e \in E} e \delta_1 \mathcal{R}, \text{ and similarly, } \mathcal{R}' = \bigoplus_{f \in F} \mathcal{R}' f \delta_1 = \bigoplus_{f \in F} f \delta_1 \mathcal{R}'.$$

By [21, Corollary 8.4], for any infinite set of indexes X , whose cardinality is bigger than or equal to those of E and F , there exists an isomorphism of k -algebras

$$(20) \quad \text{FMat}_X(\mathcal{R}) \cong \text{FMat}_X(\mathcal{R}').$$

Our purpose is to show that the above isomorphism, as established in [21], becomes an isomorphism of graded algebras if we take the gradings on $\text{FMat}_X(\mathcal{R})$ and $\text{FMat}_X(\mathcal{R}')$ which are direct extensions of those of \mathcal{R} and \mathcal{R}' . More precisely:

Theorem 6.1. *With the above notation consider the G -grading on $\text{FMat}_X(\mathcal{R})$ given by taking the g -homogeneous component of $\text{FMat}_X(\mathcal{R})$ to be $\text{FMat}_X(\mathcal{R}_g)$, $g \in G$, and consider a similar grading on $\text{FMat}_X(\mathcal{R}')$. Then (20) is an isomorphism of graded algebras.*

Proof. Theorem 8.2 of [21] gives two maps

$$\Psi : \text{FMat}_X(\mathcal{R}) \rightarrow \text{FMat}_X(\mathcal{M}) \quad \text{and} \quad \Psi' : \text{FMat}_X(\mathcal{R}') \rightarrow \text{FMat}_X(\mathcal{M}),$$

such that Ψ is an isomorphism of left $\text{FMat}_X(\mathcal{R})$ -modules, Ψ' is an isomorphism of right $\text{FMat}_X(\mathcal{R}')$ -modules, and $(A\Psi)A' = A(\Psi'A')$ for all $A \in \text{FMat}_X(\mathcal{R})$ and $A' \in \text{FMat}_X(\mathcal{R}')$. Then using Propositions 4.5 and 6.2 of [21], the isomorphism (20) is obtained as the composition $(\Psi')^{-1} \circ \Psi$. Thus it is enough to check that both Ψ and Ψ' are graded maps if one defines a grading on $\text{FMat}_X(\mathcal{M})$ by declaring the g -homogeneous component of $\text{FMat}_X(\mathcal{M})$ to be $\text{FMat}_X(M_g \delta_g)$, $g \in G$. Then it is evident that $\text{FMat}_X(\mathcal{M})$ is a G -graded $(\text{FMat}_X(\mathcal{R}), \text{FMat}_X(\mathcal{R}'))$ -bimodule.

As in [21, p. 3302] for any $f \in F$ write $f = \sum_{i=1}^{n_f} x'_i \cdot x_i$, where $x_i = x_i^{(f)} \in M$, and $x'_i = x_i^{(f)} \in M'$, $x_i = x_i f$, and $f x'_i = x'_i$ for all i . Then obviously

$$f \delta_1 = \sum_{i=1}^{n_f} x'_i \delta_1 \cdot x_i \delta_1,$$

and the epimorphism

$$\pi_f : \mathcal{R}^{n_f} \ni (r_1, r_2, \dots, r_{n_f}) \mapsto \sum r_i (x_i \delta_1) \in \mathcal{M}(f \delta_1)$$

of left \mathcal{R} -modules is clearly graded, as well as its splitting map

$$\rho_f : \mathcal{M}(f \delta_1) \ni y(f \delta_1) \mapsto (y(f x'_1 \delta_1), y(f x'_2 \delta_1), \dots, y(f x'_{n_f} \delta_1)) \in \mathcal{R}^{n_f}$$

($\rho_f \circ \pi_f = id$). Write $K_f = \text{Ker } \pi_f$ and denote by μ_f the embedding of K_f in \mathcal{R}^{n_f} . Then all maps in the exact sequences

$$0 \longrightarrow K_f \xrightarrow{\mu_f} \mathcal{R}^{n_f} \xrightarrow{\pi_f} \mathcal{M}(f \delta_1) \longrightarrow 0$$

and

$$0 \longleftarrow K_f \xleftarrow{\tau_f} \mathcal{R}^{n_f} \xleftarrow{\rho_f} \mathcal{M}(f \delta_1) \longleftarrow 0 \quad (\mu_f \circ \tau_f = id)$$

preserve the G -gradings. Next write

$$(21) \quad \mathcal{M} = \bigoplus_{e \in E} (e \delta_1) \mathcal{M} = \bigoplus_{f \in F} \mathcal{M}(f \delta_1) \quad \text{and} \quad \mathcal{M}' = \bigoplus_{f \in F} (f \delta_1) \mathcal{M}' = \bigoplus_{e \in E} \mathcal{M}'(e \delta_1).$$

Completing E or F (or both) with zeros, one assumes that $|X| = |E| = |F|$. Denote by $\mathcal{R}^{(X)}$ the direct sum of copies of \mathcal{R} indexed by the elements of X and write $\mathcal{R}^{(X)} = \bigoplus_{f \in F} \mathcal{R}^{n_f}$. Here $\mathcal{R}^{n_f} = 0$ if $f = 0$. Then the map

$$\pi : \bigoplus_{f \in F} \mathcal{R}^{n_f} \ni (\bar{r}_f) \mapsto \sum_{f \in F} (\bar{r}_f) \pi_f \in \mathcal{M} \quad (\bar{r}_f \in \mathcal{R}^{n_f}),$$

which by (21) is an epimorphism of left \mathcal{R} -modules, preserves the gradings, as well as the splitting homomorphism

$$\rho : \mathcal{M} \ni x \mapsto (x \rho_f) \in \bigoplus_{f \in F} \mathcal{R}^{n_f},$$

$\rho \circ \pi = id$. Similarly, the maps μ_f, τ_f ($f \in F$) are used to construct the left \mathcal{R} -module homomorphisms $\mu : K \rightarrow \mathcal{R}^{(X)}$ and $\tau : \mathcal{R}^{(X)} \rightarrow K = \bigoplus_{f \in F} K_f$, resulting in the exact sequences

$$(22) \quad 0 \longrightarrow K \xrightarrow{\mu} \mathcal{R}^{(X)} \xrightarrow{\pi} \mathcal{M} \longrightarrow 0 \quad \text{and} \quad 0 \longleftarrow K \xleftarrow{\tau} \mathcal{R}^{(X)} \xleftarrow{\rho} \mathcal{M} \longleftarrow 0$$

