



## Inconsistencies in Numerical Simulations of Dynamical Systems Using Interval Arithmetic

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Over the past few decades, interval arithmetic has been attracting widespread interest from the scientific community. With the expansion of computing power, scientific computing is encountering a noteworthy shift from floating-point arithmetic toward increased use of interval arithmetic. Notwithstanding the significant reliability of interval arithmetic, this paper presents a theoretical inconsistency in a simulation of dynamical systems using a well-known implementation of arithmetic interval. We have observed that two natural interval extensions present an empty intersection during a finite time range, which is contrary to the fundamental theorem of interval analysis. We have proposed a procedure to at least partially overcome this problem, based on the union of the two generated pseudo-orbits. This paper also shows a successful case of interval arithmetic application in the reduction of interval width size on the simulation of discrete map. The implications of our findings on the reliability of scientific computing using interval arithmetic have been properly addressed using two numerical examples.

*Keywords:* Simulation; modeling; interval arithmetic; interval extensions; chaos; dynamical systems; error propagation.

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## 1. Introduction

Numerical simulations play a fundamental role in the analysis of dynamical systems and have been applied to different subareas in nonlinear dynamics, such as synchronization, bifurcation and chaos, complex networks, conservative systems and nonlinear partial differential equations [Macau & Pando Lambruschini, 2014; Saberi Nik, 2015]. It is usually necessary to use recursive functions to describe and solve many types of systems and problems [Feigenbaum, 1978; Hammel *et al.*, 1987; Lozi, 2013; Nepomuceno, 2014]. Thus, the control of numerical error propagation has been considered highly important, particularly for nonlinear systems [Galias, 2013; Mendes & Nepomuceno, 2016; Nepomuceno & Martins, 2016].

Over the past few decades, interval arithmetic [Rump, 1999; Moore *et al.*, 2009] has been attracting widespread interest from the scientific community. This increasing interest has resulted in a myriad of applied and theoretical works. Regarding the theoretical investigations, many versions of interval arithmetic have been suggested in literature. One of the most respected versions is widely recognized as the standard arithmetic interval (SIA) [Moore *et al.*, 2009; Piegat & Landowski, 2017]. SIA has a long history, but it is usually attributed to Moore's development to a mature stage [Chalco-Cano *et al.*, 2014]. In this work, we are focusing our attention on SIA.

This interest on SIA is mainly motivated to evaluate the effects of approximation errors and inaccurate inputs [Bruguera, 2014]. The SIA is a tool that defines sets of intervals rather than real numbers, delimiting a range of possible values to represent some given data [Hargreaves, 2002]. Thereby, this technique allows a computer user to obtain precise results, ensuring that a true solution lies on the range of values limited by the corresponding interval [Galias, 2002]. There are libraries developed for interval arithmetic, such as C-XSC [Hofschuster & Krämer, 2004] and CoStLy [Neher & Eble, 2004], both developed for the C++ language, in addition to Intlab [Rump, 1999], developed for Matlab and Octave. With the expansion of computing power, scientific computing is encountering a noteworthy shift from floating-point arithmetic toward increased use of interval arithmetic. It has even been considered as a complete and exception-free calculus [Kulisch, 2009].

Although, the success of interval arithmetic is widely accepted, some concerns have also been reported in literature. The authors in [Piegat & Landowski, 2017, 2015] have presented a considerable list of works in which the SIA has had some faults. The main worries are related to the overestimation of results, the dependence on the form of the mathematical expression used in problem solving, the calculation of interval borders and the lack of important properties, such as distributivity and multiplicative cancellation. According to Piegat and Landowski [2017], these problems lead to controversial results.

The main principle of SIA has been well reported in [Chalco-Cano *et al.*, 2014]: “in all computations including (or most especially) those performed on a digital computer should never lose any possible value.” Although, great effort has been devoted to accomplish this principle, real implementations, such as those implemented in a digital computer, may not always be completely successful [Piegat & Landowski, 2015, 2017]. This paper reports the use of interval intersection and mean value form to reduce the interval width. In one of the numerical experiments, we have surprisingly found a theoretical inconsistency in the use of Intlab [Rump, 1999]. We were working on the simulation of a simple RLC circuit. The reduction of the interval using two pseudo-orbits (in a similar approach seen in [Nepomuceno & Martins, 2016; Nepomuceno *et al.*, 2017]) has not been possible because its intersections were empty for some time instants. This encountered problem has allowed us to make a clear parallel between floating-point and interval arithmetic. For instance, digital filters are normally implemented in hardware using the two complement arithmetic for the addition operation. The authors in [Ling *et al.*, 2006] have reported that in such situation first and second order linear filters are actually nonlinear discrete-time systems. That sort of problem in physical implementation is not different for SIA. Following these ideas, the main contribution of this paper is twofold. First, we want to revisit the reduction of interval width using the mean value form and interval intersection. We have investigated a discrete dynamical system, where the intersection has been successfully applied to reduce the interval width. The second investigated example has been focused on the simulation of RLC circuit with uncertain parameters. Our objective is to

compare the result of the RLC circuit with experimental data, which has intensive pedagogical utility [Rothwell & Cloud, 2012]. As already mentioned, we have not been able to proceed with the intersection because the simulation of analytical result considering the uncertain parameters has led us to find the empty intersection. The second contribution of this paper is a simple procedure to overcome this problem at least partially. To avoid the empty intersection, we have built a new pseudo-orbit based on the union of the pseudo-orbits generated by two different interval extensions. The results show a satisfactory agreement between experimental data and our approach.

