ON GENERAL PROPERTIES OF RETARDED FUNCTIONAL DIFFERENTIAL EQUATIONS ON MANIFOLDS

PIERLUIGI BENEVIERI, ALESSANDRO CALAMAI, MASSIMO FURI,
AND MARIA PATRIZIA PERA

ABSTRACT. We investigate general properties, such as existence and uniqueness, continuous dependence on data and continuation, of solutions to retarded functional differential equations with infinite delay on a differentiable manifold.

1. INTRODUCTION

The purpose of this paper is to analyze the general properties of retarded functional differential equations (RFDEs for short) with infinite delay on a smooth boundaryless manifold. We will discuss existence and uniqueness of solutions, continuous dependence on data and continuation of solutions to such equations.

Let $N \subseteq \mathbb{R}^k$ be a boundaryless smooth manifold. Denote by $BU((-\infty, 0], N)$ the set of bounded and uniformly continuous maps from $(-\infty, 0]$ into N with the topology of the uniform convergence and let Ω be an open subset of $\mathbb{R} \times BU((-\infty, 0], N)$. A continuous map $g \colon \Omega \to \mathbb{R}^k$ will be called a *functional field over* N provided that $g(t, \varphi) \in T_{\varphi(0)}N$ for all $(t, \varphi) \in \Omega$, where, given $p \in N$, by $T_pN \subseteq \mathbb{R}^k$ we denote the tangent space of N at p.

In this paper we will consider RFDEs of the type

$$(1.1) \qquad x'(t) = g(t, x_t),$$

where $g \colon \Omega \subseteq \mathbb{R} \times BU((-\infty, 0], N) \to \mathbb{R}^k$ is a functional field over N. By a *solution* of the above equation we mean a function $x \colon J \to N$, defined on an open real interval J with $\inf J = -\infty$, bounded and uniformly continuous on any closed half-line $(-\infty, b] \subset J$, and which verifies eventually the equality $x'(t) = g(t, x_t)$. As usual, given $t \in J$, by $x_t \in BU((-\infty, 0], N)$ we mean the function $\theta \mapsto x(t + \theta)$.

Given $(\tau, \eta) \in \Omega$, we will be interested in the initial value problem

$$(1.2) \qquad \begin{cases} x'(t) = g(t, x_t), \\ x_\tau = \eta. \end{cases}$$

A solution of problem (1.2) is a function $x \colon J \to N$, with $\sup J > \tau$, such that $x'(t) = g(t, x_t)$ for $t > \tau$, and $x_\tau = \eta$.

Given a relatively closed subset X of N, and assuming that the initial function η is X-valued, we will also study the problem in which at least one solution of (1.2), possibly the unique one, remains *confined* in X. A typical situation is when X is a manifold with boundary (a ∂-manifold) and N is the double of X. A convenient "confining" subset X of N could also be obtained by cutting away from N an appropriate open subset. This is the case considered in Example 5.1 in which we study the forced oscillations of a motion problem constrained to a compact boundaryless manifold M. Roughly speaking, in the presence of friction, we define

X as the subset of the tangent bundle $N = TM$ consisting of the elements whose "speed" does not exceed a suitable value c.

The different and related cases of RFDEs with finite delay in Euclidean spaces have been investigated by many authors. For general reference we suggest the monograph by Hale and Verduyn Lunel [9]. We refer also to the works of Gaines and Mawhin [6], Nussbaum [15, 16] and Mallet-Paret, Nussbaum and Paraskevopoulos [12]. For RFDEs with infinite delay we recommend the article of Hale and Kato [8] and the book by Hino, Murakami and Naito [10].

For RFDEs with finite delay on manifolds we cite the papers of Oliva [17, 18]. However, as far as we know, no general theory is available in literature for RFDEs on manifolds with infinite delay. Our contribution in filling this gap represents the principal motivation for this paper.

The choice of $\mathbb{R} \times BU((-\infty, 0], N)$ as the the "ambient" space containing the domain Ω of the functional field g is due to its sufficiently strong topology. This fact makes the assumption of continuity of g a weak condition. We point out that with this topology, if $x \colon J \to N$ is a solution of (1.1), then the curve $t \mapsto x_t \in BU((-\infty, 0], N)$, $t \in J$, is continuous, as it should be to ensure the continuity of $t \mapsto g(t, x_t)$.

Another motivation for chosing $\mathbb{R} \times BU((-\infty, 0], N)$ is related to some results obtained in [1, 2, 3], regarding retarded motion problems constrained to a compact boundaryless manifold. In these papers the ambient space is $\mathbb{R} \times C((-\infty, 0], N)$ with its natural compact-open topology. Since $C((-\infty, 0], N)$ induces on its subset $BU((-\infty, 0], N)$ a topology which is weaker than the one considered here, we hope that our investigation in this article could lay the groundwork for possible extensions of the results obtained in [1, 2, 3].

2. Preliminaries, initial value problem and uniqueness of solutions

Given a subset A of $\mathbb{R}^k$, we will denote by $BU((-\infty, 0], A)$ the set of bounded and uniformly continuous maps from $(-\infty, 0]$ into A with the topology of the uniform convergence. Clearly, $BU((-\infty, 0], A)$ is a metric subspace of the Banach space $BU((-\infty, 0], \mathbb{R}^k)$ and is complete, whenever A is closed. The closure and the complement of A (in $\mathbb{R}^k$) will be denoted by $\overline{A}$ and A^c, respectively. Moreover, the norm in $\mathbb{R}^k$ will be denoted by $|\cdot|$ and the norm in $BU((-\infty, 0], \mathbb{R}^k)$ by $\|\cdot\|$. For the sake of simplicity, throughout the paper any norm in an infinite dimensional vector space will be denoted by $\|\cdot\|$.

We recall that a subset Q of $BU((-\infty, 0], A)$ is precompact (i.e. totally bounded) if and only if it is bounded and given any $\varepsilon > 0$ there exists a finite covering $\mathcal{F}$ of subsets of $(-\infty, 0]$ such that the oscillation of any $\varphi \in Q$ in each $S \in \mathcal{F}$ is less than ε (see e.g. [5, Part 1, IV.6.5]). Thus, if A is closed, any precompact subset of $BU((-\infty, 0], A)$ is relatively compact.

The following is a well-known result for continuous maps between metric spaces.

Lemma 2.1. *Let* $f \colon \mathcal{X} \to \mathcal{Y}$ *be a continuous map between two metric spaces and let* K *be a compact subset of* $\mathcal{X}$. *Then, for any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$x \in \mathcal{X}, \ k \in K, \ \mathrm{dist}_{\mathcal{X}}(x, k) < \delta \implies \mathrm{dist}_{\mathcal{Y}}(f(x), f(k)) < \varepsilon \,.$$

Remark 2.2 below will be used in the proof of the existence of a global solution and in the proof of the continuous dependence of the solutions on the initial data.

Remark 2.2. Let $f \colon \mathcal{X} \to \mathcal{Y}$ be a continuous map between metric spaces and let $\{x_n\}$ be a sequence of continuous functions from a compact interval $[a, b]$ (or, more generally, from any compact space) into $\mathcal{X}$. If $\{x_n(s)\}$ converges to $x(s)$ uniformly for $s \in [a, b]$, then $f(x_n(s)) \to f(x(s))$ uniformly for $s \in [a, b]$. This assertion

follows immediately from the above lemma, by taking the compact K to be the image of the limit function $x \colon [a, b] \to \mathcal{X}$.

Let A be an arbitrary subset of $\mathbb{R}^k$. We recall the notions of tangent cone and tangent space of A at a given point p in the closure $\overline{A}$ of A. The definition of tangent cone is equivalent to the classical one introduced by Bouligand in [4].

Definition 2.3. A vector $v \in \mathbb{R}^k$ is said to be *inward* to A at $p \in \overline{A}$ if there exist two sequences $\{\alpha_n\}$ in $[0, +\infty)$ and $\{p_n\}$ in A such that

$$p_n \to p \quad \text{and} \quad \alpha_n(p_n - p) \to v.$$

The set $C_p A$ of the inward vectors to A at p is called the *tangent cone* of A at p. The *tangent space* $T_p A$ of A at p is the vector subspace of $\mathbb{R}^k$ spanned by $C_p A$. A vector v of $\mathbb{R}^k$ is said to be *tangent* to A at p if $v \in T_p A$. One can easily check that the tangent cone is always closed in $\mathbb{R}^k$.

To simplify some statements and definitions we put $C_p A = T_p A = \emptyset$ whenever $p \in \mathbb{R}^k$ does not belong to $\overline{A}$ (this can be regarded as a consequence of Definition 2.3 if one replaces the assumption $p \in \overline{A}$ with $p \in \mathbb{R}^k$). Observe that $T_p A$ is the trivial subspace $\{0\}$ of $\mathbb{R}^k$ if and only if p is an isolated point of A. In fact, if p is a limit point, then, given any $\{p_n\}$ in $A \backslash \{p\}$ such that $p_n \to p$, the sequence $\{\alpha_n(p_n - p)\}$, with $\alpha_n = 1/|p_n - p|$, admits a convergent subsequence whose limit is a unit vector.

One can show that, in the special and important case when A is a smooth manifold with (possibly empty) boundary ∂A (a ∂-*manifold* for short), this definition of tangent space is equivalent to the classical one (see for instance [13], [7]). Moreover, if $p \in \partial A$, $C_p A$ is a closed half-space in $T_p A$ (delimited by $T_p \partial A$), while $C_p A = T_p A$ if $p \in A \backslash \partial A$. Finally, a vector v is said to be *strictly inward* at $p \in \partial A$ if $v \in C_p A \backslash T_p \partial A$.

Let $g \colon \Omega \subseteq \mathbb{R} \times BU((-\infty, 0], A) \to \mathbb{R}^k$ be a continuous map on an open subset Ω of $\mathbb{R} \times BU((-\infty, 0], A)$. We say that g is a *(retarded) functional (tangent vector) field (over A)* if $g(t, \varphi) \in T_{\varphi(0)} A$ for all $(t, \varphi) \in \Omega$. In particular, g will be said *inward* (to A) if $g(t, \varphi) \in C_{\varphi(0)} A$ for all (t, φ).