($\mu \circ \tau = id$), in which all maps preserve gradings. Taking \mathcal{R}' instead of \mathcal{R} and \mathcal{M}' instead of \mathcal{M} in (22), one obtains the exact sequences of left \mathcal{R}' -modules and graded homomorphisms

$$(23) \quad 0 \longrightarrow K' \xrightarrow{\mu'} \mathcal{R}'^{(X)} \xrightarrow{\pi'} \mathcal{M}' \longrightarrow 0 \quad \text{and} \quad 0 \longleftarrow K' \xleftarrow{\tau'} \mathcal{R}'^{(X)} \xleftarrow{\rho'} \mathcal{M}' \longleftarrow 0,$$

with $\rho' \circ \pi' = id, \mu' \circ \tau' = id$. Applying the equivalence functor $\mathcal{M} \otimes_{\mathcal{R}'} -$ to (23) one comes to the exact sequences of left \mathcal{R} -modules and homomorphisms:

$$(24) \quad 0 \longrightarrow \tilde{K} \xrightarrow{\tilde{\mu}} \mathcal{M}^{(X)} \xrightarrow{\tilde{\pi}} \mathcal{R} \longrightarrow 0 \quad \text{and} \quad 0 \longleftarrow \tilde{K} \xleftarrow{\tilde{\tau}} \mathcal{M}^{(X)} \xleftarrow{\tilde{\rho}} \mathcal{R} \longleftarrow 0,$$

with $\tilde{\rho} \circ \tilde{\pi} = id, \tilde{\mu} \circ \tilde{\tau} = id$, where \tilde{K} is the submodule of $\mathcal{M}^{(X)}$ obtained from $\mathcal{M} \otimes_{\mathcal{R}'} K'$ by using $\mathcal{M} \otimes_{\mathcal{R}'} \mathcal{R}' \cong_{\mathcal{R}} \mathcal{M}$, and $\tilde{\mu}$ is the embedding of \tilde{K} . Notice that the maps

$$\tilde{\mu} = 1_{\mathcal{M}} \otimes \mu', \quad \tilde{\pi} = 1_{\mathcal{M}} \otimes \pi', \quad \tilde{\tau} = 1_{\mathcal{M}} \otimes \tau', \quad \tilde{\rho} = 1_{\mathcal{M}} \otimes \rho'$$

are all graded, as so too are μ', π', τ' and ρ' .

From (22) and (24) we obtain the isomorphisms of G -graded left \mathcal{R} -modules:

$$\pi \oplus \tau : \mathcal{R}^{(X)} \longrightarrow \mathcal{M} \oplus K, \quad \tilde{\pi} \oplus \tilde{\tau} : \mathcal{M}^{(X)} \longrightarrow \mathcal{R} \oplus \tilde{K},$$

which are used in the proof of Proposition 7.6 of [21] to establish an isomorphism of left \mathcal{R} -modules

$$\psi : \mathcal{R}^{(X)} \rightarrow \mathcal{M}^{(X)}$$

by means of the Eilenberg’s trick. The main point of Proposition 7.6 of [21] is that both ψ and its inverse are finitely determined maps in the sense of Definition 7.3 of [21] with respect to the admissible components determined by the decompositions (19) and (21). Notice that since $\pi \oplus \tau$ and $\tilde{\pi} \oplus \tilde{\tau}$ are graded and Eilenberg’s trick involves only rearrangements of indexes, it readily follows that ψ is a graded isomorphism.

In the proof of Theorem 8.2 of [21] the isomorphism ψ is interpreted as a matrix

$$[\psi] \in \text{Mat}_X(\text{Mat}_{(e,f) \in E \times F}(e\delta_1 \mathcal{M} f\delta_1)).$$

In other words, $[\psi]$ has a block structure, with the block-rows and block-columns being indexed by the elements of X , and where each block is a matrix from $\text{Mat}_{(e,f) \in E \times F}(e\delta_1 \mathcal{M} f\delta_1)$. Moreover, for fixed $i \in X$ and $e \in E$ the (“thin”) (i, e) -row is finite, as the sum of its elements is the image under ψ of $e\delta_1$ from the i -copy

of \mathcal{R} in $\mathcal{R}^{(X)}$. It is a crucial point now to observe that since ψ is G -graded the image of $e\delta_1 \in \mathcal{A}\delta_1$ under ψ must belong to the homogeneous 1-component of the graded module $\mathcal{M}^{(X)}$. Because the 1-component of \mathcal{M} is $M\delta_1$, we see now that the entries of the (i, e) -row of $[\psi]$ belong to $(eMf)\delta_1$, $(f \in F)$. Thus

$$[\psi] \in \text{Mat}_X(\text{Mat}_{(e,f) \in E \times F}(eMf\delta_1)).$$

Moreover, for any $i \in X$ and $f \in F$ the (“thin”) (i, f) -column of $[\psi]$ is also finite, as ψ is finitely determined. In particular, each block of $[\psi]$ has finite rows and finite columns; i.e. the blocks are elements of $\text{RCFMat}_{(e,f) \in E \times F}(eMf\delta_1)$, the set row and column-finite matrices from $\text{Mat}_{(e,f) \in E \times F}(eMf\delta_1)$. Furthermore, the sum of each block-row as well as that of each block-column of $[\psi]$ makes sense, so that $[\psi]$ is a so-called row and column summable $X \times X$ -matrix over $\text{RCFMat}_{(e,f) \in E \times F}(eMf\delta_1)$:

$$[\psi] \in \text{RCSumMat}_X(\text{RCFMat}_{(e,f) \in E \times F}(eMf\delta_1)).$$

In view of the decompositions (19) one readily obtains the k -algebra isomorphisms

$$\varepsilon : \mathcal{R} \cong \text{FMat}_{e,e' \in E}(e\delta_1\mathcal{R}e'\delta_1) \quad \text{and} \quad \varepsilon' : \mathcal{R}' \cong \text{FMat}_{f,f' \in E}(f\delta_1\mathcal{R}'f'\delta_1),$$

which are evidently G -graded. In a similar fashion the decomposition for \mathcal{M} in (21) gives rise to a G -graded k -linear isomorphism

$$\zeta : \mathcal{M} \cong \text{FMat}_{e \in E, f \in F}(e\delta_1\mathcal{M}f\delta_1).$$

