A discrete Bayes explanation of a failure-rate paradox

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Key Words-

Bayes, Bernoulli trials, exponential density, decreasing failure rate, predictive distribution, prior opinion.

Reader Aids-

Purpose: Explain a failure-rate paradox

Special math needed: Probability

Special math needed for applications: Statistics

Results useful to: Reliability engineers and analysts

Summary and Conclusions-

A simpler version of the explanation given by Barlow[1] for the exponential failure rate paradox is presented. The discrete counterpart of the model is used. The predictive failure rate of the model can only be decreasing.

Notation-

I indicator function

$f_X(x|\lambda)$ conditional density

$r_X(x|\lambda)$ conditional failure rate

$f_X(x)$ predictive (marginal) density

$r_X(x)$ predictive failure rate

$p(\lambda)$ prior density

1. INTRODUCTION

Suppose a lifetime X, conditional on the value of a rate λ, is judged to be exponentially distributed, i.e., $f_X(x|\lambda) = \lambda e^{-\lambda x}$ I$(x > 0)$. The conditional failure rate, $r_X(x|\lambda) = f_X(x|\lambda)/\text{P}(X > x|\lambda) = \lambda$ does not depend on x. However,the failure rate of the predictive distribution of X is decreasing on x, for any choice of a prior density for λ. Formally,

$r_X(x) = f_X(x)/\text{P}(X > x) = \frac{\int_0^{+\infty} \lambda e^{-\lambda x} p(\lambda) d\lambda}{\int_0^{+\infty} e^{-\lambda x} p(\lambda) d\lambda}$ is decreasing on x for any prior p on $(0, +\infty)$.

This fact is sometimes seen as a paradox, as the model having constant failure rate yields a predictive distribution with **decreasing** failure rate. A Bayesian explanation of this - illusory - paradox was given by Barlow[1]. He identified $r_X(x)$ to different posterior means E$(\lambda|X > x)$ of λ, given $X > x$, in order to explain the "paradox".

We present here the discrete version of the "paradox". We believe the discrete explanation adds further insight to the situation.

2. THE DISCRETE VERSION OF THE PARADOX

We will consider the geometric distribution as the discrete counterpart to the exponential density. The failure rate of a discrete random quantity N is defined as $r_N(n) =\text{P}(N = n)/\text{P}(N > n-1)$, for $n = 1, 2, ...$ The conditional failure rate of the geometric distribution, given the value of its parameter θ, is then

$$\begin{aligned} r_N(n|\theta) &= \text{P}(N = n|\theta)/\text{P}(N > n-1|\theta) \\ &= \theta(1-\theta)^{n-1} / \textstyle\sum_{i=n}^{+\infty} \theta(1-\theta)^{i-1} \\ &= \theta, \text{ for } n = 1, 2, ... \end{aligned}$$

The (constant on n) failure rate causes no surprise, as the geometric distribution may be understood in the context of Bernoulli trials. However, the predictive failure rate

$r_N(n) = \frac{\int_0^1 \theta(1-\theta)^{n-1} p(\theta) d\theta}{\int_0^1 (1-\theta)^{n-1} p(\theta) d\theta}$ is decreasing on n, for any choice of a prior density p (this fact is proved on the appendix).

3. THE EXPLANATION

We now argue that there is no paradox. Suppose a Bayesian watches the realization of consecutive Bernoulli trials. He does not know the value of the constant propensity for success, θ. His opinion, prior to the trials, is described by his personal prior density for it, $p(\theta)$. After having observed $(n-1)$ failures (and no success) his updated opinion about the possibility of a (first) success in the next trial is given by his conditional probability $\mathrm{P}(N=n|N>n-1)$, where N stands for "trial when the first success happens". We have $\mathrm{P}(N=n|N>n-1)=r_N(n)$, the **decreasing** failure rate.

But this is not surprising: As n increases, and the first success never happens, our Bayesian person is led by his current (posterior) densities $p(\theta|N>n-1)$ to increasing skepticism about the ocurrence of a success. His probabilities $\mathrm{P}(N=n|N>n-1)$ should, indeed, become smaller and smaller as time goes by and a success is never observed. Notice that this should happen regardless of which prior $p(\theta)$ he had, as the sequence of failures makes him increasingly skeptical anyway.

4. DISCUSSION

The predictive failure rate of a conditional geometric model can only be decreasing. A person, regardless of which prior opinion $p(\theta)$ he had, becomes more and more pessimistic about the occurrence of a first success when only failures are observed by him. This is the **corrected** statement of the "law of maturity" that is much known to readers of sequences of Lotto results: A sequence of failures makes one believe a success is increasingly **un**likely. The correct statement of the law also amounts to Carnap's *principle of instantial relevance* (Plato[2]) and can be shown to follow from exchangeability of trials - the full subjectivistic Bayesian description of Bernoulli trials.

REFERENCES

[1] R. E. Barlow, "A Bayes Explanation of an Apparent Failure Rate Paradox",*IEEE Trans. Reliability*, vol R-34, No. 2, 1985 June, pp 107-108.

[2] J.v. Plato, "On Partial Exchangeability as a Generalization of Symmetry Principles", *Erkenntnis*, vol 16, 1981, pp 53-59.

APPENDIX

Here we prove that $r_N(n)$ is decreasing on n, for any non-degenerate prior p.

FACT:

For any random variable Y satisfying $\mathrm{P}(0 < Y < 1) = 1$,

$$E[Y(1-Y)^{n-1}]/E[(1-Y)^{n-1}] \text{ decreases on } n = 1, 2, ...$$

PROOF:

Let us define $Z = 1 - Y$. We now aim to prove that for every $n = 1, 2, ...$

$$\mathrm{E}[(1-Z)Z^n]/\mathrm{E}[Z^n] \geq \mathrm{E}[(1-Z)Z^{n+1}]/\ \mathrm{E}[Z^{n+1}] \qquad (1)$$

But, by making $T = Z^{n/2}$ and $V = Z^{(n/2)+1}$, we can apply Cauchy-Schwarz inequality to obtain (for every $n \geq 1$)

$$E^2[Z^{n+1}] \leq E[Z^n]E[Z^{n+2}], \text{ which is equivalent to (1).}$$

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