Let us consider a *retarded functional differential equation* (*RFDE* for short) of the type

$$(2.1) \qquad\qquad\qquad x'(t) = g(t, x_t),$$

where $g \colon \Omega \subseteq \mathbb{R} \times BU((-\infty, 0], A) \to \mathbb{R}^k$ is a functional field over A. Here, as usual and whenever it makes sense, given $t \in \mathbb{R}$, by $x_t \in BU((-\infty, 0], A)$ we mean the function $\theta \mapsto x(t + \theta)$.

By a *solution* of (2.1) we mean a function $x \colon J \to A$, defined on an open real interval J with $\inf J = -\infty$, bounded and uniformly continuous on any closed half-line $(-\infty, b] \subset J$, and which verifies eventually the equality $x'(t) = g(t, x_t)$. That is, x is a solution of (2.1) if there exists τ, with $-\infty \leq \tau < \sup J$, such that $(t, x_t) \in \Omega$ for all $t \in (\tau, \sup J)$, x is C^1 on the subinterval $(\tau, \sup J)$ of J, and verifies $x'(t) = g(t, x_t)$ for all $t \in (\tau, \sup J)$. Observe that, the derivative of a solution x may not exist at $t = \tau$. However, the right derivative $D_+ x(\tau)$ of x at τ always exists and is equal to $g(\tau, x_\tau)$. Also, notice that, since x is uniformly continuous on any closed half-line $(-\infty, b]$ of J, then $t \mapsto x_t$ is a continuous curve in $BU((-\infty, 0], A)$.

A solution of (2.1) is said to be *maximal* if it is not a proper restriction of another solution to the same equation. As in the case of ODEs, Zorn's lemma implies that any solution is the restriction of a maximal solution.

Given $(\tau, \eta) \in \Omega$, we will also be interested in the following initial value problem:

$$(2.2) \qquad \begin{cases} x'(t) = g(t, x_t), \\ x_\tau = \eta. \end{cases}$$

A *solution* of problem (2.2) is a solution $x \colon J \to A$ of (2.1) such that $\sup J > \tau$, $x'(t) = g(t, x_t)$ for $t > \tau$, and $x_\tau = \eta$.

Clearly, $x \colon J \to A$ is a solution of (2.2) if and only if

$$(2.3) \qquad x(t) = \begin{cases} \eta(0) + \int_\tau^t g(s, x_s)\, ds, & \tau \le t < \sup J, \\ \eta(t - \tau), & t \le \tau. \end{cases}$$

Below, we will be concerned with the uniqueness of the solution of equation (2.1) and of problem (2.2).

In proving our uniqueness results, we will make use of the following folk result, whose proof is given here for the sake of completeness.

Lemma 2.4. *Let $\alpha : [\tau, \tau + h) \to \mathbb{R}^k$ $(0 < h \le +\infty)$ be a C^1 function such that $\alpha(\tau) = 0$ and*

$$|\alpha'(t)| \le c \sup_{\tau \le s \le t} |\alpha(s)|, \quad t \in [\tau, \tau + h)$$

for some constant $c \ge 0$. Then, $\alpha(t) = 0$ for all $t \in [\tau, \tau + h)$.

Proof. Without loss of generality, we may assume $h < +\infty$. Let $0 < \delta < h$ be such that $\delta c < 1$ and take $t_0 \in [\tau, \tau + \delta]$ satisfying the condition $|\alpha(t_0)| = \max_{\tau \le s \le \tau + \delta} |\alpha(s)|$. We have

$$|\alpha(t_0)| = |\alpha(t_0) - \alpha(\tau)| \le (t_0 - \tau) \sup_{\tau \le s \le t_0} |\alpha'(s)| \le \delta c |\alpha(t_0)|.$$

Being $\delta c < 1$, this inequality is verified if and only if $\alpha(t_0) = 0$. Thus $\alpha(t) = 0$ for any $t \in [\tau, \tau + \delta]$, and the assertion follows in a finite number of steps. $\square$

Let $g \colon \Omega \subseteq \mathbb{R} \times BU((-\infty, 0], A) \to \mathbb{R}^k$ be a functional field and let U be an open subset of Ω. We will say that g is *compactly Lipschitz in U* or, for short, *c-Lipschitz in U* if, given any compact subset Q of U, there exists $L \ge 0$ such that

$$|g(t, \varphi) - g(t, \psi)| \le L \|\varphi - \psi\|$$

for all $(t, \varphi), (t, \psi) \in Q$.

Moreover, we will say that g is *locally c-Lipschitz in Ω* if for any $(\tau, \eta) \in \Omega$ there exists an open neighborhood of (τ, η) in Ω in which g is c-Lipschitz. In spite of the fact that a locally Lipschitz map is not necessarily (globally) Lipschitz, one could actually show that if g is locally c-Lipschitz in Ω, then it is also (globally) c-Lipschitz in Ω. As a consequence, if g is C^1 or, more generally, locally Lipschitz in the second variable, then it is additionally c-Lipschitz.

Theorem 2.5 below shows that, if g is c-Lipschitz in Ω, then one gets the uniqueness of the solution of the initial value problem.

Theorem 2.5 (uniqueness). *Let A be a subset of $\mathbb{R}^k$, Ω an open subset of $\mathbb{R} \times BU((-\infty, 0], A)$ and $g \colon \Omega \to \mathbb{R}^k$ a c-Lipschitz functional field. Let $x^1 \colon J_1 \to A$, $x^2 \colon J_2 \to A$ be two maximal solutions of equation (2.1). If there exists $\tau \in J_1 \cap J_2$ such that $x^1(t) = x^2(t)$ for $t \le \tau$ and $(x^i)'(t) = g(t, x_t^i)$ for $t \in (\tau, \sup J_i)$, $i = 1, 2$, then $J_1 = J_2$ and $x^1 = x^2$.*

Proof. Let $h > 0$ be such that $[\tau, \tau + h] \subset J_1 \cap J_2$. Then, each one of the sets

$$Q_i = \left\{ (t, x_t^i) \in \Omega \subseteq \mathbb{R} \times BU((-\infty, 0], A) : \ t \in [\tau, \tau + h] \right\}, \quad i = 1, 2,$$

is compact, as the image of the continuous curve $t \mapsto (t, x_t^i) \in \Omega$ defined on $[\tau, \tau + h]$. Since g is c-Lipschitz in Ω, there exists $L \geq 0$, corresponding to the compact set $Q = Q_1 \cup Q_2$, such that for any $t \in [\tau, \tau + h]$ we have

$$|g(t, x_t^2) - g(t, x_t^1)| \leq L\|x_t^2 - x_t^1\| = L \sup_{s \leq 0} |x^2(t + s) - x^1(t + s)|$$

$$= L \sup_{s \leq t} |x^2(s) - x^1(s)| = L \sup_{\tau \leq s \leq t} |x^2(s) - x^1(s)|.$$

Now, putting $\alpha(t) = x^2(t) - x^1(t)$, $t \in [\tau, \tau + h]$, we get $\alpha(\tau) = 0$ and, by the above inequality,

$$|\alpha'(t)| = |(x^2)'(t) - (x^1)'(t)| = |g(t, x_t^2) - g(t, x_t^1)| \leq L \sup_{\tau \leq s \leq t} |\alpha(s)|,$$

for $t \in [\tau, \tau + h]$. Hence, from Lemma 2.4, $\alpha(t) = 0$, for all $t \in [\tau, \tau + h]$. This shows that x^1 and x^2 coincide in any right neighborhood of τ contained in $J_1 \cap J_2$, proving the uniqueness of the maximal solution of problem (2.2). $\square$

3. EXISTENCE OF SOLUTIONS

In this paper we will be mainly interested in RFDEs on manifolds. From now on, we will assume that A is a boundaryless smooth differentiable manifold in $\mathbb{R}^k$ which will be denoted by N.

Let $g \colon \Omega \to \mathbb{R}^k$ be a functional field over N defined on an open subset Ω of $\mathbb{R} \times BU((-\infty, 0], N)$. Given $(\tau, \eta) \in \Omega$, we will be concerned with the following initial value problem:

$$(3.1) \qquad \begin{cases} x'(t) = g(t, x_t), \\ x_\tau = \eta. \end{cases}$$

Moreover, given a relatively closed subset X of N, we will investigate under what assumptions at least one solution of (3.1), possibly the only one, remains confined into X. For instance, X could be a ∂-manifold of the type $\{p \in N : \phi(p) \leq 0\}$, where the "cutting function" $\phi : N \to \mathbb{R}$ is smooth, having $0 \in \mathbb{R}$ as a regular value. In particular, X could be an open subset of $\mathbb{R}^k$ (in this case $N = X$), or a closed subset of $\mathbb{R}^k$ (in this case N could be an open neighborhood of X, possibly coinciding with $\mathbb{R}^k$), or a compact ∂-manifold. In the last case, one may consider as N the double of X (see e.g. [11], [14]).

To be more precise, in what follows, given a relatively closed subset X of N, the problem

$$(3.2) \qquad \begin{cases} x'(t) = g(t, x_t), \\ x_\tau = \eta \in BU((-\infty, 0], X) \end{cases}$$

will be called the *confined problem* and an X-valued solution of (3.2) a *confined solution*.

In the first part of this section we are interested in obtaining existence results for the problems (3.1) and (3.2). More precisely, in Theorems 3.1 and 3.4 we will be concerned with the existence of a local solution. Then, suitable assumptions on g will ensure the existence of a global solution (see Theorems 3.7 and 3.9).

Theorem 3.1 (local existence). *Let $N \subseteq \mathbb{R}^k$ be a boundaryless smooth manifold, Ω an open subset of $\mathbb{R} \times BU((-\infty, 0], N)$ and $g \colon \Omega \to \mathbb{R}^k$ a functional field. Then, there exists $\delta > 0$ such that problem (3.1) admits at least one (N-valued) solution on $(-\infty, \tau + \delta)$.*

The proof of Theorem 3.1 makes use of a preliminary result concerning the existence of a local solution of problem (3.1) in the particular case when N is an open subset W of $\mathbb{R}^k$. For the sake of completeness, we will give below an independent proof of this well-known result (see Lemma 3.3). It should be observed that, if N is open, $BU((-\infty,0],N)$ is never open in $BU((-\infty,0],\mathbb{R}^k)$, unless $N = \mathbb{R}^k$.