Using matrix multiplication one obtains on $\text{FMat}_{e \in E, f \in F}(e\delta_1\mathcal{M}f\delta_1)$ a structure of an $(\text{FMat}_{e,e' \in E}(e\delta_1\mathcal{R}e'\delta_1), \text{FMat}_{f,f' \in E}(f\delta_1\mathcal{R}'f'\delta_1))$ -bimodule, which is compatible with the $(\mathcal{R}, \mathcal{R}')$ -bimodule structure of \mathcal{M} via $\varepsilon, \varepsilon', \zeta$ (see [21, p. 3306]). Extending $\varepsilon, \zeta, \varepsilon', \zeta'$ to finite matrices FMat_X obviously preserves the compatibility of the bimodule structures. Next taking into account that

$$\text{FMat}_X(\text{FMat}_{e,e' \in E}(e\delta_1\mathcal{R}e'\delta_1)) \cdot [\psi] \subseteq \text{FMat}_X(\text{FMat}_{E \times F}(e\delta_1\mathcal{M}f\delta_1)),$$

one defines the k -isomorphism of left $\text{FMat}_X(\mathcal{R})$ -modules

$$\Psi : \text{FMat}_X(\mathcal{R}) \ni A \mapsto A \cdot [\psi] \in \text{FMat}_X(\mathcal{M})$$

by setting $A \cdot [\psi] = \zeta^{-1}(\varepsilon(A)[\psi])$. Similarly one obtains the isomorphism

$$\Psi' : \text{FMat}_X(\mathcal{R}') \ni A' \mapsto [\psi] \cdot A' \in \text{FMat}_X(\mathcal{M})$$

of right $\text{FMat}_X(\mathcal{R}')$ -modules. Finally, since the entries of $[\psi]$ all belong to $M\delta_1$, and \mathcal{M} is a graded bimodule, it is clear now that Ψ and Ψ' are G -graded maps. \square

Keeping the notation of the above result let

$$\theta = \{\theta_g : \text{FMat}_X(\mathcal{D}_{g^{-1}}) \rightarrow \text{FMat}_X(\mathcal{D}_g) : g \in G\}$$

be the partial action of G on $\text{FMat}_X(\mathcal{A})$ such that the (x, y) -entry of $\theta_g(A)$ is $\alpha_g(a_{x,y})$, where $a_{x,y}$ is the (x, y) -entry of the matrix $A \in \text{FMat}_X(\mathcal{D}_{g^{-1}})$. Similarly define the partial action θ' of G on $\text{FMat}_X(\mathcal{A}')$ which extends α' . Then it is readily seen that

$$\text{FMat}_X(\mathcal{R}) \cong \text{FMat}_X(\mathcal{A}) \rtimes_{\theta} G \quad \text{and} \quad \text{FMat}_X(\mathcal{R}') \cong \text{FMat}_X(\mathcal{A}') \rtimes_{\theta'} G$$

as graded algebras. Then from Theorem 6.1 we directly obtain the following.

Corollary 6.2. *With the above notation, there is an isomorphism of graded algebras:*

$$\text{FMat}_X(\mathcal{A}) \rtimes_{\theta} G \cong \text{FMat}_X(\mathcal{A}') \rtimes_{\theta'} G.$$

The next example shows that partial actions which are not Morita equivalent may have isomorphic skew group rings.

Example 6.3. Let \mathcal{A} be a (non-necessarily unital) idempotent ring satisfying $\mathcal{A} \cong \mathcal{A} \times \mathcal{A}$ and $\mathcal{A} \cong \mathbb{M}_2(\mathcal{A})$, as rings. For a more concrete example, take $\mathcal{A} = \text{FMat}(\mathcal{B})$, where $\mathcal{B} = \prod_{\mathbb{N}} K$, a countable direct product of an idempotent ring K . Let G be the cyclic group of order 2 with generator g . Consider the partial action of G on \mathcal{A} with $\mathcal{D}_g = 0$. Let $\varsigma : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ be a ring isomorphism and write $\mathcal{A}_1 = \varsigma(\mathcal{A} \times 0)$ and $\mathcal{A}_2 = \varsigma(0 \times \mathcal{A})$. Then we have the decomposition of \mathcal{A} into an internal direct product $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$, and we may define an action β of G on \mathcal{A} by setting $\beta_g(a_1 + a_2) = (a_2 + a_1)$, $a_i \in \mathcal{A}_i$. Notice that $\mathcal{A} \rtimes_{\beta} G \cong \mathbb{M}_2(\mathcal{A})$ via

$$(a_1 + a_2)\delta_1 + (b_1 + b_2)\delta_g \mapsto \begin{pmatrix} a_1 & b_1 \\ b_2 & a_2 \end{pmatrix},$$

where $a_i, b_i \in \mathcal{A}_i$. Clearly, α and β cannot be Morita equivalent partial actions of G over \mathcal{A} , because α is a partial action while β is a global action. However, $\mathcal{A} \rtimes_{\alpha} G \cong \mathcal{A} \cong \mathbb{M}_2(\mathcal{A}) \cong \mathcal{A} \rtimes_{\beta} G$. If one wishes an example with unital \mathcal{A} , take K with 1 and $\mathcal{A} = \text{RFMat}(\mathcal{B})$, the endomorphism ring of $\mathcal{B}^{\mathbb{N}}$. In this case, if K is a field, then \mathcal{A} is von Neumann regular and self-injective, so that there are rings with nice properties which fit this example.

The situation changes if the isomorphism between the skew group rings is graded, as shown in Proposition 6.5 below.

Definition 6.4. Two partial actions

$$\alpha = \{\alpha_g : \mathcal{D}_{g^{-1}} \rightarrow \mathcal{D}_g : g \in G\} \quad \text{and} \quad \alpha' = \{\alpha'_g : \mathcal{D}'_{g^{-1}} \rightarrow \mathcal{D}'_g : g \in G\}$$

of a group G on algebras \mathcal{A} and \mathcal{A}' , respectively, are called isomorphic if there exists an isomorphism of algebras $\varphi : \mathcal{A} \rightarrow \mathcal{A}'$ such that for each $g \in G$:

- (i) $\varphi(\mathcal{D}_g) = \mathcal{D}'_g$ and
- (ii) $\varphi(\alpha_g(a)) = \alpha'_g(\varphi(a))$, $\forall a \in \mathcal{D}_{g^{-1}}$.

Proposition 6.5. *Let*

$$\alpha = \{\alpha_g : \mathcal{D}_{g^{-1}} \rightarrow \mathcal{D}_g : g \in G\} \quad \text{and} \quad \alpha' = \{\alpha'_g : \mathcal{D}'_{g^{-1}} \rightarrow \mathcal{D}'_g : g \in G\}$$

be s -unital partial actions of G on algebras \mathcal{A} and \mathcal{A}' , respectively. If the skew group rings $\mathcal{A} \rtimes_{\alpha} G$ and $\mathcal{A}' \rtimes_{\alpha'} G$ are isomorphic, as graded rings, then α and α' are isomorphic partial actions.