The remainder of the paper proceeds as follows. In Sec. 2, we recall some preliminary concepts of SIA and discrete dynamical systems defined by recursive functions. In Sec. 3, we present the developed method for the simulation of logistic map and an electric circuit. Section 4 is devoted to present the results, and the final remarks are given in Sec. 5.

## 2. Theoretical Foundation

In this section, we present some background that supports the method presented in the following section.

### 2.1. Recursive function

A recursive function can be defined as follows:

**Definition 2.1.** Let  $\mathbb{I}$  be a metric space such that  $\mathbb{I} \subseteq \mathbb{R}$  and  $f : \mathbb{I} \rightarrow \mathbb{R}$ . A recursive function can be defined as

$$x_n = f(x_{n-1}). \quad (1)$$

Discrete-time series can be generated by a simple iterative procedure of (1). A well-known example of recursive function is the logistic map [May, 1976] given by

$$x_{n+1} = f(x_n) = rx_n(1 - x_n). \quad (2)$$

### 2.2. Orbits and pseudo-orbits

Connected to a map, an orbit may be defined as follows:

**Definition 2.2.** An orbit is a sequence of values of a map, represented by  $\{x_n\} = [x_0, x_1, \dots, x_n]$ .

**Definition 2.3.** Let  $i \in \mathbb{N}$  represent a *pseudo-orbit*, which is defined by an initial condition, a natural

*interval extension of  $f$ , some specific hardware, software, numerical precision standard and discretization scheme.* A pseudo-orbit is an approximation of an orbit and can be represented as

$$\{\hat{x}_{i,n}\} = [\hat{x}_{i,0}, \hat{x}_{i,1}, \dots, \hat{x}_{i,n}],$$

such that

$$|x_n - \hat{x}_{i,n}| \leq \gamma_{i,n}, \quad (3)$$

where  $\gamma_{i,n} \in \mathbb{R}$  is a bound of the error and  $\gamma_{i,n} \geq 0$ .

### 2.3. Interval arithmetic

An interval is a set of real numbers such that any number that lies between two numbers in the set is also included in the set [Nepomuceno & Martins, 2016]. An interval  $X$  is denoted  $[\underline{X}, \overline{X}]$ , i.e.  $X = \{x : \underline{X} \leq x \leq \overline{X}\}$ . In a degenerated interval, we have  $\underline{X} = \overline{X}$  and such an interval amounts to a real number  $x = \underline{X} = \overline{X}$ .

For a given interval  $X = [\underline{X}, \overline{X}]$ , its width is defined by  $w(X) = (\overline{X} - \underline{X})$  and its center is  $m(X) = \frac{1}{2}(\underline{X} + \overline{X})$  [Rothwell & Cloud, 2012]. The intersection of two intervals  $X \cap Y$  is a set of real numbers which belong to both. The union  $X \cup Y$  is a set of real numbers which belong to  $X$  or  $Y$  (or both). If  $X \cap Y$  is not empty, then

$$X \cap Y = [\max(\underline{X}, \underline{Y}), \min(\overline{X}, \overline{Y})], \quad (4)$$

$$X \cup Y = [\min(\underline{X}, \underline{Y}), \max(\overline{X}, \overline{Y})]. \quad (5)$$

The intersection plays a major part in interval arithmetic. This assumption comes from the fact that if there are two distinct intervals that contain a result of interest, regardless of how they were obtained, the intersection, which may be narrower than both, will also contain the result. The above statement can be simplified through Eq. (6), which shows that the size of the intersection between two intervals is at most the smallest of the intervals

$$w(X \cap Y) \leq \min\{w(X), w(Y)\}. \quad (6)$$

Therefore, this simple algebraic operation can decrease the size of an interval which contains the solution of a problem.

Operations with intervals are the same as operations with sets. Thus, when operations are performed between two intervals, it gives a resulting interval containing all the results of operations between pairs of values of each interval. The basic

interval operations are defined by:

$$X + Y = [\underline{X} + \underline{Y}, \overline{X} + \overline{Y}], \quad (7)$$

$$X - Y = [\underline{X} - \overline{Y}, \overline{X} - \underline{Y}], \quad (8)$$

$$X \cdot Y = [\min(S), \max(S)], \quad (9)$$

where  $S = \{\underline{X}\underline{Y}, \underline{X}\overline{Y}, \overline{X}\underline{Y}, \overline{X}\overline{Y}\}$ . If 0 does not belong to  $Y$ , then  $X/Y$  is given by

$$\frac{X}{Y} = X \cdot \left(\frac{1}{Y}\right), \quad (10)$$

where  $1/Y = [1/\overline{Y}, 1/\underline{Y}]$ .

### 2.4. Natural interval extension

Let  $f$  be a function of real variable  $x$ . Moore *et al.* [2009] presented the following definition:

**Definition 2.4.** A function  $F$  is a natural *interval extension* of  $f$  if  $F$  agrees with  $f$  for degenerate interval arguments:

$$F([x, x]) = f(x).$$

Some examples and implications of natural interval extension can be found in [Moore *et al.*, 2009; Nepomuceno & Martins, 2016; Nepomuceno *et al.*, 2017; Mendes & Nepomuceno, 2016; Nepomuceno & Mendes, 2017].

### 2.5. Rational interval functions

A rational interval function can be defined as follows:

**Definition 2.5** [Moore *et al.*, 2009]. A rational interval function is an interval-valued function whose values are defined by a specific finite sequence of interval arithmetic operations.