In Lemma 3.2 below, we will consider the equation $x'(t) = \gamma(t, x_{t-r})$, where $\gamma\colon \mathbb{R} \times BU((-\infty,0],W) \to \mathbb{R}^k$ is continuous, W is an open subset of $\mathbb{R}^k$ and r is a given positive constant. We point out that this equation can be still regarded as an RFDE, since it can be written in the form $x'(t) = \gamma_r(t, x_t)$, where γ_r is the continuous map $\gamma_r(t, \varphi) = \gamma(t, \varphi_{-r})$, which is clearly defined on $\mathbb{R} \times BU((-\infty,0],W)$.

Therefore, by a solution of

$$\begin{cases} x'(t) = \gamma(t, x_{t-r}), \\ x_\tau = \eta \end{cases}$$

we mean a solution (in the RFDE meaning introduced previously) of

$$\begin{cases} x'(t) = \widetilde{\gamma}_r(t, x_t), \\ x_\tau = \eta, \end{cases}$$

where $\widetilde{\gamma}_r$ denotes the restriction of γ_r to $\mathbb{R} \times BU((-\infty,0],W)$.

Lemma 3.2. *Let W be an open subset of $\mathbb{R}^k$, let $\gamma\colon \mathbb{R} \times BU((-\infty,0],W) \to \mathbb{R}^k$ be continuous with bounded image and let $r > 0$. Then, given $(\tau, \eta) \in \mathbb{R} \times BU((-\infty,0],W)$, the problem*

$$(3.3) \qquad \begin{cases} x'(t) = \gamma(t, x_{t-r}), \\ x_\tau = \eta. \end{cases}$$

has a solution defined on the interval $(-\infty, \tau+h)$, with $h = \mathrm{dist}(\eta(0), W^c)/\mu$, where $\mu = \sup\{|\gamma(t,\varphi)| : (t,\varphi) \in \mathbb{R} \times BU((-\infty,0],W)\}$.

Proof. Let $(\tau, \eta) \in \mathbb{R} \times BU((-\infty,0],W)$. Notice that, because of the delay r, the $\mathbb{R}^k$-valued function

$$x(t) = \begin{cases} \eta(0) + \int_\tau^t \gamma(s, x_{s-r})\, ds, & \tau \le t, \\ \eta(t - \tau), & t \le \tau \end{cases}$$

is defined up to an instant t^*, provided that $\gamma(s, x_{s-r})$ is defined for all $s \in [\tau, t^*]$, i.e. if $x_{s-r} \in BU((-\infty,0],W)$ for all $s \in [\tau, t^*]$. This clearly happens if $t^* = \tau + h$, since in this case one has $|x(t) - \eta(0)| \le \mu|t - \tau| < \mu h$ and, thus, $x(t)$ cannot reach the boundary of W for all $t < t^*$. Thus, the function $x\colon (-\infty, \tau + h) \to \mathbb{R}^k$ takes values in W and is actually a solution of problem (3.3). $\qquad\square$

The following result is known (see e.g. [9, Chapter 12] and [10, Chapter 2]). Here we provide an independent proof.

Lemma 3.3 (local existence in $\mathbb{R}^k$). *Let W be an open subset of $\mathbb{R}^k$, Ω an open subset of $\mathbb{R} \times BU((-\infty,0],W)$ and let $g\colon \Omega \to \mathbb{R}^k$ be continuous. Then, there exists $\delta > 0$ such that problem (3.1) admits at least one (W-valued) solution on $(-\infty, \tau + \delta)$.*

Proof. Let V be an open neighborhood of (τ, η) in $\mathbb{R} \times BU((-\infty,0],W)$ such that $\overline{V} \subset \Omega$ and $g(\overline{V})$ is bounded. By the Tietze Extension Theorem, there exists a continuous extension $\widehat{g}\colon \mathbb{R} \times BU((-\infty,0],W) \to \mathbb{R}^k$ of the restriction $g|_{\overline{V}}$ of g to $\overline{V}$ with bounded image. Let $\{\varepsilon_n\}$ be a sequence of positive numbers converging to 0 and consider the following auxiliary problem depending on $n \in \mathbb{N}$:

$$(3.4) \qquad \begin{cases} x'(t) = \widehat{g}(t, x_{t-\varepsilon_n}), \\ x_\tau = \eta. \end{cases}$$

By applying Lemma 3.2 to problem (3.4) with $\gamma = \widehat{g}$ and $r = \varepsilon_n$, we get, for any $n \in N$, a solution $x^n \colon (-\infty, \tau + h) \to W$, with $h = \mathrm{dist}(\eta(0), W^c)/\mu$, where $\mu = \sup\{|\widehat{g}(t, \varphi)| : (t, \varphi) \in \mathbb{R} \times BU((-\infty, 0], W)\}$.

As already observed, any x^n solves the integral problem

(3.5) $$x(t) = \begin{cases} \eta(0) + \int_\tau^t \widehat{g}(s, x_{s-\varepsilon_n})\, ds, & \tau \leq t < \tau + h, \\ \eta(t - \tau), & t \leq \tau. \end{cases}$$

Since $x^n(t) = \eta(t - \tau)$ for $t \leq \tau$ and for all $n \in \mathbb{N}$, and since $\widehat{g}$ has bounded image, because of Ascoli's Theorem we may assume, without loss of generality, that the sequence $\{x^n(t)\}$ converges uniformly on any half-line $(-\infty, b] \subset (-\infty, \tau + h)$ to a continuous function $\widehat{x}(t)$ such that $\widehat{x}(t) = \eta(t - \tau)$ if $t \leq \tau$. Thus, in any such half-line, the functions x^n are equi-uniformly continuous. This implies that, for any $t > \tau$, $\|x^n_{s-\varepsilon_n} - x^n_s\| \to 0$ uniformly with respect to $s \in [\tau, t]$. Moreover, observe that $\|x^n_s - \widehat{x}_s\| \to 0$. Consequently, $\|x^n_{s-\varepsilon_n} - \widehat{x}_s\| \to 0$ uniformly for $s \in [\tau, t]$ and, thus, because of Remark 2.2, the sequence $\{\widehat{g}(s, x^n_{s-\varepsilon_n})\}$ converges uniformly to $\widehat{g}(s, \widehat{x}_s)$ on $[\tau, t]$. Therefore, passing to the limit in the integral of (3.5), we obtain

$$\widehat{x}(t) = \eta(0) + \int_\tau^t \widehat{g}(s, \widehat{x}_s)\, ds, \quad \tau \leq t < \tau + h.$$

Therefore, $\widehat{x} \colon (-\infty, \tau + h) \to W$ is a solution of $x'(t) = \widehat{g}(t, x_t)$ and $\widehat{x}(t) = \eta(t - \tau)$ for $t \leq \tau$. Moreover, since as previously observed the map $t \mapsto (t, \widehat{x}_t)$ is continuous and since $(\tau, \widehat{x}_\tau) = (\tau, \eta) \in V$, there exists $\delta > 0$ such that $(t, \widehat{x}_t) \in V$ for $\tau \leq t < \tau + \delta$. Recalling now that $\widehat{g} = g$ in $\overline{V}$, we get that the restriction $x \colon (-\infty, \tau + \delta) \to W$ of $\widehat{x}$ to $(-\infty, \tau + \delta)$ is a W-valued solution of $x'(t) = g(t, x_t)$ and satisfies $x(t) = \eta(t - \tau)$ for $t \leq \tau$. $\qquad \square$

Proof of Theorem 3.1. Let $W \subseteq \mathbb{R}^k$ be a tubular neighborhood of N with associated retraction $\rho \colon W \to N$ (if N is an open subset of $\mathbb{R}^k$, then $W = N$ and ρ is the identity). Hence, N is relatively closed in W and $BU((-\infty, 0], N)$ is a (relatively) closed subset of $BU((-\infty, 0], W)$. Therefore, there exists an open subset $\widehat{\Omega}$ of $\mathbb{R} \times BU((-\infty, 0], W)$ such that $\Omega = \widehat{\Omega} \cap (\mathbb{R} \times BU((-\infty, 0], N))$ and Ω is closed in $\widehat{\Omega}$. By the Tietze Extension Theorem, g admits a continuous extension $\widehat{g} \colon \widehat{\Omega} \to \mathbb{R}^k$. Moreover, we may assume that $\widehat{g}$ has the following additional property:

$$\widehat{g}(t, \varphi) \in T_{\rho(\varphi(0))} N \quad \text{for all } (t, \varphi) \in \widehat{\Omega}.$$

In fact, if this is not the case, it is sufficient to consider the orthogonal projection of $\widehat{g}(t, \varphi)$ onto the space $T_{\rho(\varphi(0))} N$.

Therefore, by applying Lemma 3.3 to W, $\widehat{\Omega}$ and $\widehat{g}$, we obtain the existence of $\delta > 0$ such that the problem

(3.6) $$\begin{cases} x'(t) = \widehat{g}(t, x_t), \\ x_\tau = \eta \end{cases}$$

admits a (W-valued) solution $\widehat{x}$ on $(-\infty, \tau + \delta)$. Let us show that $\widehat{x}(t) \in N$ for all $t \geq \tau$ (this could be false if $\widehat{g}$ were an arbitrary continuous extension of g). The C^1 function

$$\alpha(t) = |\widehat{x}(t) - \rho(\widehat{x}(t))|^2$$

is well defined for $\tau \leq t < \tau + \delta$ and verifies $\alpha(\tau) = 0$. Assume, by contradiction, that $\widehat{x}(t) \notin N$ for some $\tau < t < \tau + \delta$. This means that $\alpha(t) > 0$ for some $\tau < t < \tau + \delta$ and, consequently, its derivative must be positive at some $\tau < \theta < \tau + \delta$. That is,

$$\alpha'(\theta) = 2\langle \widehat{x}(\theta) - \rho(\widehat{x}(\theta)),\ \widehat{g}(\theta, \widehat{x}_\theta) - w(\theta) \rangle > 0,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^k$, and $w(\theta)$ is the derivative at $t = \theta$ of the curve $t \mapsto \rho(\widehat{x}(t))$. This is a contradiction, since both the vectors $\widehat{g}(\theta, \widehat{x}_\theta)$ and

$w(\theta)$ are tangent to N at $\rho(\widehat{x}(\theta))$ and, consequently, orthogonal to $\widehat{x}(\theta) - \rho(\widehat{x}(\theta))$. This proves that $\widehat{x}(t) \in N$ for all $t < \tau + \delta$. $\square$

In the next result we will be concerned with the local existence of the solutions in the confined case. Let X be a relatively closed subset of N and let $p \in X$ be given. We will say that the functional field $g \colon \Omega \to \mathbb{R}^k$ is *away from N at $p \in X$* if either $g(t, \varphi) \notin C_p(N \backslash X)$ for all $(t, \varphi) \in \Omega$ with $\varphi(0) = p$ or $g(t, \varphi) = 0$, for all $(t, \varphi) \in \Omega$ such that $\varphi(0) = p$. We point out that this condition is obviously satisfied whenever p, which is a point of X, is not in the topological boundary of X relative to N since, in that case, $C_p(N \backslash X) = \emptyset$. Notice that this condition is also satisfied when $X = N$, since $C_p(\emptyset) = \emptyset$.