Proof. Let $\sigma : \mathcal{A} \rtimes_{\alpha} G \rightarrow \mathcal{A}' \rtimes_{\alpha'} G$ be a graded algebra isomorphism. For each $g \in G$, we define the map $\varphi_g : \mathcal{D}_g \rightarrow \mathcal{D}'_g$ such that $\sigma(a\delta_g) = \varphi_g(a)\delta_g$. Write $\varphi = \varphi_1$.

We first show that $\varphi(\mathcal{D}_g) \subseteq \mathcal{D}'_g$ for each $g \in G$. To do this take any $a \in \mathcal{D}_g$. Let $s \in \mathcal{D}_g$ be a right unit for a and write $y = \alpha_g^{-1}(s)$. Then $a = a\alpha_g(y)$, and we compute, on one hand, $\sigma(a\delta_g \cdot y\delta_{g^{-1}}) = \sigma(\alpha_g[\alpha_{g^{-1}}(a)y]\delta_1) = \varphi[\alpha_g(y)]\delta_1 = \varphi(a)\delta_1$, and, on the other, $\sigma(a\delta_g \cdot y\delta_{g^{-1}}) = \sigma(a\delta_g) \cdot \sigma(y\delta_{g^{-1}}) = \alpha'_g(\alpha'_{g^{-1}}[\varphi_g(a)]\varphi_{g^{-1}}(y))\delta_1 = \varphi_g(a)\alpha'_g(\varphi_{g^{-1}}(y))\delta_1$. Thus $\varphi(a) = \varphi_g(a)\alpha'_g[\varphi_{g^{-1}}(y)] \in \mathcal{D}'_g$.

Next we will see that $\varphi_g(a) = \varphi(a)$ for any $a \in \mathcal{D}_g$. Consider a right unit $s \in \mathcal{D}_g$ for the set $\{a, \varphi_g^{-1}(\varphi(a))\} \subseteq \mathcal{D}_g$. Then $\varphi(a)\varphi_g(s) = \varphi(a)$ and $\varphi_g(a)\delta_g = \sigma(a\delta_g) = \sigma(as\delta_g) = \sigma(a\delta_1)\sigma(s\delta_g) = \varphi(a)\delta_1\varphi_g(s)\delta_g = \varphi(a)\varphi_g(s)\delta_g = \varphi(a)\delta_g$, as desired.

It remains to check that $\varphi(\alpha_g(a)) = \alpha'_g(\varphi(a))$ for any $a \in \mathcal{D}_{g^{-1}}$ and $g \in G$. Take any element $x \in \mathcal{D}_g$. Then the equality $\sigma(x\delta_g \cdot a\delta_{g^{-1}}) = \sigma(x\delta_g) \cdot \sigma(a\delta_{g^{-1}})$ yields $\varphi(x)\varphi(\alpha_g(a)) = \varphi(x)\alpha'_g(\varphi(a))$. Hence, $\varphi(x)[\varphi(\alpha_g(a)) - \alpha'_g(\varphi(a))] = 0$ for every $x \in \mathcal{D}_g$, which implies that $\varphi(\alpha_g(a)) = \alpha'_g(\varphi(a))$, thanks to the fact that \mathcal{D}_g is left s -unital. \square

Now we readily obtain the main result of this section:

Theorem 6.6. *Let*

$$\alpha = \{\alpha_g : \mathcal{D}_{g^{-1}} \rightarrow \mathcal{D}_g : g \in G\} \quad \text{and} \quad \alpha' = \{\alpha'_g : \mathcal{D}'_{g^{-1}} \rightarrow \mathcal{D}'_g : g \in G\}$$

be Morita equivalent s -unital partial actions of G on algebras \mathcal{A} and \mathcal{A}' , respectively. Suppose furthermore that \mathcal{A} and \mathcal{A}' have orthogonal local units. Then there is an infinite set X of indexes such that the partial actions θ and θ' are isomorphic, where θ is the extension of α to $\text{FMat}_X(\mathcal{A})$ and θ' is that of α' to $\text{FMat}_X(\mathcal{A}')$ as in Corollary 6.2.

7. MORITA EQUIVALENT PARTIAL ACTIONS ON COMMUTATIVE RINGS AND COMMUTATIVE C^* -ALGEBRAS

It is a basic fact of Morita theory that Morita equivalent commutative rings with 1 must be isomorphic. Moreover, this is also true for non-unital non-degenerate idempotent rings (see [39, Proposition 3.2]). The next result is an analogous fact in the context of partial actions.

Theorem 7.1. *Let*

$$\alpha = \{\alpha_g : \mathcal{D}_{g^{-1}} \rightarrow \mathcal{D}_g : g \in G\} \quad \text{and} \quad \alpha' = \{\alpha'_g : \mathcal{D}'_{g^{-1}} \rightarrow \mathcal{D}'_g : g \in G\}$$

be Morita equivalent s -unital partial actions of G on algebras \mathcal{A} and \mathcal{A}' , respectively. If \mathcal{A} and \mathcal{A}' are commutative rings, then α and α' are isomorphic partial actions.

Proof. By hypothesis, there exists a Morita context $\mathbf{M} = (\mathcal{A}, \mathcal{A}', M, M', \tau, \tau')$ with surjective maps τ and τ' . We begin by commenting on some maps which will be involved in the proof.

The first map that we consider is the embedding $\mathcal{A} \rightarrow \text{End}_{(\mathcal{A})}\mathcal{A}$ as the multiplication by elements of \mathcal{A} . Under this map \mathcal{A} embeds as an ideal of $\text{End}_{(\mathcal{A})}\mathcal{A}$. Similarly, \mathcal{A}' embeds as an ideal in $\text{End}_{(\mathcal{A}')}\mathcal{A}' = \text{End}_{(\mathcal{A}')}\mathcal{A}'$. The second function considered is the trace map $\tau' : M' \otimes_{\mathcal{A}} M \rightarrow \mathcal{A}'$. It is shown in the proof of [21, Proposition 6.2] that when \mathcal{A} is s -unital (which is our case), it is an isomorphism. The next map we need to deal with is $\nu : M' \otimes_{\mathcal{A}} \mathcal{A} \rightarrow M'$ given by $m' \otimes a \mapsto m'a$. It is also shown in the proof of [21, Proposition 6.2] that for s -unital rings this multiplication is an isomorphism. Our next map is the isomorphism

$$\nu \circ M' \otimes_{\mathcal{A}} - \circ \nu^{-1} : \text{End}_{(\mathcal{A})}\mathcal{A} \rightarrow \text{End}_{(\mathcal{A}')}\mathcal{A}'.$$