**Example 2.6.** Consider the function given by  $F(X) = [2, 2]X([1, 1] - X)$ . The computation of  $F(X)$  can be broken down into the finite sequence of interval arithmetic operations described by

$$\begin{aligned} T_1 &= [1, 1] - X, && \text{one interval subtraction,} \\ T_2 &= [2, 2]X, && \text{one interval multiplication,} \\ F(X_1) &= T_1T_2, && \text{one interval multiplication.} \end{aligned}$$

Therefore  $F$  is a rational interval function. According to Moore *et al.* [2009], a rational interval function usually comes from an interval extension of

some real function. In the previous example,  $F(X)$  is an interval extension of the logistic map function.

**Definition 2.7** [Moore *et al.*, 2009]. If  $F$  is a rational interval function and an interval extension of  $f$ , then

$$f(X) \subseteq F(X).$$

Definition 2.7 states that a value obtained by  $F$  contains the range of values of the corresponding real function  $f$  when the variable  $x$  is within the interval  $X$ . Therefore, for an  $x$  that belongs to the range of values of the interval  $X$ , the true answer of the calculation of a function  $f$  to  $x$ , or  $f(x)$ , will be contained in an interval obtained by an interval extension  $F(X)$  of the function  $f(x)$ .

### 2.6. The mean value form

Moore *et al.* [2009] noticed that the width of the bounds for an interval is strongly connected to the representation of the function  $f$ . They proposed an useful interval extension called the mean value form (MVF), which is defined as

$$f(X) \subseteq F_{mv}(X) = f(m) + \sum_{i=1}^n D_i F(X)(X_i - m_i) \quad (11)$$

where  $m = m(X)$ ,  $D_i F$  is an interval extension of  $\partial f / \partial x_i$ .

Consider the following example of the mean value form:

**Example 2.8.** Let the following natural interval extension:

$$G(X) = rX(1 - X). \quad (12)$$

If  $r = 3.5$  and  $X = [0.1, 0.25]$ , then the mean value form is:

$$\begin{aligned} F_{mv} &= r(m(X)(1 - m(X)) \\ &\quad + (1 - 2X)(X - m(X))), \\ F_{mv} &= r(m([0.1, 0.25])(1 - m([0.1, 0.25])) \\ &\quad + (1 - 2[0.1, 0.25])([0.1, 0.25] \\ &\quad - m([0.1, 0.25]))), \\ F_{mv} &= [0.2953, 0.7153]. \end{aligned}$$

As can be easily checked, without the mean value form, the interval would be  $[0.1312, 0.8400]$ , considerably greater than  $F_{mv} = [0.2953, 0.7153]$ .

## 2.7. Intlab

The library used in this paper for the application of interval arithmetic is the Intlab, a toolbox for Matlab which supports real and complex intervals, vectors and matrices. The Intlab concept is divided into a rapid library of interval arithmetic and an interactive programming environment for easily accessible interval operations. Arithmetical operations in Intlab are rigorously verified to be correct, including input and output and standard functions. By that, it is possible to replace every operation of a standard numerical algorithm by the corresponding interval operations. Standard functions, such as trigonometric and exponential functions, are available and are used in the usual Matlab form [Rump, 1999].

In this work, the input of an interval variable in Intlab occurred through the function `intval('x0')`, which creates an interval in which the lower and upper limits are, respectively, the floating point corresponding to the negative rounding and the floating point corresponding to the positive rounding. Thus, the range of values limited by those numbers contain  $x_0$ .

## 3. Methodology

In order to reduce the effect of overestimation, the intersection between pseudo-orbits obtained by different interval extensions is used, here denoted by IIE. This method is based on the concept presented below. Let  $\{x_0, x_1, \dots, x_n\}$  be a true orbit of  $f$  and let  $\{\hat{X}_{i,0}, \hat{X}_{i,1}, \dots, \hat{X}_{i,n}\}$  be a pseudo-orbit obtained by some rational interval extension of  $f$ , given by  $F_i$ ,  $i \in \mathbb{N}$ . From Definition 2.7 it is clear that if  $x_0 \in \hat{X}_{i,0}$ , then

$$x_n \in \hat{X}_{i,n}. \quad (13)$$

From this we may establish the following for two natural interval extensions  $F_1$  and  $F_2$ , which generate two sequences  $\hat{X}_{1,0}$  and  $\hat{X}_{2,0}$ , respectively.

**Theorem 3.1.** *If  $x_0 \in \hat{X}_{1,0}$  and  $x_0 \in \hat{X}_{2,0}$ , then  $x_n \in [\hat{X}_{1,n} \cap \hat{X}_{2,n}]$ ,  $n \in \mathbb{N}$ .*

*Proof.* Let us assume  $x_n \notin [\hat{X}_{1,n} \cap \hat{X}_{2,n}]$ ,  $n \in \mathbb{N}$ . Hence,  $x_n \notin \hat{X}_{1,n}$  or  $x_n \notin \hat{X}_{2,n}$  or both. According to (13), it means that  $\hat{X}_{1,n}$  or  $\hat{X}_{2,n}$  is not a pseudo-orbit, which is a contradiction. That completes the proof. ■

However, in some real implementation of SIA, the intersection of two or more interval extensions might result in an empty interval. Thus, Definition 3.2 has been proposed.

**Definition 3.2.** Let  $\hat{F}_{1,n}$  and  $\hat{F}_{2,n}$ ,  $n \in \mathbb{N}$ , be rational interval extensions of  $f$ , given by  $F_i$ ,  $i \in \mathbb{N}$ . If  $\hat{F}_n = [\hat{F}_{1,n} \cap \hat{F}_{2,n}] = \emptyset$ , then  $\hat{F}_n = [\hat{F}_{1,n} \cup \hat{F}_{2,n}]$ .