Theorem 3.4 (confined local existence). *Let $N \subseteq \mathbb{R}^k$ be a boundaryless smooth manifold, Ω an open subset of $\mathbb{R} \times BU((-\infty, 0], N)$ and $g \colon \Omega \to \mathbb{R}^k$ a functional field. Let X be a relatively closed subset of N. Then, there exists $\delta > 0$ such that the confined problem (3.2) admits at least one (X-valued) solution on $(-\infty, \tau + \delta)$, provided that g is away from N at $\eta(0)$.*

Proof. Let $(\tau, \eta) \in \mathbb{R} \times BU((-\infty, 0], X)$ be the initial condition of the confined problem (3.2). Consider first the case when $g(t, \varphi) = 0$, for all (t, φ) such that $\varphi(0) = \eta(0)$. We claim that the X valued function

$$x(t) = \begin{cases} \eta(t - \tau), & t \leq \tau, \\ \eta(0), & t \geq \tau \end{cases}$$

satisfies (3.2). In fact, for any $t > \tau$ one has $x(t) = x_t(0) = \eta(0)$, which clearly implies $x'(t) = 0$ and, by assumption, also $g(t, x_t) = 0$. Moreover, the initial condition $x_\tau = \eta$ is obviously verified. This shows that the above function x, which is actually defined up to $+\infty$, is the required solution.

Consider now the case $g(\tau, \varphi) \notin C_{\eta(0)}(N \backslash X)$. By Theorem 3.1, there exists $\widetilde{\delta} > 0$ such that (3.1) has an (N-valued) solution $\widetilde{x}$ on $(-\infty, \tau + \widetilde{\delta})$. Let us show that there exists a positive $\delta \leq \widetilde{\delta}$ such that $\widetilde{x}(t) \in X$ for all $t \in (\tau, \tau + \delta)$ (notice that $\widetilde{x}(t)$ already belongs to X in $(-\infty, \tau]$ since $\widetilde{x}(t) = \eta(t - \tau)$ for $t \leq \tau$). By contradiction, assume that there exists a sequence $\{t_n\}$ in $(\tau, +\infty)$ converging to τ, with $\widetilde{x}(t_n) \in N \backslash X$. We get

$$\widetilde{g}(\tau, \widetilde{x}_\tau) = D_+ \widetilde{x}(\tau) = \lim_{n \to +\infty} \frac{\widetilde{x}(t_n) - \widetilde{x}(\tau)}{t_n - \tau} \in C_{\widetilde{x}(\tau)}(N \backslash X),$$

that clearly contradicts the assumption $g(\tau, \varphi) \notin C_{\eta(0)}(N \backslash X)$. Hence, the existence of δ such that $\widetilde{x}(t) \in X$ for all $t \in (\tau, \tau + \delta)$ is ensured and, thus, the restriction of $\widetilde{x}$ to $(-\infty, \tau + \delta)$ is the required solution. $\square$

Remark 3.5. In the non-uniqueness case, the only assumption g away from N at $\eta(0) \in X$ does not guarantee that any solution of (3.1) with $\eta \in BU((-\infty, 0], X)$ is necessarily X-valued for any $t > \tau$. To see this, take, for example, $N = \mathbb{R}$, $X = (-\infty, 0]$ and $g(t, \varphi) = 3\sqrt[3]{\varphi(0)^2}$. Then, the function

$$x(t) = \begin{cases} t^3, & 0 \leq t, \\ 0, & t \leq 0, \end{cases}$$

that, clearly, does not belong to X for $t > 0$, is a solution of the problem $x'(t) = g(t, x_t) = 3\sqrt[3]{x(t)^2}$, $x_0 = \eta$, where $\eta(\theta) = 0$ for all $\theta \leq 0$.

In what follows, the continuous map $H \colon \Omega \to N$ that associates to any $(t, \varphi) \in \Omega$ the "head" $H(t, \varphi) = \varphi(0) \in N$ of φ will be called, for brevity, the *head map*.

The set

$$S_g = \{p \in N : \exists (t, \varphi) \in \Omega \text{ such that } \varphi(0) = p, \, g(t, \varphi) \neq 0\}$$

will be called the *pre-support* of g and its closure in N, i.e. $\overline{S}_g \cap N$, the *support* of g. In other words, $p \in N$ belongs to the pre-support of g if and only if $H^{-1}(p)$ is nonempty and g is not identically zero in this set.

One can easily check that, because of the continuity of g (and recalling that Ω is open), the pre-support S_g is a relatively open subset of N. This fact will be used below in proving the continuous dependence of the solutions on the initial data.

The following proposition shows how the pre-support S_g of g is related to the solutions of problem (3.1).

Proposition 3.6. *Let (τ, η) be the initial condition of problem* (3.1). *If $\eta(0) \notin \overline{S}_g$, then* (3.1) *has a unique maximal solution $x \colon \mathbb{R} \to N$ that is constantly equal to $\eta(0)$, for any $t > \tau$. If $\eta(0) \in \overline{S}_g$, then any solution $x \colon J \to N$ of* (3.1) *satisfies $x(t) \in \overline{S}_g$ for all $\tau < t < \sup J$.*

Proof. If $\eta(0) \notin \overline{S}_g$, by an argument similar to that used in the first part of the proof of Theorem 3.4, we get that the function

$$x(t) = \begin{cases} \eta(t - \tau), & t \leq \tau, \\ \eta(0), & t \geq \tau, \end{cases}$$

which is clearly defined for any $t \in \mathbb{R}$, is a solution of (3.1). Notice that this solution is unique, since $N \backslash \overline{S}_g$ is open in N. Otherwise, if $\eta(0) \in \overline{S}_g$, the fact that $g(t, \varphi) = 0$ whenever $\varphi(0)$ belongs to $N \backslash \overline{S}_g$ implies that $x(t)$ cannot enter the relatively open set $N \backslash \overline{S}_g$. $\square$

The following result shows that any maximal solution of (3.1), whose existence is ensured by Theorem 3.1 and Zorn's Lemma, is globally defined provided that g has complete support and bounded image.

Theorem 3.7 (global existence). *Let $N \subseteq \mathbb{R}^k$ be a boundaryless smooth manifold and $g \colon \mathbb{R} \times BU((-\infty, 0], N) \to \mathbb{R}^k$ a functional field. Assume that g has complete support and bounded image. Then, any maximal solution of problem* (3.1) *is defined on the whole real line.*

Proof. Let (τ, η) be the initial condition of problem (3.1). If $\eta(0) \notin \overline{S}_g$, by Proposition 3.6 the N valued function

$$x(t) = \begin{cases} \eta(t - \tau), & t \leq \tau, \\ \eta(0), & t \geq \tau \end{cases}$$

is the unique global solution of (3.1). Otherwise, if $\eta(0) \in \overline{S}_g$, we will prove that any maximal solution $\bar{x}$ of (3.1) is defined on the whole real line. By contradiction, suppose $\bar{x}$ defined up to $b > \tau$, with $b < +\infty$. Since g has bounded image, then the derivative $\bar{x}'$ is bounded in the interval (τ, b). Moreover, again by Proposition 3.6, we get $\bar{x}(t) \in \overline{S}_g$ for $\tau < t < b$ and, since $\overline{S}_g$ is complete, $\lim_{t \to b^-} \bar{x}(t)$ exists and belongs to $\overline{S}_g \subseteq N$. Consequently, the function $\bar{\eta} \colon (-\infty, 0] \to N$ given by

$$\bar{\eta}(\theta) = \begin{cases} \bar{x}(b + \theta), & \theta < 0 \\ \lim_{t \to b^-} \bar{x}(t), & \theta = 0, \end{cases}$$

is uniformly continuous. Thus, by applying Theorem 3.1 to the initial value problem

$$(3.7) \qquad \begin{cases} x'(t) = g(t, x_t), \\ x_b = \bar{\eta}, \end{cases}$$

we get the existence of a solution of (3.7) defined up to $b + \delta$, for some $\delta > 0$. This contradicts the maximality of $\bar{x}$. $\square$

Remark 3.8. The assumptions of Theorem 3.7 above are clearly satisfied if g has compact support and bounded image. Moreover, when $N = \mathbb{R}^k$, the support of g is, by definition, a closed set. Thus, as well known, any maximal solution is globally defined, provided that g has bounded image.

Our purpose now is to obtain a global existence result for the confined problem (3.2). To this end, similarly to Theorem 3.4, we need to impose an extra assumption on the functional field g. Namely, we will assume that g is *away from N in X*, meaning that g is away from N at any $p \in X$. Notice that to obtain this condition, it is enough to verify that $p \in S_g \cap X$ implies $g(t, \varphi) \notin C_p(N \backslash X)$, for all (t, φ) with $\varphi(0) = p$ (see page 8). Observe that, if g is away from N in X and $\sigma \colon N \to [0, +\infty)$ is continuous, then the product $(t, \varphi) \mapsto \sigma(\varphi(0))g(t, \varphi)$ is still away from N in X and $S_{\sigma g} \subseteq S_g$.