This is the composition of tensoring by M' (the equivalence) together with the conjugation by ν ; that is, for $f \in \text{End}_{(\mathcal{A})}\mathcal{A}$, we have $f \mapsto \nu \circ 1_{M'} \otimes f \circ \nu^{-1}$. More explicitly, for any $m' \in M'$, set $\nu^{-1}(m') = \sum m'_i \otimes a_i$. Then $(\nu \circ 1_{M'} \otimes f \circ \nu^{-1})(m') = \sum m'_i f(a_i)$. We denote this map as $f \mapsto f$. Note that if $a \in \mathcal{A} \hookrightarrow \text{End}_{(\mathcal{A})}\mathcal{A}$, then

$\bar{a}(m') = \sum m'_i(a_i \cdot a) = m'a$; that is, the composition $\mathcal{A} \hookrightarrow \text{End}({}_{\mathcal{A}'}M')$ is just “the right multiplication by a ”.

The last map that we shall consider is

$$\tau' \circ - \otimes_{\mathcal{A}'} M \circ \tau'^{-1} : \text{End}(M'_{\mathcal{A}}) \rightarrow \text{End}(\mathcal{A}'_{\mathcal{A}'}),$$

that is, the composition of the tensor product by M (the equivalence) together with the conjugation by τ' . Explicitly, for $f \in \text{End}(M'_{\mathcal{A}})$ and $a' \in \mathcal{A}'$, writing $\tau'^{-1}(a') = \sum m'_i \otimes m_i$, we have $(\tau' \circ f \otimes_{\mathcal{A}'} M \circ \tau'^{-1})(a') = \sum f(m'_i)m_i$. All these maps always exist, even in the non-commutative case. The image of \mathcal{A}' under the inverse of this isomorphism is, as above, the multiplication by a' from the left. Let us denote it again as $a' \mapsto \bar{a}'$.

Now we will make use of commutativity to see that $\text{End}(M'_{\mathcal{A}}) = \text{End}({}_{\mathcal{A}'}M')$ as rings. It is shown in the proof of [39, Proposition 3.2] that any idempotent commutative ring has a commutative endomorphism ring. So, as \mathcal{A} and \mathcal{A}' are commutative, then $\text{End}({}_{\mathcal{A}}\mathcal{A})$ and $\text{End}(\mathcal{A}'_{\mathcal{A}'})$ are also commutative. By the ring isomorphisms above we have that $\text{End}({}_{\mathcal{A}'}M')$ and $\text{End}(M'_{\mathcal{A}})$ are also commutative rings. Now we prove the equality. Take any $f \in \text{End}(M'_{\mathcal{A}})$ and any $a' \in \mathcal{A}'$; then we have that $f(a'm') = f(\bar{a}'(m')) = \bar{a}'(f(m')) = a'f(m')$, i.e. $f \in \text{End}({}_{\mathcal{A}'}M')$. The converse is completely analogous.

This equality allows us to compose all isomorphisms to get

$$\text{End}({}_{\mathcal{A}}\mathcal{A}) \cong \text{End}({}_{\mathcal{A}'}M') = \text{End}(M'_{\mathcal{A}}) \cong \text{End}(\mathcal{A}'_{\mathcal{A}'}).$$

We denote this composition by μ ; that is, for $f \in \text{End}({}_{\mathcal{A}}\mathcal{A})$ the image $\mu(f) \in \text{End}(\mathcal{A}'_{\mathcal{A}'})$ acts as follows. Take $a' \in \mathcal{A}'$ and using τ^{-1} and ν^{-1} write $a' = \sum m'_i a_i \cdot m_i$. Then $\mu(f)(a') = \sum m'_i f(a_i) \cdot m_i$. In particular, if $a \in \mathcal{A}$, we have that

$$\mu(a)(a') = \sum m'_i a_i a \cdot m_i = \sum m'_i a \cdot m_i.$$

Analogously one verifies that for any $a \in \mathcal{A}$ and $a' \in \mathcal{A}'$ with $a = \sum \bar{m}_j \bar{m}'_j = \sum \bar{m}_j a'_j \bar{m}'_j$ ($a'_j \in \mathcal{A}'$, $\bar{m}_j \in M$, $\bar{m}'_j \in M'$), we have

$$\mu^{-1}(a')(a) = \sum \bar{m}_j \cdot a'_j \bar{m}'_j = \sum \bar{m}_j \cdot a' \bar{m}'_j.$$

For the sake of brevity we shall simply write $m'm$ to refer to the elements of \mathcal{A}' . Then for $a \in \mathcal{A}$, we have $\mu(a)(m'm) = m'am$. We want to prove that $\mu(\mathcal{A}) = \mathcal{A}'$. Take any $a \in \mathcal{A}$ and, writing $a = \sum \bar{m}_j \bar{m}'_j$, there is an s -unit $t' \in \mathcal{A}'$ such that $a = \sum \bar{m}_j t' \bar{m}'_j$. Then, for any $m'm \in \mathcal{A}'$,

$$\begin{aligned} \mu(a)(m'm) &= m'am = m'(\sum \bar{m}_j t' \bar{m}'_j)m = \sum (m' \bar{m}_j)t'(\bar{m}'_j m) \\ &= \sum (m' \bar{m}_j)(\bar{m}'_j m)t' = \sum m'(\bar{m}_j \bar{m}'_j)mt' = m'amt' \\ &= [\mu(a)(m'm)]t' = \mu(a)((m'm)t') = \mu(a)(t'(m'm)) = \mu(a)(t') \cdot (m'm). \end{aligned}$$

Thus, for any $a \in \mathcal{A}$, with $a = \sum \bar{m}_j \bar{m}'_j = \sum \bar{m}_j t' \bar{m}'_j$ and $a' \in \mathcal{A}'$, we have that $\mu(a)(a') = \mu(a)(t') \cdot a'$. This means that the homomorphism $\mu(a)$ is equivalent to the left multiplication by $\mu(a)(t') \in \mathcal{A}'$, and consequently $\mu(\mathcal{A}) \subseteq \mathcal{A}'$. In a completely similar way one shows that for any $a \in \mathcal{A}$ and $a' \in \mathcal{A}'$, with $a' = \sum m'_i m_i = \sum m'_i t m_i$, $m'_i \in M'$, $m_i \in M$, $t \in \mathcal{A}$, we have that $\mu^{-1}(a')(a) = a \cdot \mu^{-1}(a')(t)$, i.e. the homomorphism $\mu^{-1}(a')$ is the right multiplication by $\mu^{-1}(a')(t) \in \mathcal{A}$. The latter means that $\mu^{-1}(\mathcal{A}') \subseteq \mathcal{A}$, and thus $\mu(\mathcal{A}) = \mathcal{A}'$.