The IIE method is summarized in the following steps:

- (1) Elaborate interval extensions  $F_i$  for the recursive function under investigation  $f$ .
- (2) Define the same parameters for all interval functions and a maximum number of iterations.
- (3) Set an initial interval  $\hat{X}_0 = [\underline{\hat{X}}_0, \overline{\hat{X}}_0]$  in such a way that  $x_0 \in \hat{X}_0$ .
- (4) Calculate the response of the current iteration for each interval extension with the previously obtained interval.
- (5) Calculate the intersection between the intervals obtained for each interval extension. The interval obtained in this step should represent the pseudo-orbit interval of the function for the current iteration.
- (6) Return to step 4 and repeat the calculations until the iterations reach a maximum number or the pseudo-orbit has converged according to tolerance criterion.

### 3.1. Discrete dynamical systems

In order to reduce the overestimation effects and the memory consumption of the computer, the mean value form for the logistic map applying (11) is given by

$$F_{n+1} = r(y_n(1 - y_n) + (1 - 2F_n)(X_n - y_n)), \quad (14)$$

where  $X_n$  is the interval which contains  $x_n$  and  $y_n$  is the midpoint of  $F_n$  [Bruguera, 2014].

We used two interval extensions of the logistic map represented in the mean value form to evaluate the proposed method. These functions are given as follows:

$$F_{1,n+1} = r(y_n(1 - y_n) + (1 - 2F_{1,n})(X_n - y_n)), \quad (15)$$

$$F_{2,n+1} = y_n(r - ry_n) + (r - 2rF_{2,n})(X_n - y_n). \quad (16)$$

The results obtained by  $F_{1,n+1} \cap F_{2,n+1}$  are compared with the results of  $F_{n+1}$ , given by Eq. (14). Besides, Intlab was used to obtain the results by interval arithmetic operations of the traditional form of the logistic map, given by Eq. (17)

$$H_{n+1} = rH_n(1 - H_n). \quad (17)$$

Four cases for computing the pseudo-orbits in the logistic map were analyzed:

- **Case 1.**  $r = 3.60$  and  $x_0 = 0.6$ ;
- **Case 2.**  $r = 3.70$  and  $x_0 = 0.6$ ;
- **Case 3.**  $r = 3.85$  and  $x_0 = 0.6$ ;
- **Case 4.**  $r = 3.95$  and  $x_0 = 0.6$ .

### 3.2. Uncertain parameters

In the second application of IIE, we analyze parametric uncertainties in a series RLC electric circuit. This circuit is composed of one resistor, one inductor and one capacitor connected in series. Initially we performed the traditional simulation of the circuit by means of Matlab software. We choose an underdamped output with ( $0 < \xi < 1$ ) and components presented in Table 1. It was also considered as a resistance of  $7.8 \Omega$  to the inductor.

The circuit was implemented in a breadboard EPB0058 and data was collected by means of an Oscilloscope DSO-X, 70 MHz, Agilent. The output voltage on the capacitor due to a unit step as input of the RLC circuit, is given by Eq. (18)

$$v_C(t) = 1 - e^{-\xi\omega t} \left( \cos \omega_d t + \frac{\xi}{\sqrt{1 - \xi^2}} \sin \omega_d t \right). \quad (18)$$

Then we compare the nominal simulation with experimental results and lastly we did the simulation using the Intlab toolbox, where the circuit components are seen as intervals, such that the bounds of the intervals are simply obtained by  $[N_v(1 - \delta/100), N_v(1 + \delta/100)]$ .

Table 1. List of components used in the circuit, nominal value and tolerance ( $\delta$ ).

Component	Nominal Value ( $N_v$ )	Tolerance ( $\delta$ ) (%)
Resistor	100 $\Omega$	5
Inductor	0.1 H	10
Capacitor	100 nF	20

In order to analyze the effect of pseudo-orbits obtained from different interval extensions of Eq. (18), we elaborate:

$$f_1(t) = 1 - e^{-\xi\omega t} \left( \cos \omega_d t + \frac{\xi}{\sqrt{1 - \xi^2}} \sin \omega_d t \right), \quad (19)$$

$$f_2(t) = 1 - e^{-\xi\omega t} \cos \omega_d t - e^{-\xi\omega t} \frac{\xi}{\sqrt{1 - \xi^2}} \sin \omega_d t. \quad (20)$$

For both applications, the first interval was obtained rounding  $x_0$  down and up, ensuring that the endpoints  $\underline{\hat{X}}_0$  and  $\overline{\hat{X}}_0$  were the lowest and the highest floating points near  $x_0$ . All calculations were done using Matlab with double precision for Windows.

### 3.3. Reduction factor

In order to give an account of how much a width of an interval has been decreased, we use the following index:

$$R_f = \log_{10} \left( \frac{w(X)}{w(Y)} \right), \quad (21)$$

where  $w(X)$  is the reference width and  $w(Y)$  is the reduced width by an application of a specific method. For instance, let  $w(X) = 0.0001$  and  $w(Y) = 0.000001$ , we have  $R_f = 2$ . When the difference is not so large, we also make use of percentage.

## 4. Results

In this section, we present the two numerical examples investigated. First, we have revisited the procedure to reduce the interval width for the logistic map. The reduction is obtained by means of the mean value form. The second example is devoted to simulate a RLC circuit with parametric uncertainties. In this case, we have to change the usual procedure as the SIA implementation has presented an inconsistent result. This result has been analyzed and a simple step has been adopted to avoid this problem, according to Algorithm 1.