As an example, let X be a compact ∂-manifold in $\mathbb{R}^k$ and N be the double of X. In this case, a functional field $g \colon \mathbb{R} \times BU((-\infty, 0], N) \to \mathbb{R}^k$ is said to be *strictly inward at a point* $p \in \partial X$ if $g(t, \varphi)$ is strictly inward at p whenever $\varphi(0) = p$. Clearly, a functional field g is away from N at $p \in \partial X$ if and only if either $p \notin S_g$, or g is strictly inward at p. In addition, an inward functional field g is said to be *strictly inward* if it is strictly inward at any $p \in \partial X$.

Theorem 3.9 below is a global existence result for the confined case. Its proof is analogous to the one of Theorem 3.7, provided that, in the contradiction argument, one replaces the local existence in N (Theorem 3.1) with the local existence in X (Theorem 3.4).

Theorem 3.9 (confined global existence). *Let X be a relatively closed subset of a boundaryless smooth manifold $N \subseteq \mathbb{R}^k$, and $g \colon \mathbb{R} \times BU((-\infty, 0], N) \to \mathbb{R}^k$ a functional field away from N in X. Assume that g has complete support in X (i.e., $\overline{S}_g \cap X$ is closed in $\mathbb{R}^k$) and bounded image. Then, any maximal solution of the confined problem* (3.2) *is defined on the whole real line.*

4. Continuous dependence on data and continuation of solutions

In what follows we will denote by $BU_\ell(\mathbb{R}, \mathbb{R}^k)$ the Fréchet space of those functions which are bounded and uniformly continuous on any left half-line, with the topology generated by the family of seminorms $\{P_b \colon b \in \mathbb{R}\}$, where $P_b(x) = \sup_{t \leq b} |x(t)|$. Clearly, for any $b \in \mathbb{R}$, one has $P_b(x) = \|x_b\|$, where, as previously, $\| \cdot \|$ denotes the norm in the Banach space $BU((-\infty, 0], \mathbb{R}^k)$. Moreover, given any subset $A \subseteq \mathbb{R}^k$, we will denote by $BU_\ell(\mathbb{R}, A)$ the subset of $BU_\ell(\mathbb{R}, \mathbb{R}^k)$ of the A-valued functions.

In Lemmas 4.1, 4.2 and 4.3 below we will be concerned with upper semicontinuous multivalued maps. We recall that a multivalued map F between two metric spaces $\mathcal{X}$ and $\mathcal{Y}$ is said to be *upper semicontinuous* if it is compact valued and for any open subset U of $\mathcal{Y}$ the upper inverse image of U, i.e. the set $F^{-1}(U) = \{x \in \mathcal{X} : F(x) \subseteq U\}$, is an open subset of $\mathcal{X}$. Equivalently, F is upper semicontinuous if and only if for any $x_0 \in \mathcal{X}$ and sequences $\{x_n\}$ in $\mathcal{X}$, $x_n \to x_0$ and $\{y_n\}$ in $\mathcal{Y}$, $y_n \in F(x_n)$ for all $n \in \mathbb{N}$, there exists a subsequence of $\{y_n\}$ converging to some element $y \in F(x_0)$. Recall that the composition of upper semicontinuous maps is upper semicontinuous.

The next lemma, in which the uniqueness of the solution of the initial value problem is not assumed, regards the continuous dependence on the initial data of the set of solutions of problem (3.1). Its statement concerning multivalued maps is justified by its application in the proof of Theorem 4.4 below. In fact, although in Theorem 4.4 the uniqueness of the solution of (3.1) is assumed for the sake of

simplicity, the use in its proof of the Tietze Extension Theorem does not guarantee the same property for the extended problem to which Lemma 4.1 is applied.

Lemma 4.1. *Let $N \subseteq \mathbb{R}^k$ be a boundaryless smooth manifold and let $g \colon \mathbb{R} \times BU((-\infty, 0], N) \to \mathbb{R}^k$ be a functional field. Assume that g has complete support and bounded image. Then, the multivalued map*

$$\Sigma \colon \mathbb{R} \times BU((-\infty, 0], N) \multimap BU_\ell(\mathbb{R}, N)$$

that associates to any (τ, η) the set $\Sigma(\tau, \eta)$ of the (global) solutions x of (3.1) is upper semicontinuous.

Proof. Take sequences $\{(\tau^n, \eta^n)\}$ in $\mathbb{R} \times BU((-\infty, 0], N)$ converging to (τ, η) and $\{x^n\}$ in $BU_\ell(\mathbb{R}, N)$ such that $x^n \in \Sigma(\tau^n, \eta^n)$ for any $n \in \mathbb{N}$. We have to prove that $\{x^n\}$ admits a subsequence converging to some $x \in \Sigma(\tau, \eta)$.

Suppose that $\{\eta^n\}$ has a subsequence, again denoted by $\{\eta^n\}$, such that $\eta^n(0) \notin \overline{S}_g$ for any n. Since, as already observed, the pre-support S_g is an open subset of N, then $\eta(0) \notin S_g$. By Proposition 3.6, the solution x^n is unique and given by

$$x^n(t) = \begin{cases} \eta^n(t - \tau^n), & t \le \tau^n, \\ \eta^n(0), & t \ge \tau^n. \end{cases}$$

Clearly, $\{x^n\}$ converges in $BU_\ell(\mathbb{R}, N)$ to the function

$$x(t) = \begin{cases} \eta^n(t - \tau^n), & t \le \tau^n, \\ \eta(0), & t \ge \tau. \end{cases}$$

Consequently, x is a solution of problem (3.1) (recall that $\eta(0) \notin S_g$). That is, $x \in \Sigma(\tau, \eta)$, proving the assertion in the case $\eta^n(0) \notin \overline{S}_g$.

Otherwise, without loss of generality, we may assume that $\eta^n(0) \in \overline{S}_g$ for any $n \in \mathbb{N}$. By the definition of Σ, any x^n is a (globally defined) solution of the problem

$$(4.1) \qquad \begin{cases} x'(t) = g(t, x_t), \\ x_{\tau^n} = \eta^n. \end{cases}$$

Therefore, as already observed, one has

$$(4.2) \qquad x^n(t) = \begin{cases} \eta^n(0) + \int_{\tau^n}^t g(s, x_s^n)\, ds, & t \ge \tau^n, \\ \eta^n(t - \tau^n), & t \le \tau^n. \end{cases}$$

Now, set $y^n(s) = x^n(\tau^n + s)$. By (4.2), for $s \ge 0$, we obtain

$$y^n(s) = x^n(\tau^n + s) = \eta^n(0) + \int_{\tau^n}^{\tau^n + s} g(\rho, x_\rho^n)\, d\rho = \eta^n(0) + \int_{\tau^n}^{\tau^n + s} g(\rho, y_{\rho - \tau^n}^n)\, d\rho.$$

Therefore, by a change of variable in the above integral, we have

$$(4.3) \qquad y^n(s) = \begin{cases} \eta^n(0) + \int_0^s g(\tau^n + \sigma, y_\sigma^n)\, d\sigma, & s \ge 0, \\ \eta^n(s), & s \le 0. \end{cases}$$

Since g has bounded image, say $|g(t, \varphi)| \le c$, we get

$$|(y^n)'(s)| \le |g(\tau^n + s, y_s^n)| \le c, \quad s > 0.$$

Hence, by Ascoli's Theorem, $\{y^n\}$ has a subsequence converging to a function $y \in BU_\ell(\mathbb{R}, \mathbb{R}^k)$ that clearly satisfies $y(s) = \eta(s)$ for $s \le 0$. Without loss of generality we may assume that $\{y^n\}$ itself converges to y.

Define $x(t) = y(t - \tau)$. Let us show that $x^n \to x$ in $BU_\ell(\mathbb{R}, \mathbb{R}^k)$. This follows from the inequality

$$|x^n(t) - x(t)| = |y^n(t - \tau^n) - y(t - \tau)| \le |y^n(t - \tau^n) - y(t - \tau^n)| + |y(t - \tau^n) - y(t - \tau)|,$$

taking into account that y belongs to $BU_\ell(\mathbb{R}, \mathbb{R}^k)$ and $y^n \to y$ (in $BU_\ell(\mathbb{R}, \mathbb{R}^k)$). Observe now that, since we are assuming $\eta^n(0) \in \overline{S}_g$, by Proposition 3.6 we get

$x^n(t) \in \overline{S}_g$ for any t. Thus, the limit $x(t)$ belongs to $\overline{S}_g$ as well. Now, from the completeness of $\overline{S}_g$, we get $\overline{S}_g \subseteq N$, so that $x(t) \in N$ for any t. Moreover, the same fact obviously holds for any y^n and for y. To complete our proof, it remains to show that x is a solution of problem (3.1).

Since, for $0 \leq \sigma \leq s$, $\{y_\sigma^n\}$ converges to y_σ as $n \to \infty$, and again using the fact that g has bounded image, by the Lebesgue Theorem, we get in (4.3)

$$\int_0^s g(\tau^n + \sigma, y_\sigma^n) \, d\sigma \longrightarrow \int_0^s g(\tau + \sigma, y_\sigma) \, d\sigma.$$

On the other hand, the convergence of $\{y^n\}$ to y in $BU_\ell(\mathbb{R}, N)$ implies, in particular, that $\{y^n(s)\}$ converges to $y(s)$ for any s. Thus, we obtain

$$(4.4) \qquad y(s) = \begin{cases} \eta(0) + \int_0^s g(\tau + \sigma, y_\sigma) \, d\sigma, & s \geq 0, \\ \eta(s), & s \leq 0. \end{cases}$$

Therefore, for $t \geq \tau$ and again by a change of variable, we get

$$x(t) = y(t - \tau) = \eta(0) + \int_0^{t-\tau} g(\tau + \sigma, y_\sigma) \, d\sigma = \eta(0) + \int_\tau^t g(\rho, y_{\rho-\tau}) \, d\rho \, .$$

Consequently, (4.4) becomes

$$(4.5) \qquad x(t) = \begin{cases} \eta(0) + \int_\tau^t g(\rho, x_\rho) \, d\rho, & t \geq \tau, \\ \eta(t - \tau), & t \leq \tau. \end{cases}$$

This proves that x is a solution of problem (3.1), that is $x \in \Sigma(\tau, \eta)$. $\qquad \square$

Our purpose now is to remove the assumptions ensuring that the solutions of (3.1) are globally defined. To this end, we need the two preliminary results Lemma 4.2 and 4.3 below. The first one is a folk result and its simple proof will be omitted.