Moreover, if $a \in \mathcal{D}_g$ with $a = \sum \bar{m}_j \bar{m}'_j = \sum \bar{m}_j t' \bar{m}'_j$, then writing $t' = \sum n'_j n_j$ we have that $\mu(a)(t') = \sum n'_j a n_j \in M' \mathcal{D}_g M = \mathcal{D}'_g$, and hence $\mu(\mathcal{D}_g) \subseteq \mathcal{D}'_g$. Similarly, taking $\mathcal{D}_g \ni a' = \sum m'_i m_i = \sum m'_i t m_i$, $t = \sum \bar{n}_j \bar{n}'_j$, $n_j \in M, \bar{n}_j \in M'$, we see that $\mu^{-1}(a')(t) = \sum \bar{n}_j a' \bar{n}'_j \in M \mathcal{D}'_g M' = \mathcal{D}_g$, giving $\mu^{-1}(\mathcal{D}'_g) \subseteq \mathcal{D}_g$. Consequently, $\mu(\mathcal{D}_g) = \mathcal{D}'_g$, and it remains to show that $\mu(\alpha_g(a)) = \alpha'_g(\mu(a))$, for any $a \in \mathcal{D}_{g^{-1}}$.

Write $\mathcal{D}_{g^{-1}} \ni a = \sum \bar{m}_j \bar{m}'_j = \sum \bar{m}_j t' \bar{m}'_j$, $\bar{m}_j \in M_{g^{-1}}, \bar{m}'_j \in M'_{g^{-1}}, t' \in \mathcal{D}'_{g^{-1}}$. Then for any $m' m \in \mathcal{D}'_g$, $m' \in M'_g, m \in M_g$, we have,

$$\begin{aligned} (m' m) \alpha'_g(\mu(a)[t']) &= \alpha'_g(\alpha'_{g^{-1}}(m' m) \mu(a)[t']) = \alpha'_g(\mu(a)[\alpha'_{g^{-1}}(m' m)]) \\ &= \alpha'_g(\mu(a)[\gamma'_{g^{-1}}(m') \gamma_{g^{-1}}(m)]) = \alpha'_g(\gamma'_{g^{-1}}(m') a \gamma_{g^{-1}}(m)) \\ &= \alpha'_g(\gamma'_{g^{-1}}(m' \alpha_g[a]) \gamma_{g^{-1}}(m)) = \alpha'_g(\alpha'_{g^{-1}}[m' \alpha_g(a) m]) \\ &= m' \alpha'_g(a) m. \end{aligned}$$

Now, note that

$$\alpha_g(a) = \alpha_g(\sum \bar{m}_j \bar{m}'_j) = \sum \gamma_g(\bar{m}_j) \gamma_g(\bar{m}'_j)$$

and

$$\alpha_g(a) = \alpha_g(\sum \bar{m}_j t' \bar{m}'_j) = \sum \gamma_g(\bar{m}_j) \gamma_g(t' \bar{m}'_j) = \sum \gamma_g(\bar{m}_j) \alpha'_g(t') \gamma_g(\bar{m}'_j),$$

and so $\mu(\alpha_g(a))(a') = a' \cdot \mu[\alpha_g(a)][\alpha'_g(t')]$. Hence, $(m' m) \mu[\alpha_g(a)][\alpha'_g(t')] = m' \alpha'_g(a) m$, for any $m' m \in \mathcal{A}'$, and this yields that

$$(m' m) \alpha'_g(\mu(a)[t']) = m' \alpha'_g(a) m = (m' m) \mu[\alpha_g(a)][\alpha'_g(t')]$$

for all $m' m \in \mathcal{D}'_g$. Setting $d' = \alpha'_g(\mu(a)[t']) - \mu[\alpha_g(a)][\alpha'_g(t')]$, we have that $\mathcal{D}'_g d' = 0$. Furthermore, $d' \in \mathcal{D}'_g$ as $\alpha'_g(\mu(a)[t'])$ and $\mu[\alpha_g(a)][\alpha'_g(t')]$ belong to \mathcal{D}'_g . Finally, since \mathcal{D}'_g is s -unital, it follows that the equality $\mathcal{D}'_g d' = 0$ implies $d' = 0$, and we conclude that $\alpha'_g(\mu(a)[t']) = \mu[\alpha_g(a)][\alpha'_g(t')]$. \square

We will show next that Theorem 7.1 is also true in the category of C^* -algebras. First, it will be convenient to recall some facts about strong Morita equivalence of C^* -algebras (that we will refer to just as Morita equivalence), and also about partial actions on commutative C^* -algebras.

Suppose that \mathcal{A} and \mathcal{A}' are Morita equivalent C^* -algebras, and let \mathcal{X} be an $\mathcal{A} - \mathcal{A}'$ -imprimitivity bimodule. Let $\mathcal{I}(\mathcal{A}')$ be the lattice of closed ideals of \mathcal{A}' and $\mathcal{I}(\mathcal{X})$ the lattice of closed $\mathcal{A} - \mathcal{A}'$ -submodules of \mathcal{X} . As it is well known (see for instance [2, Proposition 4.2]), we have a lattice isomorphism $\rho_{\mathcal{A}'} : \mathcal{I}(\mathcal{A}') \rightarrow \mathcal{I}(\mathcal{X})$, given by $\rho_{\mathcal{A}'}(J) := \mathcal{X}J$; its inverse sends an $\mathcal{A} - \mathcal{A}'$ -submodule \mathcal{Y} of \mathcal{X} to the ideal $\mathcal{Y}^r := \overline{\text{span}}\{\langle y, y' \rangle_{\mathcal{A}'} : y, y' \in \mathcal{Y}\}$. Similarly, we have a lattice isomorphism $\lambda_{\mathcal{A}} : \mathcal{I}(\mathcal{A}) \rightarrow \mathcal{I}(\mathcal{X})$ such that $\lambda_{\mathcal{A}}(I) = I\mathcal{X}$, whose inverse is given by $\mathcal{Y} \mapsto \mathcal{Y}^l := \overline{\text{span}}\{\langle y, y' \rangle_{\mathcal{A}} y, y' \in \mathcal{Y}\}$. Thus the composition $\lambda_{\mathcal{A}}^{-1} \circ \rho_{\mathcal{A}'}$, called the Rieffel correspondence, is an isomorphism from $\mathcal{I}(\mathcal{A}')$ onto $\mathcal{I}(\mathcal{A})$. Let $h_{\mathcal{X}} : \text{Prim } \mathcal{A}' \rightarrow \text{Prim } \mathcal{A}$ be the restriction-corestriction of the isomorphism $\lambda_{\mathcal{A}}^{-1} \circ \rho_{\mathcal{A}'}$ to the primitive ideal spaces of \mathcal{A}' and \mathcal{A} . Then $h_{\mathcal{X}}$ is a homeomorphism known as the Rieffel homeomorphism (see for example [50] for more information). When \mathcal{A} is a commutative C^* -algebra the space $\text{Prim } \mathcal{A}$ determines \mathcal{A} up to isomorphism via the Gelfand transform, so in case \mathcal{A} and \mathcal{A}' are Morita equivalent commutative C^* -algebras, they are isomorphic ([50, Corollary 3.33]). If J is an ideal in \mathcal{A}' , then $\mathcal{O}_J := \{Q \in \text{Prim } \mathcal{A}' : Q \cap J \neq 0\}$ is an open subset of $\text{Prim } \mathcal{A}'$, and the map $r : \mathcal{O}_J \rightarrow \text{Prim } J$ given by $r(Q) := Q \cap J$ is a homeomorphism.