### 4.1. Width reduction for the logistic map

The results obtained for iterations 1, 5, 15 and 25 by the IIE of Eqs. (15) and (16) are compared with

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**Algorithm 1** Pseudo-code of the IIE method.

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1: input Number of iterations (N), parameters and initial conditions
2:  $X_0 \leftarrow \text{intval}('x_0')$  ▷ Initial conditions
3: for  $n=1:N$  do
4:    $\hat{F}_{1,n+1} \leftarrow F(\hat{X}_n)$  ▷ Interval Extension 1
5:    $\hat{F}_{2,n+1} \leftarrow F(\hat{X}_n)$  ▷ Interval Extension 2
6:    $F_{n+1} \leftarrow \hat{F}_{1,n+1} \cap \hat{F}_{2,n+1}$ 
7:   if  $F_{n+1} = \emptyset$  then
8:      $F_{n+1} \leftarrow \hat{F}_{1,n+1} \cup \hat{F}_{2,n+1}$ 
9:   end if
10: end for

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the calculation of the logistic map in the mean value form, using Eq. (14), represented by  $F_{n+1}$  and the calculation of  $H_{n+1}$  using Eq. (17). These comparisons are presented in Table 2, which presents the reduction on the size of the interval obtained at iteration  $n$  for each case.

Analyzing Table 2 we can see that the mean value form presents a reduction on the size of the intervals with respect to the traditional form  $H_{n+1}$ . Besides that, for all cases, the intervals obtained by the IIE present a reduction on their sizes in relation to the mean value form.

Although the method of the MVF [Moore & Moore, 1979; Lohner, 1993] reduces the effects of overestimation, after some iterations, the intervals obtained tend to increase in such a way as to extrapolate the limits of representation of the variable  $x_n$  and diverge to infinite. Figure 1 shows the evolution of the size of intervals for  $F_{n+1}$  and  $F_{1,n+1} \cap F_{2,n+1}$  with  $r = 3.85$  and  $x_0 = 0.6$ , where it can be observed that the effect of the overestimation for the proposed method is much smaller than for the method of mean value form. The reduction, in percentage, of the proposed method with respect to the

Table 2. Comparison of width size for the simulation of logistic map. The reference is the width produced by means of Intlab [Rump, 1999]. We show the results using the mean value form (MVF) presented by [Moore & Moore, 1979; Lohner, 1993] and our method of intersection of interval extensions (IIE). The reduction is calculated using a  $\log_{10}$  scale compared to Intlab reference, according to Eq. (21). We have studied four cases, of which the parameters and initial conditions are as follows: (1):  $r = 3.60$  and  $x_0 = 0.6$ . (2):  $r = 3.70$  and  $x_0 = 0.6$ . (3):  $r = 3.85$  and  $x_0 = 0.6$ . (4):  $r = 3.95$  and  $x_0 = 0.6$ .

Case	$n$	Intlab	Mean Value Form		IIE	
		Width	Width	$R_f$	Width	$R_f$
(1)	1	$1.110 \times 10^{-16}$	$1.110 \times 10^{-16}$	0	$1.110 \times 10^{-16}$	0
	5	$3.486 \times 10^{-14}$	$1.610 \times 10^{-15}$	1.34	$6.661 \times 10^{-16}$	1.72
	15	$1.279 \times 10^{-08}$	$3.009 \times 10^{-14}$	5.63	$1.121 \times 10^{-14}$	6.06
	25	0.0046	$5.884 \times 10^{-14}$	10.90	$2.243 \times 10^{-14}$	11.32
(2)	1	$1.110 \times 10^{-16}$	$1.110 \times 10^{-16}$	0	$1.110 \times 10^{-16}$	0
	5	$3.664 \times 10^{-14}$	$2.776 \times 10^{-15}$	1.12	$1.221 \times 10^{-15}$	1.48
	15	$1.760 \times 10^{-08}$	$8.6385 \times 10^{-14}$	5.31	$4.041 \times 10^{-14}$	5.64
	25	0.0085	$6.698 \times 10^{-13}$	10.10	$3.129 \times 10^{-13}$	10.43
(3)	1	$1.110 \times 10^{-16}$	$1.110 \times 10^{-16}$	0	$1.110 \times 10^{-16}$	0
	5	$4.119 \times 10^{-14}$	$3.331 \times 10^{-15}$	1.09	$1.332 \times 10^{-15}$	1.49
	15	$2.954 \times 10^{-08}$	$1.332 \times 10^{-12}$	4.35	$5.393 \times 10^{-13}$	4.74
	25	0.0211	$2.378 \times 10^{-10}$	7.95	$9.629 \times 10^{-11}$	8.34
(4)	1	$1.1102 \times 10^{-16}$	$1.1102 \times 10^{-16}$	0	$1.1102 \times 10^{-16}$	0
	5	$4.330 \times 10^{-14}$	$2.443 \times 10^{-15}$	1.25	$1.110 \times 10^{-15}$	1.59
	15	$4.009 \times 10^{-08}$	$3.759 \times 10^{-12}$	4.03	$1.765 \times 10^{-12}$	4.36
	25	0.0370	$1.347 \times 10^{-10}$	8.44	$6.325 \times 10^{-11}$	8.77

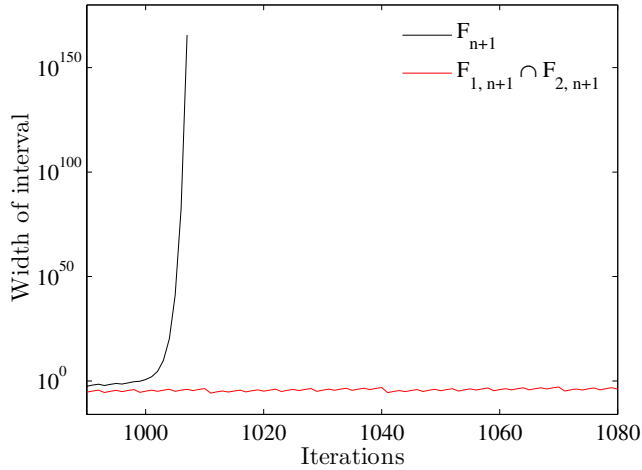


Fig. 1. Width of intervals obtained with  $r = 3.85$  and  $x_0 = 0.6$  by  $F_{n+1}$  and  $F_{1,n+1} \cap F_{2,n+1}$ .

mean value form, along the iterations, can be seen in Fig. 2.