Lemma 4.2. *Let $C(\mathcal{X}, \mathcal{Y})$ be the metric space of the continuous maps from a compact metric space $\mathcal{X}$ to a metric space $\mathcal{Y}$. Then, the multivalued map*

$$\mathrm{Img} \colon C(\mathcal{X}, \mathcal{Y}) \multimap \mathcal{Y}$$

that associates to any $f \in C(\mathcal{X}, \mathcal{Y})$ its image $\mathrm{Img}(f) \subseteq \mathcal{Y}$ is upper semicontinuous.

Lemma 4.3. *Let $N \subseteq \mathbb{R}^k$ be a boundaryless smooth manifold and let $\widehat{g} \colon \mathbb{R} \times BU((-\infty, 0], N) \to \mathbb{R}^k$ be a functional field. Assume that $\widehat{g}$ has complete support and bounded image. Then, given a compact real interval I, the multivalued map*

$$\widehat{K}_I \colon \mathbb{R} \times BU((-\infty, 0], N) \multimap \mathbb{R} \times BU((-\infty, 0], N)$$

that associates to any (τ, η) the set

$$\widehat{K}_I(\tau, \eta) = \{(t, \widehat{x}_t) : t \in I, \, \widehat{x} \text{ solution of } x'(t) = \widehat{g}(t, x_t), \, x_\tau = \eta\}$$

is upper semicontinuous.

Proof. Let $A_I \colon BU_\ell(\mathbb{R}, \mathbb{R}^k) \to C(I, \mathbb{R} \times BU((-\infty, 0], \mathbb{R}^k))$ be the single valued map that associates to any $y \in BU_\ell(\mathbb{R}, \mathbb{R}^k)$ the (continuous) curve $t \in I \mapsto (t, y_t) \in \mathbb{R} \times BU((-\infty, 0], \mathbb{R}^k)$. Let us show that A_I is continuous. In fact, one has

$$\|A_I(z) - A_I(y)\| = \sup_{t \in I} \|z_t - y_t\| = \sup_{t \in I} \sup_{\theta \leq 0} |z_t(\theta) - y_t(\theta)|$$

$$= \sup_{t \in I} \sup_{\tau \leq t} |z(\tau) - y(\tau)| \leq \|z_b - y_b\| = P_b(z - y),$$

where $b = \max I$. This proves the continuity of A_I. Consequently, by applying Lemma 4.1 and Lemma 4.2 with $\mathcal{X} = I$ and $\mathcal{Y} = \mathbb{R} \times BU((-\infty, 0], N)$, we get that the map

$$\mathrm{Img} \circ A_I \circ \Sigma \colon \mathbb{R} \times BU((-\infty, 0], N) \multimap \mathbb{R} \times BU((-\infty, 0], N),$$

being composition of upper semicontinuous maps, is upper semicontinuous. Now, it is enough to observe that

$$\widehat{K}_I = \mathrm{Img} \circ A_I \circ \Sigma.$$

This completes our proof. $\qquad\square$

We are now in a position to state our result on the continuous dependence of the solutions of problem (3.1) on the initial data, in the general case when g is a functional field over N defined on an open subset Ω of $\mathbb{R} \times BU((-\infty, 0], N)$.

Theorem 4.4 (continuous dependence). *Let N be a boundaryless smooth manifold, Ω an open subset of $\mathbb{R} \times BU((-\infty, 0], N)$ and $g \colon \Omega \to \mathbb{R}^k$ a functional field. Assume that problem (3.1) has a unique maximal solution for any $(\tau, \eta) \in \Omega$. Then, given $b \in \mathbb{R}$, the set*

$$\mathcal{D}_b = \{(\tau, \eta) \in \Omega : \tau < b \text{ and the maximal solution of (3.1) is defined up to } b\}$$

is an open subset of Ω. Moreover, the map $s_b \colon \mathcal{D}_b \to BU((-\infty, 0], N)$ given by $s_b(\tau, \eta) = x_b$, where $x(\cdot)$ is the restriction to $(-\infty, b]$ of the unique maximal solution of problem (3.1), is continuous.

Proof. Let $(\tau^0, \eta^0) \in \mathcal{D}_b$ and fix $a < \tau^0$. Denote by x^0 the restriction to $(-\infty, b]$ of the unique maximal solution of problem (3.1) with initial conditions (τ^0, η^0). Set

$$K^0 = \{(t, x_t^0) \in \Omega : t \in [a, b]\}.$$

Observe that K^0 is compact, being the image of $[a, b]$ under the (continuous) curve $t \mapsto (t, x_t^0)$. Let V be an open neighborhood of K^0 in Ω such that $\overline{V} \subset \Omega$ and $g(\overline{V})$ is bounded. Because of the Tietze Extension Theorem, there exists a continuous extension $\widetilde{g} \colon \mathbb{R} \times BU((-\infty, 0], N) \to \mathbb{R}^k$ of the restriction $g|_{\overline{V}}$ of g to $\overline{V}$ with bounded image. We may also assume that $\widetilde{g}$ is a functional field over N, by taking if necessary the orthogonal projection onto the tangent space. As previously, let H denote the head map $(t, \varphi) \mapsto \varphi(0)$ and consider the set

$$C^0 = H(K^0) = \{x_t^0(0) \in N : t \in [a, b]\}.$$

Clearly C^0 is compact since H is continuous.

Let U_1, U_2 be open subsets of N such that $C^0 \subset U_1$ and $\overline{U}_1 \subset U_2$. Since N is locally compact, we may also assume that U_2 is a relatively compact subset of N. Let $\sigma \colon N \to [0, 1]$ be a continuous function such that $\sigma(p) = 1$ if $p \in \overline{U}_1$, $\sigma(p) = 0$ if $p \notin U_2$. Define $\widehat{g} \colon \mathbb{R} \times BU((-\infty, 0], N) \to \mathbb{R}^k$ by $\widehat{g}(t, \varphi) = \sigma(\varphi(0))\, \widetilde{g}(t, \varphi)$. The map $\widehat{g}$ is clearly continuous being $\widehat{g}(t, \varphi) = \sigma(H(t, \varphi))\, \widetilde{g}(t, \varphi)$, and has bounded image, since $\widetilde{g}$ itself has bounded image. Moreover, its support, that is the closure (in $\mathbb{R}^k$) of the set

$$S_{\widehat{g}} = \big\{p \in N : \exists\, (t, \varphi) \in \mathbb{R} \times BU((-\infty, 0], N) \text{ such that } \varphi(0) = p,\, \widehat{g}(t, \varphi) \neq 0\big\},$$

is clearly contained in $\overline{U}_2$ and, thus, is compact. Therefore, by applying Lemma 4.3 to the N-functional field $\widehat{g}$ with $I = [a, b]$, we obtain that the multivalued map

$$\widehat{K}_I \colon \mathbb{R} \times BU((-\infty, 0], N) \multimap \mathbb{R} \times BU((-\infty, 0], N)$$

given by

$$\widehat{K}_I(\tau, \eta) = \{(t, \widehat{x}_t) : t \in I,\, \widehat{x} \text{ solution of } x'(t) = \widehat{g}(t, x_t),\, x_\tau = \eta\}$$

is upper semicontinuous. Let us show now that

$$(4.6) \qquad\qquad\qquad K^0 = \widehat{K}_I(\tau^0, \eta^0).$$

Clearly, $K^0 \subseteq \widehat{K}_I(\tau^0, \eta^0)$. On the other hand, since $g = \widehat{g}$ on the neighborhood $V \cap H^{-1}(U_1)$ of K^0, any solution $\widehat{x}$ of $x'(t) = \widehat{g}(t, x_t)$ with initial condition (τ^0, η^0)

coincides with x^0 for $t \in [a, b]$. Therefore, $\widehat{K}_I(\tau^0, \eta^0) \subseteq K^0$, proving (4.6). Consequently, from the equality (4.6), $V \cap H^{-1}(U_1)$ is actually an open neighborhood of $\widehat{K}_I(\tau^0, \eta^0)$ and, thus, by the upper semicontinuity of $\widehat{K}_I$, $\widehat{K}_I^{-1}(V \cap H^{-1}(U_1))$ is an open subset of $\mathbb{R} \times BU((-\infty, 0], N)$ containing (τ^0, η^0). Let us prove that the set

$$ W := \widehat{K}_I^{-1}(V \cap H^{-1}(U_1)) \cap \{(\tau, \varphi) \in \mathbb{R} \times BU((-\infty, 0], N) : a < \tau < b\}, $$

that is clearly an open neighborhood of (τ^0, η^0) in $\mathbb{R} \times BU((-\infty, 0], N)$, is in fact contained in $\mathcal{D}_b$. To this end, take $(\tau, \eta) \in W$. Then, for $t \in [a, b]$, we have $(t, \widehat{x}_t) \in V$ and $\widehat{x}_t(0) \in U_1$, where $\widehat{x}(\cdot)$ satisfies $x'(t) = \widehat{g}(t, x_t)$, $x_\tau = \eta$. Recalling again that $g = \widehat{g}$ on $V \cap H^{-1}(U_1)$, we get $\widehat{g}(t, \widehat{x}_t) = g(t, \widehat{x}_t)$ for $t \in [a, b]$. Hence, in particular, $\widehat{g}(t, \widehat{x}_t) = g(t, \widehat{x}_t)$ for $t \in [\tau, b]$. On the other hand, because of the uniqueness assumption, $\widehat{x}$ must coincide with the solution of problem (3.1) that, therefore, is defined up to b. Thus, $(\tau, \eta) \in \mathcal{D}_b$. This proves that $\mathcal{D}_b$ is an open subset of Ω, as claimed.