Moreover, $h_{\mathcal{X}J} : \text{Prim } J \rightarrow \text{Prim } h_{\mathcal{X}}(J)$ is the restriction of $h_{\mathcal{X}}$ to $\text{Prim } J$ under the identifications $\text{Prim } J \sim \mathcal{O}_J$ and $\text{Prim } h_{\mathcal{X}}(J) \sim \mathcal{O}_{h_{\mathcal{X}}(J)}$. More precisely, $r(h_{\mathcal{X}}(Q)) = h_{\mathcal{X}J}(r(Q))$, $\forall Q \in \mathcal{O}_J$.

Suppose that \mathcal{X} is an $\mathcal{A} - \mathcal{A}'$ -imprimitivity bimodule with inner products $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ and $\langle \cdot, \cdot \rangle_{\mathcal{A}'}$ respectively, and that \mathcal{Y} is a $\mathcal{B} - \mathcal{B}'$ -imprimitivity bimodule with inner products $\langle \cdot, \cdot \rangle_{\mathcal{B}}$ and $\langle \cdot, \cdot \rangle_{\mathcal{B}'}$ respectively. Suppose moreover that $\phi : \mathcal{X} \rightarrow \mathcal{Y}$ is an isomorphism of C^* -trings, that is, a bijective linear map such that $\phi(\langle x_1, x_2 \rangle_{\mathcal{A}} x_3) = \langle \phi(x_1), \phi(x_2) \rangle_{\mathcal{B}} \phi(x_3)$, $\forall x_1, x_2, x_3$ (therefore we also have that $\phi(x_1) \langle \phi(x_2), \phi(x_3) \rangle_{\mathcal{B}'} = \phi(x_1 \langle x_2, x_3 \rangle_{\mathcal{A}'})$, $\forall x_1, x_2, x_3$). Then there exist (unique) isomorphisms $\phi^l : \mathcal{A} \rightarrow \mathcal{B}$ and $\phi^r : \mathcal{A}' \rightarrow \mathcal{B}'$ such that $\phi^l(\langle x_1, x_2 \rangle_{\mathcal{A}}) = \langle \phi(x_1), \phi(x_2) \rangle_{\mathcal{B}}$ and $\phi^r(\langle x_1, x_2 \rangle_{\mathcal{A}'}) = \langle \phi(x_1), \phi(x_2) \rangle_{\mathcal{B}'}$, $\forall x_1, x_2 \in \mathcal{X}$ (thus $\phi(ax) = \phi^l(a)\phi(x)$ and $\phi(xa') = \phi(x)\phi^r(a')$, $\forall a \in \mathcal{A}, x \in \mathcal{X}$ and $a' \in \mathcal{A}'$ ([2, Proposition 4.1]). These isomorphisms satisfy $\phi^l(Z^l) = \phi(Z)^l$ and $\phi^r(Z^r) = \phi(Z)^r$, for every $Z \in \mathcal{I}(\mathcal{X})$.

Lemma 7.2. *In the conditions above the following square is commutative:*

$$\begin{CD} \text{Prim } \mathcal{A} @<h_{\mathcal{X}}<< \text{Prim } \mathcal{A}' \\ @V\bar{\phi}^lVV @VV\bar{\phi}^rV \\ \text{Prim } \mathcal{B} @<h_{\mathcal{Y}}<< \text{Prim } \mathcal{B}' \end{CD}$$

where $\bar{\phi}^l : \text{Prim } \mathcal{A} \rightarrow \text{Prim } \mathcal{B}$ is the homeomorphism given by $\bar{\phi}^l(P) := \phi^l(P)$, and similarly $\bar{\phi}^r$.

Proof. Let $Q \in \text{Prim } \mathcal{A}'$. Then

$$\bar{\phi}^l h_{\mathcal{X}}(Q) = \phi^l((\mathcal{X}Q)^l) = \phi(\mathcal{X}Q)^l = (\phi(\mathcal{X})\phi^r(Q))^l = (\mathcal{Y}\bar{\phi}^r(Q))^l = h_{\mathcal{Y}}\bar{\phi}^r(Q).$$

□

If \mathcal{A} is a commutative C^* -algebra with spectrum $\hat{\mathcal{A}}$, the map $\kappa_{\mathcal{A}} : \hat{\mathcal{A}} \rightarrow \text{Prim } \mathcal{A}$ such that $\kappa_{\mathcal{A}}(\psi) = \ker \psi$ is a homeomorphism, and if $\phi : \mathcal{A} \rightarrow \mathcal{B}$ is an isomorphism of commutative C^* -algebras and $\hat{\phi} : \hat{\mathcal{A}} \rightarrow \hat{\mathcal{B}}$ is given by $\hat{\phi}(\psi) := \psi \circ \phi^{-1}$, then $\hat{\phi}$ is a homeomorphism, and it is readily verified that the following diagram commutes:

$$\begin{CD} \text{Prim } \mathcal{A} @<\kappa_{\mathcal{A}}<< \hat{\mathcal{A}} \\ @V\bar{\phi}VV @VV\hat{\phi}V \\ \text{Prim } \mathcal{B} @<\kappa_{\mathcal{B}}<< \hat{\mathcal{B}} \end{CD}$$

Note that if J is an ideal in \mathcal{A} , then $\kappa_{\mathcal{A}}^{-1}(\mathcal{O}_J) = \{\psi \in \hat{\mathcal{A}} : \psi|_J \neq 0\}$. Thus the map $\{\psi \in \hat{\mathcal{A}} : \psi|_J \neq 0\} \rightarrow \hat{J}$ given by $\psi \mapsto \psi|_J$ is a homeomorphism.