#### 4.2. Parametric uncertainties

Figure 3 presents a comparison between the simulation of Eqs. (19) and (20) obtained by Matlab and the response obtained experimentally. We can observe from Fig. 3 that all curves do not totally coincide and present an apparently considerable error. Considering the existing errors associated with the inherent properties of the components, the finite accuracy of the measuring equipment and the errors propagated during the simulation, it is difficult to say which one faithfully represents the characteristic of the circuit. Therefore, it is more reliable to use a simulation that contains the associated errors by means of SIA.

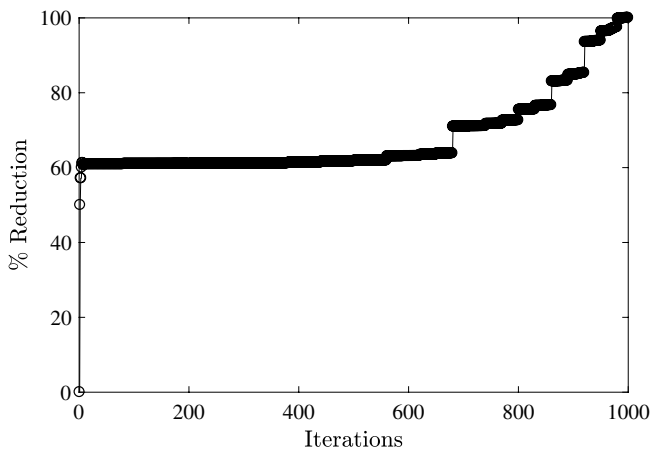


Fig. 2. Reduction of interval obtained with  $r = 3.85$  and  $x_0 = 0.6$  by  $F_{n+1}$  and  $F_{1,n+1} \cap F_{2,n+1}$ .

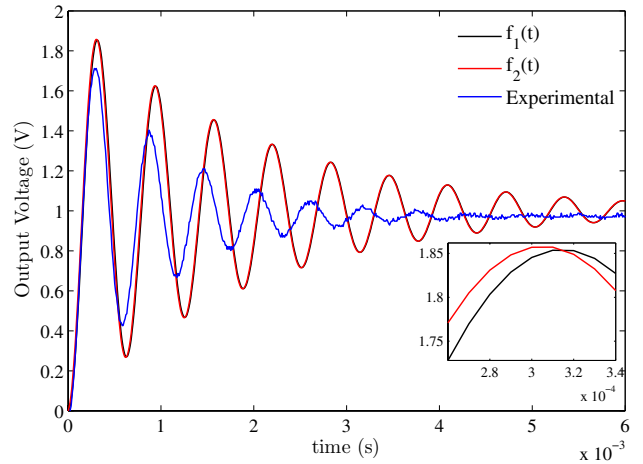


Fig. 3. Step response RLC circuit: Comparison between nominal simulation, according to Eqs. (19) and (20), and experimental data.

Proper treatment of measurement errors is a crucial aspect of careful laboratory procedure. We should have tools to identify sources of experimental errors, establish uncertainties in measurements, and propagate those uncertainties through calculations leading to the final results [Rothwell & Cloud, 2012]. The curve presented in Fig. 4 was obtained also using the Intlab toolbox. It is possible to notice that the response found through the toolbox encompasses the traditional simulation and the response obtained experimentally. In this case, the estimation of the errors were satisfied.

In order to reduce the width of the interval, the IIE was applied from different interval extensions of Eq. (18), according to Eqs. (19) and (20). It is possible to notice in Fig. 5 that the reduction

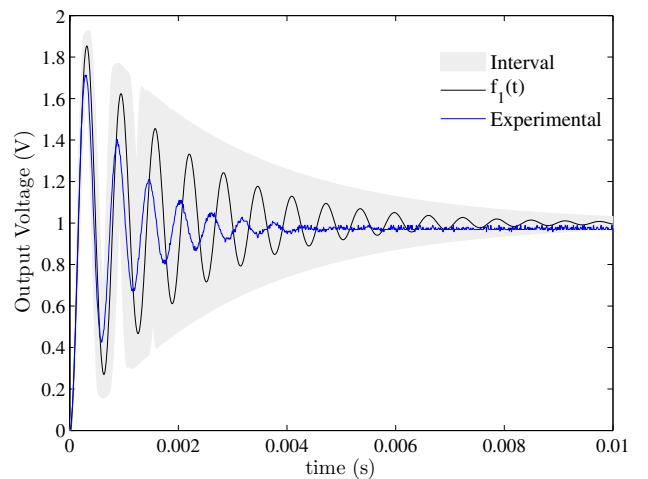


Fig. 4. Comparison between the interval simulation, nominal simulation and experimental by step input.