It remains to show that the map s_b is continuous in D_b. To this end, given any compact interval $I \subset (-\infty, b]$ and $(\tau, \eta) \in \mathcal{D}_b$ define

$$ K_I(\tau, \eta) = \{(t, x_t) : t \in I, \ x \text{ the maximal solution of } (3.1)\}. $$

Clearly, by the previous argument, if $(\tau, \eta) \in W$, then $\widehat{K}_I(\tau, \eta) = K_I(\tau, \eta)$. Thus, by taking in particular $I = \{b\}$, we obtain that $\widehat{K}_{\{b\}}(\tau, \eta)$ is the singleton $\{(b, s_b(\tau, \eta))\}$. Now, the continuity of s_b follows from Theorem 4.3. $\square$

The proof of the continuous dependence in the confined case (Corollary 4.5 below) is a straightforward consequence of Theorem 4.4. Indeed, the assumption that g is away from N in X together with the uniqueness of the solutions of problem (3.1) imply that any solution of (3.1) with $\eta \in BU((-\infty, 0], X)$ is actually a confined solution (recall Remark 3.5).

Corollary 4.5 (confined continuous dependence)**.** *Let N, Ω, g be as in Theorem 4.4 and assume that problem (3.1) has a unique maximal solution for any $(\tau, \eta) \in \Omega$. Let X be a relatively closed subset of N and assume that g is away from N in X. Then, given $b \in \mathbb{R}$, the restriction of the continuous map s_b to the relatively open subset $\mathcal{D}_b \cap (\mathbb{R} \times BU((-\infty, 0], X))$ of $\mathbb{R} \times BU((-\infty, 0], X)$ takes values in $BU((-\infty, 0], X)$.*

Let us now discuss the continuation property of the solutions. Analogous results in Euclidean spaces can be found e.g. in [9, Chapter 12] and [10, Chapter 2].

Our first result states that, given a solution $x \colon J \to N$ of equation (2.1), if the curve $t \mapsto (t, x_t)$ lies eventually in a bounded and complete subset $\mathcal{C}$ of Ω such that $g(\mathcal{C})$ is bounded, then x is not a maximal solution.

Theorem 4.6 (continuation of solutions)**.** *Let N be a boundaryless smooth manifold, Ω an open subset of $\mathbb{R} \times BU((-\infty, 0], N)$ and $g \colon \Omega \to \mathbb{R}^k$ a functional field. Let $x \colon (-\infty, b) \to N$, $b < +\infty$, be a solution of equation (2.1) such that (t, x_t) belongs eventually to a complete subset $\mathcal{C}$ of Ω. If $g(\mathcal{C})$ is bounded, then x is continuable.*

Proof. Since (t, x_t) belongs eventually to $\mathcal{C}$, $g(t, x_t)$ is eventually bounded, and so is $x'(t)$. Thus,

$$ \lim_{t \to b^-} x(t) $$

exists and is finite. Therefore, the continuous function $t \mapsto x(t)$ can be extended to the uniformly continuous function

$$ \bar{x}(t) = \begin{cases} x(t), & t < b, \\ \lim_{s \to b^-} x(s), & t = b. \end{cases} $$

This implies that

$$\lim_{t \to b^-} (t, x_t) = (b, \bar{x}_b).$$

Since $\mathcal{C}$ is complete, $(b, \bar{x}_b)$ belongs to $\mathcal{C}$ and, thus, to Ω. Consequently, by applying Theorem 3.1 with $(\tau, \eta) = (b, \bar{x}_b)$, we get that x is continuable on the right hand side of b. $\square$

The following two corollaries are straightforward consequences of Theorem 4.6. Therefore, their proofs will be omitted.

Corollary 4.7. *Let N be a boundaryless smooth manifold, Ω an open subset of $\mathbb{R} \times BU((-\infty, 0], N)$ and $g \colon \Omega \to \mathbb{R}^k$ a functional field. Let $x \colon J \to N$ be a solution of equation (2.1) and assume that (t, x_t) belongs eventually to a compact subset of Ω. Then x is continuable.*

Corollary 4.8. *Let N be a boundaryless smooth manifold, and*

$$g \colon \mathbb{R} \times BU((-\infty, 0], N) \to \mathbb{R}^k$$

a functional field. Assume that N is closed as a subset of $\mathbb{R}^k$ and that g sends bounded subsets of $\mathbb{R} \times BU((-\infty, 0], N)$ into bounded subsets of $\mathbb{R}^k$. If $x \colon J \to N$ is a solution of equation (2.1) such that (t, x_t) belongs eventually to a closed and bounded subset of $\mathbb{R} \times BU((-\infty, 0], N)$, then x is continuable.

Theorem 4.9 below is the confined analogue of the continuation result proved in Theorem 4.6.

Theorem 4.9 (continuation of confined solutions)**.** *Let X be a relatively closed subset of a boundaryless smooth manifold $N \subseteq \mathbb{R}^k$, Ω an open subset of the space $\mathbb{R} \times BU((-\infty, 0], N)$ and $g \colon \Omega \to \mathbb{R}^k$ a functional field away from N in X. Let $x \colon (-\infty, b) \to X$, $b < +\infty$, be a solution of equation (2.1) such that (t, x_t) belongs eventually to a complete subset $\mathcal{C}$ of Ω. If $g(\mathcal{C})$ is bounded, then x is continuable.*

Proof. The proof is analogous to the one of Theorem 4.6, except that here one applies Theorem 3.4 (confined local existence) instead of Theorem 3.1. Moreover, since X is a relatively closed subset of N, it should be observed that now $\lim_{t \to b^-} x(t)$ belongs to X. $\square$

The following consequence of Theorem 4.6 can be regarded as a Kamke-type result for RFDEs.

Corollary 4.10. *Let N be a boundaryless smooth manifold, and*

$$g \colon \mathbb{R} \times BU((-\infty, 0], N) \to \mathbb{R}^k$$

a functional field. Assume that g sends bounded subsets of $\mathbb{R} \times BU((-\infty, 0], N)$ into bounded subsets of $\mathbb{R}^k$. If $x \colon J \to N$ is a solution of equation (2.1) whose graph-curve $t \mapsto (t, x(t))$ belongs eventually to a compact subset of $\mathbb{R} \times N$, then x is continuable.

Proof. Suppose that there exists $\tau \in J$ and a compact subset K of $\mathbb{R} \times N$ such that $(t, x(t)) \in K$ when $\tau \leq t < \sup J$. Clearly $b = \sup J < +\infty$. Consider the following subset of $\mathbb{R} \times BU((-\infty, 0], \mathbb{R}^k)$:

$$\mathcal{C} = \{(t, \varphi) : \tau \leq t \leq b, \ (t, \varphi(0)) \in K, \ \varphi_\theta = x_\tau \ \text{for some} \ \theta \in [\tau - b, 0]\}.$$

Notice that $\mathcal{C}$ is closed in $\mathbb{R} \times BU((-\infty, 0], \mathbb{R}^k)$ and contained in $\mathbb{R} \times BU((-\infty, 0], N)$. Thus, it is complete. Moreover $g(\mathcal{C})$ is bounded, since so is $\mathcal{C}$. Observe, finally, that $(t, x_t) \in \mathcal{C}$ for $t \geq \tau$. Therefore, all the assumptions in Theorem 4.6 are satisfied, so that x is continuable. $\square$

The following example shows that, if the assumption that g sends bounded sets into bounded sets is removed, then the continuation property of the solutions may fail.

Example 4.11. Let $N = \mathbb{R}$ and let $x \colon (-\infty, 1) \to \mathbb{R}$ given by

$$x(t) = \sin \frac{1}{t-1}.$$

Consider the continuous injective curve $\gamma \colon [0, 1) \to BU((-\infty, 0], \mathbb{R})$ that to any $t \in [0, 1)$ associates x_t. We claim that the image

$$\Gamma = \{x_t : t \in [0, 1)\}$$

of γ is a closed subset of $BU((-\infty, 0], \mathbb{R})$, and that γ is a homeomorphism onto Γ. We need to show that, if $C \subseteq [0, 1)$ is relatively closed in $[0, 1)$, then $\gamma(C)$ is closed in $BU((-\infty, 0], \mathbb{R})$. If $\sup C < 1$, then C is compact and, thus, $\gamma(C)$ is closed. Therefore, it is sufficient to show that, if $\{t_n\}$ is a sequence in $[0, 1)$ converging to 1, then $\{\gamma(t_n)\}$ is divergent (i.e. it does not admit a convergent subsequence). Given such a $\{t_n\}$, for any $\theta < 0$ we get

$$\gamma(t_n)(\theta) = x_{t_n}(\theta) \to \sin \frac{1}{\theta}, \quad \text{as } n \to \infty.$$

Thus, $\{\gamma(t_n)\}$ is divergent, since the function $\theta \mapsto \sin \frac{1}{\theta}$ is not uniformly continuous on $(-\infty, 0)$, and this proves our claim.

Define the continuous function $g \colon \Gamma \to \mathbb{R}$ by $g(\varphi) = x'(\gamma^{-1}(\varphi))$. Since Γ is closed, by the Tietze Extension Theorem g can be extended to a continuous map $\widehat{g}$ on $BU((-\infty, 0], \mathbb{R})$. Obviously, x is a solution of $x'(t) = \widehat{g}(x_t)$ and its graph belongs to $[0, 1] \times [-1, 1]$ for $t \geq 0$. However x is noncontinuable.

We point out that this fact does not contradicts the assertion of Corollary 4.10 since $\widehat{g}(\Gamma)$ is not bounded and, thus, $\widehat{g}$ does not send bounded subsets of $BU((-\infty, 0], \mathbb{R})$ into bounded subsets of $\mathbb{R}$.

5. Reduction to a confined problem: an example

We close the paper with an example in which we illustrate how the results about the confined problem (3.2) can be applied to retarded functional motion equations. In particular we would like to highlight how the relatively closed subset X, introduced in the confined problem, will be interpreted in Example 5.1. In fact, as we will show below, the difficulty arising from the noncompactness of the tangent bundle will be removed by restricting the search of T-periodic solutions to a convenient compact manifold with boundary that plays exactly the role that the set X has in the general context.

Example 5.1. Let $M \subseteq \mathbb{R}^s$ be a smooth boundaryless manifold and let

$$TM = \big\{(q, v) \in \mathbb{R}^s \times \mathbb{R}^s : q \in M, \, v \in T_q M\big\}$$

be the tangent bundle of M. Given $q \in M$, let $(T_q M)^\perp \subseteq \mathbb{R}^s$ denote the normal space of M at q. Since $\mathbb{R}^s = T_q M \oplus (T_q M)^\perp$, any vector $u \in \mathbb{R}^s$ can be uniquely decomposed into the sum of the parallel (or tangential) component $u_\pi \in T_q M$ of u at q and the normal component $u_\nu \in (T_q M)^\perp$ of u at q.