Recall now that the Gelfand duality determines a one-to-one correspondence from the set of partial actions of G on a commutative C^* -algebra \mathcal{A} onto the set of partial actions of G on its spectrum $\hat{\mathcal{A}}$ ([3, Proposition 1.5]). Let us recall one of the directions of this correspondence. So suppose that $\alpha = \{\alpha_g : \mathcal{D}_{g^{-1}} \rightarrow \mathcal{D}_g : g \in G\}$ is a partial action of G on the commutative C^* -algebra \mathcal{A} . As mentioned above, we identify $\hat{\mathcal{D}}_g$ with $\{\varphi \in \hat{\mathcal{A}} : \varphi|_{\mathcal{D}_g} \neq 0\}$, which is an open subset of $\hat{\mathcal{A}}$. Now let $\hat{\alpha}_g : \hat{\mathcal{D}}_{g^{-1}} \rightarrow \hat{\mathcal{D}}_g$, such that $\hat{\alpha}_g(\varphi)$ is defined to be the unique extension of $\varphi \circ \alpha_{g^{-1}} : \mathcal{D}_g \rightarrow \mathbb{C}$ to a homomorphism defined on all of \mathcal{A} . It follows that $\hat{\alpha} := \{\hat{\alpha}_g : \hat{\mathcal{D}}_{g^{-1}} \rightarrow \hat{\mathcal{D}}_g : g \in G\}$ is a partial action of G on $\hat{\mathcal{A}}$.

Finally, recall ([2, Definition 4.4]) that the partial actions $\alpha = \{\alpha_g : \mathcal{D}_{g^{-1}} \rightarrow \mathcal{D}_g : g \in G\}$ and $\alpha' = \{\alpha'_g : \mathcal{D}'_{g^{-1}} \rightarrow \mathcal{D}'_g : g \in G\}$ of G on the C^* -algebras \mathcal{A} and \mathcal{A}' (in the category of C^* -algebras) are Morita equivalent if there exist an imprimitivity $\mathcal{A} - \mathcal{A}'$ -bimodule \mathcal{X} and a partial action $\phi = \{\phi_g : \mathcal{X}_{g^{-1}} \rightarrow \mathcal{X}_g : g \in G\}$ on the C^* -tring \mathcal{X} such that $\mathcal{X}_g^l = \mathcal{D}_g = h_{\mathcal{X}}(\mathcal{D}'_g) = h_{\mathcal{X}}(\mathcal{X}_g^r)$, $\alpha_g = \phi_g^l$ and $\alpha'_g = \phi_g^r$, $\forall g \in G$.

Theorem 7.3. *Let $\alpha = \{\alpha_g : \mathcal{D}_{g^{-1}} \rightarrow \mathcal{D}_g : g \in G\}$ and $\alpha' = \{\alpha'_g : \mathcal{D}'_{g^{-1}} \rightarrow \mathcal{D}'_g : g \in G\}$ be Morita equivalent partial actions of G on the commutative C^* -algebras \mathcal{A} and \mathcal{A}' respectively. Then:*

- (1) *If \mathcal{X} is an imprimitivity bimodule giving a Morita equivalence between α and α' , then the Rieffel homeomorphism $h_{\mathcal{X}}$ is an isomorphism between the partial actions $\hat{\alpha}$ and $\hat{\alpha}'$ defined by α and α' on the spectra of \mathcal{A} and \mathcal{A}' .*
- (2) *The partial actions α and α' are isomorphic.*

Proof. Let $\phi = \{\phi_g : \mathcal{X}_{g^{-1}} \rightarrow \mathcal{X}_g : g \in G\}$ be a partial action of G on the C^* -tring \mathcal{X} implementing the Morita equivalence between α and α' . Then $\mathcal{X}_g^l = \mathcal{D}_g = h_{\mathcal{X}}(\mathcal{D}'_g)$, $\mathcal{D}'_g = \mathcal{X}_g^r$, $\alpha_g = \phi_g^l$ and $\alpha'_g = \phi_g^r$, $\forall g \in G$. We will denote the homeomorphism $\kappa_{\mathcal{A}'}^{-1} h_{\mathcal{X}} \kappa_{\mathcal{A}} : \hat{\mathcal{A}} \rightarrow \hat{\mathcal{A}'}$ also by $h_{\mathcal{X}}$, and similarly with every $h_{\mathcal{X}_g}$. Since for every $g \in G$ we have $h_{\mathcal{X}_g} = h_{\mathcal{X}}|_{\mathcal{X}_g}$ under the identifications $\hat{\mathcal{D}}_g \sim \{\psi \in \hat{\mathcal{A}} : \psi|_{\mathcal{D}_g} \neq 0\}$ and $\hat{\mathcal{D}}'_g \sim \{\psi' \in \hat{\mathcal{A}'} : \psi'|_{\mathcal{D}'_g} \neq 0\}$, Lemma 7.2 implies that we have a commutative diagram

$$\begin{array}{ccc} \hat{\mathcal{D}}_{g^{-1}} & \xleftarrow{h_{\mathcal{X}}} & \hat{\mathcal{D}}'_{g^{-1}} \\ \hat{\alpha}_g \downarrow & & \downarrow \hat{\alpha}'_g \\ \mathcal{D}_g & \xleftarrow{h_{\mathcal{X}}} & \mathcal{D}'_g \end{array}$$

for each $g \in G$, which shows that $h_{\mathcal{X}} : \hat{\alpha}' \rightarrow \hat{\alpha}$ is an isomorphism of partial actions. This proves the first assertion of the theorem. It also follows from this fact that the partial actions α_* and α'_* induced by $\hat{\alpha}$ and $\hat{\alpha}'$, on $C_0(\hat{\mathcal{A}})$ and $C_0(\hat{\mathcal{A}'})$ respectively (see [3]), are isomorphic. Since clearly the Gelfand transforms give isomorphisms $\alpha \cong \alpha_*$ and $\alpha' \cong \alpha'_*$, we conclude that $\alpha \cong \alpha'$. \square

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