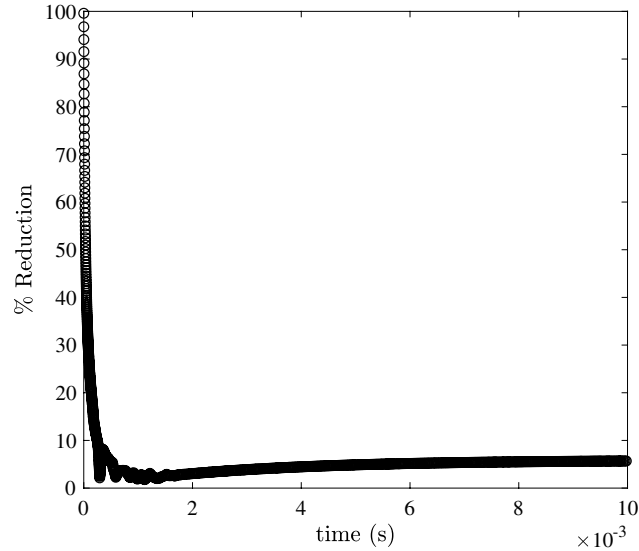


Fig. 5. Reduction in the width of the step response interval using the interval intersection of Eq. (18).

Table 3. Analysis of intersection of Eqs. (19) and (20), with a step-size  $h = 1e^{-6}$ . The last column presents the result obtained by means of Definition 3.2.

$t$	$f_1(t)$	$f_2(t)$	$f_1(t) \cap f_2(t)$	$f_1(t) \cup f_2(t)$
1	0	0	0	0
2	$[-0.0003, 0.0004]$	$[0.0007, 0.0015]$	$\emptyset$	$[-0.0003, 0.0015]$
3	$[-0.0006, 0.0010]$	$[0.0016, 0.0031]$	$\emptyset$	$[-0.0006, 0.0031]$
4	$[-0.0007, 0.0017]$	$[0.0024, 0.0049]$	$\emptyset$	$[-0.0007, 0.0049]$
5	$[-0.0008, 0.0026]$	$[0.0034, 0.0068]$	$\emptyset$	$[-0.0008, 0.0068]$
6	$[-0.0009, 0.0035]$	$[0.0045, 0.0088]$	$\emptyset$	$[-0.0009, 0.0088]$
7	$[-0.0008, 0.0047]$	$[0.0056, 0.0110]$	$\emptyset$	$[-0.0008, 0.0110]$
8	$[-0.0007, 0.0059]$	$[0.0068, 0.0133]$	$\emptyset$	$[-0.0007, 0.0133]$
9	$[-0.0005, 0.0073]$	$[0.0080, 0.0158]$	$\emptyset$	$[-0.0005, 0.0158]$
10	$[-0.0002, 0.0088]$	$[0.0094, 0.0183]$	$\emptyset$	$[-0.0002, 0.0183]$
11	$[0.0002, 0.0105]$	$[0.0108, 0.0211]$	$\emptyset$	$[0.0002, 0.0211]$
12	$[0.0006, 0.0123]$	$[0.0122, 0.0239]$	$[0.0122, 0.0123]$	—

reaches up to 95% and in the steady state regime it is around 5%, which is significant as the tolerance of the components presents similar magnitude. However, as can be seen from Table 3, the intersection of Eqs. (19) and (20) results in an empty interval for a given time period. From this, we applied Definition 3.2 and the last column presents the result. We believe that, although Eqs. (19) and (20) are not recursive, they use trigonometrical functions, which are intrinsic algorithms that use recursive functions.

## 5. Conclusion

Interval arithmetic has been considered a reliable alternative to the conventional floating-point arithmetic [Bruguera, 2014]. It provides a great

diversity of algorithms and methods for a wide range of applications. Among the great number of versions, the standard interval arithmetic (SIA) has been considered the most trustworthy [Moore *et al.*, 2009]. However, care must be taken that there is no overestimation of the interval width. Moreover, less attention has been paid for physical implementation of SIA in digital computer. This paper has reported such a situation when an inconsistent theoretical result has been found in an implementation of SIA.

First, we have investigated the reduction of interval width in the simulation of the logistic map. The results obtained by the intersection of interval extensions (IIE) have been compared to values obtained by the use of Intlab and the mean

value form. The IIE has overwhelmed both strategies showing a reduction factor equal to or smaller than all four investigated cases. Furthermore, the IIE has also been able to avoid pseudo-orbit divergence, contrary to what happened for the mean value form approach. This is an important aspect that has been highlighted by Rothwell and Cloud [2012] as a desirable feature of refined algorithm for interval arithmetic.

In the second application, the IIE has been applied for the treatment of uncertainties in the simulation of a RLC circuit. Traditional simulations and data collected from laboratory experiments show considerable differences due to the errors associated with simulation and inherent errors of the practical circuit, as can be seen in Fig. 3. In order to reduce the interval width, IIE extensions of the step response equation were used. The reduction in the interval width has been observed as above — 95% in transient regime and 5% in permanent regime. This is a type of improvement expected in [Rothwell & Cloud, 2012]. However, for some time instants (see Table 3), the IIE resulted in empty interval. According to interval analysis this is an unacceptable fact, as both intervals derived from the two natural interval extensions [see Eqs. (19) and (20)] should contain the real values. This problem has been overcome by introducing a test in each iteration, and if it is necessary, a union operation, as shown in Algorithm 1. This simple approach has been shown to be satisfactory to ensure that the true response is contained in the interval obtained.