Consider the retarded functional motion equation on the constraint M

$$(5.1) \qquad\qquad x_\pi''(t) = F(t, x_t) - \varepsilon x'(t),$$

where $x_\pi''(t)$ stands for the parallel component of the acceleration $x''(t) \in \mathbb{R}^s$ at the point $x(t)$, the parameter $\varepsilon > 0$ is the frictional coefficient, and the map $F \colon \mathbb{R} \times BU((-\infty, 0], M) \to \mathbb{R}^s$ is a (continuous) functional field, T-periodic in the first

variable, with bounded image, and which verifies conditions ensuring the uniqueness of the associated initial value problems. Any T-periodic solution of (5.1) is called a *forced oscillation*.

In [2] we proved that the equation (5.1) admits at least one forced oscillation, provided that the constraint M is compact with nonzero Euler–Poincaré characteristic and assuming the stronger hypothesis of the continuity of the functional field F on $\mathbb{R} \times C((-\infty, 0], M)$ instead on $\mathbb{R} \times BU((-\infty, 0], M)$. We believe that the same result is true with the sole assumption that F is continuous on $\mathbb{R} \times BU((-\infty, 0], M)$ – recall that $BU((-\infty, 0], M)$ has a finer topology than $C((-\infty, 0], M)$. Unfortunately the proof in [2] does not fit in this context. However, in the attempt of proving the existence of a forced oscillation of (5.1), one could still define a Poincaré-type operator, acting on a suitable topological space, and with the property that its fixed points correspond to the T-periodic solutions of the equation, and then prove the existence of a fixed point of such an operator by means of topological methods. Let us sketch now this argument.

A crucial step to define a Poincaré-type operator is to write (5.1) as a confined first order RFDE, and then use the confined global existence theorem (Theorem 3.9) together with the uniqueness theorem (Theorem 2.5) proved above. The possibility of reducing (5.1) to a first order RFDE is motivated by the fact that any second order differential equation on M is equivalent to a first order system on the tangent bundle TM of M. To be more precise, observe that the equation (5.1) can be equivalently written as

$$(5.2) \qquad x''(t) = R(x(t), x'(t)) + F(t, x_t) - \varepsilon x'(t),$$

where $R \colon TM \to \mathbb{R}^s$ is a smooth map – the so-called reactive force (or inertial reaction) – with the following properties:

(a) $R(q, v) \in (T_q M)^\perp$ for any $(q, v) \in TM$;
(b) R is quadratic in the second variable;
(c) given $(q, v) \in TM$, $R(q, v)$ is the unique vector such that $(v, R(q, v))$ belongs to $T_{(q,v)}(TM)$;
(d) any C^2 curve $\gamma \colon (a, b) \to M$ verifies the condition $\gamma''_\nu(t) = R(\gamma(t), \gamma'(t))$ for any $t \in (a, b)$, i.e. for each $t \in (a, b)$, the normal component $\gamma''_\nu(t)$ of $\gamma''(t)$ at $\gamma(t)$ equals $R(\gamma(t), \gamma'(t))$.

Now, the second order equation (5.2) can be transformed in the first order system

$$(5.3) \qquad \begin{cases} x'(t) = y(t), \\ y'(t) = R(x(t), y(t)) + F(t, x_t) - \varepsilon y(t) \end{cases}$$

and is equivalent to (5.3) in the following sense: a function $x \colon J \to M$ is a solution of (5.2) if and only if the pair (x, x') is a solution of (5.3). Observe that system (5.3) is actually a first order RFDE on the noncompact manifold TM, since it can be written as

$$(x'(t), y'(t)) = G(t, (x_t, y_t)),$$

where the map $G \colon \mathbb{R} \times BU((-\infty, 0], TM) \to \mathbb{R}^s \times \mathbb{R}^s$ is the functional field over TM given by

$$G(t, (\varphi, \psi)) = (\psi(0), R(\varphi(0), \psi(0)) + F(t, \varphi) - \varepsilon \psi(0)).$$

Given $c > 0$, consider the closed subset

$$X_c = \{(q, v) \in TM : \|v\| \le c\}$$

of TM. It is not difficult to show that X_c is a ∂-manifold in $\mathbb{R}^s \times \mathbb{R}^s$ with boundary

$$\partial X_c = \{(q, v) \in X_c : \|v\| = c\}.$$

The choice of such a manifold is suggested by *a priori* estimates on the set of all the possible T-periodic solutions of the equation (5.1). These estimates are made possible by the compactness of M and the presence of the positive frictional coefficient ε.

As in [2], one can show that if c is sufficiently large, then G is strictly inward to X_c and, thus, away from TM in X_c (observe that the tangent cone of X_c at $(q, v) \in \partial X_c$ is the half subspace of $T_{(q,v)} X_c$ given by

$$C_{(q,v)} X_c = \big\{ (\dot{q}, \dot{v}) \in T_{(q,v)}(TM) : \langle v, \dot{v} \rangle \leq 0 \big\},$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^s$). Consequently, we are reduced to the context of the confined problem (3.2) with $\mathbb{R}^k = \mathbb{R}^s \times \mathbb{R}^s$, $N = TM$, $g = G$ and the confining set X given by the ∂-manifold X_c.

Now, given $(\varphi, \psi) \in BU((-\infty, 0], X_c)$, consider the following initial value problem depending on a real parameter λ:

(5.4)
$$\begin{cases} (x'(t), y'(t)) = \lambda G(t, (x_t, y_t)), \\ (x_0, y_0) = (\varphi, \psi). \end{cases}$$

Let $\lambda \geq 0$ be given. Theorems 2.5 and 3.9 yield a unique global X_c-valued solution (x, y) of (5.4). Define the Poincaré-type operator $\mathcal{P}_\lambda \colon BU((-\infty, 0], X_c) \to BU((-\infty, 0], X_c)$ by

$$\mathcal{P}_\lambda(\varphi, \psi)(s) = \big(x(s + T), y(s + T) \big), \qquad s \in (-\infty, 0].$$

Our conjecture is that the assumption $\varepsilon > 0$ ensures the existence of an unbounded connected subset of

$$\big\{ (\lambda, (\varphi, \psi)) \in [0, +\infty) \times BU((-\infty, 0], X_c) : \mathcal{P}_\lambda(\varphi, \psi) = (\varphi, \psi) \big\}$$

emanating from the slice $\{0\} \times BU((-\infty, 0], X_c)$. This would imply the existence of a fixed point of $\mathcal{P}_1$, due to the boundedness of $BU((-\infty, 0], X_c)$. As a consequence one would get the existence of a forced oscillation of the motion equation (5.1) previously considered.

References

[1] P. Benevieri, A. Calamai, M. Furi, and M.P. Pera, *Retarded functional differential equations on manifolds and applications to motion problems for forced constrained systems*, Adv. Nonlinear Stud. **9** (2009), 199–214.

[2] P. Benevieri, A. Calamai, M. Furi, and M.P. Pera, *A continuation result for forced oscillations of constrained motion problems with infinite delay*, to appear in Adv. Nonlinear Stud.

[3] P. Benevieri, A. Calamai, M. Furi, and M.P. Pera, *On the existence of forced oscillations for the spherical pendulum acted on by a retarded periodic force*, to appear in Journal of Dynamics and Differential Equations.

[4] G. Bouligand, *Introduction à la géométrie infinitésimale directe*, Gauthier-Villard, Paris, 1932.

[5] N. Dunford and J.T. Schwartz, *Linear Operators*, Wiley & Sons, Inc., New York, 1957.

[6] R. Gaines and J. Mawhin, *Coincidence degree and nonlinear differential equations*, Lecture Notes in Math., **568**, Springer Verlag, Berlin, 1977.

[7] V. Guillemin and A. Pollack, *Differential Topology*, Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1974.

[8] J.K. Hale and J. Kato, *Phase Space for Retarded Equations with Infinite Delay*, Funkc. Ekvac. **21** (1978), 11–41.

[9] J.K. Hale and S.M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Springer Verlag, New York, 1993.

[10] Y. Hino, S. Murakami and T. Naito, *Functional-differential equations with infinite delay*, Lecture Notes in Math., **1473**, Springer Verlag, Berlin, 1991.

[11] M.W. Hirsch, *Differential Topology*, Graduate Texts in Math., Vol. 33, Springer Verlag, Berlin, 1976.

[12] J. Mallet-Paret, R.D. Nussbaum and P. Paraskevopoulos, *Periodic solutions for functional-differential equations with multiple state-dependent time lags*, Topol. Methods Nonlinear Anal. **3** (1994), 101–162.

[13] J.M. Milnor, *Topology from the differentiable viewpoint*, Univ. Press of Virginia, Charlottesville, 1965.

[14] J.R. Munkres, *Elementary Differential Topology*, Princeton University Press, Princeton, New Jersey, 1966.

[15] R.D. Nussbaum, *Periodic solutions of some nonlinear autonomous functional differential equations*, Ann. Mat. Pura Appl. **101** (1974), 263–306.

[16] R.D. Nussbaum, *The fixed point index and fixed point theorems*, Topological methods for ordinary differential equations (Montecatini Terme, 1991), 143–205, Lecture Notes in Math., **1537**, Springer Verlag, Berlin, 1993.

[17] W.M. Oliva, *Functional differential equations on compact manifolds and an approximation theorem*, J. Differential Equations **5** (1969), 483–496.

[18] W.M. Oliva, Functional differential equations–generic theory. Dynamical systems (Proc. Internat. Sympos., Brown Univ., Providence, R.I., 1974), Vol. I, pp. 195–209. Academic Press, New York, 1976.

Pierluigi Benevieri, Massimo Furi, and Maria Patrizia Pera,
Dipartimento di Sistemi e Informatica
Università degli Studi di Firenze
Via S. Marta 3
I-50139 Firenze, Italy

Alessandro Calamai
Dipartimento di Scienze Matematiche
Università Politecnica delle Marche
Via Brecce Bianche
I-60131 Ancona, Italy.

e-mail addresses:
pierluigi.benevieri@unifi.it
calamai@dipmat.univpm.it
massimo.furi@unifi.it
mpatrizia.pera@unifi.it