Finally, we would like to point out the importance of a critical judgement of the results. Even though the SIA has been investigated as an approach to take rigor to a computer simulation, its physical implementation, particularly for functions other than the simple arithmetic operations, in digital computer requires care and a critical observation of the results. As a matter of fact, new challenges of science, such as the Big Data, do not only require a hardware revolution as suggested in [Editorial Nature, 2018], but it will need further investigation to properly operate these super power machines, which should certainly be finite, and therefore, subject to some sort of error.

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## References

- Bruguera, J. D. [2014] “Optimizing the representation of intervals,” *Sci. Comput. Program.* **90**, 21–33.
- Chalco-Cano, Y., Lodwick, W. & Bede, B. [2014] “Single level constraint interval arithmetic,” *Fuzzy Sets Syst.* **257**, 146–168.
- Editorial Nature [2018] “Big data needs a hardware revolution,” *Nature* **554**, 145–146.
- Feigenbaum, M. J. [1978] “Quantitative universality for a class of nonlinear transformations,” *J. Statist. Phys.* **19**, 25–52.
- Galias, Z. [2002] “Proving the existence of long periodic orbits in 1D maps using interval Newton method and backward shooting,” *Topol. Appl.* **124**, 25–37.
- Galias, Z. [2013] “The dangers of rounding errors for simulations and analysis of nonlinear circuits and systems — And how to avoid them,” *IEEE Circuits Syst. Mag.* **13**, 35–52.
- Hammel, S. M., Yorke, J. A. & Grebogi, C. [1987] “Do numerical orbits of chaotic dynamical processes represent true orbits?” *J. Compl.* **3**, 136–145.
- Hargreaves, G. I. [2002] “Interval analysis in Matlab,” *Analysis* **416**, 1–49.
- Hofschuster, W. & Krämer, W. [2004] “C-XSC 2.0 A C++ Library for extended scientific computing,” *Numerical Software with Result Verification*, eds. Alt, R., Frommer, A., Kearfott, R. B. & Luther, W. (Springer, Berlin, Heidelberg), pp. 15–35.
- Kulisch, U. W. [2009] “Complete interval arithmetic and its implementation on the computer,” *Numerical Validation*, Lecture Notes in Computer Science, Vol. 5492 (Springer, Berlin, Heidelberg), pp. 7–26.
- Ling, B. W.-K., Ho, C. Y.-F. & Tam, P.-K. S. [2006] “Nonlinear behaviors of first- and second-order complex digital filters with two’s complement arithmetic,” *IEEE Trans. Sign. Process.* **54**, 4052–4055.
- Lohner, R. J. [1993] “Interval arithmetic in staggered correction format,” *Scientific Computing with Automatic Result Verification*, eds. Adams, E. & Kulisch, U. (Elsevier), pp. 301–321.
- Lozi, R. [2013] “Can we trust in numerical computations of chaotic solutions of dynamical systems?” *Topology and Dynamics of Chaos*, World Scientific Series in Nonlinear Science Series A, Vol. 84 (World Scientific), pp. 63–98.
- Macau, E. E. N. & Pando Lambruschini, C. L. [2014] “Advanced computational and experimental techniques in nonlinear dynamics,” *Eur. Phys. J. Special Topics* **223**, 2645–2648.

- May, R. M. [1976] “Simple mathematical models with very complicated dynamics,” *Nature* **261**, 459–467.
- Mendes, E. M. A. M. & Nepomuceno, E. G. [2016] “A very simple method to calculate the (positive) largest Lyapunov exponent using interval extensions,” *Int. J. Bifurcation and Chaos* **26**, 1650226-1–7.
- Moore, R. E. & Moore, R. E. [1979] *Methods and Applications of Interval Analysis*, Studies in Applied and Numerical Mathematics, Vol. 2 (SIAM).
- Moore, R. E., Kearfott, R. B. & Cloud, M. J. [2009] *Introduction to Interval Analysis* (SIAM).
- Neher, M. & Eble, I. [2004] “Costly: A validated library for complex functions,” *Proc. Appl. Math. Mech.* **4**, 594–595.
- Nepomuceno, E. G. [2014] “Convergence of recursive functions on computers,” *J. Engin.*, 1–3.
- Nepomuceno, E. G. & Martins, S. A. M. [2016] “A lower bound error for free-run simulation of the polynomial NARMAX,” *Syst. Sci. Contr. Engin.* **4**, 50–58.
- Nepomuceno, E. G. & Mendes, E. M. [2017] “On the analysis of pseudo-orbits of continuous chaotic nonlinear systems simulated using discretization schemes in a digital computer,” *Chaos Solit. Fract.* **95**, 21–32.
- Nepomuceno, E., Martins, S., Amaral, G. & Riveret, R. [2017] “On the lower bound error for discrete maps using associative property,” *Syst. Sci. Contr. Engin.* **5**, 462–473.
- Piegat, A. & Landowski, M. [2015] “Is the conventional interval arithmetic correct?” *J. Theoret. Appl. Comput. Sci.* **6**, 27–44.
- Piegat, A. & Landowski, M. [2017] “Is an interval the right result of arithmetic operations on intervals?” *Int. J. Appl. Math. Comput. Sci.* **27**.
- Rothwell, E. J. & Cloud, M. J. [2012] “Automatic error analysis using intervals,” *IEEE Trans. Education* **55**, 9–15.
- Rump, S. [1999] “INTLAB — INTerval LABoratory,” *Developments in Reliable Computing*, ed. Csendes, T. (Kluwer Academic Publishers, Dordrecht), pp. 77–104.
- Saberi Nik, H. [2015] “Finite-time chaos control for a chaotic complex-variable system,” *J. Contr. Autom. Electr. Syst.* **26**, 371–379.